The Relationship between the Equivalence Scale and the Inequality Index and Its Impact on the Measurement of Income Inequality

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THE RELATIONSHIP BETWEEN THE EQUIVALENCE SCALE AND THE INEQUALITY INDEX AND ITS IMPACT ON THE MEASUREMENT OF INCOME INEQUALITY

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The paper discusses the ∪-shaped relationship between the equivalence scale n^{ε} and the Gini index instead of considering the equivalence scale's relationship to the generalised entropy measures, which was studied by Coulter, *et al*. (1992). An end-point condition is given for the ∪-shaped relationship, which corresponds to a condition for that of the generalised entropy measures. Additionally, using a mixture of log-normal distributions approach, five factors are shown to be required for a convex relationship between size elasticity ε and the Gini index. Empirically, income distributions satisfy those factors. Thus, the end-point condition essentially determines the shape of the relationship.

1. Objective

Coulter *et al*. (1992) studied how index values such as the generalised entropy inequality measures and the Foster-Greer-Thorbecke (FGT) poverty indices relate to parameter ε of an equivalence scale specification of the form $v(n, \varepsilon)$, where n denotes the number of household members, and the function v monotonically increases with n and a scalar parameter ε. They derived an approximate condition for the relationship (called the '*e-i* curve' hereafter) to be ∪-shaped and asserted that the approximate condition is usually satisfied. However, their empirical study only used data from the UK Family Expenditure Survey, and they were unable to analyse the relationship to the Gini index, the most popular inequality index, using the method they employed. They supposed, however, that there is a ∪-shaped relationship between ε and the Gini index as long as changes in the rankings induced by changes in ε are relatively small.

They also discussed the impact of a ∪-shaped *e-i* curve on measurement practices. The minimum of the ∪-shaped *e-i* curve is reached at approximately $\varepsilon = 0.6$ in the UK when using $v(n, \varepsilon) = n^{\varepsilon}$, a scale class proposed by Buhman *et al*. (1988). As the official scale derived by the McClements method corresponds to $\varepsilon \fallingdotseq 0.6$, Coulter *et al*. suggested that the official scale provides lower estimates of the extents of inequality and poverty than do other scales. Banks and Johnson (1994) argued that the results of Coulter *et al*. are dependent on particular years and a particular equivalence scale specification and are not robust to other choices of years and equivalence scale specifications.[1]

This study has three objectives. The first is to derive the conditions for a ∪-shaped Gini index *e-i* curve when using $v(n, \varepsilon) = n^{\varepsilon}$. The next is to observe the actual *e-i* curves of major inequality indices including the generalised entropy measures for many countries to explore the generality of the ∪-shaped relationship. The last

[1] Jenkins and Cowell (1994) refuted the argument of Banks and Johnson. Nevertheless, the argument does not appear to be completely off the point, regardless of whether Banks and Johnson properly comprehended the objective of the empirical illustration provided by Coulter *et al*. A detailed explanation is given in Section 4.

objective is to provide an illustrative example in which the ∪-shaped relationship has significant impacts on the measurement of income inequality for a significant period, and similar effects are observed even if a more general specification of the equivalence scales is used.

The subsequent sections are organised as follows. In Section 2, regarding the equivalent scale specification $\mathrm{v}(n,\varepsilon)=n^{\varepsilon}$, an end-point condition for the ∪-shaped relationship of parameter ε to the Gini index is presented instead of those of the generalised entropy measures for which Coulter et al. (1992) derived an approximate condition. It is also shown that, unlike those of the generalised entropy measures, the Gini index *e-i* curve may diverge from a ∪-shape depending on the income distributions within groups of equal household size, even if the end-point condition is satisfied. To address this issue, five factors necessary for the convexity of the *e-i* curve are specified by using a set of log-normal distributions in which each within-group income distribution is approximated with an appropriate log-normal distribution. In Section 3, using recent disposable income data from 34 countries contained in the Luxembourg Income Study (LIS) database, the *e-i* curves of the Gini index are shown to be convex in all 34 countries, meaning that practically the end-point condition determines whether the curve is ∪-shaped. The condition is not satisfied or comes close to not being satisfied for several low-income countries. The *e-i* curves of the Theil index (Theil) and the Squared Coefficient of Variation (SCV) are also investigated and are found to differ from a ∪-shape for several low-income countries, although the Coulter *et al.*'s (1992) approximate condition hold. Regarding SCV in particular, these negative cases also emerge for some high-income countries, leading to a rejection of the generality of the ∪-shaped relationship even for high-income countries.[2] Section 4 is devoted to an example that shows the impact of the ∪-shaped *e-i* curve on the measurement of income inequality in Japan, where equivalent scales have changed substantially for the last two decades. The scale of Buhman *et al.* with a fixed parameter $\varepsilon=0.5$, which the OECD uses for international comparisons, is shown to significantly underestimate the recent rise in income inequality among households with two or more persons compared to the current equivalence scales derived from several procedures because of the ∪-shaped Gini index *e-i* curve. The significance of this result is made more robust because a similar tendency is observed when using a more general specification of the equivalence scales. Section 5 concludes the discussion and provides final remarks.

2. Conditions for a ∪-Shaped Relationship between Size elasticity and the Gini Index

2.1. *End-Point Condition and Counter-Examples*

Let Γ_n denote a group consisting of all *n*-person households. Its population share, average household income, and cumulative distribution function for household income relative to the within-group average are denoted as p_n, y_n, and F_n, respectively. In this paper, the equivalised income of each household member in a *n*-person household that earns an amount y is expressed as y/n^{ε}, where $0\le\varepsilon\le1$. Parameter ε is called 'size elasticity' hereafter. The overall average $\mu^{(\varepsilon)}$ and overall cumulative distribution function $F^{(\varepsilon)}$ of the equivalised incomes are expressed as follows:

$$\mu^{(\varepsilon)}=\sum_n p_n\frac{y_n}{n^{\varepsilon}},\qquad F^{(\varepsilon)}(x)=\sum_n p_nF_n\left(\frac{n^{\varepsilon}}{y_n}x\right).$$

[2] Note that Coulter *et al.* (1992) did not explicitly discuss the applicability of their approximate condition and the generality of the ∪-shaped relationship for countries other than the UK.

The Gini index $G^{(\varepsilon)}$ of the overall equivalised income distribution is expressed as follows:

$$G^{(\varepsilon)} = \frac{1}{2\mu^{(\varepsilon)}} \iint |x-y| dF^{(\varepsilon)}(x) dF^{(\varepsilon)}(y) = \frac{1}{\mu^{(\varepsilon)}} \int F^{(\varepsilon)}(x)\big(1-F^{(\varepsilon)}(x)\big)dx.$$

For simplicity, F_n is assumed to be continuously differentiable with the density function F_n' for any group Γ_n, hereafter. The derivative of $G^{(\varepsilon)}$ with respect to size elasticity ε is expressed as follows:

$$\frac{\partial G^{(\varepsilon)}}{\partial \varepsilon} = \sum_n s_n^{(\varepsilon)}\big(\log n - \overline{\log n}\big) \int \left(1 - 2F^{(\varepsilon)}\left(\frac{y_n}{n^\varepsilon} z\right)\right) z F_n'(z) dz = \underset{n}{\mathrm{COV}}\big(\log n, D_n^{(\varepsilon)}\big), \tag{1}$$

where $s_n^{(\varepsilon)} \coloneqq p_n \frac{y_n/n^\varepsilon}{\mu^{(\varepsilon)}}$: the share of Γ_n in terms of equivalised income, $\overline{\log n} \coloneqq \sum_n s_n^{(\varepsilon)} \log n$: the average of $\log n$ with weight $s_n^{(\varepsilon)}$ (the $s_n^{(\varepsilon)}$-weighted average of the variable *X* is denoted $\bar{X}$ hereafter), and $D_n^{(\varepsilon)} \coloneqq \int \left(1 - 2F^{(\varepsilon)}\left(\frac{y_n}{n^\varepsilon} z\right)\right) z F_n'(z) dz$. Notation $\mathrm{COV}_n(\cdot,\cdot)$ on the right-hand side of the equation expresses the covariance with weight $s_n^{(\varepsilon)}$. Using the overlap index (Yitzhaki and Lerman, 1991) of the equivalised income distribution within Γ_n over the overall equivalised income distribution $O_n^{(\varepsilon)} \coloneqq 2\int \big(F^{(\varepsilon)}(x) - 1/2\big) x dF_n\left(\frac{n^\varepsilon}{y_n}x\right) / 2\int \left(F_n\left(\frac{n^\varepsilon}{y_n}x\right) - 1/2\right) x dF_n\left(\frac{n^\varepsilon}{y_n}x\right)$ and the Gini index of the within-group (equivalised) income distribution $G_n = 2\int \left(F_n\left(\frac{n^\varepsilon}{y_n}x\right) - \frac{1}{2}\right) x dF_n\left(\frac{n^\varepsilon}{y_n}x\right) / \frac{y_n}{n^\varepsilon} = 2\int \left(F_n(x) - \frac{1}{2}\right) x dF_n(x)$, the derivative of $G^{(\varepsilon)}$ in (1) can be expressed as follows:

$$\frac{\partial G^{(\varepsilon)}}{\partial \varepsilon} = -\underset{n}{\mathrm{COV}}\big(\log n, O_n^{(\varepsilon)} G_n\big)$$

because of the equality $D_n^{(\varepsilon)} = -O_n^{(\varepsilon)} G_n$.

When F_ns are identical to any other (the Identical Income Distributions condition, the IID), if $y_1 < \frac{y_2}{2^\varepsilon} < \cdots < \frac{y_n}{n^\varepsilon} < \cdots$, then the inequalities $D_1 > D_2 > \cdots > D_n > \cdots$ and $\partial G^{(\varepsilon)}/\partial\varepsilon = \mathrm{COV}_n\big(\log n, D_n^{(\varepsilon)}\big) < 0$ hold due to the increasing-monotonicity of $F^{(\varepsilon)}$. In addition to the IID condition, if y_n is proportional to n^{ε_0} for some ε_0, where $0 < \varepsilon_0 < 1$, that is, $y_n \propto n^{\varepsilon_0}$ or $\log y_n = a_0 + \varepsilon_0 \log n$ for some a_0 (the Log-Linearity condition, the LL), the inequality $\partial G^{(\varepsilon)}/\partial\varepsilon < 0$ holds if $\varepsilon < \varepsilon_0$, and $\partial G^{(\varepsilon)}/\partial\varepsilon > 0$ holds if $\varepsilon > \varepsilon_0$. Thus, the *e-i* curve of $G^{(\varepsilon)}$ is ∪-shaped with the minimum at ε_0. In particular, at the end points $\varepsilon = 0$, 1, the following inequalities are satisfied:

$$\underset{n}{\mathrm{COV}}\big(\log n, D_n^{(0)}\big) < 0, \qquad \underset{n}{\mathrm{COV}}\big(\log n, D_n^{(1)}\big) > 0. \tag{2}$$

The ∪-shaped *e-i* relationship under the IID and LL conditions is made intuitive by the following subgroup decomposition of the Gini index (Okamoto, 2009):

$$G^{(\varepsilon)} = \sum_n s_n^{(\varepsilon)} G_n + \frac{1}{\mu^{(\varepsilon)}} \sum_{n<m} p_n p_m \int \left(F_n\left(\frac{n^\varepsilon}{y_n}x\right) - F_m\left(\frac{m^\varepsilon}{y_m}x\right)\right)^2 dx$$

$$= G_1 + \frac{1}{\mu^{(\varepsilon)}} \sum_{n<m} s_n^{(\varepsilon)} p_m \int \left(F_1(z) - F_1\left(\left(\frac{m}{n}\right)^{\varepsilon-\varepsilon_0} z\right)\right)^2 dz.$$

In the above decomposition, the first term, which represents within-group inequality, is independent of ε, and the second term, which represents between-group inequality, is equal to zero if $\varepsilon = \varepsilon_0$ and positive otherwise.

The further ε is from ε_0, the larger the integrand in the second term is. Although the dependencies of $s_n^{(\varepsilon)}$ and $\mu^{(\varepsilon)}$on ε need to be taken into account to strictly prove the ∪-shaped relationship, the above decomposition is expected to be helpful for an intuitive understanding of the ∪-shaped relationship.

In general, if the *e-i* curve is ∪-shaped, then the end-point condition (2) holds. The condition corresponds to the following approximate condition for the ∪-shaped *e-i* curve of the generalised entropy measures and the FGT poverty measures derived by Coulter *et al*. (1992):

$$\mathrm{COV}(\log n\,, y_n) > 0, \mathrm{COV}(\log n\,, y_n/n) < 0. \tag{2}'$$

The covariance in (2)′ is calculated using the population weight p_n. Condition (2) and (2)′ are similar to each other and are generally considered to agree but are not equivalent. Illustrative examples for this inconsistency are presented below, and an empirical example is given in Section 3.

In the case of the Gini index, the *e-i* curve may be non-∪-shaped when the dispersions of the within-group income distributions are very small, when the within-group income distributions differ from each other substantially, when pairs of $\log n$ and $\log y_n$ deviate substantially from a linear-relationship, and when the range of household sizes is very wide, as shown in the following examples.

EXAMPLE 1. *Suppose the universe consists of one-person, two-person and four-person households. Group* Γ_1, Γ_2, and Γ_4 *have population shares of* 0.1, 0.8, *and* 0.1, *and incomes of* 1, $2^{0.99} \fallingdotseq 1.986$, and $4^{0.5}$=2 on average, *respectively. In addition, the within-group income distributions follow log-normal distributions with* $\sigma = 0.01$. *Thus, the IID condition is satisfied, but the LL condition is not satisfied. In this case, the Gini index e-i curve is non-*∪*-shaped even though the end-point condition holds, as shown in the upper-left panel of Figure 1.*

EXAMPLE 2. *However, if the within-group dispersion is made larger such as* $\sigma = 0.09$, *then the e-i curve becomes* ∪*-shaped, as shown in the upper-right panel.*

EXAMPLE 3. *Even if the dispersion parameter* σ *remains at* 0.01, *by making the pairs of* $\log n$ *and* $\log y_n$ *closer to having a linear-relationship such that the average income of* Γ_2 *is changed from* $2^{0.99}$ *to* $2^{0.55} \fallingdotseq 1.464$, *the e-i curve becomes* ∪*-shaped, as shown in the middle-left panel.*

EXAMPLE 4. *In cases where the IID condition is not satisfied, the e-i curve may be non-*∪*-shaped, even if the average within-group dispersion is not small. For example, let the dispersion parameters for* Γ_1, Γ_2, Γ_4 *be* $\sigma_1 = 0.01$, $\sigma_2 = \sigma_4 = 0.089/0.9 \fallingdotseq 0.099$, *respectively, in Example 1; then, the e-i curve has two local minima at* $\varepsilon \fallingdotseq 0.5$ *and* 0.77, *as shown in the middle-right panel, although the average dispersion (with population weights) is* 0.09, *and the end-point condition is satisfied.*

EXAMPLE 5. *The lower-left panel shows the e-i curve after the average incomes of the three groups in Example 2 are changed to* 1, $20^{0.99} \fallingdotseq 19.410$, *and* $40^{0.806} \fallingdotseq 19.555$, *respectively. The minimum point of the e-i curve approaches zero, but the curve is still* ∪*-shaped.*

EXAMPLE 6. *A wider range of household sizes may also cause singularity. In Example* 5, *if the household sizes of the two non-single household groups are changed from* 2, 4 *to* $2^5 = 32$, $4^5 = 1024$, *respectively, then, as shown in the lower-right panel, the e-i curve becomes non-*∪*-shaped, although the end-point condition holds.*

EXAMPLE 7. *If the within-group income averages are replaced by those in Example 6 divided by the square roots of the household sizes, then the e-i curve becomes* ∪*-shaped (this chart is omitted). This example indicates the* *'slope' of the relationship between* $\log n$ *and* $\log y_n$ *(e.g., the slope when* $\log y_n$ *is regressed on* $\log n$*) may affect the shape of the e-i curve.*

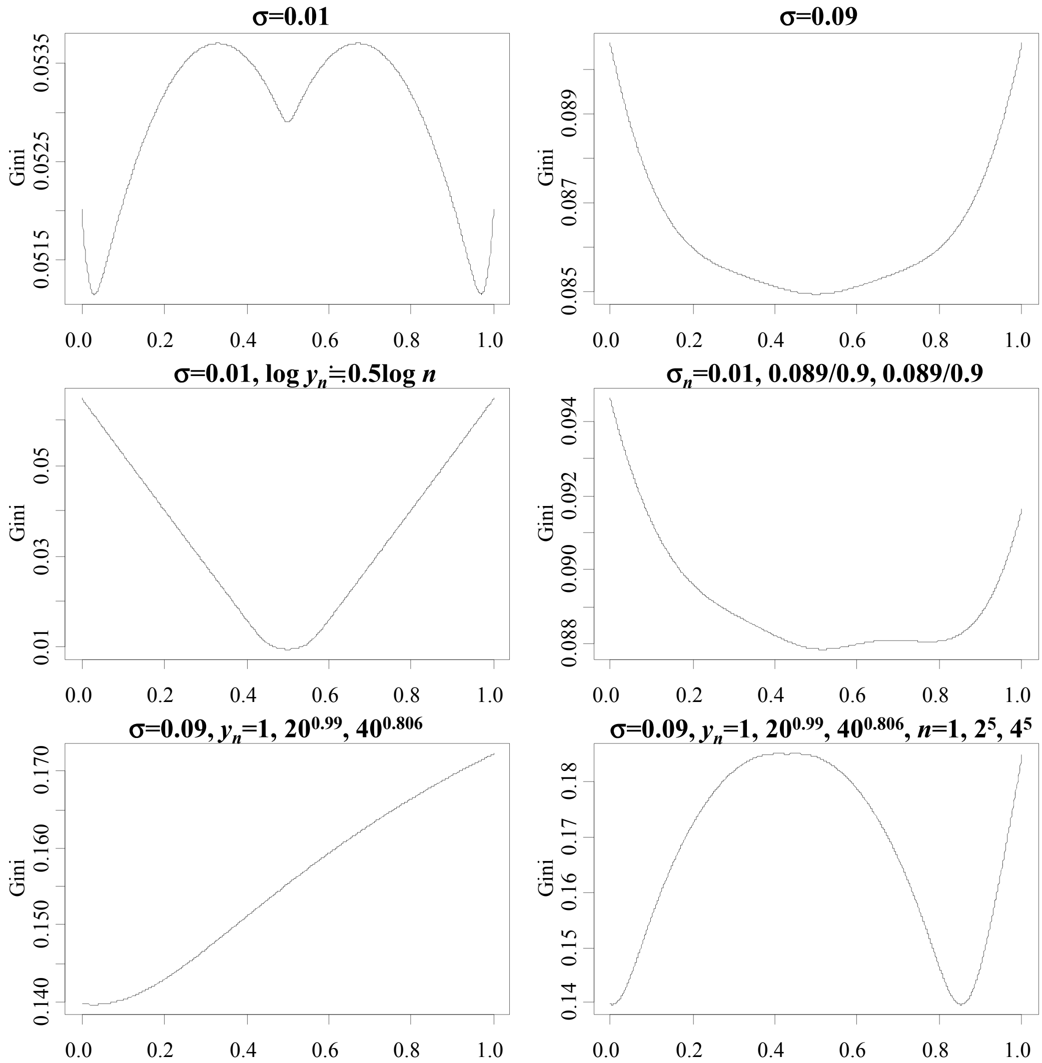

Fig. 1. *Examples e-i curves of the Gini Index*

In a sense, the actual income distributions are sufficiently close to the IID and LL conditions, with a slight relaxation of the LL condition to allow for the cases $\varepsilon_0 \leq 0$ and $\varepsilon_0 \geq 1$, as shown in the subsequent sections.

2.2. *Conditions for a Convex Relationship between Size elasticity and Gini Index*

2.2.1. *A mixture of log-normal distributions approach*

In the above-mentioned examples, the *e-i* curves of the Mean Logarithmic Deviation (MLD), Theil, and SCV would be ∪-shaped in Examples 1 – 4, whereas the MLD curves would be non-∪-shaped in Examples 5 and 7, and the Theil and SCV curves would be non-∪-shaped in Example 6. As condition (2)′ does not hold in Examples 5 and 7, condition (2)′ is only consistent with a ∪-shaped *e-i* curve in the case of MLD. With respect to the Gini index, the *e-i* curve is necessary to test whether there is a singularity due to the within-group income distributions, as shown in Examples 1 – 4. However, as it is difficult to clarify and test additional factors for a ∪-shaped *e-i* curve analytically without specifying the shape of the income distribution, let the within-group income distributions be approximated by log-normal distributions with the same averages and dispersions. By this approximation, the overall income distribution is replaced by the mixture of log-normal distributions (MLN). Empirically, the MLN approach yields sufficiently accurate approximations, as shown in Section 3.

When the relative income within group Γ_n follows $LN(-\sigma_n^2/2, \sigma_n^2)$, its CDF is $F_n(x) = \Phi\left(\frac{\log x - \sigma_n^2/2}{\sigma_n}\right)$, and the derivative of the Gini index *e-i* curve is expressed as follows:

$$\frac{\partial G^{(\varepsilon)}}{\partial \varepsilon} = -2\sum_n s_n^{(\varepsilon)}\left(\log n - \overline{\log n}\right)\sum_m p_m \int F_m\left(a_{mn}^{(\varepsilon)} z\right) z F_n'(z) dz = -2\sum_{n,m} s_n^{(\varepsilon)} p_m \left(\log n - \overline{\log n}\right)\Phi_{mn}^{(\varepsilon)}, \quad (3)$$

where $a_{mn}^{(\varepsilon)} := \frac{m^\varepsilon}{y_m}\frac{y_n}{n^\varepsilon}$, and $\Phi_{mn}^{(\varepsilon)} := \Phi\left(\frac{\log(y_n/y_m) + \varepsilon\log(m/n) + \sigma_{mn}^2}{\sqrt{2}\sigma_{mn}}\right)$, ($\Phi(\cdot)$ denotes the CDF of the standard normal distribution, and $\sigma_{mn}^2 := (\sigma_m^2 + \sigma_n^2)/2$). The second-order derivative of the *e-i* curve is expressed as follows:

$$\begin{aligned}
\frac{\partial^2 G^{(\varepsilon)}}{\partial \varepsilon^2} &= 2\sum_{n,m} s_n^{(\varepsilon)} p_m \left[\left(\log n - \overline{\log n}\right)^2 - \overline{\left(\log n - \overline{\log n}\right)^2}\right]\Phi_{mn} \\
&\quad - 2\sum_{n,m} s_n^{(\varepsilon)} p_m \left(\log n - \overline{\log n}\right)\phi_{mn}\frac{\log(m/n)}{\sqrt{2}\sigma_{mn}} \\
&= 2\sum_{n,m} s_n^{(\varepsilon)} p_m \left[\left(\log n - \overline{\log n}\right)^2 - \overline{\left(\log n - \overline{\log n}\right)^2}\right]\left(\Phi_{mn} - \tilde{\Phi}\right) \\
&\quad + \frac{\sqrt{2}}{\bar{\bar{\sigma}}}\sum_{n,m} s_n^{(\varepsilon)} p_m \log\frac{n}{m}\left(\log n - \overline{\log n}\right)\left(\phi_{mn} - \tilde{\phi}\right) \\
&\quad + \sqrt{2}\sum_{n,m} s_n^{(\varepsilon)} p_m \log\frac{n}{m}\left(\log n - \overline{\log n}\right)\left(\phi_{mn} - \tilde{\phi}\right)\left(\frac{1}{\sigma_{mn}} - \frac{1}{\bar{\bar{\sigma}}}\right) \\
&\quad + \sqrt{2}\tilde{\phi}\sum_{n,m} s_n^{(\varepsilon)} p_m \log\frac{n}{m}\left(\log n - \overline{\log n}\right)\left(\frac{1}{\sigma_{mn}} - \frac{1}{\bar{\bar{\sigma}}}\right) + \frac{\sqrt{2}}{\bar{\bar{\sigma}}}\tilde{\phi}\sum_n s_n^{(\varepsilon)}\left(\log n - \overline{\log n}\right)^2,
\end{aligned} \quad (4)$$

where $\phi_{mn} := \phi\left(\frac{\log(y_n/y_m) + \varepsilon\log(m/n) + \sigma_{mn}^2}{\sqrt{2}\sigma_{mn}}\right)$ ($\phi(\cdot)$ denotes the density function of the standard normal distribution), $\tilde{\Phi} := \sum_{n,m} s_n^{(\varepsilon)} p_m \Phi_{mn}$, $\tilde{\phi} := \sum_{n,m} s_n^{(\varepsilon)} p_m \phi_{mn}$, and $\bar{\bar{\sigma}} := \frac{1}{\sum_{n,m} s_n^{(\varepsilon)} p_m/\sigma_{mn}}$. Using the following notation for covariance and variance $\mathrm{COV}_{m,n}(X_{mn}, Y_{mn}) = \sum_{n,m} s_n^{(\varepsilon)} p_m \left(X_{mn} - \widetilde{X_{mn}}\right)\left(Y_{mn} - \widetilde{Y_{mn}}\right)$, and $\mathrm{VAR}_n(X_n) = \sum_n s_n^{(\varepsilon)}(X_n - \bar{X})^2 = \overline{(X_n - \bar{X})^2}$ ($\widetilde{X_{mn}} := \sum_{n,m} s_n^{(\varepsilon)} p_m X_{mn}$, and $\bar{X} = \sum_n s_n^{(\varepsilon)} X_n$), the second-order derivative in (4) is expressed as follows:

$$\frac{\partial^2 G^{(\varepsilon)}}{\partial \varepsilon^2} = 2 \underset{m,n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \Phi_{mn}\right) + \frac{\sqrt{2}}{\bar{\bar{\sigma}}} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right), \phi_{mn}\right) \tag{4$'$}$$

$$+ \sqrt{2} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right)\left(\frac{1}{\sigma_{mn}} - \frac{1}{\bar{\bar{\sigma}}}\right), \phi_{mn}\right)$$

$$+ \sqrt{2}\tilde{\phi} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right), \frac{1}{\sigma_{mn}}\right) + \frac{\sqrt{2}}{\bar{\bar{\sigma}}} \tilde{\phi} \underset{n}{\mathrm{VAR}}(\log n).$$

If $\partial^2 G^{(\varepsilon)}/\partial \varepsilon^2 > 0$ for $0 \le \forall \varepsilon \le 1$, then the *e-i* curve is convex at any possible size elasticity. Under convexity, the end-point condition (2) is a necessary and sufficient condition for a ∪-shaped *e-i* curve.

The first, second, and third terms in formula (4)′ are affected by deviations from the LL condition via fluctuation among Φ_{mn}, ϕ_{mn} and their interrelations with $\log n$. The first trough fourth terms are affected by deviations from the IID condition via fluctuation among σ_{mn}, Φ_{mn}, and ϕ_{mn} and their interrelations with $\log n$. Only the fifth term is independent of those fluctuations. As formula (4)′ does not allow the contributions of individual factors to be distinguished, it will next be approximated to derive factor decomposition.

2.2.2. *Type I approximation*

By applying the linear approximations $\Phi_{mn} \fallingdotseq \tilde{\Phi} + \phi\left(\Phi^{-1}(\tilde{\Phi})\right)(q_{mn} - \widetilde{q_{mn}})$, and $\phi_{mn} \fallingdotseq \tilde{\phi}\left[1 - \left(q_{mn}^2 - \widetilde{q_{mn}^2}\right)/2\right]$, where $q_{mn} \coloneqq \frac{\log(y_n/y_m) + \varepsilon \log(m/n) + \sigma_{mn}^2}{\sqrt{2}\sigma_{mn}}$, $\widetilde{q_{mn}} \coloneqq \sum_{n,m} s_n^{(\varepsilon)} p_m q_{mn}$, and $\widetilde{q_{mn}^2} \coloneqq \sum_{n,m} s_n^{(\varepsilon)} p_m q_{mn}^2$, an approximation (called the type I approximation hereafter) of formula (4)′ is derived as follows:

$$\frac{\partial^2 G^{(\varepsilon)}}{\partial \varepsilon^2} \fallingdotseq \sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right) \underset{m,n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, q_{mn}\right) - \frac{1}{\sqrt{2}\bar{\bar{\sigma}}} \tilde{\phi} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right), q_{mn}^2\right) \tag{5}$$

$$- \frac{1}{\sqrt{2}} \tilde{\phi} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right)\left(\frac{1}{\sigma_{mn}} - \frac{1}{\bar{\bar{\sigma}}}\right), q_{mn}^2\right)$$

$$+ \sqrt{2}\tilde{\phi} \underset{m,n}{\mathrm{COV}}\left(\log \frac{n}{m}\left(\log n - \overline{\log n}\right), \frac{1}{\sigma_{mn}}\right) + \frac{\sqrt{2}}{\bar{\bar{\sigma}}} \tilde{\phi} \underset{n}{\mathrm{VAR}}(\log n) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where I_1 through I_5 on the right-hand side denote the first through fifth terms of the type I approximation, respectively. Similar notations are used for other approximations. The range of q_{mn} must be sufficiently narrow for approximation (5) to be sufficiently accurate. As mentioned in Section 3, in the lowest-income-inequality countries, such as some northern European countries, σ_{mn} in the denominator of q_{mn} is so small that the accuracy of approximation (5) is substantially reduced. Nevertheless, approximation (5) remains consistent with formula (4) in terms of sign, that is, the approximation is positive when $\partial^2 G^{(\varepsilon)}/\partial \varepsilon^2 > 0$ for the recent income distributions in all of the countries studied. Thus, the approximation is valid for the verification of convexity.

The first term I_1 is further decomposed as follows:

$$I_1 = \sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\left[\frac{\varepsilon-\widehat{\varepsilon_0}}{\overline{\overline{\sigma}}}\underset{m,n}{\mathrm{COV}}\left(\left(\log n-\overline{\log n}\right)^2,\log\left(\frac{m}{n}\right)\right)\right. \tag{6}$$

$$+(\varepsilon-\widehat{\varepsilon_0})\underset{m,n}{\mathrm{COV}}\left(\left(\log n-\overline{\log n}\right)^2,\log\left(\frac{m}{n}\right)\left(\frac{1}{\sigma_{mn}}-\frac{1}{\overline{\overline{\sigma}}}\right)\right)$$

$$\left.+\underset{m,n}{\mathrm{COV}}\left(\left(\log n-\overline{\log n}\right)^2,\frac{\widehat{\psi_n}-\widehat{\psi_m}}{\sigma_{mn}}\right)+\underset{m,n}{\mathrm{COV}}\left(\left(\log n-\overline{\log n}\right)^2,\sigma_{mn}\right)\right]$$

$$= I_{11}+I_{12}+I_{13}+I_{14},$$

where $\widehat{\varepsilon_0}$ and $\widehat{\psi_n}$ denote the estimated coefficient of the covariate and the residual when a log-linear regression model $\log y_n \sim \widehat{\alpha_0}+\widehat{\varepsilon_0}\log n$ is applied with $s_n^{(\varepsilon)}$-weights. Note that $\widehat{\varepsilon_0}$ and $\widehat{\psi_n}$ depend on the size elasticity ε because of the $s_n^{(\varepsilon)}$-weighted regression. Estimate $\widehat{\varepsilon_0}$ is usually close to ε_0, the point at which the minimum of the *e-i* curve is located when the curve is ∪-shaped.

The second term I_2 is further decomposed as follows:

$$I_2 = -\frac{1}{\sqrt{2}\overline{\overline{\sigma}}}\tilde{\phi}\left[\frac{(\varepsilon-\widehat{\varepsilon_0})^2}{2\overline{\overline{\sigma^2}}}\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\log\left(\frac{m}{n}\right)^2\right)\right. \tag{7}$$

$$+\frac{(\varepsilon-\widehat{\varepsilon_0})^2}{2}\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\log\left(\frac{m}{n}\right)^2\left(\frac{1}{\sigma_{mn}^2}-\frac{1}{\overline{\overline{\sigma^2}}}\right)\right)$$

$$+\frac{1}{2}\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\frac{\left(\widehat{\psi_n}-\widehat{\psi_m}\right)^2}{\sigma_{mn}^2}\right)+\frac{1}{2}\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\sigma_{mn}^2\right)$$

$$+(\varepsilon-\widehat{\varepsilon_0})\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\log\left(\frac{m}{n}\right)\frac{\widehat{\psi_n}-\widehat{\psi_m}}{\sigma_{mn}^2}\right)$$

$$+(\varepsilon-\widehat{\varepsilon_0})\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\log\left(\frac{m}{n}\right)\right)$$

$$\left.+\underset{m,n}{\mathrm{COV}}\left(\log\frac{n}{m}\left(\log n-\overline{\log n}\right),\widehat{\psi_n}-\widehat{\psi_m}\right)\right] = I_{21}+I_{22}+I_{23}+I_{24}+I_{25}+I_{26}+I_{27},$$

where $\overline{\overline{\sigma^2}} := \frac{1}{\sum_{n,m} s_n^{(\varepsilon)} p_m/\sigma_{mn}^2}$.

Among the terms in type I approximations (5) – (7), I_{11}, I_{21}, I_{26} and I_5 involve the variance or higher moments of $\log n$; I_{12}, I_{14}, I_{22}, I_{24} and I_4 involve the covariance of polynomials of $\log n$ with σ_{mn}, σ_{mn}^2 or their reciprocals; and I_{13}, I_{23}, I_{25} and I_{27} involve the covariance of polynomials of $\log n$ with $\widehat{\psi_n}$ or $\widehat{\psi_n}$ relative to σ_{mn}. Let the sums of the respective terms be denoted $K_0 := I_{11}+I_{21}+I_{26}+I_5$, $K_\sigma := I_{12}+I_{14}+I_{22}+I_{24}+I_4$, and $K_\psi := I_{13}+I_{23}+I_{25}+I_{27}$. The rest term I_3 involves $\log n$, σ_{mn}, and $\widehat{\psi_n}$. As approximations (5) – (7) are expressed in double summation form, it is difficult to understand the contributions of the factors and their interactions; hence, let the type I approximation be converted to single summation form by further approximations.

2.2.3. *Type II approximation*

In the derivation of the type II approximation, $\sum_n p_n \log n$, the population-weighted average of $\log n$ is approximated by $\overline{\log n}$, an $s_n^{(\varepsilon)}$-weighted average of $\log n$, to obtain an approximation of K_0 as follows:

$$K_0 = I_{11} + I_{21} + I_{26} + I_5 \tag{8}$$
$$\fallingdotseq -\sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\frac{\varepsilon - \widehat{\varepsilon_0}}{\bar{\bar{\sigma}}}\overline{\left(\log n - \overline{\log n}\right)^3}$$
$$-\frac{1}{2\sqrt{2}}\tilde{\phi}\frac{(\varepsilon - \widehat{\varepsilon_0})^2}{\bar{\bar{\sigma}}\cdot\overline{\overline{\sigma^2}}}\left[\overline{\left(\log n - \overline{\log n}\right)^4} + \overline{\left(\log n - \overline{\log n}\right)^2}^2\right]$$
$$+\frac{1}{\sqrt{2}}\tilde{\phi}\frac{\varepsilon - \widehat{\varepsilon_0}}{\bar{\bar{\sigma}}}\overline{\left(\log n - \overline{\log n}\right)^3} + \frac{\sqrt{2}}{\bar{\bar{\sigma}}}\tilde{\phi}\overline{\left(\log n - \overline{\log n}\right)^2}.$$

The first and fourth terms in approximation (8) are identical to I_{11} and I_5, respectively. Using the approximations $1/\bar{\bar{\sigma}} \fallingdotseq 1/\bar{\sigma}$, $1/\overline{\overline{\sigma^2}} \fallingdotseq 1/\bar{\sigma}^2$, and $1/\sigma_{mn}^k = [(\sigma_m^2 + \sigma_n^2)/2]^{-k/2} \fallingdotseq \left[1 - \frac{(\sigma_m^2 - \bar{\sigma}^2) + (\sigma_n^2 - \bar{\sigma}^2)}{4\bar{\sigma}^2}\right]/\bar{\sigma}^k$, where $k = -1, 1, 2$ and $\bar{\sigma} \coloneqq \left(\sum_n s_n^{(\varepsilon)}\sigma_n^2\right)^{1/2}$, in addition to $\sum_n p_n \log n \fallingdotseq \overline{\log n}$, K_σ is replaced as follows:

$$K_\sigma = I_{12} + I_{14} + I_{22} + I_{24} + I_4 \tag{9}$$
$$\fallingdotseq \sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\frac{(\varepsilon - \widehat{\varepsilon_0})}{\bar{\sigma}}\underset{n}{\mathrm{COV}}\left(\left[\left(\log n - \overline{\log n}\right)^2 - \overline{\left(\log n - \overline{\log n}\right)^2}\right]\left(\log n - \overline{\log n}\right), \frac{\sigma_n^2}{4\bar{\sigma}^2}\right) + \sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\bar{\sigma}\,\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)$$
$$+\frac{1}{\sqrt{2}}\tilde{\phi}\frac{(\varepsilon - \widehat{\varepsilon_0})^2}{\bar{\bar{\sigma}}\cdot\bar{\sigma}^2}\left[\underset{n}{\mathrm{COV}}\left(\left(\left(\log n - \overline{\log n}\right)^2 - \overline{\left(\log n - \overline{\log n}\right)^2}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)\right.$$
$$- 4\overline{\left(\log n - \overline{\log n}\right)^3}\underset{n}{\mathrm{COV}}\left(\log n, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)$$
$$\left.+ 6\overline{\left(\log n - \overline{\log n}\right)^2}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)\right]$$
$$-\frac{1}{\sqrt{2}}\tilde{\phi}\frac{\bar{\sigma}^2}{\bar{\bar{\sigma}}}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right) - \sqrt{2}\tilde{\phi}\frac{1}{\bar{\sigma}}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right).$$

By further applying the approximations $\phi\left(\Phi^{-1}(\tilde{\Phi})\right) \fallingdotseq \tilde{\phi}$ and $\bar{\sigma} \fallingdotseq \bar{\bar{\sigma}}$ with an integration of the terms corresponding to I_{14}, I_{24} and I_4, approximation (9) is shortened as follows:

$$\sqrt{2}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\frac{(\varepsilon - \widehat{\varepsilon_0})}{\bar{\sigma}}\left[\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^3, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right) - \overline{\left(\log n - \overline{\log n}\right)^2}\underset{n}{\mathrm{COV}}\left(\log n, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)\right] \tag{9$'$}$$
$$+\frac{1}{\sqrt{2}}\tilde{\phi}\frac{(\varepsilon - \widehat{\varepsilon_0})^2}{\bar{\bar{\sigma}}\cdot\bar{\sigma}^2}\left[\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^4, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right) - 4\overline{\left(\log n - \overline{\log n}\right)^3}\underset{n}{\mathrm{COV}}\left(\log n, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)\right.$$
$$\left.+ 4\overline{\left(\log n - \overline{\log n}\right)^2}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right)\right]$$
$$-\frac{1}{\sqrt{2}}\tilde{\phi}\frac{2 - \bar{\sigma}^2}{\bar{\bar{\sigma}}}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \frac{\sigma_n^2}{4\bar{\sigma}^2}\right).$$

With respect to K_ψ, the additional approximations $(\widehat{\psi_n} - \widehat{\psi_m})/\sigma_{mn} \fallingdotseq (\widehat{\psi_n} - \widehat{\psi_m})/\bar{\bar{\sigma}}$, $(\widehat{\psi_n} - \widehat{\psi_m})^2/\sigma_{mn}^2 \fallingdotseq (\widehat{\psi_n} - \widehat{\psi_m})^2/\overline{\overline{\sigma^2}}$ and $(\widehat{\psi_n} - \widehat{\psi_m})/\sigma_{mn}^2 \fallingdotseq (\widehat{\psi_n} - \widehat{\psi_m})/\overline{\overline{\sigma^2}}$ yield to the following replacement:

$$K_\psi = I_{13} + I_{23} + I_{25} + I_{27} \tag{10}$$

$$\fallingdotseq \frac{\sqrt{2}}{\bar{\bar{\sigma}}}\phi\left(\Phi^{-1}(\tilde{\Phi})\right)\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \widehat{\psi_n}\right)$$

$$-\frac{1}{2\sqrt{2\overline{\overline{\sigma^2}}}}\tilde{\phi}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \left(\widehat{\psi_n} - \check{\psi}\right)^2\right)$$

$$+\frac{1}{\sqrt{2}}\tilde{\phi}\frac{\varepsilon - \widehat{\varepsilon_0}}{\overline{\overline{\sigma^2}}}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^3, \widehat{\psi_n}\right) - \frac{1}{\sqrt{2}}\tilde{\phi}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \widehat{\psi_n}\right),$$

where $\check{\psi} := \sum_n p_n \widehat{\psi_n}$. By further applying approximations $\phi\left(\Phi^{-1}(\tilde{\Phi})\right) \fallingdotseq \tilde{\phi}$ and $\overline{\overline{\sigma^2}} \fallingdotseq \bar{\bar{\sigma}}^2$ with an integration of the terms correrspondіng to I_{13} and I_{27}, approximation (10) is shortened as follows:

$$\frac{1}{\sqrt{2}}\tilde{\phi}\frac{2-\bar{\bar{\sigma}}}{\bar{\bar{\sigma}}}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \widehat{\psi_n}\right) - \frac{1}{2\sqrt{2}\bar{\bar{\sigma}}^2}\tilde{\phi}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^2, \left(\widehat{\psi_n} - \check{\psi}\right)^2\right) \tag{10$'$}$$

$$+\frac{1}{\sqrt{2}}\tilde{\phi}\frac{\varepsilon - \widehat{\varepsilon_0}}{\bar{\bar{\sigma}}^2}\underset{n}{\mathrm{COV}}\left(\left(\log n - \overline{\log n}\right)^3, \widehat{\psi_n}\right).$$

The rest term I_3, an interaction term of $\log n$, $1/\sigma_{mn}$ and q_{mn}^2, is ignored because I_3 is usually very small. The sum of approximations (8) – (10) is defined as the type II approximation. As mentioned in Section 3, the type II approximation is relatively less accurate than the type I approximation around $\varepsilon = 1$ in low-income countries; however, it is not particularly important for the verification of convexity.

The fourth term I_5 of K_0's approximation in (8), consisting of the variance of $\log n$ multiplied by a positive value $\sqrt{2}\tilde{\phi}/\bar{\bar{\sigma}}$, is always positive and usually the largest contributor to the overall approximation. The first through third terms in approximation (8), corresponding to I_{11}, I_{21} and I_{26}, involve the centred third and fourth moments of $\log n$ and the square of the variance of $\log n$. The second term is always negative. As the inequality $\phi\left(\Phi^{-1}(\tilde{\Phi})\right) > \tilde{\phi}$ holds, although they usually have similar values, the absolute value of the first term is larger than that of the third term. In addition, as the centred third moment of $\log n$ is usually negative, the sum of the first and third terms is usually negative around $\varepsilon = 0$, whereas the sum is positive around $\varepsilon = 1$. Thus, approximation (8) reveals that the range of $\log n$ needs to be sufficiently narrow and the shape of $\log n$'s size distribution needs to be moderate (not extremely two-sided) such that the centred higher moments of $\log n$ and the square of the variance of $\log n$ are sufficiently small relative to the variance of $\log n$ to satisfy $K_0 > 0$. Furthermore, because $\overline{\overline{\sigma^2}}$ is included in the denominator of the multiplier for the second term, approximation (8) also reveals that the average within-group income dispersion needs to be sufficiently large. K_σ's approximation in (9) tells us that the covariance of $\sigma_n^2/4\bar{\sigma}^2$ with $\log n$ and its higher moments should be sufficiently small when $K_\sigma < 0$; that is, a deviation from the IID condition relative to the average income dispersion $\bar{\sigma}^2$ and its interactions with household size are allowed to a limited extent. Similarly, K_ψ's approximation in (10) tells us that a deviation $\widehat{\psi_n}$ from the LL condition and its interactions with household size are allowed as long as the overall type II approximation remains positive.

The average within-group income dispersion can be considered a required factor for the convex *e-i* curve. The factor affects the overall type II approximation via the multipliers for I_{21} in K_0; I_{22} and $I_{14} + I_{24} + I_4$ in K_σ; and I_{23}, I_{25} and I_{27} in K_ψ. If the IID and LL conditions hold, that is, the σ_ns are equal and $\psi_n = 0$, then $K_\sigma = K_\psi = 0$. In this case, restrictions on the range of $\log n$ and the average magnitude of σ_n are still necessary to satisfy $K_0 > 0$. However, the restrictions are for convexity. The *e-i* curve is ∪-shaped without the restrictions (except for the constraints $0 < \varepsilon_0 < 1$), as mentioned in the paragraph above on the end-point condition (2). Thus, the average of σ_n can only be one of the factors for a ∪-shaped relation when either the IID or LL condition is not satisfied. It also should be noted that the *e-i* curve may be convex even if $K_0 \leq 0$ because it is an approximate condition.

In the above discussion, four factors are specified to ensure that the *e-i* curve is convex. They concern the range of $\log n$, the average of σ_n, the relative fluctuation of σ_n and the fluctuation of $\widehat{\psi_n}$. An additional factor that should be included when seeking completeness concerns $\widehat{\varepsilon_0}$, the slope of regression line when $\log y_n$ is regressed on $\log n$. In the type II approximation, the slope $\widehat{\varepsilon_0}$ affects K_0, K_σ, and K_ψ via $\varepsilon - \widehat{\varepsilon_0}$ in the multipliers for the numbers of the terms. The change in the slope in Example 7 of Subsection 2.1 is equivalent to shifting the *e-i* curve 0.5 to left. To the extent that the MLN approach is valid, $\widehat{\varepsilon_0}$ should be regarded as a shifter of the *e-i* curve. When the *e-i* curve is non-∪-shaped, if the range of size elasticity is extended beyond [0, 1] to certain degree, a change of $\widehat{\varepsilon_0}$ may cause the non-∪-shaped curve. Similar to the average of σ_n and the range and shape of $\log n$, the slope $\widehat{\varepsilon_0}$ can only be one of the factors for the ∪-shaped *e-i* curve when either the IID or LL condition is not satisfied.

Because of the complex interdependency between the five factors, it is difficult to specify a permissible range of each factor independently or in simple formulas without losing practicality. The next section empirically demonstrates that actual income distributions satisfy the five factors in the sense that the type I and II approximations are positive, and the Gini index *e-i* curves are convex.

3. Relationships between Size elasticity and Income Inequality Index in the Thirty-Four LIS Countries

3.1. *Empirical Relationships between Size elasticity and Major Inequality Indices*

Datasets for 32 countries for 2004 or around 2004 (Wave VI) and those of two additional countries, Belgium and Russia, for 2000 (Wave V) from the LIS database are selected for the empirical study. Many high-income countries, such as Western European and North American countries, South Korea, and Taiwan, are included. Although the coverage of the LIS database has been expanding rapidly, there are fewer participating countries from Eastern Europe and the low- or middle-income country group than those from the high-income country group in Wave VI. The 34 countries from the LIS include the Czech Republic, Estonia, Hungary, Poland, Russia and Slovenia from Eastern Europe or the former Soviet Union, and Brazil, Columbia, Guatemala, Mexico, Peru and Uruguay, which represent low- or middle-income countries of other regions (abbreviated LMI6 hereafter).

The size elasticity at the minimum of the *e-i* Gini index curve, that is, the point at which the Gini index value for individual equivalised disposable income reaches its minimum, ranges from 0.2 to 0.8 except for LMI6, as shown in Table 1. The curves are ∪-shaped without other minimal points. Among the LMI6, the minimum point of the curve is located near $\varepsilon = 0$ for Brazil (0.06), Columbia (0.09) and Mexico (0.13),[3] whereas the minimum point of the curve for Peru (0.36) and Uruguay (0.22) is inner than 0.2. Those five countries also have

[3] Household consumption data is only available for 9 countries of the 34 countries. The minimum point of the *e-i* curve for consumption is close to that for disposable income in those countries. In Guatemala, the curve for consumption also reaches the minimum at $\varepsilon = 0$ and is strictly non-∪-shaped.

∪-shaped *e-i* curves in the sense that the minimum point is located inside of the interval [0, 1]. However, in Guatemala, the Gini index value reaches the minimum at $\varepsilon = 0$, and the *e-i* curve is strictly non-∪-shaped. The LMI6 consists of six countries that are located in Central and South America. It is too early to draw the conclusion that the minimum point of the *e-i* curve tend to be located at or near $\varepsilon = 0$ among low- or middle-income countries.

Table 1

Location of the Minimum Point of the e-i curve of Major Inequality Measures for Disposable Income

Country	Year	ε_0				$\mathrm{COV}(\log n, y_n)$	$\mathrm{COV}\left(\log n, \frac{y_n}{n}\right)$
		Gini	MLD	Theil	SCV		
Slovenia	2004	0.67	0.71	0.65	0.59	+	-
Denmark	2004	0.68	0.69	0.65	0.46	+	-
Sweden	2005	0.63	0.65	0.60	0.00	+	-
Finland	2004	0.63	0.65	0.55	0.01	+	-
Czech Rep	2004	0.69	0.66	0.64	0.53	+	-
Austria	2004	0.55	0.57	0.53	0.49	+	-
Luxembourg	2004	0.47	0.47	0.43	0.37	+	-
Switzerland	2004	0.35	0.39	0.32	0.21	+	-
Netherlands	2004	0.50	0.47	0.42	0.22	+	-
France	2005	0.46	0.48	0.45	0.43	+	-
Norway	2004	0.63	0.68	0.64	0.00	+	-
Germany	2004	0.53	0.56	0.46	0.00	+	-
Hungary	2005	0.60	0.57	0.52	0.25	+	-
Taiwan	2005	0.56	0.61	0.50	0.35	+	-
South Korea	2006	0.52	0.63	0.48	0.32	+	-
Belgium	2000	0.66	0.62	0.22	0.00	+	-
Australia	2003	0.52	0.58	0.50	0.44	+	-
Canada	2004	0.57	0.60	0.56	0.51	+	-
Ireland	2004	0.61	0.63	0.57	0.34	+	-
Spain	2004	0.55	0.56	0.51	0.44	+	-
Poland	2004	0.44	0.40	0.46	0.70	+	-
Greece	2004	0.66	0.66	0.64	0.60	+	-
Italy	2004	0.48	0.43	0.40	0.23	+	-
Estonia	2004	0.81	0.78	0.76	0.70	+	-
UK	2004	0.53	0.55	0.50	0.45	+	-
US	2004	0.40	0.42	0.38	0.31	+	-
Israel	2005	0.20	0.25	0.21	0.08	+	-
Russia	2000	0.75	0.71	0.54	0.00	+	-
Urguay	2004	0.22	0.20	0.29	0.47	+	-
Mexico	2004	0.13	0.20	0.00	0.00	+	-
Brazil	2006	0.06	0.08	0.01	0.00	+	-
Guatemala	2006	0.00	0.06	0.00	0.00	+	-
Peru	2004	0.36	0.40	0.30	0.16	+	-
Colombia	2004	0.09	0.13	0.00	0.00	+	-

Note: Countries are listed in ascending order of the Gini indices at size elasticity $\varepsilon = 0.5$.

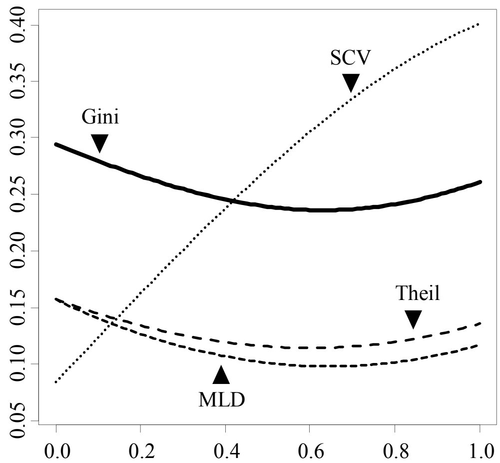

Fig. 2. *The e-i curves of Major Inequality Measures for Sweden, 2005*
Note: The curve for SCV is a linear transformation of the *e-i* curve by the formula (index values − 2.3)/4.

The end-point condition holds for all countries except for Guatemala, whereas the approximate condition (2)′ holds for all 34 countries, as shown in Table 1. However, the Theil and SCV *e-i* curves reach the minimum at $\varepsilon = 0$ in Guatemala. In the case of Theil, the minimum is also attained at $\varepsilon = 0$ in Columbia and Mexico. SCV violates condition (2)′ much more than Theil. In fact, the inconsistency is found in nine countries, including Norway and Sweden. Thus, regarding the SCV measure, the generality of the ∪-shaped *e-i* curve is denied even among high-income countries. As shown in Figure 2, the SCV *e-i* curve is concave for Sweden. The shape of the MLD *e-i* curve is consistent with condition (2)′ in all 34 countries.

The Gini index *e-i* curve is non-∪-shaped for Guatemala, as mentioned above, whereas its convexity is satisfied in all 34 countries. Thus, the empirical study based on the recent income distributions in LIS countries reveals that the end-point condition essentially determines whether the *e-i* curve is ∪-shaped.

3.2. *The Overall Accuracy of the MLN, Type I and II Approximations*

Using the MLN approach, the disposable income distribution within group Γ_n, consisting of all *n*-person households, is replaced with the log-normal distribution $LN(y_n - \sigma_n^2/2\,, \sigma_n^2)$ that has the same average y_n and the same Gini index G_n, where $G_n = 2\Phi(\sigma_n/\sqrt{2}) - 1$ (*cf.* Kleiber and Kotz, 2003). This approach verifies that the approximate conditions for the convex *e-i* curve in Section 2 hold for the 34 countries. Taking the sample sizes and household size distributions into consideration, households with 12 or more persons are classified into a single group in Guatemala and Peru, those with 9 or more are grouped together in Brazil, Columbia, Israel, Mexico, Taiwan and Uruguay, and those with 6 or more are grouped together in the remindar of the countries. The *e-i* curves for the distributions fitted using the MLN approach (the MLN *e-i* curves) are compared with the original curves in the upper-left panel, their derivatives are compared in the upper-right panel, and their second-order derivatives and the type I and II approximations are compared in the lower-left panel in Figures 3a – 3d for Denmark, Hungary, the USA and Uruguay, respectively, and in Annex 3 for all 34 countries. The first and second-order derivatives $\partial G^{(\varepsilon)}/\partial\varepsilon$ and $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ of the original *e-i* curves are numerically derived from the Gini indices $G^{(\varepsilon)}$s at size elasticity values of $\varepsilon = k/100$, where $k = 0, \cdots, 100$ (calculations using a larger number of elasticity values corresponding to more minute subdivisions of the interval [0, 1] cause large fluctuations in $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$). When comparing the countries' Gini index values at $\varepsilon = 0.5$, Denmark has the lowest value 0.2328 among the high-income countries and the second lowest value after Slovenia (0.2313) among all 34 countries. Guatemala (0.5115) has the third highest value after Columbia (0.5339) and Peru (0.5251) among all 34 countries. Furthermore, Guatemala is the only country that has a non-∪-shaped *e-i* curve because of the failure to satisfy the end-point condition (2). The USA (0.3747) has the highest income inequality among the high-income countries except Israel (0.3770). Hungary (0.2914) is approximately at the average income inequality among Eastern Europe and the former Soviet Union.

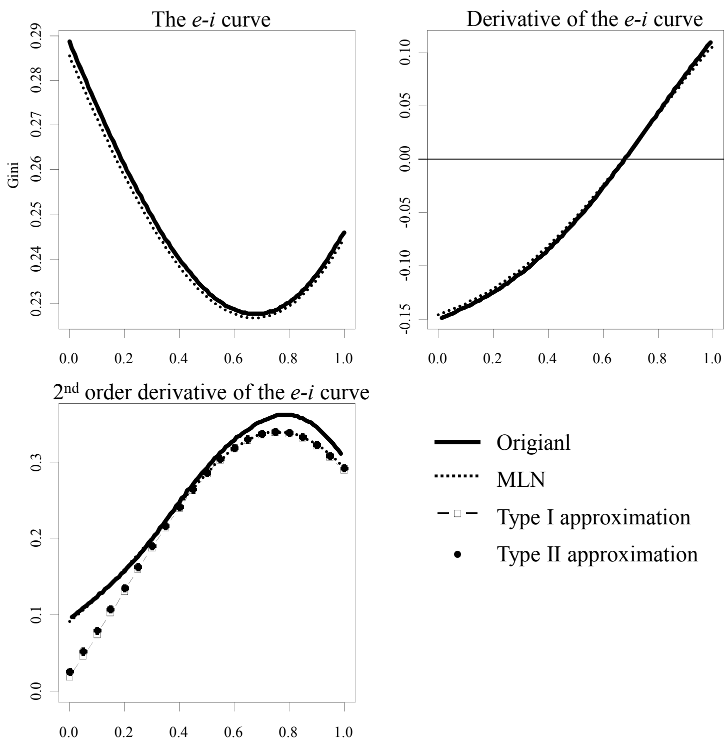

Fig. 3a. *The e-i curve and Its Approximations for Denmark, 2004*

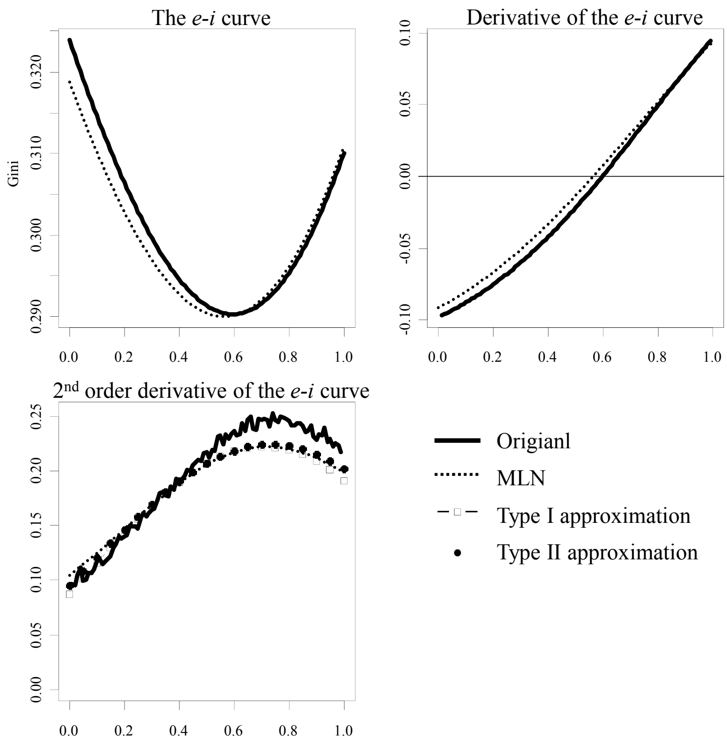

Fig. 3b. *The e-i curve and Its Approximations for Hungary, 2005*

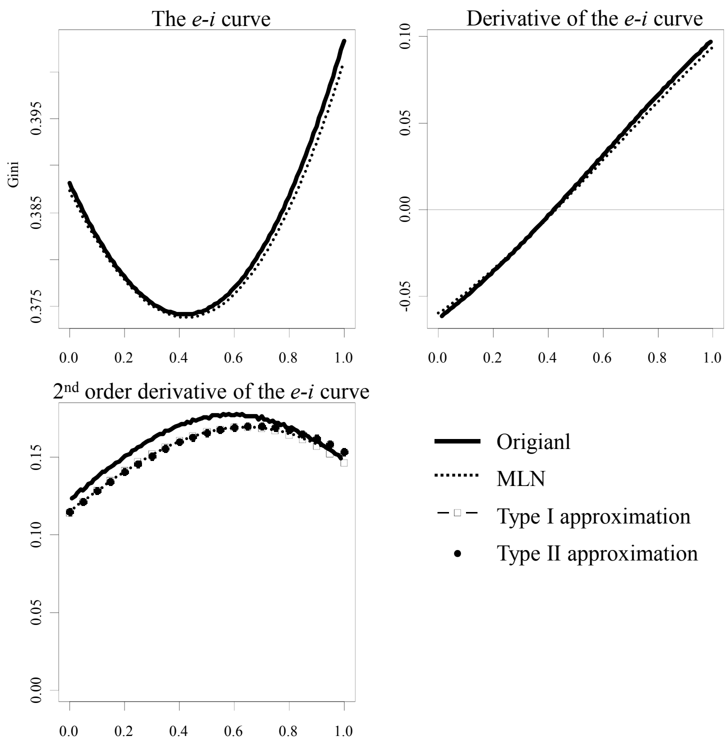

Fig. 3c. *The e-i curve and Its Approximations for the USA, 2004*

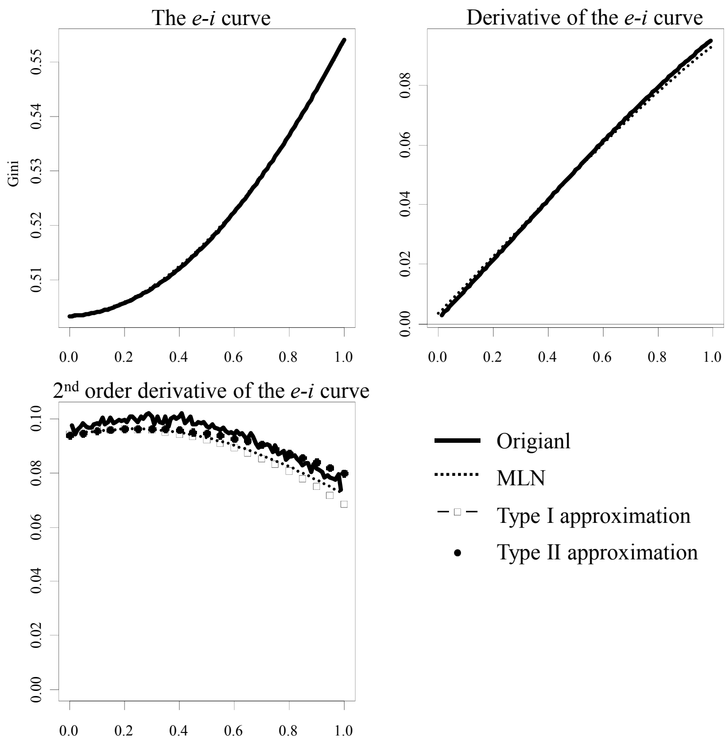

Fig. 3d. *The e-i curve and Its Approximations for Guatemala, 2006*

The maximum absolute error in the MLN *e-i* curves as an approximation of the original curve is 0.0105, the maximum absolute error for each country averaged 0.0029, the Mean Absolute Error (MAE) is 0.0013, and the square Root of the Mean Square Error (RMSE) is 0.0019. In terms of the absolute error rate with respect to the original curve, the corresponding figures are 2.9%, 0.8%, 0.4% and 0.6%, respectively. The absolute difference of the minimum point of the MLN *e-i* curve from that of the original curve is 0.08 at maximum, 0.02 in terms of MAE, and 0.03 in terms of RMSE. The MLN *e-i* curve and the original curve reach their minimum at $\varepsilon = 0$ for Guatemala, and at an inner point for other countries. Thus, the MLN *e-i* curve and the original are non-∪-shaped for Guatemala. The absolute error rate in the derivative of the MLN *e-i* curve relative to the range of the original $\partial G^{(\varepsilon)}/\partial\varepsilon$, i.e. relative to its maximum minus its minimum for a given country, is 11.5% at maximum, 3.5% in terms of the mean of the maximum values for the individual countries, 0.4% in terms of MAE, and 0.6% in terms of RMSE. With respect to the absolute error rate in the second-order derivative of the MLN *e-i* curve relative to the original $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$, the corresponding figures are 27.2%, 9.7%, 4.8% and 6.2%, respectively. In all cases, the maximum errors occur in Belgium, where the income dispersion is extremely large within the two-person household group, and some singularity exists in the income distribution.

The type I approximation of the second-order derivative of the MLN *e-i* curve tends to be inferior around $\varepsilon = 0$ in low-income-inequality countries such as Denmark, as shown in the lower-left panel of Figure 3a. The absolute error rate as an approximation of $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ for the MLN *e-i* curve is 80% at maximum (51% at maximum in Slovenia). In addition to low-income-inequality countries such as Denmark, Sweden (a Gini index value of 0.2392 at $\varepsilon = 0.5$), Slovenia, and Finland (0.2646), Norway (0.2837) and Belgium (0.3176) also suffer from large approximation errors around $\varepsilon = 0$. Among the remaining 28 countries, the absolute error rate is much lower, 18.9% at maximum, 6.9% in terms of the average of the maximum values for the individual countries, 1.1% in terms of MAE, and 2.3% in terms of RMSE. Similarly, the absolute error rates as approximations of the original $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ are summarised as 16.6%, 10.2%, 4.8%, and 5.7%, respectively. In Norway, as well as in other northern European countries, the one-person household group has a large population share. Furthermore, the average income of one-person households relative to that of other households is lower than in other countries, resulting in a larger deviation from the LL condition, which is seemingly the main cause for inaccuracy in the type I approximation. In Belgium, extremely large income dispersion within the two-person household group seemingly causes the inaccuracy.

The type II approximation of the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ tends to differ from the type I approximation with a relatively large magnitude around $\varepsilon = 1$ in high-income-inequality countries such as Guatemala, as shown in the lower-left panel of Figure 3d. Apart from this difference, the type II approximation inherits accuracy and inaccuracy from type I in that the absolute error rate as an approximation of the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ is large around $\varepsilon = 0$ in low-income-inequality countries, Norway, and Belgium, being 71.5% at maximum. If these six countries are excluded, however, the absolute error rate is much lower; 15.7% at maximum, 5.6% in terms of the average of the maximum values for the individual countries, 1.4% in terms of MAE, and 2.3% in terms of RMSE. The corresponding figures for the absolute error rates as approximations of the original $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ are 17.7%, 10.0%, 4.5%, and 5.6%, respectively. Thus, type II is not necessarily inferior to type I.

Although it is difficult to completely avoid inaccurate approximations of higher-order derivatives, the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ and its type I and II approximations are always positive in all 34 countries. To verify that the five factors for a convex *e-i* curve described in Section 2 are empirically satisfied, the MLN approach provides sufficiently well-fitted parametric distributions, and the type I and II approximations are valid.

3.3. *Results of Factor Decompositions by the Type I and II Approximations*

The results of factor decompositions by the type I and II approximations are presented in Tables 2 and 3 for Denmark, Hungary, the USA and Uruguay, and in Annex 1 and 2 for all 34 countries. The term I_5, which is the variance of $\log n$ with a multiplier $\sqrt{2}\tilde{\phi}/\bar{\bar{\sigma}}$, makes the largest positive contribution in the type I and II approximations. To observe the degree to which the magnitude of the contribution of I_5 is reduced by the household size distribution, the fluctuation of σ_n and ψ_n and their interrelations, the ratios of terms such as K_0, K_σ and K_ψ to I_5 (called the 'relative contribution' hereafter) are presented in parenthesis (),[4] and the ratios of the components such as the centred higher moments of $\log n$ and the variances of $\sigma_n/4\overline{\overline{\sigma^2}}$ and ψ_n to the variance of $\log n$ are presented in braces {}.

The overall type II approximations for Denmark relative to I_5 are 10.7%, 83.2%, and 84.0% at $\varepsilon = 0, 0.5,$ and 1, respectively (the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ relative to I_5 are 37.6%, 83.0%, and 84.4%, respectively). The ratios are 96.4%, 94.2% and 82.1%, respectively, for Guatemala (96.4%, 92.6%, and 74.1%, respectively, for the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$). At $\varepsilon = 0$, the ratio of the overall type II to I_5 for Guatemala is higher than that for Denmark. Among all 34 countries, the ratio is lowest for Denmark and highest for Guatemala, and higher income-inequality countries tend to have higher ratios. At $\varepsilon = 0.5$, the lowest value is 77.2% (80.7% for the

[4] For simplicity, the approximations of K_0, K_σ, K_ψ, and I_k in the type II approximation are denoted K_0, K_σ, K_ψ, and I_k, ignoring distinctions hereafter.

MLN) for Slovenia, and the highest is 98.5% (98.8%) for Peru. The range of the ratio is relatively narrow at $\varepsilon = 0.5$. At $\varepsilon = 1$, the lowest value is 67.4% (68.7%) for Switzerland, and the highest is 97.8% (93.7%) for Peru. A clear trend in the relationship between the ratio and the level of income inequality is not observed in case $\varepsilon = 1$. The lack of a clear tendency is due to differences in the location of the minimum point of the *e-i* curve, as will be explained later. Although the ratio's range is wider than that at $\varepsilon = 0.5$, the overall approximations are greater than 2/3 of I_5 in all countries. The country with the lowest or highest ratio is identical among the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$, type I and II approximations, except for the country with the highest ratio at $\varepsilon = 1$. In the $\varepsilon = 1$ case, Greece has the highest ratio for the MLN $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ (96.9%) and type I but has a slightly lower ratio (97.7%) than Peru for the type II approximation. The ratio of the original $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ to I_5 also exhibits a similar tendency concerning its range and its relationship to the level of income inequality, although this result is omitted here.

The decompositions into K_0, K_σ, and K_ψ reveal that K_0 is always positive for all 34 countries. However, if I_5 is removed, it becomes negative around $\varepsilon = 0$ and $\varepsilon = 1$ for most countries. In particular, around $\varepsilon = 0$, the absolute value $|K_0 - I_5|$ is larger than $|K_\sigma|$ and $|K_\psi|$ except for low-income countries. Although $|K_\sigma|$ is larger than $|K_\psi|$ at $\varepsilon = 1$ in some low-income countries, $|K_\psi|$ is larger than $|K_\sigma|$ in most cases. In particular, around $\varepsilon = 0$, K_σ is positive for many countries and much smaller than $|K_\psi|$. For example, in Denmark, one of the lowest income-inequality countries, $K_0 - I_5$ is -79.4% of I_5 in the type I approximation, and K_σ and K_ψ are 5.1% and -20.0% of I_5, respectively, at $\varepsilon = 0$. In the type II approximation, the corresponding figures are -70.8%, 5.0% and -23.5%, respectively. In Guatemala, a low-income country with the third-highest income inequality, the corresponding figures are -0.3%, 0.4% and -2.9% in type I approximations, and -0.3%, 0.6% and -3.9% in type II approximations at $\varepsilon = 0$. The rest term I_3 in type I approximations is 1.9% of I_5 for Denmark and -0.3% for Guatemala at $\varepsilon = 0$.

In summary, the empirical study using the LIS 34 countries reveals that the sensitive points are around $\varepsilon = 0$ in the lowest-income-inequality countries (or non-high-income-inequality countries with some singularity in their income distributions) with respect to the possibility of a non-convex *e-i* curve. The most likely contributor to a negative $\partial^2 G^{(\varepsilon)}/\partial\varepsilon^2$ is the shape of (the logarithm of) household size distribution and a less likely contributor is the magnitude of the deviation from the LL condition and its interrelation with household size if the contribution of the level of income inequality is set aside.

Table 2

The MLN e-i curve and Its Factor Decompositions by the Type I and II Approximations

	Denmark, 2004			Hungary, 2005			USA, 2004			Guatemala, 2006		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.2888	0.2328	0.2460	0.3240	0.2914	0.3101	0.3882	0.3747	0.4033	0.4982	0.5115	0.5491
Original $\partial^2 G/\partial\varepsilon^2$	0.0981	0.2926	0.3112	0.0963	0.2170	0.2173	0.1237	0.1756	0.1493	0.0978	0.0995	0.0763
MLN $\partial^2 G/\partial\varepsilon^2$	0.0921	0.2851	0.2929	0.1044	0.2058	0.1986	0.1150	0.1659	0.1492	0.0939	0.0929	0.0721
	(37.6)	(83.0)	(84.4)	(56.8)	(88.8)	(83.0)	(74.4)	(91.0)	(82.2)	(96.4)	(92.6)	(74.1)
Type I approximation	0.0184	0.2842	0.2886	0.0875	0.2065	0.1910	0.1141	0.1662	0.1463	0.0944	0.0922	0.0685
	(7.5)	(82.8)	(83.2)	(47.6)	(89.1)	(79.8)	(73.9)	(91.2)	(80.6)	(96.8)	(91.9)	(70.3)
I_5	0.2448	0.3434	0.3470	0.1838	0.2317	0.2394	0.1545	0.1822	0.1816	0.0975	0.1003	0.0974
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(20.6)	(93.1)	(88.4)	(66.9)	(98.7)	(89.2)	(83.0)	(100.3)	(88.5)	(99.7)	(96.6)	(76.6)
deducting I_5	(-79.4)	(-6.9)	(-11.6)	(-33.1)	(-1.3)	(-10.8)	(-17.0)	(0.3)	(-11.5)	(-0.3)	(-3.4)	(-23.4)
K_σ	(5.1)	(-1.7)	(-1.2)	(0.2)	(1.1)	(0.0)	(0.6)	(-2.1)	(-2.8)	(0.4)	(-2.3)	(-4.6)
K_ψ	(-20.0)	(-8.5)	(-4.5)	(-18.5)	(-10.7)	(-8.6)	(-10.0)	(-6.9)	(-5.9)	(-2.9)	(-2.6)	(-2.2)
I_3	(1.9)	(-0.2)	(0.4)	(-1.1)	(0.1)	(-0.9)	(0.2)	(-0.1)	(0.7)	(-0.3)	(0.2)	(0.5)
Type II approximation	0.0263	0.2857	0.2915	0.0952	0.2069	0.2020	0.1148	0.1658	0.1536	0.0939	0.0945	0.0799
	(10.7)	(83.2)	(84.0)	(51.8)	(89.3)	(84.4)	(74.3)	(91.0)	(84.6)	(96.4)	(94.2)	(82.1)
K_0	(29.2)	(93.8)	(90.5)	(71.7)	(98.8)	(92.6)	(85.8)	(100.3)	(93.1)	(99.7)	(98.9)	(89.4)
deducting I_5	(-70.8)	(-6.2)	(-9.5)	(-28.3)	(-1.2)	(-7.4)	(-14.2)	(0.3)	(-6.9)	(-0.3)	(-1.1)	(-10.6)
K_σ	(-23.5)	(-9.1)	(-6.4)	(0.1)	(1.1)	(0.3)	(0.2)	(-1.9)	(-1.8)	(0.6)	(-1.1)	(-3.3)
K_ψ	(7.5)	(82.8)	(83.2)	(-19.9)	(-10.6)	(-8.5)	(-11.7)	(-7.4)	(-6.7)	(-3.9)	(-3.7)	(-4.1)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinctions.

Table 3

Components in Factor Decomposition of the Second-Order Derivative of the MLN e-i curve by the Type II Approximation

	Denmark, 2004			Hungary, 2005			USA, 2004			Guatemala, 2006		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\hat{\varepsilon}_0$	0.616	0.670	0.721	0.487	0.565	0.636	0.408	0.471	0.534	0.040	0.080	0.151
$\tilde{\phi}$	0.2894	0.3626	0.3439	0.3246	0.3609	0.3338	0.3326	0.3497	0.3206	0.3130	0.3012	0.2671
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3731	0.3820	0.3801	0.3667	0.3722	0.3683	0.3509	0.3542	0.3474	0.3166	0.3119	0.2985
RMS of σ_n ($\bar{\sigma}$)	0.3865	0.3946	0.4036	0.5079	0.5119	0.5153	0.6717	0.6818	0.6938	0.9575	0.9738	0.9889
$\bar{\bar{\sigma}}$	0.3867	0.3909	0.3957	0.5082	0.5103	0.5121	0.6749	0.6797	0.6856	0.9473	0.9560	0.9642
$\overline{\overline{\sigma^2}}$	0.1481	0.1513	0.1552	0.2579	0.2601	0.2619	0.4546	0.4610	0.4688	0.8920	0.9088	0.9249
$\overline{\log n}$	1.0831	0.9596	0.8230	1.2061	1.0974	0.9744	1.1902	1.0721	0.9406	1.7247	1.6165	1.4983
$\mathrm{VAR}_n(\log n)$	0.2313	0.2617	0.2823	0.2035	0.2317	0.2596	0.2217	0.2504	0.2745	0.2086	0.2252	0.2485
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-28.2}	{-20.5}	{-9.1}	{-26.5}	{-24.9}	{-19.9}	{-26.4}	{-21.9}	{-14.3}	{-13.7}	{-17.3}	{-22.1}
Centred 4th moment of $\log n$	{65.8}	{63.5}	{59.4}	{67.5}	{69.6}	{67.5}	{67.1}	{68.0}	{66.0}	{69.5}	{79.5}	{88.8}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^2\right)$	{42.7}	{37.3}	{31.2}	{47.2}	{46.4}	{41.6}	{44.9}	{43.0}	{38.6}	{48.6}	{57.0}	{64.0}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^3\right)$	{62.7}	{51.0}	{41.4}	{73.3}	{68.0}	{57.2}	{71.3}	{64.0}	{53.8}	{93.9}	{116.4}	{133.2}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^4\right)$	{60.8}	{35.2}	{22.3}	{90.3}	{65.9}	{41.3}	{84.6}	{57.4}	{35.5}	{190.1}	{227.5}	{237.7}
$\mathrm{VAR}_n\left(\sigma_n / 4\overline{\overline{\sigma^2}}\right)$	{2.08}	{1.62}	{1.24}	{0.33}	{0.24}	{0.17}	{0.43}	{0.46}	{0.48}	{1.35}	{1.08}	{0.82}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{3.39}	{3.42}	{3.28}	{6.57}	{5.99}	{5.07}	{4.28}	{4.45}	{4.43}	{8.15}	{8.63}	{9.49}
$\mathrm{VAR}_n\left(\left(\widehat{\psi_n} - \breve{\psi}\right)^2\right)$	{0.041}	{0.029}	{0.032}	{0.187}	{0.148}	{0.151}	{0.145}	{0.089}	{0.066}	{0.897}	{1.246}	{1.128}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.5672	-0.6629	-0.7147	-0.7377	-0.6434	-0.5073	-0.9380	-0.9583	-0.9728	-0.7281	-0.6895	-0.6532
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \sigma_n^2\right)$	0.4657	0.2757	0.0056	-0.2205	-0.3749	-0.5743	0.6434	0.5446	0.4062	-0.1045	-0.1113	-0.0932
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^3, \sigma_n^2\right)$	-0.2494	-0.3072	-0.4011	-0.3072	-0.2826	-0.2754	-0.8070	-0.8278	-0.8448	-0.4211	-0.4339	-0.4500
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^4, \sigma_n^2\right)$	0.3952	0.3898	0.2188	-0.1121	-0.2603	-0.4901	0.6611	0.6334	0.4887	0.0687	0.0546	0.0361
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \widehat{\psi_n}\right)$	-0.8896	-0.9322	-0.9788	-0.9005	-0.9062	-0.9243	-0.8933	-0.9191	-0.9510	-0.2476	-0.5059	-0.6879
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \left(\widehat{\psi_n} - \breve{\psi}\right)^2\right)$	0.9452	0.7544	0.3799	0.8296	0.7600	0.5587	0.9398	0.9527	0.6809	0.5673	0.6301	0.6793
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^3, \widehat{\psi_n}\right)$	0.4262	0.2649	0.0193	0.3649	0.2469	0.0781	0.4450	0.3331	0.1539	0.4887	0.5189	0.4869

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

3.4. *Effects of the Wide Range of Household Sizes in Low-Income Countries*

As households with large families represent a significant share of the population in low-income countries, higher moments of $\log n$ may be so large relative to the variance of $\log n$ that the positive contribution of I_5 is cancelled out. However, in practice, the relative contributions of K_0, K_σ, and K_ψ in the type I and II approximations relative to I_5 do not show clear effects of the wider range of household sizes. As shown in Table 3, the variance of $\log n$ in low-income countries does not differ a great deal from that in high-income countries. The former actually tends to be slightly lower than the latter, whereas the centred fourth moment of $\log n$ and the variance of $\left(\log n - \overline{\log n}\right)^k$, where $k = 2,3,4$, tend to be larger in low-income countries; in particular, the gap between the high- and low-income country groups is wider around $\varepsilon = 1$. The centred third moment of $\log n$ tends to be larger in the absolute value in high-income countries around $\varepsilon = 0$, whereas it tends to be larger in low-income countries around $\varepsilon = 1$. However, the contributions of those relative increases in low-income countries are generally offset by the rise of the average within-group income dispersion represented by $\bar{\bar{\sigma}}$, $\overline{\overline{\sigma^2}}$ or something similar, as explained below.

Regarding the contribution of K_0, the centred fourth moment of $\log n$ included in I_{21} is 69.5% relative to the variance of $\log n$ in Guatemala, which is only 1.06 times higher than that in Denmark (65.8%) at $\varepsilon = 0$, whereas the $\overline{\overline{\sigma^2}}$ included in the denominator of the multiplier for I_{21} is 6.02 times higher in Guatemala (0.8920) than in Denmark (0.1481). Furthermore, $\widehat{\varepsilon_0}$ is much closer to zero in Guatemala (0.040) than in Denmark (0.616) because $\widehat{\varepsilon_0}$ is located near the minimum point of the *e-i* curve, resulting in a substantially lower relative contribution of $|I_{21}|$ (the ratio of $|I_{21}|$ to I_5) due to $(\varepsilon - \widehat{\varepsilon_0})^2$ being included in the numerator of the multiplier for I_{21}. The absolute value of the centred third moment of $\log n$ included in $I_{11} + I_{26}$ relative to the variance of $\log n$ is lower in Guatemala (-13.7%) than in Denmark (-28.2%). As $|\varepsilon - \widehat{\varepsilon_0}|$ is also much smaller, the relative contribution of $|I_{11} + I_{26}|$ is also substantially lower. Thus, the relative contribution of $|K_0 - I_5|$ is much smaller at $\varepsilon = 0$ in Guatemala. At $\varepsilon = 1$, although $\overline{\overline{\sigma^2}}$ is 5.96 times larger in Guatemala (0.9249) than in Denmark (0.1552), the centred fourth moment of $\log n$ relative to the variance of $\log n$ is 1.49 times higher in Guatemala (88.8%) than in Denmark (59.4%), and $(\varepsilon - \widehat{\varepsilon_0})^2$ is 9.25 times larger because $\widehat{\varepsilon_0}$ is much closer to zero in Guatemala (0.151) than in Denmark (0.721), resulting in a doubling of the relative contribution of I_{21}. The contribution of $I_{11} + I_{26}$ is positive at $\varepsilon = 1$. As the absolute value of the centred third moment of $\log n$ relative to the variance of $\log n$ is 2.42 times higher in Guatemala (-22.1%) than in Denmark (-9.1%) and $\varepsilon - \widehat{\varepsilon_0}$ is 3.04 times larger, the relative contribution of $I_{11} + I_{26}$ is approximately 7 times larger, almost offsetting the increase in $|I_{21}|$. Thus, the relative contribution of K_0 does not differ considerably between Denmark and Guatemala at $\varepsilon = 1$. Similar tendencies are observed in comparisons between the high- and low-income country groups. Switzerland has the lowest relative contribution of K_0, that is, the highest relative contribution of $|K_0 - I_5|$, among the 34 countries because of its relatively low income inequality (its Gini index of 0.2704 at $\varepsilon = 0.5$ is the 8th lowest), and its relatively leftward location of the minimum point of the *e-i* curve (at 0.35) among the high-income countries.

Regarding the contribution of K_ψ, the variance of $\left(\log n - \overline{\log n}\right)^2$ and the variance of $\widehat{\psi_n}$ relative to the variance of $\log n$ are higher for low-income countries than for high-income countries, causing increases of the relative contribution of $|K_\psi|$, whereas the statistics representing the average of the within-group income dispersions, such as $\bar{\bar{\sigma}}$, show increases, the negative correlation between $\widehat{\psi_n}$ and $\left(\log n - \overline{\log n}\right)^2$ weakens, and the variance of $\left(\log n - \overline{\log n}\right)^3$ relative to the variance of $\log n$ increases and essentially makes a positive contribution when $\widehat{\varepsilon_0}$ comes close to zero for the low-income countries, causing decreases of the relative contribution of $|K_\psi|$. On balance, the relative contribution of $|K_\psi|$ tends to be smaller for low-income countries at any ε, including $\varepsilon = 1$. For example, comparing Denmark and Guatemala, the variance of $\left(\log n - \overline{\log n}\right)^2$ relative to that of $\log n$ is 1.14 times higher for Guatemala (48.6%) than for Denmark (42.7%) at $\varepsilon = 0$ and is 2.05 times higher at $\varepsilon = 1$ (31.2% and 64.0%, respectively). The variance of $\widehat{\psi_n}$ relative to that of $\log n$ is 2.41 times higher at $\varepsilon = 0$ (3.39% and 8.15%) and 2.89 times higher at $\varepsilon = 1$ (3.28% and 9.49%). However, the $\bar{\bar{\sigma}}$ increases from 0.3867 for Guatemala to 0.9473 for Denmark at $\varepsilon = 0$ and from 0.3957 to 0.9642 at $\varepsilon = 1$. The increase in $\bar{\bar{\sigma}}$ restrains the increase in the relative contribution of $I_{13} + I_{27}$ to 1.1 – 1.6 times due to the inclusion of $2 - \bar{\bar{\sigma}}$ in the numerator of the multiplier for the 1st term in (10)′ (if the correlation between $\widehat{\psi_n}$ and $\left(\log n - \overline{\log n}\right)^2$ is unchanged). In practice, the correlations are -0.8896 and -0.2476 at $\varepsilon = 0$, -0.9788 and -0.6879 at $\varepsilon = 1$, respectively; that is, the correlation is 0.28 times weaker at $\varepsilon = 0$ and 0.70 times weaker at $\varepsilon = 1$ in Guatemala, meaning that the relative contribution of $I_{13} + I_{27}$ is reduced 0.3 times at $\varepsilon = 0$ and increases by no more than 1.1 times at $\varepsilon = 1$. With respect to term I_{23}, the variance of $\left(\widehat{\psi_n} - \breve{\psi}\right)^2$relative to that of $\log n$ is 0.041% for Denmark and 0.897% for Guatemala at $\varepsilon = 0$, 0.032% and 1.128%, respectively, at $\varepsilon = 1$; that is, much higher for Guatemala than for Denmark, meaning that I_{23} makes a much larger relative contribution in Guatemala even if the correlation between $\left(\widehat{\psi_n} - \breve{\psi}\right)^2$ and $\left(\log n - \overline{\log n}\right)^2$ is taken into account. However, because the variance of $\left(\widehat{\psi_n} - \breve{\psi}\right)^2$ is much smaller than that of $\widehat{\psi_n}$, I_{23} does not affect the overall change significantly. With respect to the rest term I_{25}, the variance of $\left(\log n - \overline{\log n}\right)^3$ relative to that of $\log n$ is 62.7% for Denmark and 93.9% for Guatemala (1.50 times higher) at $\varepsilon = 0$, 41.4% and 133.2% (3.22 times higher), respectively, at $\varepsilon = 1$, and the correlation between $\widehat{\psi_n}$ and $\left(\log n - \overline{\log n}\right)^3$ is 0.4262 and 0.4887 (1.15 times stronger) at $\varepsilon = 0$, 0.0193 and 0.4869 (25.2 times stronger) at $\varepsilon = 1$, respectively. However, as $\widehat{\varepsilon_0}$ approaches zero in Guatemala in addition to the increase in $\bar{\bar{\sigma}}$, $(\varepsilon - \widehat{\varepsilon_0})/\bar{\bar{\sigma}}$ in the multiplier reduces the relative contribution of $|I_{25}|$ substantially at $\varepsilon = 0$. At $\varepsilon = 1$, the relative contribution of I_{25} is positive and shows a substantial increase with a slight magnification by the multiplier $(\varepsilon - \widehat{\varepsilon_0})/\bar{\bar{\sigma}}$ (1.25 times larger), offsetting the negative changes in the other terms. Similar tendencies are observed in comparisons between the high- and low-income country groups.

The relative contribution of K_σ ranges from -5.4% to 13.2% (1.9% in terms of MAE) at $\varepsilon = 0$, from -8.5% to 4.6% (2.2%) at $\varepsilon = 1$ in the type II approximation. The relative contribution is often positive or smaller in absolute value than that of $|K_\psi|$, which ranges from -26.5% to 0.8% (12.7%) at $\varepsilon = 0$, from -10.6% to 0.2% (4.7%) at $\varepsilon = 1$, although $|K_\sigma|$ exceeds $|K_\psi|$ at $\varepsilon \geq 0.5$ in some low-income countries. The minor contribution of K_σ is attributed to the smaller magnitude of the variance of $\sigma_n/4\overline{\overline{\sigma^2}}$. For example, the variance of $\sigma_n/4\overline{\overline{\sigma^2}}$ relative to that of $\log n$ is 2.08% at $\varepsilon = 0$ and 1.24% at $\varepsilon = 1$ in Denmark and is 1.35% and 0.82%, respectively, in Guatemala, which is less than the variance of $\widehat{\psi_n}$ relative to that of $\log n$; the variance of $\widehat{\psi_n}$ relative to that of $\log n$ at $\varepsilon = 0$ and $\varepsilon = 1$ is 3.39% and 3.28%, respectively, in Denmark and 8.15% and 9.49%, respectively, in Guatemala. The generally weaker correlations of σ_n with $\log n$ and higher moments of $\log n$ compared to those of $\widehat{\psi_n}$ (although the magnitude varies considerably among countries) also cause a reduction in the relative contribution of K_σ. The variance of $\sigma_n/4\overline{\overline{\sigma^2}}$ does not significantly differ between the high- and low-income country groups. In addition, the increase in $\bar{\sigma}^2$ restrains the increases of the relative contributions of I_{22} and $I_{14}+I_{24}+I_4$ in low-income countries (although this is actually somewhat complicated because the magnitude of the correlation between σ_n and higher moments of $\log n$ varies substantially among countries). With respect to the relative contribution of I_{12}, as $\bar{\sigma}^2$ or similar statistics does not affect I_{12}, a rise in the variance of $\left(\log n - \overline{\log n}\right)^3$ relative to that of $\log n$ increases its absolute value by 3 – 5 times in low-income countries. The relative contribution of I_{12} is further amplified by the inclusion of $\varepsilon - \widehat{\varepsilon_0}$ in the multiplier at $\varepsilon = 1$ because $\widehat{\varepsilon_0}$ comes much closer to zero in low-income countries. However, its increase does not considerably affect the overall relative contribution of K_σ.

4. Illustrative Example for Impact of the ∪-Shaped Relationship between Size elasticity and the Gini Index on the Measurement of Income Inequality

4.1. *Ways to Ascertain the Impact on Measurement Practices*

Coulter *et al*. (1992) showed that the *e-i* curve is ∪-shaped with a minimum point around 0.6 in the UK using the 1986 UK Family Expenditure Survey. Additionally, they mentioned that the results suggest that the official equivalence scale (McClements, 1977), which corresponds to $\varepsilon \fallingdotseq 0.6$ in the parametric equivalence scale $\mathrm{v} = n^\varepsilon$ of Buhman *et al*. (1988), tends to yield lower index values relative to other equivalence scales. Banks and Johnson (1994) argued that their results are not robust on the grounds that the minimum point of the *e-i* curve is not always located around 0.6. For that reason, index values based on the McClements scale are not necessarily lower than those based on other scales for some years, such as 1979 in the case of the Gini index. The other argument for the lack of robustness is that, even when the year is limited to 1986, the minimum point of the *e-i* curve moves away from 0.6 as η decreases when using the two-parameter scale $\nu(\varepsilon,\eta) = (n_A + \eta \cdot n_C)^\varepsilon$, where n_A/n_C denotes a number of adult/child household members and $0 \le \varepsilon, \eta \le 1$. Thus, the same negative conclusion holds. Jenkins and Cowell (1994) responded negatively to these comments by saying that Banks and Johnson exaggerated the instability of the *e-i* curve while admitting that their results regarding the McClements scale are not immutable. Additionally, Jenkins and Cowell (1994) suggest that, when using $\nu(\varepsilon,\eta) = (n_A + \eta \cdot n_C)^\varepsilon$, the correct parameter set corresponding to the McClements scale is the pair of $\eta \fallingdotseq 0.53$ and $\varepsilon \fallingdotseq 0.77$. The value $\varepsilon \fallingdotseq 0.77$ is close to the minimum point of the *e-i* curve when $\eta \fallingdotseq 0.53$; hence, their suggestion is correct even if the more general specification is used.

With respect to the inter-temporal stability of the impact, the conclusion depends on the time-span being considered. The stability against the different equivalence scale specification holds if η is fixed at 0.53 or if it is limited to a certain range around 0.53, as explained by Jenkins and Cowell. However, if η is allowed to vary in a wide range, the index values could be lower than those derived from the McClements scale according to Figure 1b of Jenkins and Cowell (1994). It seems difficult to discover the cases in which a ∪-shaped *e-i* curve has affected the measurement of income inequality in the same manner for a long period, irrespective of the choice of equivalence scale specification, without imposing strict (but reasonable) constraints on the parameters of the equivalence scale function. In the example given below, by choosing several procedures for parameter estimation, the parameter values being compared are restricted to a few numbers. Under the restriction, the stable effect of the ∪-shaped *e-i* curve on the measurement of trends in income inequality can be observed for a long period in the example.[5] Furthermore, a similar tendency can be observed even if a more general type of equivalence scale specification is used. As the selected procedures are either in practical use or considered to be appropriate from a theoretical perspective, and both types of procedures exhibit similar tendencies, the example is expected to have practical importance. The example would be at least useful to understand what may happen under various procedures due to the shape of the *e-i* curve because inter-temporal changes in income inequality are usually measured continuously by one specific procedure for equivalence scale estimation.

4.2. *Data and Methods for Estimating Equivalence Scales*

Japanese survey data from 1989 to 2009, cross-tabulated by the number of household members and annual income class, is used as an illustration. The data were obtained from the National Survey of Family Income and Expenditures (NSFIE), a large family budget survey of approximately 60,000 sample households (of which, approximately 50,000 households contain two or more persons) conducted quinquennially by the Statistics Bureau of the Ministry of Internal Affairs and Communications. In the statistical table, six-or-more person households are classified into a single group. Gross income before the deduction of direct taxes and social insurance premiums is used due to data availability.

First, the popular Engel method is applied for equivalence scale estimation for reference purposes. The method assumes that the standard of living is higher when the budget share of food is lower in a homogenous household group, and the standard of living is equivalent between two different household groups if the budget share of food is the same. The following Working-Leser model is used for the estimation:

$$w \sim \alpha + \beta \log C + \log \phi(n|\gamma), \tag{11}$$

where C and w denote consumption expenditures and food's share of consumption, and $\phi(\cdot|\gamma)$ is a function with the parameter γ for determining the equivalence scale. The equivalence scale for *n*-person households is derived as $\nu = -\phi(n|\gamma)/\beta$. The Engel method is simple and popular, whereas the derived equivalence scales for households with children are argued to be overestimated because children are relatively food intensive. Then, a variant of the Engel method is applied. In the variant used by Phipps and Garner (1994), the budget share of food is replaced by the budget share of necessities including non-food items. Their method is also used for the estimation of Low-Income-Cutoffs (LICOs) in Canada. Five categories ('food', 'clothes and footwear', 'fuel, light and water charges', 'housing', and 'furniture and household utensils') are classified as necessities here. This variant intuitively appears to be more suitable for high-income countries; however, it does not have a firm theoretical background. To address the lack of a firm theoretical background, an estimation method based on a complete demand system is applied. Ray (1983) proposed applying his price scaling method to a non-separable extension of the Linear Expenditure System (LES) studied by Blundell and Ray (1982). His method has the advantage being solvable in a one-time cross-section data setting without suffering from identification problems in addition to its ability to allow for substitutions among expenditure categories. The derived equations are represented, as follows:

[5] Jenkins and Cowell (1994) mentioned the possibility of the *e-i* relationship's impact on trends in index values. However, they only described differences in estimated index values between 1987 and 1988/89.

$$w_i \sim \alpha_i + \tau(n|\delta_i) + \beta_i \frac{\phi(n|\gamma)}{C}, \quad (12)$$

where w_i denotes the budget share of category i, and $\tau(\cdot|\delta)$ is an intercept shifter variable with the parameter δ. The shifter τ varies its value according to household size n. Three constraints on the parameters in (12) are imposed: $\sum_i \alpha_i = 1$, $\sum_i \beta_i = 0$ and $\sum_i \tau(n|\delta_i) = 0$, so that the predicted shares of the categories sum to unity. The resulting equivalence scale is $\nu = \phi(n|\gamma)$. In the example, expenditures are aggregated into five categories: 'food', 'clothes and footwear', 'fuel, light and water charges', 'housing, furniture and household utensils', and 'others'.[6]

Although there are several methods based on demand systems such as the Prais-Houthakker method and its variant, the McClements method, the parameters are intrinsically not uniquely determined. The procedures employed for parameter estimation are not clearly justified from a theoretical point of view. Muellbauer (1980) proposed eliminating the identification problem with prior information such as a nutrition-based food scale, but it is not easy to choose an appropriate food scale because there are different views about nutrition-based measurements. Furthermore, those methods do not allow substitutions among categories. Barten's scaling method is popular in methods based on complete demand systems but it requires repeated cross-sectional data to avoid the identification problem. As quinquennial data are used, and equivalence scales cannot be regarded as constant during the period studied, the Barten method is unsuitable for the example presented here. It also should be noted that quasi-price substitution effects are overestimated when a child enters a family, resulting in the underestimation of the equivalence scale (Muellbauer, 1977). The Gorman-Barten method addresses the bias problem by adding a fixed child cost (Deaton and Muellbauer, 1986). However, the assumption of a fixed child cost, independent of income level, appears to be inappropriate for high-income countries.

As the functional form of $\phi(n|\gamma)$ in (11) and (12), $\phi(n|\gamma) = \sum_{j>2} \gamma_j I(n = j)$, which has a dummy variable for each household group, and $\phi(n|\gamma) = n^\gamma$, studied by Buhman *et al.* (1988), are used (denoted as 'form 1' and 'form 2', respectively). In form 2, γ corresponds to the size elasticity ε. The functional forms with separate parameters for adult and child members such as $(n_A + \eta \cdot n_C)^\varepsilon$ are not used here because of data availability. Form 1 with separate parameters for each household-size group is expected to address this limitation to certain extent. In equation (12) for Ray's method, a simple form, $\tau(n|\delta_i) = \delta_i n$, is used as the intercept shifter variable for household size. The parameters α_i, β_i, δ_i and γ are estimated by the iterative non-linear SUR techniques.

4.3. *Differences in the Trends of the Gini Indices among the Estimation Procedures (the case of households with two-or-more persons)*

As appropriate equivalence scales may change over time, it seems desirable to use the current equivalence scales for the respective years if we have an appropriate procedure for estimating equivalence scales, rather than to use a fixed set of scales that are usually determined based on past investigations. From this point of view,

[6] The Almost Ideal Demand System (AIDS) is probably preferred to the LES and its generalization at present. However, it is not possible to estimate the parameters in a one-time cross-section data setting when applying the AIDS with Ray's price scaling method. If the shifter $\tau(\cdot|\delta)$ is excluded, the identification problem can be avoided. However, in that case, the resulting equivalence scales are almost the same as those derived from equations (12) without shifter $\tau(\cdot|\delta)$ in the example presented here. It also should be noted that the equivalent standard of living can be attained if and only if the budget shares of all categories are identical (the property is called 'Engel exactness'); thus, no substitution is allowed among different household compositions when excluding the shifter.

measurement results derived from a fixed set of scales (which can also be regarded as a procedure for equivalence-scale estimation) should be compared with those derived from the current scales. Such comparisons are made in Figure 4 and in Tables 4 and 5. The procedures for obtaining the current scales using the Engel, Phipps and Garner, and Ray methods are denoted 'Eng', 'PG', and 'Ray', respectively, along with an attached symbol ('1' or '2') depending on the functional form of $\phi(n|\gamma)$, e.g., 'Eng1' and 'Ray2'. Another type of procedure, using the equivalence scale specification $\nu = n^{\varepsilon}$ with a fixed size-elasticity value, is denoted 'ε0', 'ε0.5', and 'ε1' depending on the elasticity value. Procedure ε0.5 is used by the OECD and was adopted for the official tabulation of the NSFIE.

The Gini index *e-i* curves for 1989 – 2009 are presented in the upper-left panel of Figure 4. The minimum point of the *e-i* curve consistently moved to the right from 0.34 in 1989 to 0.36 in 1994, 0.42 in 1999, 0.45 in 2004, and 0.47 in 2009. The size elasticity values estimated by the three procedures using form 2 of $\phi(n|\gamma)$ are listed in Table 4. All three estimates declined to below a half of 1989 for 20 years. The elasticity values derived from Eng2 are higher than those derived from PG2 and Ray2. The result confirms that the Engel method tends to yield higher equivalence scales. When comparing the elasticity values derived from PG2 and Ray2, higher values are obtained from the former than from the latters; however, the estimated Gini indices are similar, with differences of less than 0.001, as shown in Table 5. Ray2 continuously resulted in a size elasticity further from the minimum point of the *e-i* curve for 20 years, meaning that the more recent the date, the higher the estimated Gini index for equivalised income relative to that derived from the size elasticity corresponding the minimum point of the *e-i* curve. PG2 also placed the size elasticity further from the minimum point of the *e-i* curve continuously, and it created a similar effect on the income-inequality estimations from 1994 to 2009. In contrast, Eng2 made the size elasticity closer to the minimum point of the *e-i* curve from 1989 to 1994 and produced an opposite effect to the other procedures during that period. Among the procedures with fixed size elasticity, ε0.5 and ε1 placed the (fixed) size elasticity closer to the minimum point of the *e-i* curve for 20 years, whereas ε0 placed it further from the minimum point, consequently bringing about the corresponding effects.

Comparisons of the estimated Gini indices for equivalised income in Table 5 reveal that the estimates for 1989 from PG2 and Ray2 were below that by ε0.5, with relatively small differences of less than 0.002, whereas PG2 and Ray2 produced higher estimates for 2009, with differences larger than 0.004. In the case of Eng2, the estimate for 1989 was approximately 0.01 larger than the estimate by ε0.5, and both estimates for 2009 were approximately the same. Procedure ε0.5 showed an increase of 0.0138 over 20 years, whereas Eng2, PG2 and Ray2 showed increases of 0.0049, 0.0196 and 0.0211, respectively, during the same period, that is, 0.0091 smaller and 0.0056 and 0.0072 larger than the increase shown by ε0.5. The excess increases in the latter two procedures, caused by the shape of the *e-i* curves, are statistically significant.[7] Strictly speaking, changes in the shape of the *e-i* curve other than the location of the minimum point may affect the estimations. For this reason, a counterfactual distribution analysis is performed, as follows: if the income distributions for each year were replaced by that for 1989, but the current size elasticity values were used, the 20-year increase in the Gini index would be estimated at -0.0109 by Eng2, 0.0012 by PG2 and 0.0022 by Ray2. Similarly, if the income distributions for each year were replaced by that for 2009, but the current size elasticity values were used, the

[7] Calculation of a 95% confidence interval is made using the standard deviation of the Gini index computed from the estimated parameters summed with 50,000 sets of multivariate normal noises generated from the error variance and covariance matrix of the parameters.

corresponding figures would be -0.0057, 0.0045 and 0.0049, respectively (note that ε0.5 created no changes in these settings). In the counterfactual settings, the (excess) increases estimated by PG2 and Ray2 would be smaller because of exclusion of the *e-i* curve's shape change effect, but the increases purely due to the changes in the size elasticity are statistically significant. In another setting in which the estimated size elasticity for 1989 (0.421 for PG2 and 0.328 for Ray2) was used for 20 years instead of 0.5 (but the current income distributions were used for each year), the increases estimated by PG2 and Ray2 would still be significantly larger than those by ε0.421 and ε0.328, respectively.

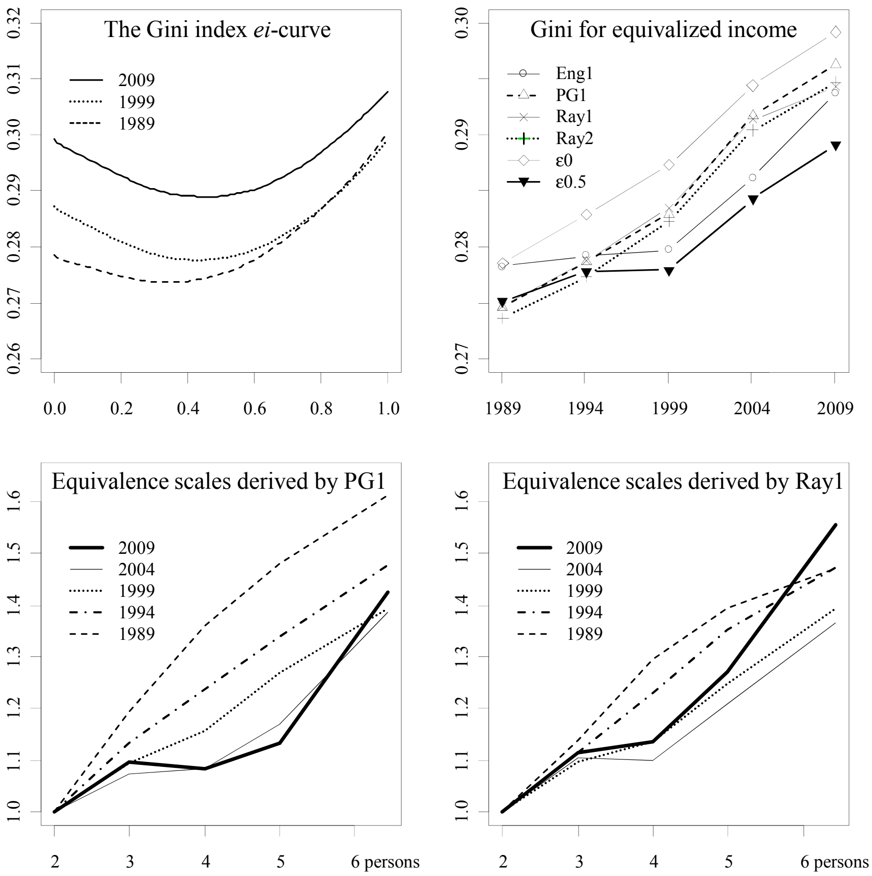

Fig. 4. *The Gini Index e-i curve, Estimated Ginis for Equivalised Income and Equivalence Scales for Japan, Two-or-More-Person Households*

Table 4

Size elasticity by the Estimation Procedure for Japan, Two-or-More-Person Households

Year	Eng2	PG2	Ray2
1989	0.759	0.421	0.328
1994	0.690	0.324	0.260
1999	0.602	0.256	0.152
2004	0.515	0.187	0.116
2009	0.358	0.172	0.134

Table 5
The Gini Indices for Equivalised Income for Japan, Two-or-More-Person Households

Year	Eng1	PG1	Ray1	Eng2	PG2	Ray2	ε0	ε0.5	ε1
1989	0.2783	0.2746	0.2747	0.2845	0.2740	0.2736	0.2785	0.2751	0.3003
2009	0.2937	0.2963	0.2943	0.2894	0.2936	0.2947	0.2992	0.2891	0.3078
Difference	0.0155	0.0217	0.0196	0.0049	0.0196	0.0211	0.0207	0.0139	0.0074

As shown in Table 5, similar results are obtained when form 1 of $\phi(n|\gamma)$ is used instead of form 2. The excess increases in PG1 and Ray1 would also be statistically significant in the counterfactual settings and in a setting of comparison with scales fixed using the starting-year values. Thus, the effects corresponding to the ∪-shaped *e-i* curve are considered to work on the estimated Gini indices for equivalised income. It is notable that discrepancies caused by the different forms of $\phi(n|\gamma)$ are small in the case of Ray's procedure, with differences of less than 0.001, as shown in Table 5 and in the upper-right panel of Figure 4. The equivalence scales for the individual household sizes, which are normalised to the unity of two-person households, estimated using PG1 and Ray1 are presented in the lower two panels of Figure 4. Both procedures showed declining tendencies in the equivalence scales, accompanied with the changes in the shape of the relationship between household size and equivalent scale from concave to convex.

In summary, substantial declines in the current equivalence scales were observed for Japan in recent years.[8] In combination with the effects of a ∪-shaped *e-i* curve or corresponding effects in a more general equivalence scale specification, this trend has caused a procedure that have a fixed size elasticity at 0.5 (ε0.5) and those that use fixed scales estimated for an early year to have the tendency to underestimate increases in income inequality among households with two or more persons.

4.4. *Differences in the Trends of the Gini Indices among the Estimation Procedures (the case of all households)*

As the consumption structure of one-person households is much different from that of two-or-more person households, procedures for estimating equivalence scales based on consumption structure such as Eng, PG, and Ray may be inappropriate when one-person households are included in a study. In fact, PG1 yielded equivalence scales for one-person households that were higher than those for two-person households. Using Ray's procedure, if the form of the intercept shifter for household size is modified to incorporate separate dummy variables for male and female single households, that is, $\tau\left(n|\delta_i,\delta_i^{sm},\delta_i^{sf}\right)=\delta_i nI(n\geq 2)+\delta_i^{sm}I\big((n=1)\wedge(gender=male)\big)+\delta_i^{sf}I\big((n=1)\wedge(gender=female)\big)$, with the additional constraints $\sum_i\delta_i^{sm}=\sum_i\delta_i^{sf}=0$, then Ray1 yields equivalence scales for one-person households in the range of 0.5 to 1 of those for two-person households during 1989 – 2009 but slightly below 0.5 in 1984 and earlier. These results indicate that the derived scales are not fully reliable. Nevertheless, Ray's procedure is applied here because, unlike Eng1 and PG1, the

8 The reason for substantial declines in equivalence scales for households with three or more persons relative to two-person households is not known at present. One speculation is that major retailers set the prices of large-sized products significantly lower relative to those of the same small-sized products relative to the prices set by traditional small-scale retailers. Major retailers operating supermarkets and volume sales specialty stores have substantially expanded their share in the retail market. As a consequence, economies of scale would have improved dramatically if the speculation were correct. However, no clear evidence has been found thus far. This issue needs to be investigated elsewhere because it is beyond the scope of this paper.

derived relative scales for one-person households exhibit an upward tendency, which is consistent with those for three or more person households exhibiting downward tendencies, and the relative scales for one-person households range between 0.5 and 1 from 1989 to 2009.

The results for all households, including one-person households, are listed in Table 6. The minimum point of the Gini index *e-i* curve remained at nearly the same location; 0.48 in 1989 and 0.50 in 2009. The size elasticity derived by Ray2 showed a decrease from 0.540 in 1989 to 0.379 in 2009, but this decrease is smaller than the 0.194 decrease found when one-person households are excluded. The size elasticity came close to the minimum point of the *e-i* curve from 1989 to 1999, whereas it was further from the minimum point after 1999. The 20-year rise in the estimated Gini index for equivalised income was 0.0194, which is larger than the 0.0179 increase estimated by ε0.5, but the gap is statistically insignificant. Although the estimate using Ray1 exhibited a rise of 0.0221, which is significantly larger than that by ε0.5, the excess increase is mainly caused by the different forms of $\phi(n|\gamma)$ used. The increase is not significantly larger than that estimated using the procedure with the scales fixed using 1989 values. Because of the reliability issues of the procedures, the appropriateness of equivalence scale specification n^{ε} should not only be judged using the results for the present example when one-person households are included in a study; rather, a comparison with the results from Ray2 does not clearly deny the appropriateness of ε0.5 for calculating the Gini index for equivalised income distributions. However, when focusing on specific subgroups such as households with two or more persons, procedures with a fixed set of scales such as ε0.5 may cause biases in the measurement of income inequality within the specific groups and in the identification of poverty.

Table 6

Equivalence Scales and the Gini Indices for Equivalised Income for Japan, All households

Year	Relative scale* for one-person households	Size** elasticity	Gini				
			Ray1	Ray2	ε0	ε0.5	ε1
1989	0.548	0.540	0.2787	0.2808	0.2972	0.2804	0.3117
2009	0.619	0.379	0.3008	0.3002	0.3219	0.2983	0.3267
Difference			0.0221	0.0194	0.0247	0.0179	0.0151

Notes: * Estimates using the Ray1 procedure, normalized to the scale for two-person households = 1.
** Estimates using the Ray2 procedure.

5. Concluding Remarks

The U-shaped relationship between size elasticity and index value (the U-shaped *e-i* curve) suggested by Coulter *et al*. (1992) is common among high-income countries when the Gini index, MLD, or Theil are used. Among low-income countries, a non-U-shaped *e-i* curve and nearly J-shaped *e-i* curves with minimum points close to zero are found. However, as low- and middle-income countries contained in the LIS Wave VI database are few and are concentrated in specific regions, the generality of the shape of the curve should be further investigated in the future. Using the Mixture of Log-Normal distributions approach and its approximations, five factors for the convexity of the Gini index *e-i* curve are derived. The factors concern the range and shape of household size distribution, the average within-group income dispersions, the magnitude of fluctuation of within-group income dispersions, the magnitude of deviation from a log-linear relationship between household size and within-group average income, and the slope of the log-linear relationship. Disposable income distributions in the 34 LIS countries satisfy the five factors, and their *e-i* curves are shown to be convex. Thus, the U-shaped *e-i* curve is empirically determined by the end-point condition, which corresponds to the approximate condition suggested by Coulter *et al*. (1992) for the generalised entropy measures and the FGT poverty measures. Although the author has no intention to deny the possibility that heavier upper-tails in the distributions of income or other economic variables may affect the shapes of the *e-i* curves for the respective size distributions, this empirical study shows the validity of the MLN approach for income distributions in many countries.

The *e-i* curve is derived from the application of a specific class of equivalence scale: $v = n^{\varepsilon}$. However, the example for Japan presented in this paper shows that effects similar to those of the U-shaped *e-i* curve are observed when a more general class of equivalence scale is used. Although the author does not know whether similar phenomena have been arising in other countries, it is expected that this study on the relationship between size elasticity and index value and its impact will be useful for the measurement of income inequality and other economic inequality.

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Annex 1. The MLN *e-i* Curve and Its Factor Decompositions by the Type I and II Approximations

	Slovenia, 2004			Denmark, 2004			Sweden, 2005			Finland, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.2711	0.2313	0.2413	0.2888	0.2328	0.2460	0.2944	0.2392	0.2609	0.3114	0.2646	0.2820
Original $\partial^2 G/\partial\varepsilon^2$	0.0861	0.2026	0.2528	0.0981	0.2926	0.3112	0.1016	0.3367	0.2902	0.1210	0.2732	0.2546
MLN $\partial^2 G/\partial\varepsilon^2$	0.0850	0.2032	0.2423	0.0921	0.2851	0.2929	0.1022	0.3235	0.2784	0.1236	0.2677	0.2348
	(46.3)	(80.7)	(86.3)	(37.6)	(83.0)	(84.4)	(39.3)	(88.5)	(79.6)	(51.6)	(87.1)	(79.5)
Type I approximation	0.0413	0.2028	0.2401	0.0184	0.2842	0.2886	0.0417	0.3235	0.2675	0.0844	0.2680	0.2284
	(22.5)	(80.5)	(85.5)	(7.5)	(82.8)	(83.2)	(16.0)	(88.5)	(76.5)	(35.2)	(87.2)	(77.3)
I_5	0.1835	0.2518	0.2809	0.2448	0.3434	0.3470	0.2603	0.3657	0.3497	0.2398	0.3075	0.2955
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(25.5)	(91.3)	(93.3)	(20.6)	(93.1)	(88.4)	(30.8)	(95.8)	(84.5)	(46.2)	(96.8)	(87.7)
deducting I_5	(-74.5)	(-8.7)	(-6.7)	(-79.4)	(-6.9)	(-11.6)	(-69.2)	(-4.2)	(-15.5)	(-53.8)	(-3.2)	(-12.3)
K_σ	(10.6)	(-1.8)	(-5.3)	(5.1)	(-1.7)	(-1.2)	(2.0)	(-0.2)	(-1.1)	(3.8)	(-0.6)	(-2.0)
K_ψ	(-18.3)	(-9.3)	(-2.8)	(-20.0)	(-8.5)	(-4.5)	(-17.0)	(-7.1)	(-6.7)	(-16.0)	(-8.9)	(-8.5)
I_3	(4.7)	(0.4)	(0.3)	(1.9)	(-0.2)	(0.4)	(0.2)	(-0.1)	(-0.3)	(1.2)	(-0.1)	(0.0)
Type II approximation	0.0353	0.1945	0.2450	0.0263	0.2857	0.2915	0.0494	0.3245	0.2781	0.0907	0.2684	0.2369
	(19.2)	(77.2)	(87.2)	(10.7)	(83.2)	(84.0)	(19.0)	(88.7)	(79.5)	(37.8)	(87.3)	(80.2)
K_0	(32.4)	(91.8)	(94.9)	(29.2)	(93.8)	(90.5)	(38.1)	(96.3)	(87.4)	(54.4)	(97.2)	(90.3)
deducting I_5	(-67.6)	(-8.2)	(-5.1)	(-70.8)	(-6.2)	(-9.5)	(-61.9)	(-3.7)	(-12.6)	(-45.6)	(-2.8)	(-9.7)
K_σ	(13.2)	(-2.3)	(-4.7)	(5.0)	(-1.5)	(-0.1)	(1.9)	(-0.0)	(-0.3)	(3.3)	(-0.4)	(-0.8)
K_ψ	(-26.5)	(-12.3)	(-2.9)	(-23.5)	(-9.1)	(-6.4)	(-21.0)	(-7.5)	(-7.5)	(-20.0)	(-9.5)	(-9.3)

	Czech, 2004			Austria, 2004			Luxembourg, 2004			Switzerland, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3038	0.2668	0.2744	0.3081	0.2691	0.3004	0.2958	0.2701	0.3110	0.2826	0.2704	0.3235
Original $\partial^2 G/\partial\varepsilon^2$	0.0760	0.1809	0.2462	0.1318	0.3028	0.2530	0.1371	0.2930	0.2266	0.1845	0.2824	0.1657
MLN $\partial^2 G/\partial\varepsilon^2$	0.0827	0.1879	0.2148	0.1289	0.2870	0.2482	0.1417	0.2771	0.2154	0.1721	0.2596	0.1669
	(52.4)	(88.1)	(90.6)	(57.1)	(93.9)	(83.1)	(65.6)	(97.2)	(78.1)	(73.2)	(93.4)	(68.7)
Type I approximation	0.0690	0.1882	0.2131	0.1109	0.2871	0.2414	0.1320	0.2772	0.1994	0.1680	0.2594	0.1498
	(43.7)	(88.3)	(89.9)	(49.1)	(94.0)	(80.8)	(61.1)	(97.2)	(72.3)	(71.4)	(93.3)	(61.7)
I_5	0.1578	0.2132	0.2370	0.2258	0.3056	0.2989	0.2160	0.2853	0.2757	0.2351	0.2780	0.2429
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(50.5)	(95.4)	(93.4)	(50.5)	(98.5)	(82.5)	(65.9)	(100.4)	(75.7)	(80.1)	(99.9)	(67.4)
deducting I_5	(-49.5)	(-4.6)	(-6.6)	(-49.5)	(-1.5)	(-17.5)	(-34.1)	(0.4)	(-24.3)	(-19.9)	(-0.1)	(-32.6)
K_σ	(2.0)	(-1.8)	(-1.8)	(2.5)	(-2.5)	(-1.8)	(1.5)	(-0.9)	(-2.3)	(0.0)	(-2.6)	(-2.5)
K_ψ	(-10.6)	(-5.3)	(-2.3)	(-5.8)	(-2.1)	(-0.6)	(-5.9)	(-2.3)	(-1.1)	(-8.9)	(-3.9)	(-3.8)
I_3	(1.8)	(0.0)	(0.6)	(1.9)	(-0.0)	(0.6)	(-0.5)	(0.0)	(-0.0)	(0.2)	(-0.1)	(0.6)
Type II approximation	0.0698	0.1874	0.2151	0.1134	0.2867	0.2521	0.1354	0.2773	0.2163	0.1677	0.2622	0.1638
	(44.2)	(87.9)	(90.7)	(50.2)	(93.8)	(84.3)	(62.7)	(97.2)	(78.4)	(71.3)	(94.3)	(67.4)
K_0	(54.9)	(95.6)	(95.3)	(54.7)	(98.6)	(86.1)	(68.4)	(100.4)	(81.6)	(82.1)	(100.1)	(74.8)
deducting I_5	(-45.1)	(-4.4)	(-4.7)	(-45.3)	(-1.4)	(-13.9)	(-31.6)	(0.4)	(-18.4)	(-17.9)	(0.1)	(-25.2)
K_σ	(2.4)	(-1.9)	(-1.8)	(2.7)	(-2.4)	(-1.0)	(1.2)	(-0.8)	(-1.7)	(0.0)	(-2.6)	(-2.5)
K_ψ	(-13.1)	(-5.9)	(-2.7)	(-7.2)	(-2.4)	(-0.9)	(-6.9)	(-2.3)	(-1.4)	(-8.9)	(-3.9)	(-3.8)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinction.

Annex 1. The MLN *e-i* Curve and Its Factor Decompositions by the Type I and II Approximations (*Continued*)

	Netherland, 2004			France, 2005			Norway, 2004			Germany, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3042	0.2736	0.3114	0.3029	0.2807	0.3169	0.3305	0.2837	0.3019	0.3195	0.2876	0.3151
Original $\partial^2 G/\partial\varepsilon^2$	0.1392	0.2982	0.2190	0.1373	0.2519	0.1944	0.0902	0.2861	0.2520	0.1250	0.2562	0.2029
MLN $\partial^2 G/\partial\varepsilon^2$	0.1479	0.2635	0.2002	0.1348	0.2407	0.1958	0.0890	0.2269	0.2407	0.1230	0.2369	0.1982
	(64.7)	(92.2)	(75.7)	(65.3)	(93.1)	(78.7)	(43.5)	(84.0)	(86.4)	(61.0)	(94.2)	(82.6)
Type I approximation	0.1411	0.2635	0.1897	0.1272	0.2410	0.1856	0.0540	0.2275	0.2378	0.1100	0.2371	0.1932
	(61.8)	(92.2)	(71.7)	(61.6)	(93.2)	(74.6)	(26.4)	(84.2)	(85.4)	(54.5)	(94.3)	(80.5)
I_5	0.2285	0.2857	0.2646	0.2064	0.2587	0.2488	0.2045	0.2703	0.2785	0.2017	0.2515	0.2400
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(72.8)	(100.3)	(75.1)	(72.1)	(100.2)	(79.9)	(48.4)	(95.5)	(92.5)	(66.9)	(99.3)	(85.8)
deducting I_5	(-27.2)	(0.3)	(-24.9)	(-27.9)	(0.2)	(-20.1)	(-51.6)	(-4.5)	(-7.5)	(-33.1)	(-0.7)	(-14.2)
K_σ	(-0.1)	(-2.6)	(-0.7)	(0.6)	(-1.4)	(-1.6)	(0.9)	(-0.2)	(-0.5)	(0.4)	(0.4)	(-0.5)
K_ψ	(-11.8)	(-5.3)	(-4.0)	(-11.3)	(-5.6)	(-4.1)	(-22.4)	(-10.9)	(-6.7)	(-11.8)	(-5.3)	(-4.2)
I_3	(0.9)	(-0.1)	(1.3)	(0.2)	(-0.1)	(0.3)	(-0.5)	(-0.2)	(-0.0)	(-0.9)	(-0.1)	(-0.5)
Type II approximation	0.1417	0.2632	0.1986	0.1290	0.2412	0.1968	0.0635	0.2285	0.2429	0.1173	0.2382	0.2021
	(62.0)	(92.1)	(75.0)	(62.5)	(93.3)	(79.1)	(31.1)	(84.6)	(87.2)	(58.2)	(94.7)	(84.2)
K_0	(75.9)	(100.3)	(81.2)	(75.2)	(100.2)	(85.2)	(55.7)	(96.1)	(94.6)	(71.1)	(99.4)	(89.1)
deducting I_5	(-24.1)	(0.3)	(-18.8)	(-24.8)	(0.2)	(-14.8)	(-44.3)	(-3.9)	(-5.4)	(-28.9)	(-0.6)	(-10.9)
K_σ	(-0.2)	(-2.5)	(-0.2)	(0.2)	(-1.1)	(-1.0)	(0.2)	(-0.1)	(0.4)	(-0.3)	(1.0)	(0.4)
K_ψ	(-13.6)	(-5.7)	(-5.9)	(-12.9)	(-5.9)	(-5.1)	(-24.9)	(-11.4)	(-7.8)	(-12.6)	(-5.6)	(-5.3)

	Hungary, 2005			Taiwan, 2005			South Korea, 2006			Belgium, 2000		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3240	0.2914	0.3101	0.3231	0.3054	0.3219	0.3274	0.3124	0.3286	0.3583	0.3176	0.3315
Original $\partial^2 G/\partial\varepsilon^2$	0.0963	0.2170	0.2173	0.0879	0.1415	0.1514	0.0722	0.1276	0.1475	0.0638	0.2310	0.2670
MLN $\partial^2 G/\partial\varepsilon^2$	0.1044	0.2058	0.1986	0.0875	0.1376	0.1511	0.0669	0.1194	0.1475	0.0792	0.2114	0.2167
	(56.8)	(88.8)	(83.0)	(67.4)	(86.9)	(86.1)	(61.0)	(83.8)	(87.9)	(43.5)	(90.2)	(90.3)
Type I approximation	0.0875	0.2065	0.1910	0.0836	0.1377	0.1505	0.0652	0.1195	0.1473	0.0401	0.2122	0.2114
	(47.6)	(89.1)	(79.8)	(64.4)	(87.0)	(85.8)	(59.5)	(83.8)	(87.8)	(22.0)	(90.5)	(88.1)
I_5	0.1838	0.2317	0.2394	0.1299	0.1582	0.1755	0.1096	0.1425	0.1678	0.1822	0.2344	0.2399
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(66.9)	(98.7)	(89.2)	(68.0)	(97.7)	(94.7)	(62.6)	(96.0)	(98.3)	(58.4)	(97.5)	(91.6)
deducting I_5	(-33.1)	(-1.3)	(-10.8)	(-32.0)	(-2.3)	(-5.3)	(-37.4)	(-4.0)	(-1.7)	(-41.6)	(-2.5)	(-8.4)
K_σ	(0.2)	(1.1)	(0.0)	(4.4)	(-5.2)	(-8.0)	(4.4)	(-7.2)	(-10.4)	(-4.0)	(2.8)	(2.5)
K_ψ	(-18.5)	(-10.7)	(-8.6)	(-9.2)	(-5.2)	(-2.8)	(-8.6)	(-4.6)	(-2.2)	(-22.6)	(-9.5)	(-3.6)
I_3	(-1.1)	(0.1)	(-0.9)	(1.2)	(-0.3)	(1.8)	(1.1)	(-0.4)	(2.1)	(-9.8)	(-0.3)	(-2.3)
Type II approximation	0.0952	0.2069	0.2020	0.0821	0.1357	0.1528	0.0639	0.1178	0.1490	0.0722	0.2177	0.2240
	(51.8)	(89.3)	(84.4)	(63.2)	(85.8)	(87.1)	(58.2)	(82.6)	(88.8)	(39.6)	(92.9)	(93.3)
K_0	(71.7)	(98.8)	(92.6)	(71.6)	(97.8)	(96.8)	(65.6)	(96.2)	(99.9)	(63.7)	(97.7)	(94.2)
deducting I_5	(-28.3)	(-1.2)	(-7.4)	(-28.4)	(-2.2)	(-3.2)	(-34.4)	(-3.8)	(-0.1)	(-36.3)	(-2.3)	(-5.8)
K_σ	(0.1)	(1.1)	(0.3)	(4.7)	(-5.6)	(-6.6)	(5.6)	(-7.5)	(-8.5)	(-5.4)	(3.9)	(4.6)
K_ψ	(-19.9)	(-10.6)	(-8.5)	(-13.1)	(-6.5)	(-3.1)	(-13.0)	(-6.1)	(-2.7)	(-18.7)	(-8.7)	(-5.5)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinction.

Annex 1. The MLN *e-i* Curve and Its Factor Decompositions by the Type I and II Approximations (*Continued*)

	Australia, 2003			Canada, 2004			Ireland, 2004			Spain, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3397	0.3181	0.3401	0.3471	0.3204	0.3388	0.3490	0.3213	0.3346	0.3405	0.3215	0.3373
Original $\partial^2 G/\partial\varepsilon^2$	0.1058	0.1803	0.1833	0.1002	0.1884	0.1925	0.0873	0.1687	0.2051	0.0851	0.1418	0.1611
MLN $\partial^2 G/\partial\varepsilon^2$	0.0988	0.1761	0.1819	0.0992	0.1785	0.1859	0.0787	0.1666	0.1943	0.0863	0.1432	0.1561
	(63.7)	(89.2)	(87.4)	(63.5)	(89.1)	(87.3)	(54.6)	(87.4)	(90.4)	(33.2)	(39.2)	(44.6)
Type I approximation	0.0938	0.1765	0.1801	0.0944	0.1789	0.1841	0.0708	0.1672	0.1926	0.0841	0.1434	0.1548
	(60.4)	(89.4)	(86.5)	(60.4)	(89.3)	(86.4)	(49.1)	(87.7)	(89.6)	(32.3)	(39.2)	(44.3)
I_5	0.1552	0.1974	0.2082	0.1563	0.2003	0.2130	0.1443	0.1907	0.2150	0.2603	0.3657	0.3497
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(67.5)	(98.4)	(92.8)	(66.5)	(98.0)	(93.3)	(62.5)	(96.6)	(95.8)	(34.4)	(42.0)	(46.4)
deducting I_5	(-32.5)	(-1.6)	(-7.2)	(-33.5)	(-2.0)	(-6.7)	(-37.5)	(-3.4)	(-4.2)	(-65.6)	(-58.0)	(-53.6)
K_σ	(2.0)	(-3.2)	(-4.1)	(2.4)	(-2.6)	(-3.7)	(1.7)	(-0.7)	(-1.9)	(0.8)	(-1.1)	(-1.8)
K_ψ	(-10.0)	(-5.5)	(-2.9)	(-9.7)	(-5.9)	(-3.9)	(-15.3)	(-8.1)	(-4.5)	(-3.1)	(-1.7)	(-0.8)
I_3	(0.9)	(-0.2)	(0.8)	(1.3)	(-0.1)	(0.8)	(0.2)	(-0.2)	(0.1)	(0.2)	(-0.1)	(0.5)
Type II approximation	0.0937	0.1756	0.1857	0.0945	0.1769	0.1880	0.0724	0.1664	0.1978	0.0848	0.1425	0.1577
	(60.3)	(89.0)	(89.2)	(60.5)	(88.3)	(88.2)	(50.2)	(87.3)	(92.0)	(32.6)	(39.0)	(45.1)
K_0	(71.3)	(98.4)	(95.5)	(70.7)	(98.1)	(95.8)	(66.9)	(96.9)	(98.1)	(35.7)	(42.1)	(47.6)
deducting I_5	(-28.7)	(-1.6)	(-4.5)	(-29.3)	(-1.9)	(-4.2)	(-33.1)	(-3.1)	(-1.9)	(-64.3)	(-57.9)	(-52.4)
K_σ	(1.8)	(-3.1)	(-2.7)	(2.6)	(-3.0)	(-3.2)	(1.3)	(-0.6)	(-1.2)	(0.7)	(-1.2)	(-1.5)
K_ψ	(-12.7)	(-6.4)	(-3.6)	(-12.9)	(-6.8)	(-4.4)	(-18.0)	(-8.9)	(-4.9)	(-3.9)	(-1.9)	(-1.0)

	Poland, 2004			Greece, 2004			Italy, 2004			Estonia, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3473	0.3292	0.3655	0.3528	0.3304	0.3372	0.3639	0.3476	0.3727	0.3915	0.3506	0.3455
Original $\partial^2 G/\partial\varepsilon^2$	0.1326	0.2334	0.1865	0.0477	0.1205	0.1606	0.0912	0.1814	0.1764	0.0619	0.1427	0.2191
MLN $\partial^2 G/\partial\varepsilon^2$	0.1212	0.2082	0.1857	0.0516	0.1161	0.1562	0.1001	0.1643	0.1644	0.0628	0.1552	0.2040
	(69.3)	(95.0)	(83.0)	(54.4)	(88.8)	(96.9)	(73.4)	(94.4)	(88.0)	(46.4)	(84.4)	(95.0)
Type I approximation	0.1205	0.2082	0.1769	0.0499	0.1163	0.1556	0.1001	0.1643	0.1604	0.0509	0.1552	0.2038
	(69.0)	(95.0)	(79.0)	(52.6)	(88.9)	(96.5)	(73.4)	(94.5)	(85.9)	(37.6)	(84.4)	(95.0)
I_5	0.1747	0.2191	0.2238	0.0949	0.1308	0.1613	0.1364	0.1740	0.1869	0.1353	0.1840	0.2146
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(83.0)	(100.8)	(79.4)	(58.6)	(94.2)	(100.4)	(80.4)	(100.9)	(88.6)	(45.5)	(90.0)	(99.6)
deducting I_5	(-17.0)	(0.8)	(-20.6)	(-41.4)	(-5.8)	(0.4)	(-19.6)	(0.9)	(-11.4)	(-54.5)	(-10.0)	(-0.4)
K_σ	(-1.5)	(0.3)	(2.3)	(1.4)	(-1.2)	(-2.1)	(-1.0)	(-2.9)	(-1.6)	(1.4)	(-0.0)	(-0.8)
K_ψ	(-12.9)	(-6.2)	(-2.8)	(-7.7)	(-4.0)	(-2.2)	(-6.4)	(-3.6)	(-2.7)	(-9.4)	(-5.5)	(-3.9)
I_3	(0.4)	(0.1)	(0.2)	(0.3)	(-0.1)	(0.4)	(0.3)	(0.0)	(1.5)	(0.1)	(-0.1)	(0.1)
Type II approximation	0.1246	0.2085	0.1887	0.0509	0.1163	0.1575	0.1010	0.1638	0.1667	0.0580	0.1562	0.2054
	(71.3)	(95.2)	(84.3)	(53.7)	(88.9)	(97.7)	(74.0)	(94.2)	(89.2)	(42.9)	(84.9)	(95.7)
K_0	(84.9)	(100.9)	(86.8)	(61.4)	(94.5)	(101.7)	(82.2)	(100.9)	(93.7)	(53.0)	(91.0)	(100.3)
deducting I_5	(-15.1)	(0.9)	(-13.2)	(-38.6)	(-5.5)	(1.7)	(-17.8)	(0.9)	(-6.3)	(-47.0)	(-9.0)	(0.3)
K_σ	(-0.9)	(0.2)	(1.9)	(1.4)	(-1.3)	(-1.9)	(-0.9)	(-2.9)	(-1.2)	(1.4)	(-0.2)	(-0.6)
K_ψ	(-12.7)	(-6.0)	(-4.3)	(-9.2)	(-4.3)	(-2.2)	(-7.4)	(-3.9)	(-3.3)	(-11.6)	(-5.9)	(-4.0)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinction.

Annex 1. The MLN *e-i* Curve and Its Factor Decompositions by the Type I and II Approximations (*Continued*)

	UK, 2004			USA, 2004			Israel, 2005			Russia, 2000		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.3788	0.3567	0.3764	0.3882	0.3747	0.4033	0.3732	0.3770	0.4187	0.4534	0.4272	0.4273
Original $\partial^2 G/\partial\varepsilon^2$	0.1068	0.1735	0.1660	0.1237	0.1756	0.1493	0.1312	0.1553	0.1336	0.0624	0.1052	0.1407
MLN $\partial^2 G/\partial\varepsilon^2$	0.1070	0.1659	0.1569	0.1150	0.1659	0.1492	0.1236	0.1542	0.1339	0.0673	0.1115	0.1345
	(70.5)	(91.1)	(85.2)	(74.4)	(91.0)	(82.2)	(77.9)	(85.0)	(72.8)	(66.2)	(89.4)	(96.1)
Type I approximation	0.1044	0.1663	0.1547	0.1141	0.1662	0.1463	0.1248	0.1544	0.1167	0.0640	0.1121	0.1342
	(68.8)	(91.3)	(84.0)	(73.9)	(91.2)	(80.6)	(78.6)	(85.0)	(63.5)	(63.0)	(89.9)	(95.9)
I_5	0.1518	0.1822	0.1841	0.1545	0.1822	0.1816	0.1588	0.1815	0.1839	0.1016	0.1247	0.1400
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(77.3)	(99.4)	(91.7)	(83.0)	(100.3)	(88.5)	(93.9)	(99.9)	(77.1)	(70.0)	(95.4)	(99.9)
deducting I_5	(-22.7)	(-0.6)	(-8.3)	(-17.0)	(0.3)	(-11.5)	(-6.1)	(-0.1)	(-22.9)	(-30.0)	(-4.6)	(-0.1)
K_σ	(0.9)	(-1.2)	(-1.9)	(0.6)	(-2.1)	(-2.8)	(0.5)	(-2.4)	(-3.0)	(1.2)	(0.3)	(-0.4)
K_ψ	(-9.8)	(-6.8)	(-6.0)	(-10.0)	(-6.9)	(-5.9)	(-16.0)	(-12.3)	(-10.9)	(-7.5)	(-5.6)	(-3.6)
I_3	(0.3)	(-0.1)	(0.3)	(0.2)	(-0.1)	(0.7)	(-0.2)	(0.2)	(-0.3)	(-0.7)	(-0.1)	(0.0)
Type II approximation	0.1067	0.1660	0.1602	0.1148	0.1658	0.1536	0.1230	0.1577	0.1381	0.0695	0.1126	0.1359
	(70.3)	(91.1)	(87.0)	(74.3)	(91.0)	(84.6)	(77.5)	(86.9)	(75.1)	(68.4)	(90.3)	(97.1)
K_0	(81.1)	(99.4)	(94.7)	(85.8)	(100.3)	(93.1)	(94.7)	(100.8)	(87.9)	(75.6)	(95.9)	(100.8)
deducting I_5	(-18.9)	(-0.6)	(-5.3)	(-14.2)	(0.3)	(-6.9)	(-5.3)	(0.8)	(-12.1)	(-24.4)	(-4.1)	(0.8)
K_σ	(0.6)	(-1.2)	(-1.2)	(0.2)	(-1.9)	(-1.8)	(-0.0)	(-2.2)	(-2.1)	(0.6)	(0.1)	(0.1)
K_ψ	(-11.4)	(-7.1)	(-6.4)	(-11.7)	(-7.4)	(-6.7)	(-17.2)	(-11.8)	(-10.6)	(-7.8)	(-5.7)	(-3.8)

	Uruguay, 2004			Mexico, 2004			Brazil, 2006			Guatemala, 2006		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.4485	0.4505	0.4843	0.4673	0.4732	0.5048	0.4960	0.5043	0.5351	0.4982	0.5115	0.5491
Original $\partial^2 G/\partial\varepsilon^2$	0.1075	0.1312	0.1121	0.0877	0.1050	0.0905	0.0805	0.0942	0.0828	0.0978	0.0995	0.0763
MLN $\partial^2 G/\partial\varepsilon^2$	0.1052	0.1290	0.1149	0.0897	0.0988	0.0908	0.0811	0.0887	0.0803	0.0939	0.0929	0.0721
	(85.5)	(91.4)	(79.5)	(91.8)	(92.2)	(80.7)	(91.2)	(91.3)	(79.7)	(96.4)	(92.6)	(74.1)
Type I approximation	0.1063	0.1290	0.1020	0.0898	0.0991	0.0896	0.0815	0.0889	0.0763	0.0944	0.0922	0.0685
	(86.4)	(91.5)	(70.5)	(92.0)	(92.5)	(79.6)	(91.6)	(91.5)	(75.8)	(96.8)	(91.9)	(70.3)
I_5	0.1230	0.1411	0.1446	0.0977	0.1071	0.1125	0.0890	0.0971	0.1007	0.0975	0.1003	0.0974
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(95.8)	(100.2)	(80.8)	(96.9)	(100.4)	(86.1)	(99.1)	(99.5)	(84.2)	(99.7)	(96.6)	(76.6)
deducting I_5	(-4.2)	(0.2)	(-19.2)	(-3.1)	(0.4)	(-13.9)	(-0.9)	(-0.5)	(-15.8)	(-0.3)	(-3.4)	(-23.4)
K_σ	(1.1)	(0.9)	(0.5)	(0.1)	(-4.9)	(-7.4)	(0.7)	(-1.2)	(-2.5)	(0.4)	(-2.3)	(-4.6)
K_ψ	(-11.0)	(-9.1)	(-9.0)	(-4.6)	(-4.0)	(-3.3)	(-8.1)	(-7.0)	(-6.3)	(-2.9)	(-2.6)	(-2.2)
I_3	(0.4)	(-0.5)	(-1.7)	(-0.3)	(1.0)	(4.3)	(0.0)	(0.2)	(0.4)	(-0.3)	(0.2)	(0.5)
Type II approximation	0.1069	0.1323	0.1215	0.0887	0.0989	0.0964	0.0813	0.0908	0.0877	0.0939	0.0945	0.0799
	(86.9)	(93.8)	(84.1)	(90.8)	(92.3)	(85.7)	(91.4)	(93.5)	(87.1)	(96.4)	(94.2)	(82.1)
K_0	(96.3)	(101.3)	(91.5)	(97.3)	(101.3)	(95.1)	(99.1)	(101.3)	(95.4)	(99.7)	(98.9)	(89.4)
deducting I_5	(-3.7)	(1.3)	(-8.5)	(-2.7)	(1.3)	(-4.9)	(-0.9)	(1.3)	(-4.6)	(-0.3)	(-1.1)	(-10.6)
K_σ	(1.2)	(0.6)	(0.3)	(-0.4)	(-4.4)	(-5.8)	(0.5)	(-0.9)	(-1.9)	(0.6)	(-1.1)	(-3.3)
K_ψ	(-10.6)	(-8.1)	(-7.8)	(-6.0)	(-4.6)	(-3.7)	(-8.3)	(-6.9)	(-6.4)	(-3.9)	(-3.7)	(-4.1)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinction.

Annex 1. The MLN *e-i* Curve and Its Factor Decompositions by the Type I and II Approximations (*Continued*)

	Peru, 2004			Columbia, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
Original Gini	0.5302	0.5251	0.5453	0.5259	0.5339	0.5663
Original $\partial^2 G/\partial\varepsilon^2$	0.0804	0.1038	0.1055	0.0863	0.1014	0.0787
MLN $\partial^2 G/\partial\varepsilon^2$	0.0791	0.0982	0.1027	0.0828	0.0924	0.0847
	(90.5)	(98.8)	(93.7)	(95.6)	(95.9)	(83.2)
Type I approximation	0.0795	0.0985	0.1022	0.0829	0.0927	0.0827
	(91.0)	(99.0)	(93.2)	(95.7)	(96.2)	(81.2)
I_5	0.0874	0.0994	0.1097	0.0866	0.0963	0.1019
	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)	(100.0)
K_0	(87.9)	(100.8)	(96.1)	(97.8)	(100.7)	(86.0)
deducting I_5	(-12.1)	(0.8)	(-3.9)	(-2.2)	(0.7)	(-14.0)
K_σ	(0.8)	(-2.6)	(-5.0)	(0.8)	(-3.0)	(-5.5)
K_ψ	(1.2)	(0.3)	(-0.3)	(-2.7)	(-2.4)	(-2.1)
I_3	(1.0)	(0.4)	(2.4)	(-0.2)	(0.9)	(2.8)
Type II approximation	0.0798	0.0980	0.1072	0.0825	0.0937	0.0924
	(91.3)	(98.5)	(97.8)	(95.3)	(97.2)	(90.7)
K_0	(90.0)	(100.8)	(101.1)	(98.0)	(102.2)	(97.1)
deducting I_5	(-10.0)	(0.8)	(1.1)	(-2.0)	(2.2)	(-2.9)
K_σ	(0.6)	(-2.5)	(-3.5)	(0.5)	(-2.3)	(-4.0)
K_ψ	(0.8)	(0.2)	(0.2)	(-3.3)	(-2.7)	(-2.4)

Notes: Figures in parentheses () are the ratios to I_5 (in percent).

The approximations of K_0, K_σ, and K_ψ in the type II approximation are denoted K_0, K_σ, and K_ψ, ignoring distinction.

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation

	Slovenia, 2004			Denmark, 2004			Sweden, 2005			Finland, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.636	0.685	0.736	0.616	0.670	0.721	0.577	0.631	0.681	0.561	0.628	0.691
$\tilde{\phi}$	0.3137	0.3702	0.3545	0.2894	0.3626	0.3439	0.2902	0.3629	0.3320	0.3035	0.3626	0.3377
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3759	0.3821	0.3805	0.3731	0.3820	0.3801	0.3721	0.3811	0.3777	0.3691	0.3771	0.3737
RMS of σ_n $(\bar{\sigma})$	0.3906	0.4000	0.4112	0.3865	0.3946	0.4036	0.3989	0.4080	0.4170	0.4437	0.4564	0.4693
$\bar{\bar{\sigma}}$	0.3937	0.3982	0.4034	0.3867	0.3909	0.3957	0.4012	0.4057	0.4105	0.4465	0.4529	0.4596
$\overline{\overline{\sigma^2}}$	0.1540	0.1574	0.1616	0.1481	0.1513	0.1552	0.1600	0.1638	0.1677	0.1978	0.2034	0.2096
$\overline{\log n}$	1.2609	1.1726	1.0684	1.0831	0.9596	0.8230	1.0385	0.9021	0.7524	1.0904	0.9599	0.8204
$\mathrm{VAR}_n(\log n)$	0.1629	0.1915	0.2260	0.2313	0.2617	0.2823	0.2544	0.2891	0.3057	0.2495	0.2715	0.2843
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-31.0}	{-33.4}	{-32.0}	{-28.2}	{-20.5}	{-9.1}	{-31.3}	{-19.0}	{-2.8}	{-19.8}	{-13.6}	{-4.3}
Centred 4th moment of $\log n$	{65.8}	{70.4}	{70.2}	{65.8}	{63.5}	{59.4}	{66.4}	{61.4}	{56.7}	{69.4}	{67.8}	{63.9}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^2\right)$	{49.5}	{51.2}	{47.6}	{42.7}	{37.3}	{31.2}	{40.9}	{32.4}	{26.1}	{44.5}	{40.7}	{35.5}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^3\right)$	{76.4}	{76.5}	{67.0}	{62.7}	{51.0}	{41.4}	{59.3}	{44.4}	{37.0}	{67.5}	{58.3}	{50.6}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^4\right)$	{107.5}	{91.7}	{63.4}	{60.8}	{35.2}	{22.3}	{51.9}	{24.9}	{17.7}	{62.5}	{41.8}	{35.2}
$\mathrm{VAR}_n(\sigma_n/4\overline{\overline{\sigma^2}})$	{1.90}	{1.82}	{1.73}	{2.08}	{1.62}	{1.24}	{1.03}	{0.81}	{0.63}	{1.50}	{1.27}	{1.04}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{2.48}	{2.80}	{2.86}	{3.39}	{3.42}	{3.28}	{4.17}	{3.97}	{3.71}	{4.40}	{4.49}	{4.31}
$\mathrm{VAR}_n\left((\widehat{\psi_n}-\breve{\psi})^2\right)$	{0.092}	{0.055}	{0.025}	{0.041}	{0.029}	{0.032}	{0.053}	{0.057}	{0.070}	{0.087}	{0.071}	{0.084}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.8623	-0.8660	-0.8635	-0.5672	-0.6629	-0.7147	-0.8623	-0.8660	-0.8635	-0.9106	-0.9257	-0.9335
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \sigma_n^2\right)$	0.2656	0.0505	-0.2686	0.4657	0.2757	0.0056	0.2656	0.0505	-0.2686	0.2874	0.1254	-0.0790
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \sigma_n^2\right)$	-0.5709	-0.6596	-0.7683	-0.2494	-0.3072	-0.4011	-0.5709	-0.6596	-0.7683	-0.6176	-0.6797	-0.7502
$\mathrm{COR}_n\left((\log n-\overline{\log n})^4, \sigma_n^2\right)$	0.3654	0.2164	-0.2220	0.3952	0.3898	0.2188	0.3654	0.2164	-0.2220	0.3743	0.2418	-0.0434
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \widehat{\psi_n}\right)$	-0.8519	-0.9229	-0.9797	-0.8896	-0.9322	-0.9788	-0.8519	-0.9229	-0.9797	-0.9474	-0.9705	-0.9920
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, (\widehat{\psi_n}-\breve{\psi})^2\right)$	0.8551	0.9186	0.9022	0.9452	0.7544	0.3799	0.7178	0.3651	0.1197	0.9242	0.7835	0.4946
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \widehat{\psi_n}\right)$	0.3761	0.1846	-0.1036	0.4262	0.2649	0.0193	0.3761	0.1846	-0.1036	0.3342	0.1564	-0.0816

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Condtinued*)

	Czcch, 2004			Austria, 2004			Luxembourg, 2004			Switzerland, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon}_0$	0.605	0.645	0.681	0.553	0.567	0.586	0.450	0.472	0.495	0.352	0.395	0.438
$\tilde{\phi}$	0.3228	0.3701	0.3538	0.3189	0.3722	0.3311	0.3374	0.3741	0.3180	0.3474	0.3674	0.3031
$\phi\left(\Phi^{-1}(\widetilde{\Phi})\right)$	0.3706	0.3769	0.3751	0.3691	0.3760	0.3706	0.3716	0.3759	0.3683	0.3739	0.3759	0.3665
RMS of σ_n ($\bar{\sigma}$)	0.4695	0.4683	0.4683	0.4757	0.4830	0.4920	0.4797	0.4872	0.4951	0.4683	0.4816	0.4953
$\bar{\bar{\sigma}}$	0.4681	0.4676	0.4676	0.4770	0.4806	0.4851	0.4806	0.4845	0.4887	0.4678	0.4744	0.4815
$\overline{\overline{\sigma^2}}$	0.2188	0.2183	0.2183	0.2267	0.2301	0.2344	0.2301	0.2339	0.2380	0.2172	0.2233	0.2301
$\overline{\log n}$	1.1635	1.0756	0.9726	1.1564	1.0268	0.8790	1.1832	1.0635	0.9228	1.0473	0.9276	0.7953
$\mathrm{VAR}_n(\log n)$	0.1618	0.1904	0.2215	0.2388	0.2790	0.3097	0.2176	0.2612	0.2996	0.2239	0.2538	0.2728
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-32.6}	{-32.0}	{-27.7}	{-34.4}	{-26.8}	{-14.1}	{-39.3}	{-32.9}	{-20.9}	{-29.3}	{-20.3}	{-8.2}
Centred 4th moment of $\log n$	{60.9}	{62.9}	{61.3}	{73.3}	{70.5}	{64.6}	{73.3}	{70.8}	{64.7}	{61.1}	{58.9}	{55.7}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^2\right)$	{44.8}	{43.8}	{39.1}	{49.4}	{42.6}	{33.7}	{51.6}	{44.7}	{34.8}	{38.7}	{33.5}	{28.4}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^3\right)$	{63.7}	{59.5}	{49.5}	{78.3}	{62.4}	{48.3}	{82.2}	{66.4}	{49.3}	{54.7}	{43.5}	{35.2}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^4\right)$	{77.2}	{59.8}	{37.9}	{89.5}	{49.6}	{26.1}	{102.3}	{60.9}	{29.3}	{50.4}	{27.4}	{15.9}
$\mathrm{VAR}_n(\sigma_n/4\overline{\overline{\sigma^2}})$	{0.59}	{0.46}	{0.38}	{0.71}	{0.63}	{0.55}	{0.72}	{0.51}	{0.37}	{1.66}	{1.39}	{1.13}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{2.20}	{2.05}	{1.80}	{1.45}	{1.27}	{1.15}	{0.67}	{0.72}	{0.73}	{2.71}	{2.75}	{2.71}
$\mathrm{VAR}_n\left((\widehat{\psi_n}-\breve{\psi})^2\right)$	{0.022}	{0.014}	{0.010}	{0.002}	{0.001}	{0.001}	{0.003}	{0.002}	{0.001}	{0.028}	{0.020}	{0.020}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	0.2808	0.1100	-0.1032	-0.6466	-0.7741	-0.8574	-0.8200	-0.8537	-0.8873	-0.9469	-0.9474	-0.9447
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \sigma_n^2\right)$	0.4056	0.5744	0.7429	0.6724	0.5660	0.3567	0.1959	0.1849	0.0875	0.3818	0.1983	-0.0530
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \sigma_n^2\right)$	-0.1206	-0.2366	-0.3482	-0.5038	-0.5751	-0.6379	-0.6161	-0.7266	-0.8451	-0.6406	-0.7087	-0.7982
$\mathrm{COR}_n\left((\log n-\overline{\log n})^4, \sigma_n^2\right)$	0.3364	0.5036	0.6927	0.5937	0.6376	0.4889	0.3130	0.3061	0.1413	0.4503	0.3637	0.0494
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \widehat{\psi_n}\right)$	-0.8040	-0.8101	-0.8312	-0.2619	-0.4640	-0.6459	-0.7390	-0.7922	-0.8701	-0.8408	-0.9003	-0.9557
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, (\widehat{\psi_n}-\breve{\psi})^2\right)$	0.7339	0.6723	0.4888	0.7940	0.5424	-0.0537	0.9637	0.7999	0.2646	0.9317	0.6330	0.1639
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \widehat{\psi_n}\right)$	0.4308	0.3496	0.2264	0.4234	0.4057	0.2449	0.5318	0.4272	0.2369	0.4478	0.3004	0.0546

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	Netherland, 2004			France, 2005			Norway, 2004			Germany, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.411	0.462	0.512	0.432	0.481	0.529	0.576	0.648	0.720	0.502	0.549	0.596
$\tilde{\phi}$	0.3363	0.3681	0.3149	0.3369	0.3684	0.3219	0.3041	0.3543	0.3390	0.3243	0.3663	0.3316
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3711	0.3752	0.3679	0.3701	0.3741	0.3675	0.3651	0.3732	0.3709	0.3669	0.3726	0.3675
RMS of σ_n ($\bar{\sigma}$)	0.4823	0.4859	0.4914	0.4929	0.5003	0.5083	0.4814	0.4907	0.5000	0.5062	0.5171	0.5270
$\bar{\bar{\sigma}}$	0.4825	0.4844	0.4871	0.4946	0.4982	0.5023	0.4822	0.4869	0.4917	0.5050	0.5107	0.5161
$\overline{\overline{\sigma^2}}$	0.2321	0.2339	0.2366	0.2441	0.2477	0.2517	0.2313	0.2357	0.2405	0.2531	0.2589	0.2645
$\overline{\log n}$	1.0977	0.9730	0.8336	1.1276	1.0121	0.8812	1.0945	0.9713	0.8336	1.0140	0.8962	0.7677
$\mathrm{VAR}_n(\log n)$	0.2318	0.2658	0.2894	0.2143	0.2474	0.2745	0.2293	0.2626	0.2857	0.2221	0.2479	0.2642
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-31.2}	{-22.9}	{-10.5}	{-31.3}	{-25.4}	{-15.5}	{-31.0}	{-22.6}	{-10.4}	{-25.5}	{-17.9}	{-7.0}
Centred 4th moment of $\log n$	{66.8}	{64.1}	{59.7}	{66.3}	{65.0}	{61.0}	{65.5}	{63.1}	{59.2}	{61.3}	{58.9}	{55.1}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^2\right)$	{43.7}	{37.5}	{30.7}	{44.8}	{40.2}	{33.6}	{42.6}	{36.9}	{30.6}	{39.1}	{34.2}	{28.7}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^3\right)$	{65.2}	{52.1}	{41.4}	{67.0}	{56.0}	{44.3}	{63.4}	{50.8}	{40.5}	{52.4}	{42.3}	{35.1}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^4\right)$	{66.2}	{37.1}	{21.4}	{73.1}	{45.0}	{25.1}	{64.2}	{35.9}	{20.4}	{43.7}	{25.1}	{17.4}
$\mathrm{VAR}_n(\sigma_n/4\overline{\overline{\sigma^2}})$	{0.63}	{0.54}	{0.47}	{0.51}	{0.47}	{0.41}	{1.25}	{1.04}	{0.84}	{1.72}	{1.36}	{1.05}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{3.21}	{3.34}	{3.29}	{3.05}	{2.97}	{2.79}	{6.77}	{7.05}	{7.01}	{3.18}	{3.44}	{3.50}
$\mathrm{VAR}_n\left((\widehat{\psi_n}-\breve{\psi})^2\right)$	{0.043}	{0.022}	{0.020}	{0.050}	{0.036}	{0.034}	{0.166}	{0.110}	{0.117}	{0.034}	{0.020}	{0.020}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.3066	-0.4881	-0.6363	-0.9187	-0.9205	-0.9158	-0.7312	-0.7215	-0.6837	-0.7475	-0.7101	-0.6445
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \sigma_n^2\right)$	0.6542	0.6352	0.4942	0.4302	0.2822	0.0734	0.2608	0.0369	-0.2508	0.0817	-0.1646	-0.4523
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \sigma_n^2\right)$	-0.3276	-0.3775	-0.4144	-0.5967	-0.6302	-0.6919	-0.3375	-0.3585	-0.4263	-0.3035	-0.3501	-0.4467
$\mathrm{COR}_n\left((\log n-\overline{\log n})^4, \sigma_n^2\right)$	0.4967	0.5997	0.5533	0.4247	0.3746	0.1924	0.2306	0.1377	-0.0958	0.1491	0.0097	-0.2617
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \widehat{\psi_n}\right)$	-0.8608	-0.9120	-0.9699	-0.8595	-0.8938	-0.9427	-0.8471	-0.8992	-0.9561	-0.8191	-0.8767	-0.9292
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, (\widehat{\psi_n}-\breve{\psi})^2\right)$	0.9285	0.7435	0.2129	0.9598	0.8084	0.4576	0.9243	0.6608	0.1673	0.7789	0.4418	-0.0599
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \widehat{\psi_n}\right)$	0.4778	0.3170	0.0657	0.4282	0.3120	0.1163	0.4718	0.3243	0.0842	0.4671	0.2939	0.0373

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	Hungary, 2005			Taiwan, 2005			South Korea, 2006			Belgium, 2000		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.487	0.565	0.636	0.545	0.594	0.641	0.576	0.624	0.665	0.543	0.602	0.661
$\tilde{\phi}$	0.3246	0.3609	0.3338	0.3395	0.3643	0.3460	0.3411	0.3636	0.3499	0.3134	0.3548	0.3336
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3667	0.3722	0.3683	0.3660	0.3694	0.3661	0.3647	0.3678	0.3649	0.3604	0.3668	0.3624
RMS of σ_n ($\bar{\sigma}$)	0.5079	0.5119	0.5153	0.5334	0.5483	0.5663	0.5481	0.5656	0.5872	0.5604	0.5866	0.6076
$\bar{\bar{\sigma}}$	0.5082	0.5103	0.5121	0.5349	0.5414	0.5496	0.5479	0.5554	0.5651	0.5265	0.5390	0.5502
$\overline{\overline{\sigma^2}}$	0.2579	0.2601	0.2619	0.2833	0.2900	0.2985	0.2960	0.3039	0.3142	0.2646	0.2770	0.2887
$\overline{\log n}$	1.2061	1.0974	0.9744	1.4327	1.3553	1.2649	1.2528	1.1835	1.0974	1.1300	1.0128	0.8794
$\mathrm{VAR}_n(\log n)$	0.2035	0.2317	0.2596	0.1447	0.1663	0.1971	0.1245	0.1539	0.1917	0.2164	0.2518	0.2799
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-26.5}	{-24.9}	{-19.9}	{-24.0}	{-31.3}	{-36.0}	{-40.6}	{-43.8}	{-43.0}	{-33.5}	{-26.4}	{-15.2}
Centred 4th moment of $\log n$	{67.5}	{69.6}	{67.5}	{65.0}	{72.4}	{77.2}	{63.1}	{68.4}	{69.2}	{65.9}	{64.1}	{60.2}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^2\right)$	{47.2}	{46.4}	{41.6}	{50.5}	{55.8}	{57.5}	{50.7}	{53.0}	{50.1}	{44.3}	{38.9}	{32.2}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^3\right)$	{73.3}	{68.0}	{57.2}	{81.8}	{94.0}	{96.3}	{75.6}	{77.8}	{69.7}	{67.1}	{55.0}	{42.9}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^4\right)$	{90.3}	{65.9}	{41.3}	{137.0}	{148.8}	{135.7}	{110.6}	{101.6}	{76.0}	{74.5}	{44.5}	{23.6}
$\mathrm{VAR}_n\left(\sigma_n/4\overline{\overline{\sigma^2}}\right)$	{0.33}	{0.24}	{0.17}	{3.73}	{3.77}	{3.52}	{6.02}	{5.52}	{4.78}	{12.93}	{10.08}	{8.10}
$\mathrm{VAR}_n\left(\widehat{\psi_n}\right)$	{6.57}	{5.99}	{5.07}	{4.50}	{4.29}	{3.68}	{4.03}	{3.36}	{2.59}	{4.62}	{4.91}	{5.02}
$\mathrm{VAR}_n\left(\left(\widehat{\psi_n}-\breve{\psi}\right)^2\right)$	{0.187}	{0.148}	{0.151}	{0.031}	{0.013}	{0.009}	{0.031}	{0.027}	{0.027}	{0.130}	{0.089}	{0.073}
$\mathrm{COR}_n(\log n,\sigma_n^2)$	-0.7377	-0.6434	-0.5073	-0.9072	-0.9249	-0.9357	-0.9294	-0.9511	-0.9648	-0.6347	-0.5155	-0.3562
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2,\sigma_n^2\right)$	-0.2205	-0.3749	-0.5743	0.5603	0.5389	0.4966	0.6802	0.6483	0.6039	-0.0290	-0.2919	-0.5786
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^3,\sigma_n^2\right)$	-0.3072	-0.2826	-0.2754	-0.6437	-0.6248	-0.6213	-0.6576	-0.6850	-0.7234	-0.1079	-0.0443	-0.0379
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^4,\sigma_n^2\right)$	-0.1121	-0.2603	-0.4901	0.4222	0.4293	0.4369	0.5540	0.5906	0.6216	-0.0999	-0.2493	-0.4543
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2,\widehat{\psi_n}\right)$	-0.9005	-0.9062	-0.9243	-0.6201	-0.6408	-0.6183	-0.7149	-0.6739	-0.6443	-0.8014	-0.8559	-0.9219
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2,\left(\widehat{\psi_n}-\breve{\psi}\right)^2\right)$	0.8296	0.7600	0.5587	0.7080	0.6289	0.2334	0.7619	0.4815	0.2250	0.9426	0.7284	0.2603
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^3,\widehat{\psi_n}\right)$	0.3649	0.2469	0.0781	0.3751	0.2873	0.1815	0.3732	0.3024	0.2107	0.5062	0.3965	0.1962

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	Australia, 2003			Canada, 2004			Ireland, 2004			Spain, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon}_0$	0.531	0.576	0.623	0.542	0.591	0.640	0.548	0.613	0.674	0.524	0.557	0.590
$\tilde{\phi}$	0.3322	0.3629	0.3399	0.3282	0.3614	0.3407	0.3260	0.3589	0.3416	0.3400	0.3646	0.3442
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3623	0.3670	0.3627	0.3607	0.3663	0.3627	0.3606	0.3662	0.3631	0.3625	0.3663	0.3627
RMS of σ_n ($\bar{\sigma}$)	0.5589	0.5720	0.5874	0.5694	0.5752	0.5836	0.5666	0.5746	0.5832	0.5754	0.5816	0.5896
$\bar{\bar{\sigma}}$	0.5619	0.5680	0.5755	0.5723	0.5748	0.5786	0.5672	0.5712	0.5755	0.5776	0.5805	0.5842
$\overline{\overline{\sigma^2}}$	0.3138	0.3206	0.3290	0.3269	0.3297	0.3339	0.3205	0.3249	0.3299	0.3330	0.3363	0.3405
$\overline{\log n}$	1.1792	1.0782	0.9611	1.1770	1.0725	0.9520	1.3166	1.2189	1.1013	1.2408	1.1601	1.0648
$\mathrm{VAR}_n(\log n)$	0.1857	0.2185	0.2493	0.1927	0.2253	0.2558	0.1775	0.2146	0.2561	0.1486	0.1752	0.2067
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-34.7}	{-30.0}	{-22.2}	{-33.1}	{-28.9}	{-21.3}	{-37.9}	{-37.4}	{-32.6}	{-32.0}	{-33.5}	{-31.9}
Centred 4th moment of $\log n$	{61.0}	{62.2}	{60.9}	{63.5}	{64.4}	{62.3}	{69.7}	{73.6}	{72.7}	{59.8}	{64.1}	{65.1}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^2\right)$	{42.4}	{40.4}	{36.0}	{44.2}	{41.8}	{36.7}	{52.0}	{52.1}	{47.1}	{45.0}	{46.6}	{44.5}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^3\right)$	{65.0}	{58.1}	{47.3}	{67.6}	{60.0}	{48.6}	{88.3}	{85.5}	{72.3}	{66.7}	{67.7}	{61.0}
$\mathrm{VAR}_n\left(\left(\log n-\overline{\log n}\right)^4\right)$	{81.9}	{57.5}	{32.6}	{83.2}	{57.6}	{32.6}	{138.4}	{112.5}	{72.0}	{92.1}	{81.2}	{58.9}
$\mathrm{VAR}_n\left(\sigma_n/4\overline{\overline{\sigma^2}}\right)$	{1.55}	{1.47}	{1.33}	{0.43}	{0.54}	{0.67}	{1.21}	{0.99}	{0.78}	{0.60}	{0.64}	{0.66}
$\mathrm{VAR}_n\left(\widehat{\psi_n}\right)$	{2.62}	{2.81}	{2.87}	{3.42}	{3.29}	{3.10}	{5.27}	{4.85}	{4.13}	{1.59}	{1.50}	{1.41}
$\mathrm{VAR}_n\left(\left(\widehat{\psi_n}-\breve{\psi}\right)^2\right)$	{0.063}	{0.037}	{0.019}	{0.071}	{0.050}	{0.041}	{0.121}	{0.077}	{0.062}	{0.028}	{0.023}	{0.016}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.9242	-0.9474	-0.9623	-0.6573	-0.7381	-0.8034	-0.6925	-0.6780	-0.6664	-0.8128	-0.8619	-0.9011
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2, \sigma_n^2\right)$	0.6497	0.5413	0.3953	0.8807	0.8688	0.7972	0.2105	0.1616	0.1098	0.6723	0.6766	0.6532
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^3, \sigma_n^2\right)$	-0.6826	-0.7175	-0.7718	-0.9127	-0.9202	-0.8969	-0.3632	-0.4183	-0.5053	-0.7396	-0.7805	-0.8170
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^4, \sigma_n^2\right)$	0.5762	0.5844	0.5390	0.9146	0.9191	0.8605	0.2580	0.2861	0.2947	0.6329	0.6752	0.6978
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2, \widehat{\psi_n}\right)$	-0.8395	-0.8754	-0.9231	-0.8118	-0.8457	-0.8928	-0.8021	-0.8002	-0.8146	-0.7858	-0.8047	-0.8298
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^2, \left(\widehat{\psi_n}-\breve{\psi}\right)^2\right)$	0.9471	0.9582	0.7639	0.9348	0.8872	0.6279	0.7657	0.6534	0.3993	0.8958	0.8678	0.7349
$\mathrm{COR}_n\left(\left(\log n-\overline{\log n}\right)^3, \widehat{\psi_n}\right)$	0.5677	0.4807	0.3287	0.4455	0.3753	0.2382	0.4586	0.3683	0.2390	0.4525	0.4416	0.3851

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	Poland, 2004			Greece, 2004			Italy, 2004			Estonia, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.346	0.403	0.462	0.623	0.661	0.689	0.402	0.442	0.475	0.720	0.767	0.806
$\tilde{\phi}$	0.3418	0.3586	0.3130	0.3329	0.3600	0.3516	0.3417	0.3578	0.3256	0.3035	0.3497	0.3533
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3611	0.3644	0.3567	0.3596	0.3644	0.3628	0.3576	0.3606	0.3543	0.3510	0.3605	0.3610
RMS of σ_n ($\bar{\sigma}$)	0.5996	0.5911	0.5826	0.5916	0.5945	0.5983	0.6314	0.6313	0.6335	0.6126	0.6162	0.6203
$\bar{\bar{\sigma}}$	0.5947	0.5906	0.5865	0.5933	0.5947	0.5966	0.6299	0.6299	0.6310	0.6151	0.6169	0.6189
$\overline{\overline{\sigma^2}}$	0.3529	0.3480	0.3432	0.3519	0.3535	0.3557	0.3958	0.3957	0.3972	0.3781	0.3803	0.3829
$\overline{\log n}$	1.2905	1.1732	1.0351	1.1788	1.1111	1.0247	1.1442	1.0459	0.9275	1.1919	1.0863	0.9623
$\mathrm{VAR}_n(\log n)$	0.2150	0.2552	0.2965	0.1196	0.1528	0.1935	0.1778	0.2165	0.2561	0.1940	0.2295	0.2658
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-34.9}	{-33.0}	{-26.0}	{-48.3}	{-48.9}	{-44.7}	{-40.5}	{-37.4}	{-28.7}	{-34.1}	{-32.4}	{-25.5}
Centred 4th moment of $\log n$	{76.8}	{79.2}	{75.4}	{64.5}	{66.6}	{64.2}	{68.8}	{67.3}	{61.6}	{70.7}	{71.2}	{66.8}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^2\right)$	{55.3}	{53.6}	{45.8}	{52.6}	{51.3}	{44.8}	{51.1}	{45.7}	{36.0}	{51.3}	{48.2}	{40.2}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^3\right)$	{96.9}	{87.3}	{69.4}	{73.4}	{69.7}	{57.4}	{73.6}	{61.9}	{46.0}	{78.8}	{69.4}	{54.4}
$\mathrm{VAR}_n\left(\left(\log n - \overline{\log n}\right)^4\right)$	{142.5}	{99.7}	{54.9}	{97.3}	{81.7}	{55.0}	{87.5}	{58.3}	{30.3}	{98.5}	{68.1}	{38.1}
$\mathrm{VAR}_n\left(\sigma_n/4\overline{\overline{\sigma^2}}\right)$	{0.60}	{0.46}	{0.34}	{0.15}	{0.15}	{0.16}	{0.70}	{0.55}	{0.46}	{0.15}	{0.13}	{0.11}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{3.20}	{3.55}	{3.58}	{3.99}	{2.88}	{2.01}	{3.23}	{2.37}	{1.67}	{3.12}	{2.46}	{1.82}
$\mathrm{VAR}_n\left(\left(\widehat{\psi_n} - \breve{\psi}\right)^2\right)$	{0.117}	{0.052}	{0.017}	{0.194}	{0.145}	{0.097}	{0.034}	{0.038}	{0.036}	{0.027}	{0.031}	{0.035}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	0.8422	0.8383	0.8450	-0.9035	-0.9287	-0.9488	0.1307	-0.0965	-0.3234	-0.7317	-0.7669	-0.8021
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \sigma_n^2\right)$	-0.0100	0.0121	0.0680	0.7245	0.7627	0.7569	0.5820	0.7113	0.8014	0.3150	0.3078	0.2643
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^3, \sigma_n^2\right)$	0.5273	0.6370	0.7769	-0.8123	-0.8425	-0.8716	-0.1819	-0.2784	-0.3450	-0.5348	-0.6176	-0.7120
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^4, \sigma_n^2\right)$	-0.2093	-0.1885	-0.0619	0.6752	0.7338	0.7700	0.4211	0.5746	0.7396	0.3866	0.4269	0.3946
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \widehat{\psi_n}\right)$	-0.8219	-0.8501	-0.8924	-0.5861	-0.5376	-0.5088	-0.6835	-0.6937	-0.7366	-0.7962	-0.7981	-0.8237
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^2, \left(\widehat{\psi_n} - \breve{\psi}\right)^2\right)$	0.8638	0.8950	0.7023	0.2678	0.1637	0.0924	0.5739	0.4261	0.3586	0.7103	0.5675	0.4687
$\mathrm{COR}_n\left(\left(\log n - \overline{\log n}\right)^3, \widehat{\psi_n}\right)$	0.5901	0.4648	0.2776	0.2415	0.1902	0.1158	0.2652	0.1863	0.0569	0.2958	0.1887	0.0296

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	UK, 2004			USA, 2004			Israel, 2005			Russia, 2000		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.487	0.547	0.603	0.408	0.471	0.534	0.197	0.303	0.397	0.662	0.697	0.739
$\tilde{\phi}$	0.3279	0.3538	0.3320	0.3326	0.3497	0.3206	0.3413	0.3381	0.2964	0.3108	0.3370	0.3336
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3537	0.3587	0.3541	0.3509	0.3542	0.3474	0.3547	0.3539	0.3434	0.3345	0.3411	0.3400
RMS of σ_n ($\bar{\sigma}$)	0.6367	0.6438	0.6520	0.6717	0.6818	0.6938	0.6578	0.6640	0.6717	0.7742	0.7819	0.7895
$\bar{\bar{\sigma}}$	0.6394	0.6429	0.6469	0.6749	0.6797	0.6856	0.6571	0.6604	0.6643	0.7766	0.7805	0.7844
$\overline{\overline{\sigma^2}}$	0.4083	0.4127	0.4178	0.4546	0.4610	0.4688	0.4304	0.4348	0.4400	0.6021	0.6081	0.6142
$\overline{\log n}$	1.1039	0.9930	0.8707	1.1902	1.0721	0.9406	1.4193	1.3029	1.1675	1.2927	1.1969	1.0878
$\mathrm{VAR}_n(\log n)$	0.2094	0.2341	0.2537	0.2217	0.2504	0.2745	0.2161	0.2507	0.2915	0.1795	0.2042	0.2327
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-24.5}	{-19.6}	{-12.1}	{-26.4}	{-21.9}	{-14.3}	{-27.7}	{-30.8}	{-28.4}	{-24.8}	{-26.3}	{-25.5}
Centred 4th moment of $\log n$	{61.4}	{61.7}	{59.8}	{67.1}	{68.0}	{66.0}	{82.7}	{86.1}	{83.9}	{64.1}	{69.4}	{70.9}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^2\right)$	{40.4}	{38.3}	{34.4}	{44.9}	{43.0}	{38.6}	{61.1}	{61.0}	{54.7}	{46.2}	{49.0}	{47.6}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^3\right)$	{58.6}	{51.5}	{43.2}	{71.3}	{64.0}	{53.8}	{114.4}	{113.2}	{97.6}	{73.2}	{75.4}	{69.2}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^4\right)$	{59.6}	{39.4}	{25.0}	{84.6}	{57.4}	{35.5}	{188.6}	{160.8}	{110.9}	{104.3}	{90.7}	{65.7}
$\mathrm{VAR}_n\left(\sigma_n/4\overline{\overline{\sigma^2}}\right)$	{0.30}	{0.31}	{0.32}	{0.43}	{0.46}	{0.48}	{0.55}	{0.43}	{0.37}	{0.49}	{0.42}	{0.34}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{4.78}	{4.36}	{3.92}	{4.28}	{4.45}	{4.43}	{10.76}	{9.24}	{7.32}	{1.96}	{2.39}	{2.75}
$\mathrm{VAR}_n\left((\widehat{\psi_n}-\breve{\psi})^2\right)$	{0.130}	{0.122}	{0.122}	{0.145}	{0.089}	{0.066}	{0.666}	{0.488}	{0.411}	{0.068}	{0.055}	{0.031}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.8880	-0.9119	-0.9321	-0.9380	-0.9583	-0.9728	-0.5330	-0.6293	-0.7387	-0.7738	-0.7484	-0.7010
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \sigma_n^2\right)$	0.5373	0.4600	0.3338	0.6434	0.5446	0.4062	0.2167	0.3970	0.4930	0.1086	0.0144	-0.1179
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \sigma_n^2\right)$	-0.7671	-0.7935	-0.8128	-0.8070	-0.8278	-0.8448	-0.6318	-0.6880	-0.7273	-0.3815	-0.3504	-0.3317
$\mathrm{COR}_n\left((\log n-\overline{\log n})^4, \sigma_n^2\right)$	0.5683	0.5342	0.3752	0.6611	0.6334	0.4887	0.2370	0.3442	0.4123	0.0855	0.0264	-0.0662
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \widehat{\psi_n}\right)$	-0.8665	-0.8995	-0.9361	-0.8933	-0.9191	-0.9510	-0.8594	-0.8515	-0.8607	-0.6522	-0.7215	-0.7777
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, (\widehat{\psi_n}-\breve{\psi})^2\right)$	0.8748	0.7425	0.5253	0.9398	0.9527	0.6809	0.8030	0.7716	0.5829	0.8475	0.9064	0.8966
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \widehat{\psi_n}\right)$	0.3229	0.2264	0.0582	0.4450	0.3331	0.1539	0.2965	0.2257	0.1243	0.5428	0.5173	0.4332

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

	Uruguay, 2004			Mexico, 2004			Brazil, 2006			Guatemala, 2006		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\widehat{\varepsilon_0}$	0.177	0.254	0.316	0.178	0.227	0.281	0.072	0.144	0.210	0.040	0.080	0.151
$\tilde{\phi}$	0.3279	0.3249	0.2899	0.3261	0.3210	0.2916	0.3172	0.3086	0.2789	0.3130	0.3012	0.2671
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3342	0.3337	0.3235	0.3287	0.3263	0.3158	0.3192	0.3162	0.3054	0.3166	0.3119	0.2985
RMS of σ_n ($\bar{\sigma}$)	0.8302	0.8230	0.8147	0.8742	0.8928	0.9151	0.9393	0.9449	0.9505	0.9575	0.9738	0.9889
$\bar{\bar{\sigma}}$	0.8244	0.8210	0.8171	0.8706	0.8794	0.8899	0.9376	0.9405	0.9434	0.9473	0.9560	0.9642
$\overline{\overline{\sigma^2}}$	0.6780	0.6725	0.6661	0.7539	0.7692	0.7873	0.8782	0.8838	0.8893	0.8920	0.9088	0.9249
$\overline{\log n}$	1.3106	1.1931	1.0581	1.5536	1.4561	1.3440	1.3632	1.2647	1.1525	1.7247	1.6165	1.4983
$\mathrm{VAR}_n(\log n)$	0.2188	0.2520	0.2881	0.1843	0.2075	0.2427	0.1860	0.2093	0.2410	0.2086	0.2252	0.2485
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-27.4}	{-28.5}	{-24.0}	{-19.6}	{-27.8}	{-34.2}	{-20.2}	{-26.5}	{-29.0}	{-13.7}	{-17.3}	{-22.1}
Centred 4th moment of $\log n$	{79.6}	{80.8}	{77.0}	{74.7}	{86.0}	{92.7}	{74.4}	{79.7}	{80.1}	{69.5}	{79.5}	{88.8}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^2\right)$	{57.7}	{55.6}	{48.2}	{56.2}	{65.2}	{68.4}	{55.8}	{58.7}	{56.0}	{48.6}	{57.0}	{64.0}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^3\right)$	{101.8}	{94.7}	{78.0}	{106.8}	{126.5}	{130.2}	{94.8}	{99.1}	{90.7}	{93.9}	{116.4}	{133.2}
$\mathrm{VAR}_n\left((\log n-\overline{\log n})^4\right)$	{146.1}	{113.2}	{73.3}	{205.6}	{224.0}	{199.3}	{143.1}	{133.7}	{102.9}	{190.1}	{227.5}	{237.7}
$\mathrm{VAR}_n\left(\sigma_n/4\overline{\overline{\sigma^2}}\right)$	{0.49}	{0.39}	{0.30}	{1.41}	{1.43}	{1.45}	{0.22}	{0.15}	{0.12}	{1.35}	{1.08}	{0.82}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{7.11}	{5.24}	{3.61}	{2.57}	{2.77}	{2.73}	{5.01}	{4.65}	{3.95}	{8.15}	{8.63}	{9.49}
$\mathrm{VAR}_n\left((\widehat{\psi_n}-\breve{\psi})^2\right)$	{0.543}	{0.526}	{0.420}	{0.119}	{0.097}	{0.055}	{0.204}	{0.142}	{0.102}	{0.897}	{1.246}	{1.128}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	0.5012	0.6105	0.6609	-0.9019	-0.9092	-0.9223	-0.7070	-0.7047	-0.7377	-0.7281	-0.6895	-0.6532
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \sigma_n^2\right)$	-0.3677	-0.2067	-0.0062	0.3567	0.4594	0.5340	-0.1829	0.0069	0.2275	-0.1045	-0.1113	-0.0932
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \sigma_n^2\right)$	0.1070	0.1881	0.2857	-0.7590	-0.7990	-0.8302	-0.6580	-0.7742	-0.8957	-0.4211	-0.4339	-0.4500
$\mathrm{COR}_n\left((\log n-\overline{\log n})^4, \sigma_n^2\right)$	-0.3064	-0.2836	-0.2003	0.4691	0.5381	0.5877	0.1044	0.2164	0.3243	0.0687	0.0546	0.0361
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, \widehat{\psi_n}\right)$	-0.8400	-0.8188	-0.8208	-0.7372	-0.7837	-0.7984	-0.8575	-0.8499	-0.8432	-0.2476	-0.5059	-0.6879
$\mathrm{COR}_n\left((\log n-\overline{\log n})^2, (\widehat{\psi_n}-\breve{\psi})^2\right)$	0.6133	0.4922	0.4061	0.6707	0.7253	0.7284	0.7909	0.8209	0.6817	0.5673	0.6301	0.6793
$\mathrm{COR}_n\left((\log n-\overline{\log n})^3, \widehat{\psi_n}\right)$	0.1451	0.0683	-0.0464	0.4694	0.4289	0.3622	0.3045	0.2555	0.1832	0.4887	0.5189	0.4869

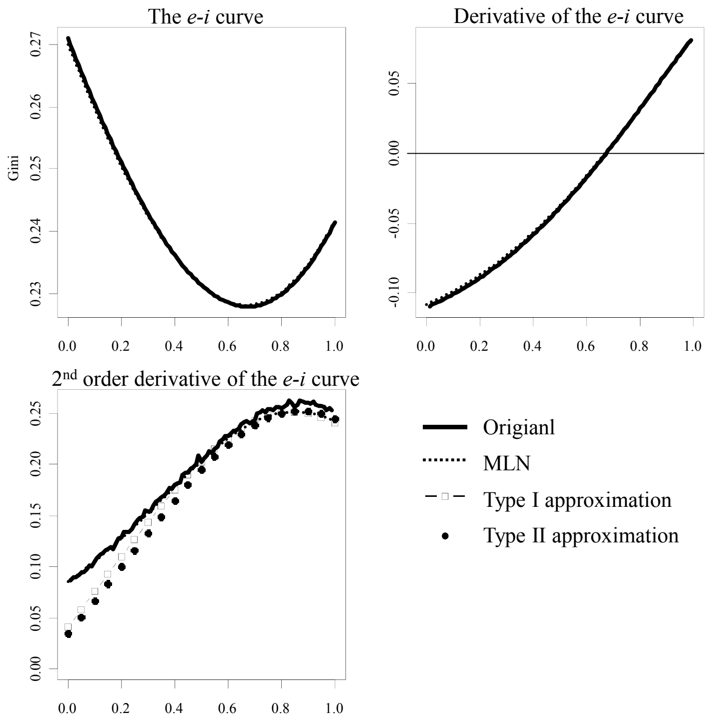
Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 2. Components in Factor Decomposition of the Second-Order Derivative of the MLN *e-i* Curve by the Type II Approximation (*Conditnued*)

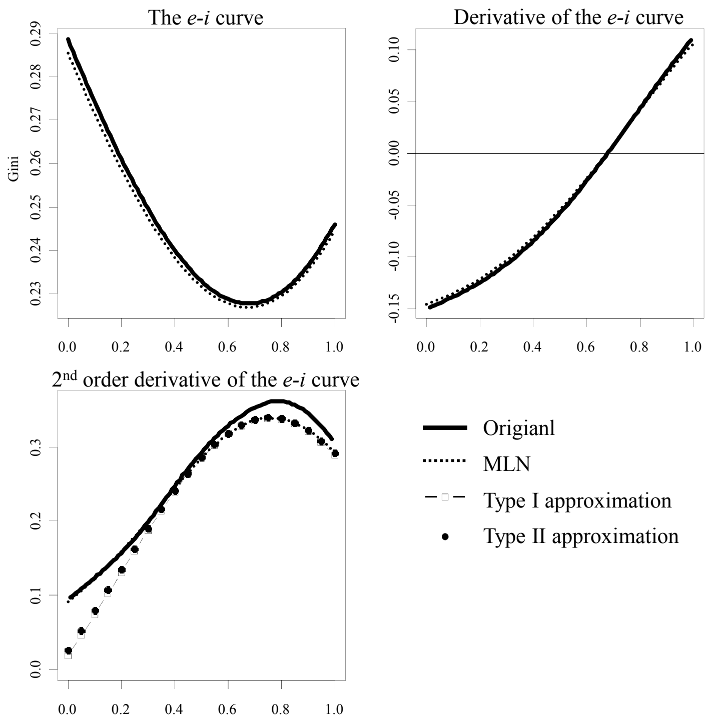
	Peru, 2004			Columbia, 2004		
	ε=0	ε=0.5	ε=1	ε=0	ε=0.5	ε=1
$\hat{\varepsilon}_0$	0.458	0.434	0.442	0.131	0.161	0.195
$\tilde{\phi}$	0.2987	0.3041	0.2858	0.3078	0.3010	0.2722
$\phi\left(\Phi^{-1}(\tilde{\Phi})\right)$	0.3074	0.3089	0.3015	0.3091	0.3060	0.2942
RMS of σ_n ($\bar{\sigma}$)	0.9946	1.0052	1.0210	1.0082	1.0227	1.0387
$\bar{\bar{\sigma}}$	0.9887	0.9940	1.0017	1.0040	1.0116	1.0198
$\overline{\overline{\sigma^2}}$	0.9735	0.9839	0.9990	1.0038	1.0193	1.0360
$\overline{\log n}$	1.6821	1.5741	1.4495	1.5221	1.4154	1.2912
$\mathrm{VAR}_n(\log n)$	0.2045	0.2298	0.2718	0.1997	0.2289	0.2699
	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}	{100.0}
Centred 3rd moment of $\log n$	{-18.5}	{-28.6}	{-37.8}	{-23.7}	{-30.7}	{-34.3}
Centred 4th moment of $\log n$	{82.2}	{98.4}	{109.3}	{80.1}	{90.0}	{93.8}
$\mathrm{VAR}_n\left((\log n - \overline{\log n})^2\right)$	{61.7}	{75.4}	{82.2}	{60.1}	{67.2}	{66.8}
$\mathrm{VAR}_n\left((\log n - \overline{\log n})^3\right)$	{137.6}	{170.7}	{179.8}	{118.2}	{131.9}	{126.0}
$\mathrm{VAR}_n\left((\log n - \overline{\log n})^4\right)$	{312.4}	{356.1}	{322.4}	{224.4}	{222.4}	{175.9}
$\mathrm{VAR}_n(\sigma_n/4\overline{\overline{\sigma^2}})$	{0.97}	{0.88}	{0.82}	{0.89}	{0.69}	{0.57}
$\mathrm{VAR}_n(\widehat{\psi_n})$	{18.26}	{12.71}	{8.46}	{1.55}	{1.49}	{1.42}
$\mathrm{VAR}_n\left((\widehat{\psi_n} - \breve{\psi})^2\right)$	{2.793}	{1.890}	{0.935}	{0.040}	{0.034}	{0.022}
$\mathrm{COR}_n(\log n, \sigma_n^2)$	-0.4141	-0.6001	-0.7488	-0.7498	-0.7679	-0.8089
$\mathrm{COR}_n\left((\log n - \overline{\log n})^2, \sigma_n^2\right)$	0.5551	0.5697	0.5903	0.0443	0.2061	0.3701
$\mathrm{COR}_n\left((\log n - \overline{\log n})^3, \sigma_n^2\right)$	-0.3529	-0.4994	-0.6310	-0.6225	-0.7354	-0.8406
$\mathrm{COR}_n\left((\log n - \overline{\log n})^4, \sigma_n^2\right)$	0.4425	0.5377	0.6177	0.2859	0.3903	0.4807
$\mathrm{COR}_n\left((\log n - \overline{\log n})^2, \widehat{\psi_n}\right)$	0.2561	0.0440	-0.1583	-0.5499	-0.6540	-0.7201
$\mathrm{COR}_n\left((\log n - \overline{\log n})^2, (\widehat{\psi_n} - \breve{\psi})^2\right)$	0.3363	0.3023	0.2484	0.5827	0.6540	0.6347
$\mathrm{COR}_n\left((\log n - \overline{\log n})^3, \widehat{\psi_n}\right)$	0.2246	0.2756	0.2844	0.4120	0.4134	0.3672

Note: Figures in parentheses () are the ratios to $\mathrm{VAR}_n(\log n)$ (in percent).

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations

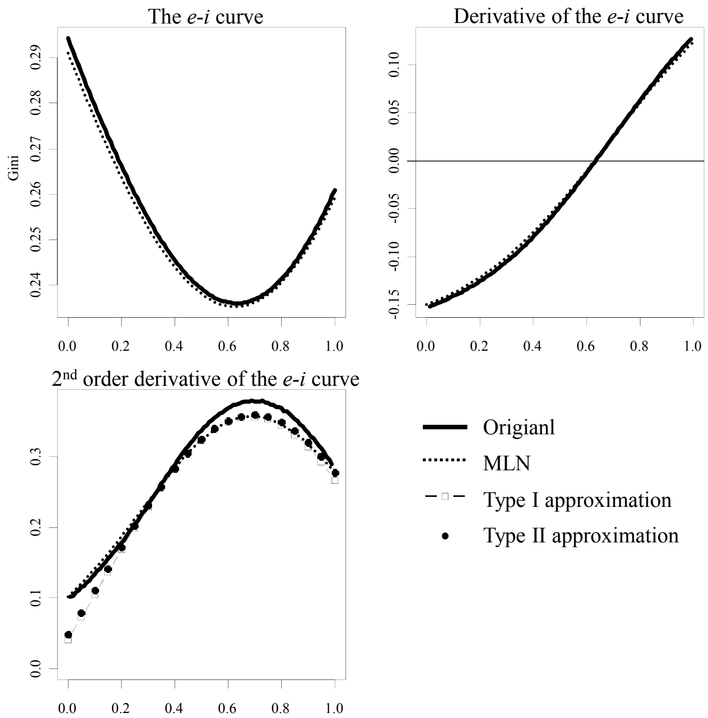


Slovenia, 2004

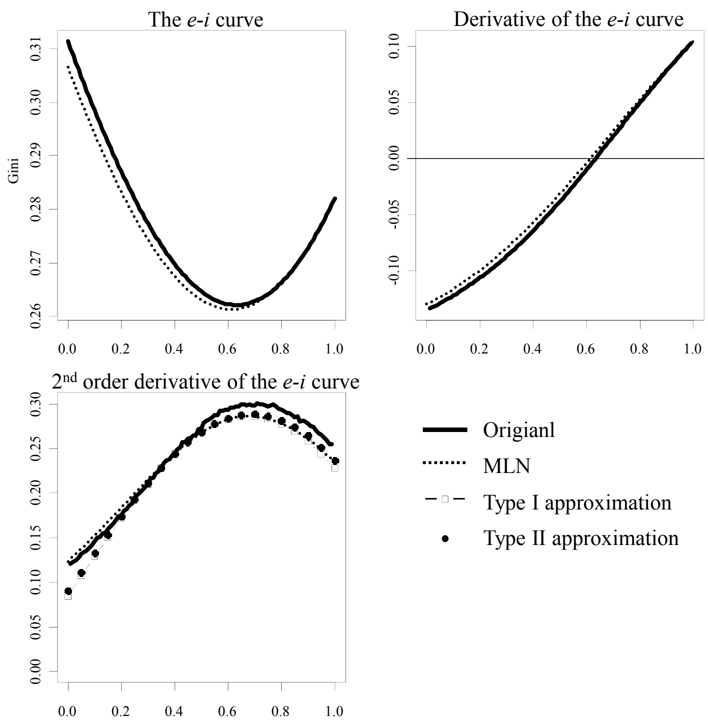


Denmark, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

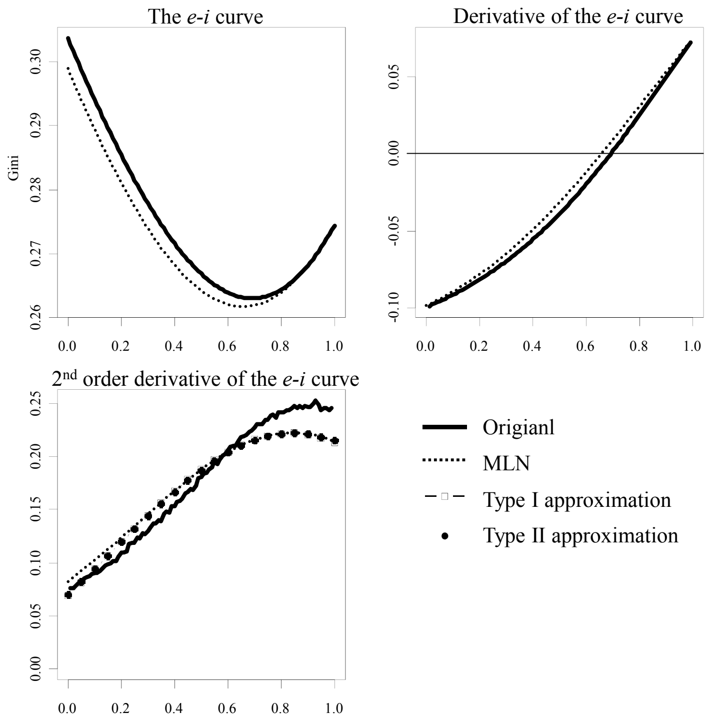


Sweden, 2005

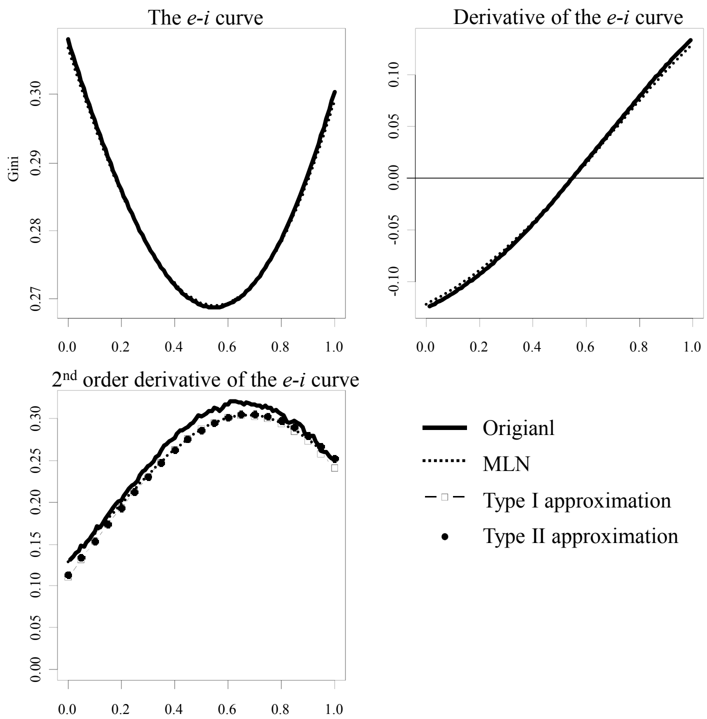


Finland, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

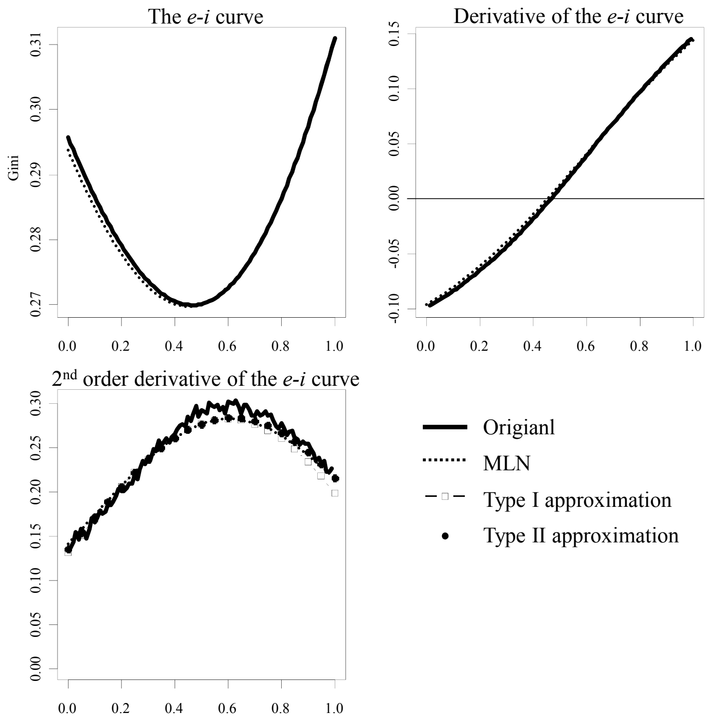


Czech Rep, 2004

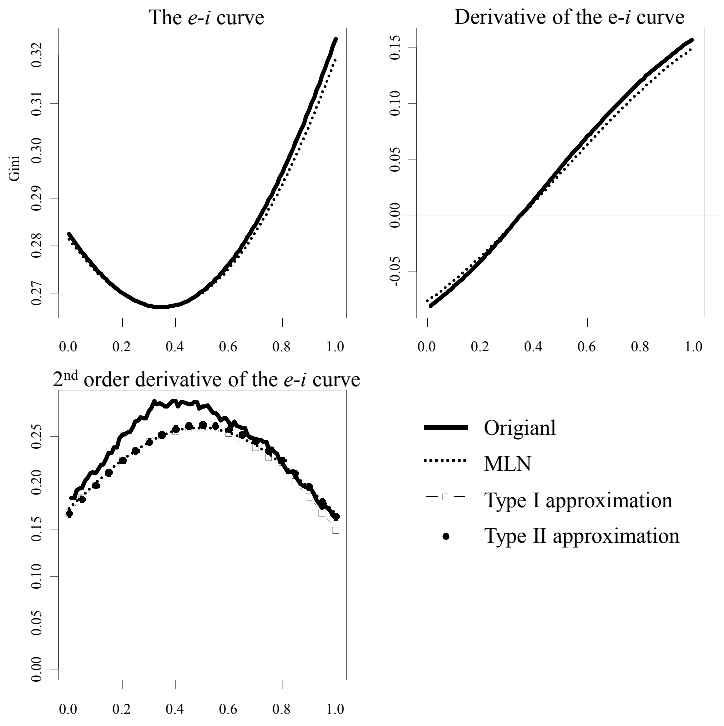


Austria, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

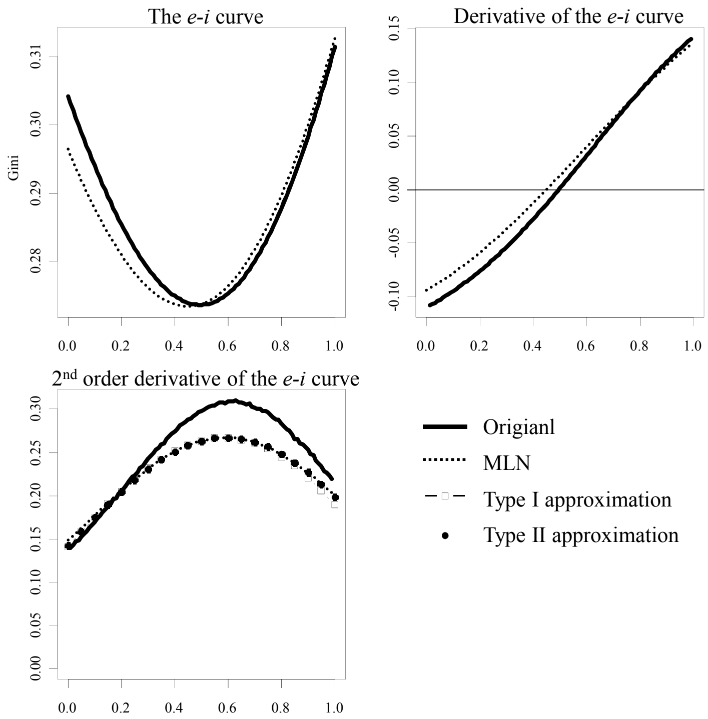


Luxembourg, 2004

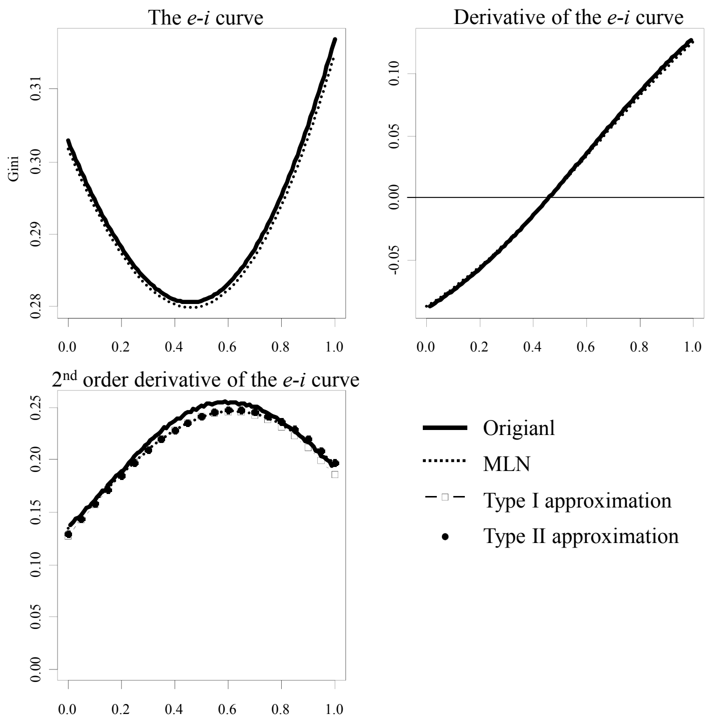


Switzerland, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

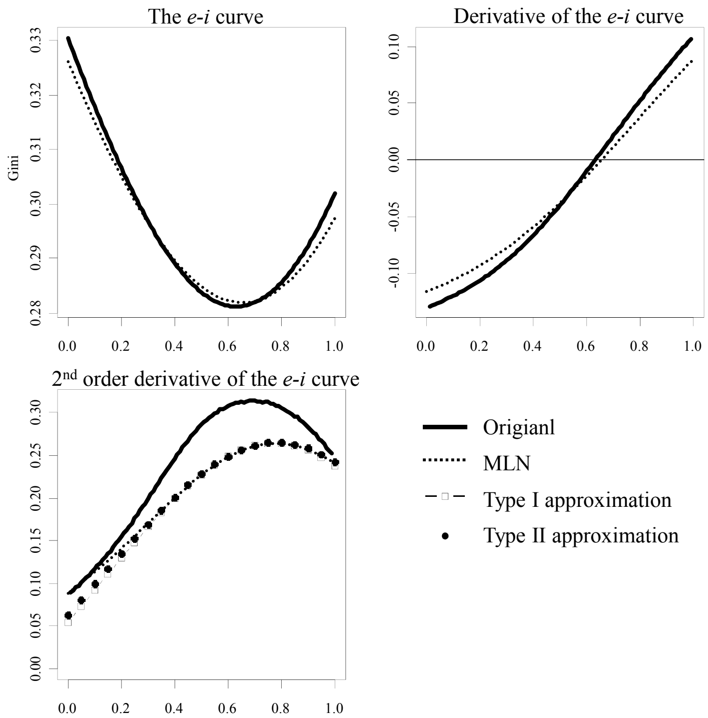


Netherland, 2004

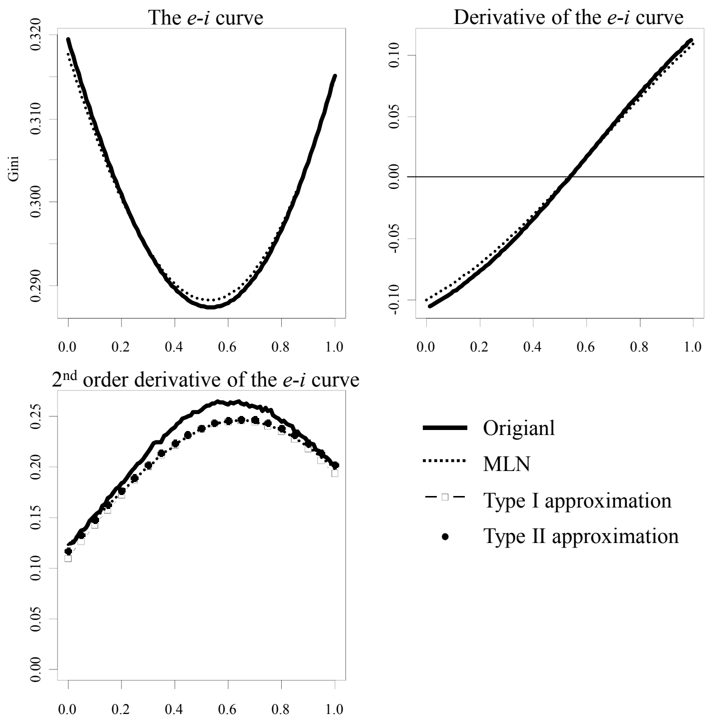


France, 2005

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

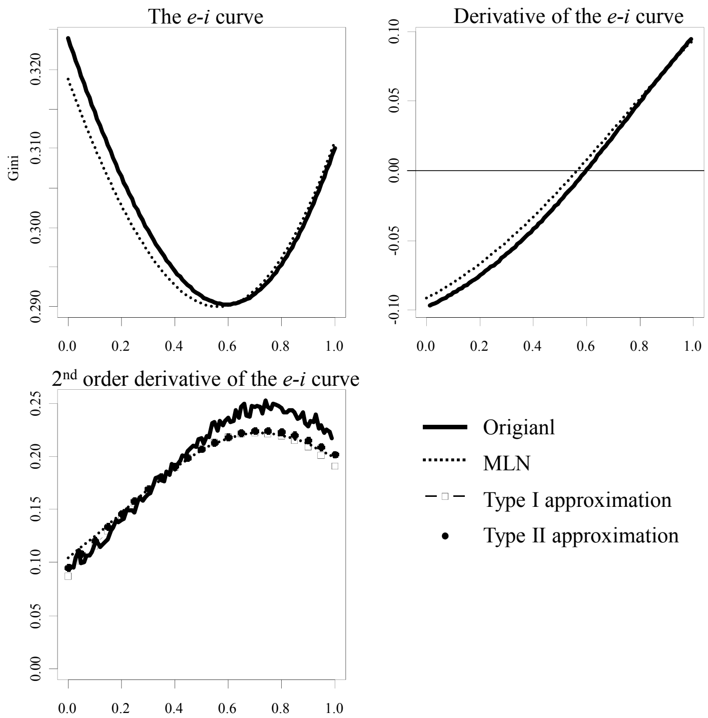


Norway, 2004

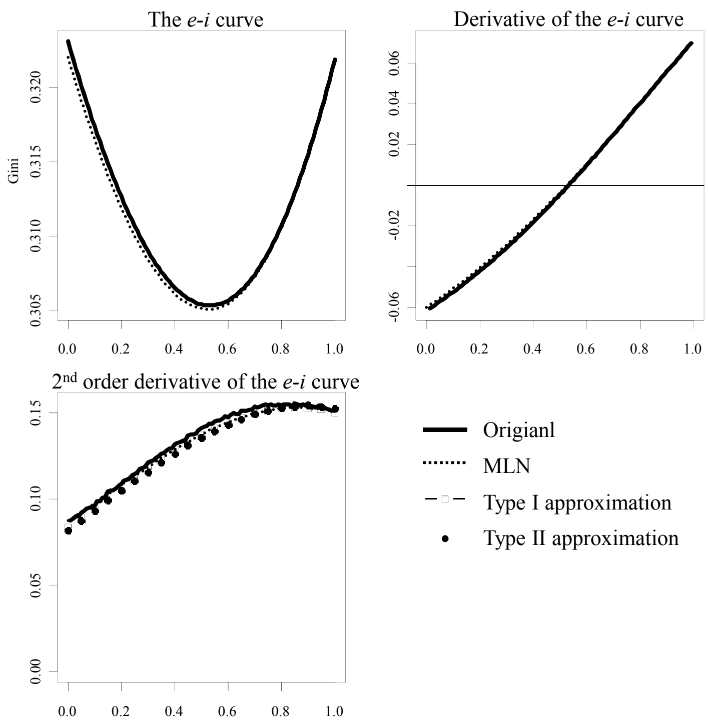


Germany, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

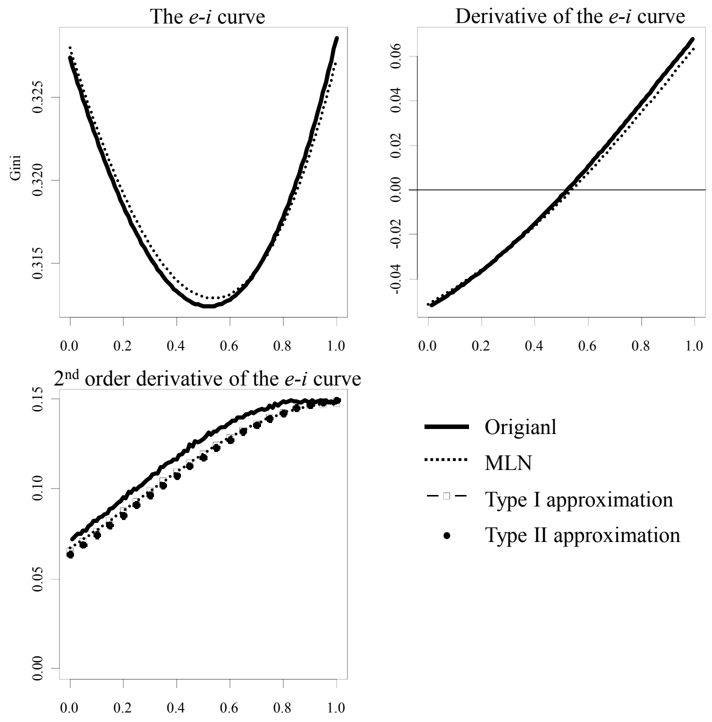


Hungary, 2005

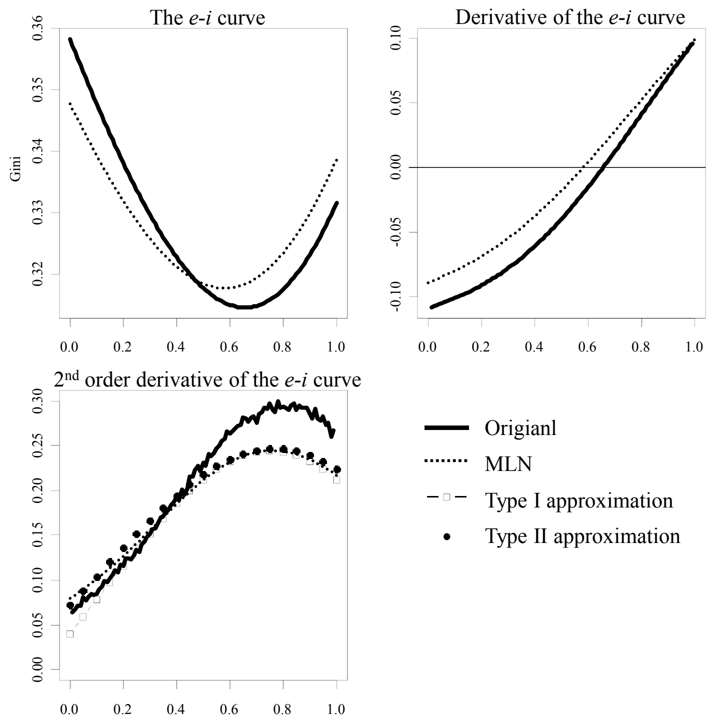


Taiwan, 2005

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

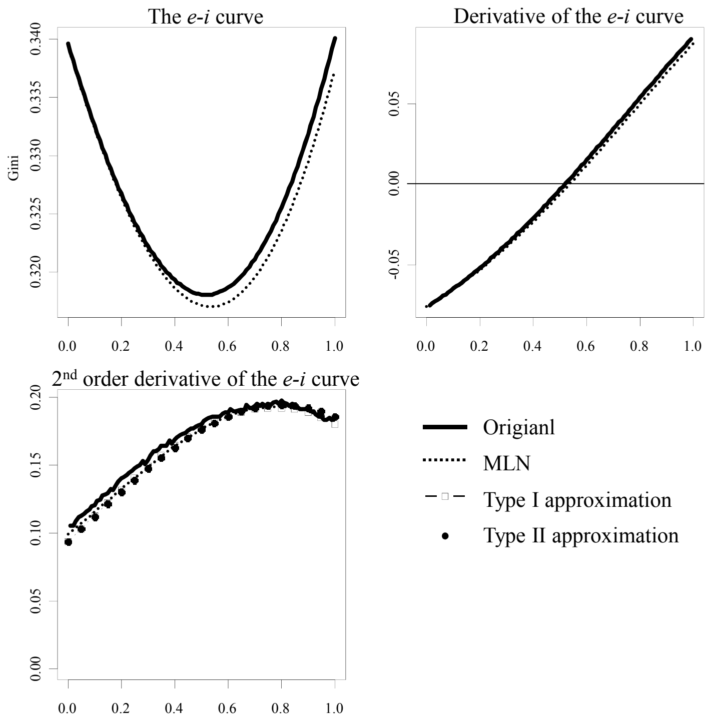


South Korea, 2006

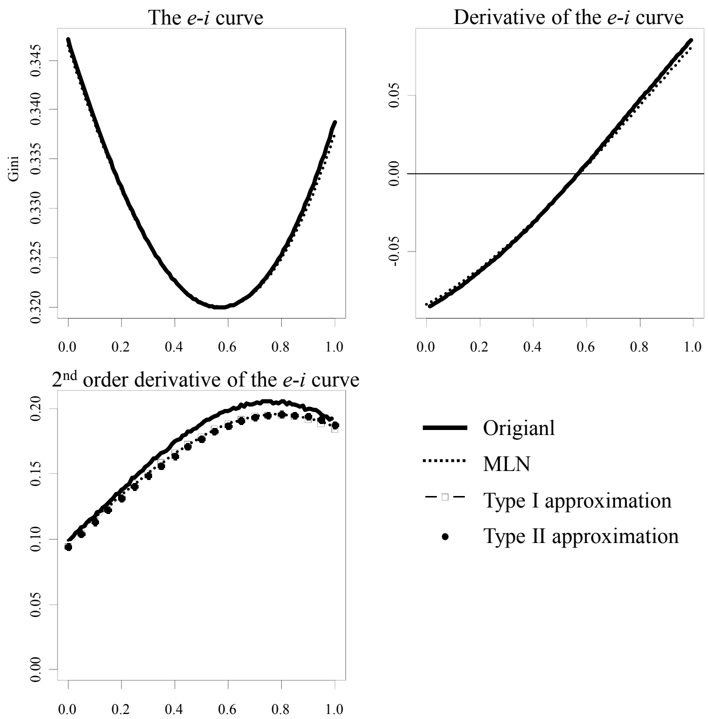


Belgium, 2000

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

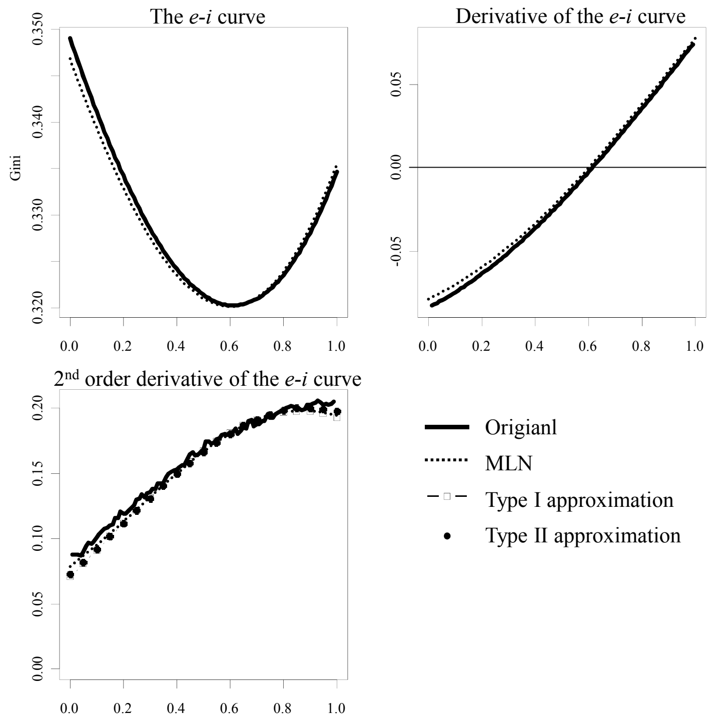


Australia, 2003

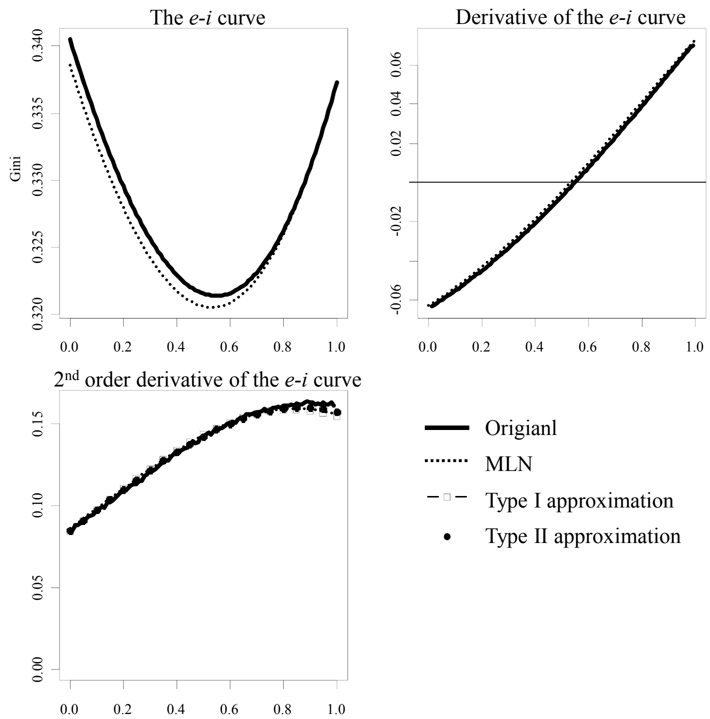


Canada, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

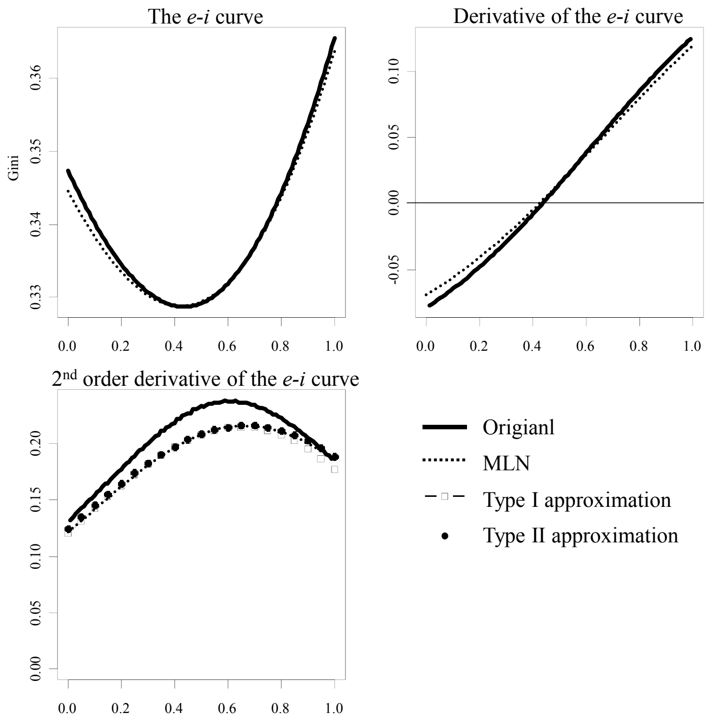


Ireland, 2004

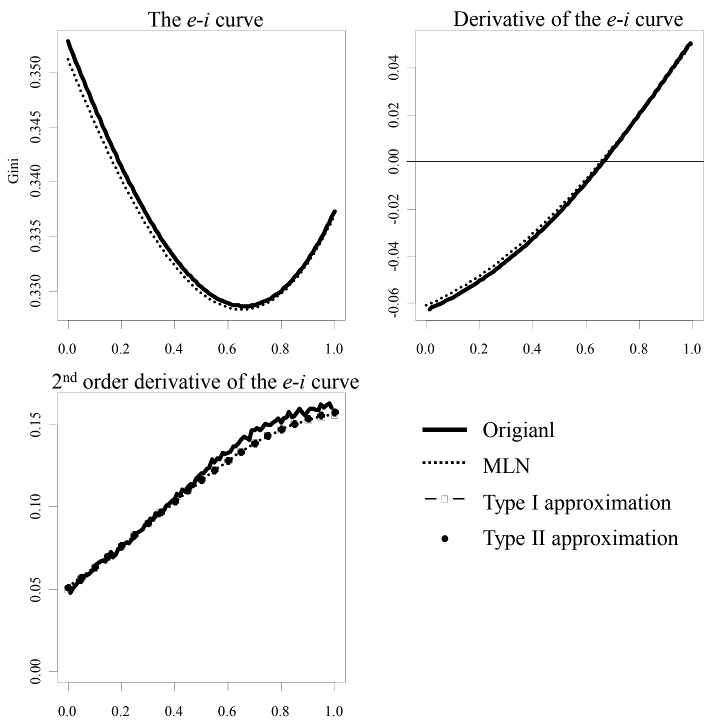


Spain, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

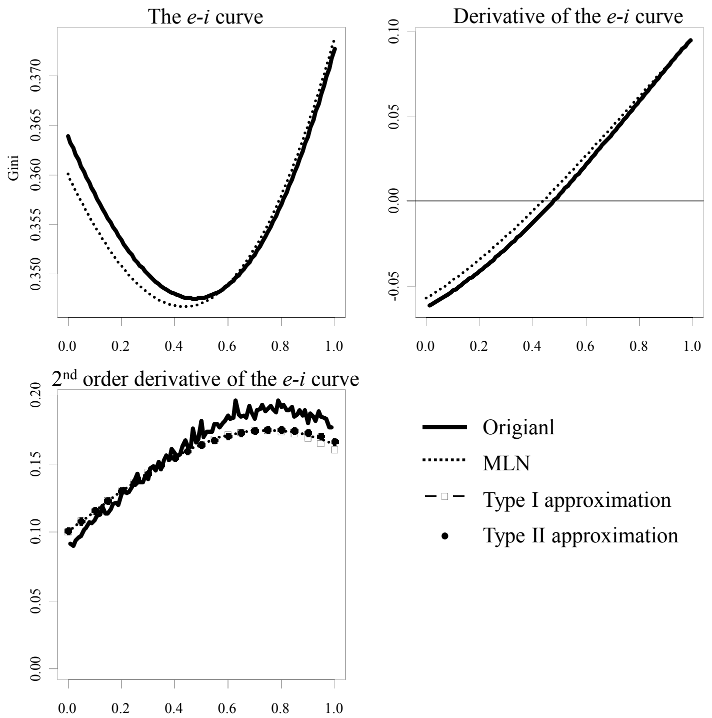


Poland, 2004

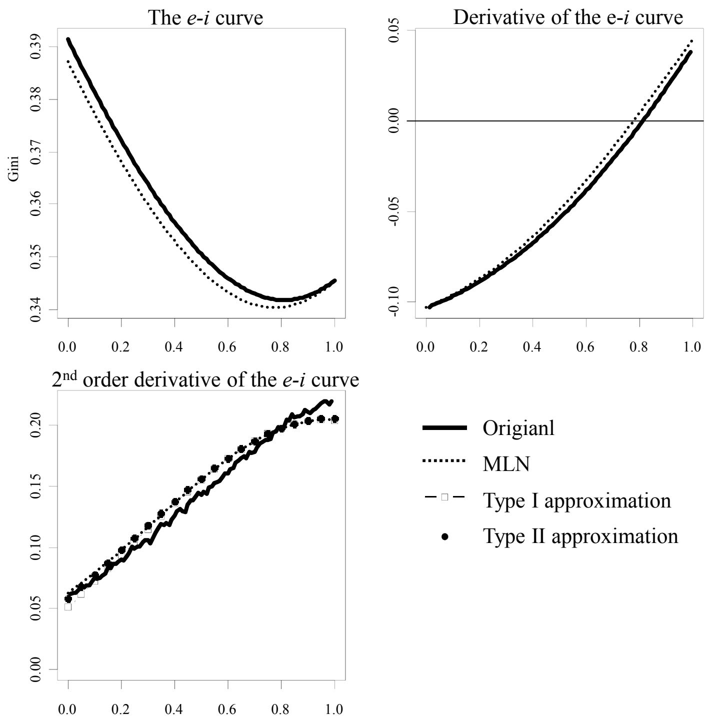


Greece, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

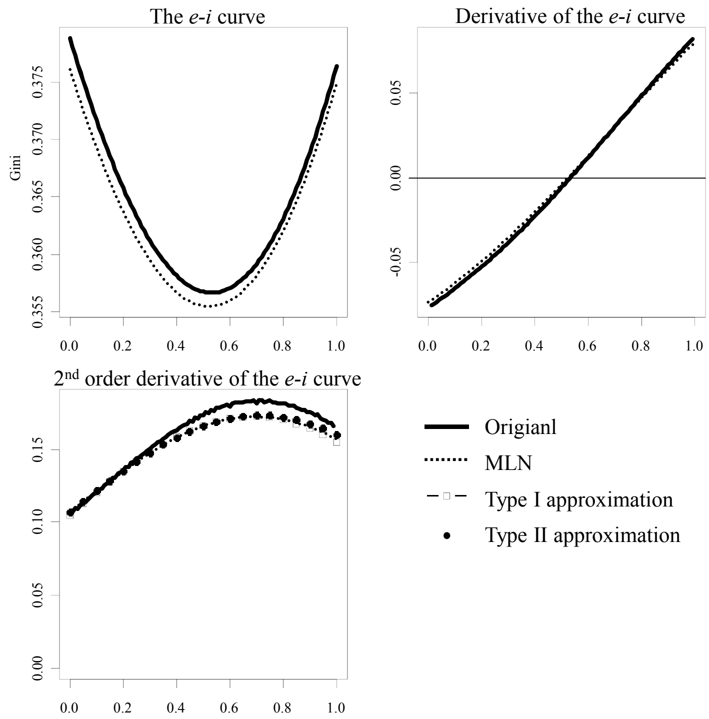


Italy, 2004

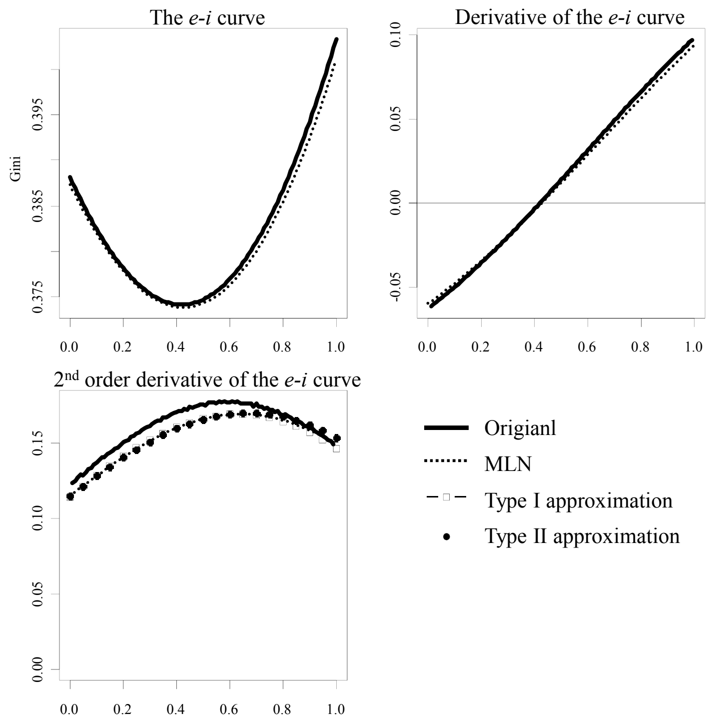


Estonia, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

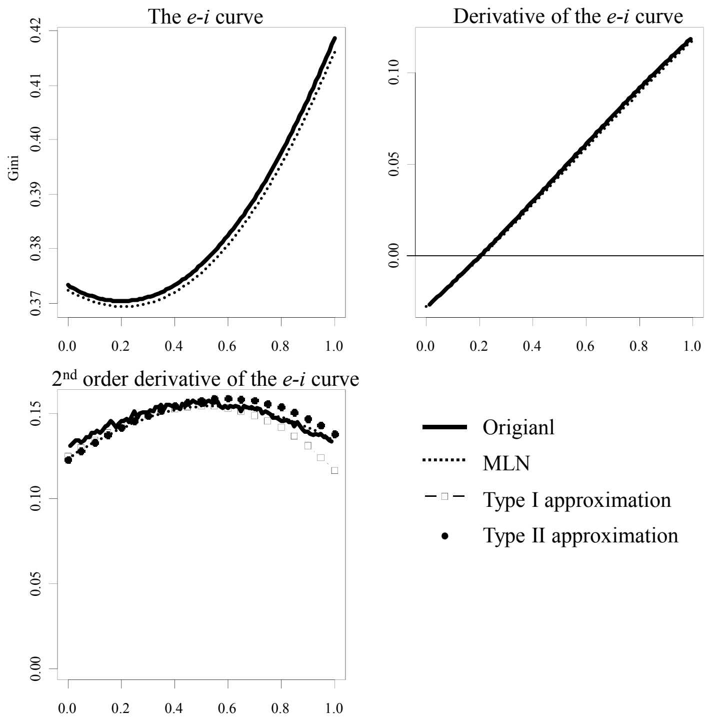


UK, 2004

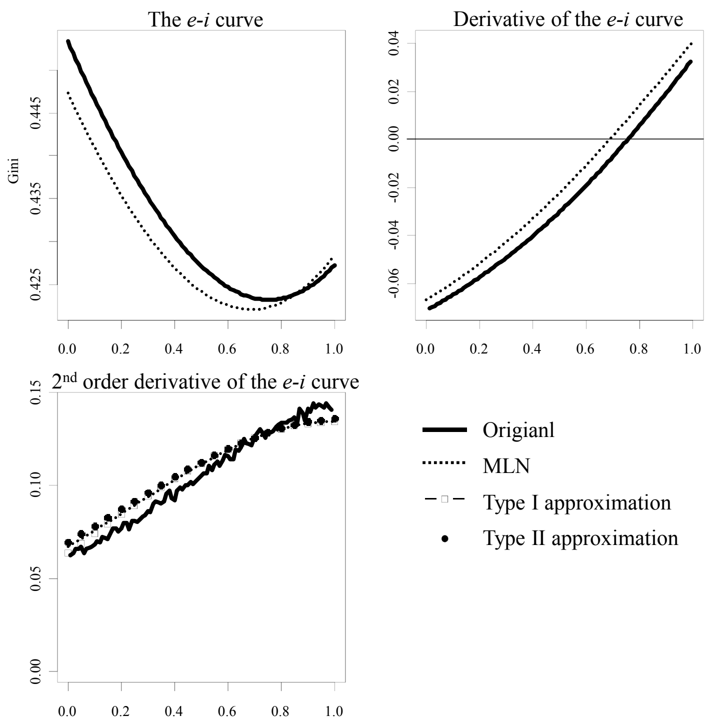


USA, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

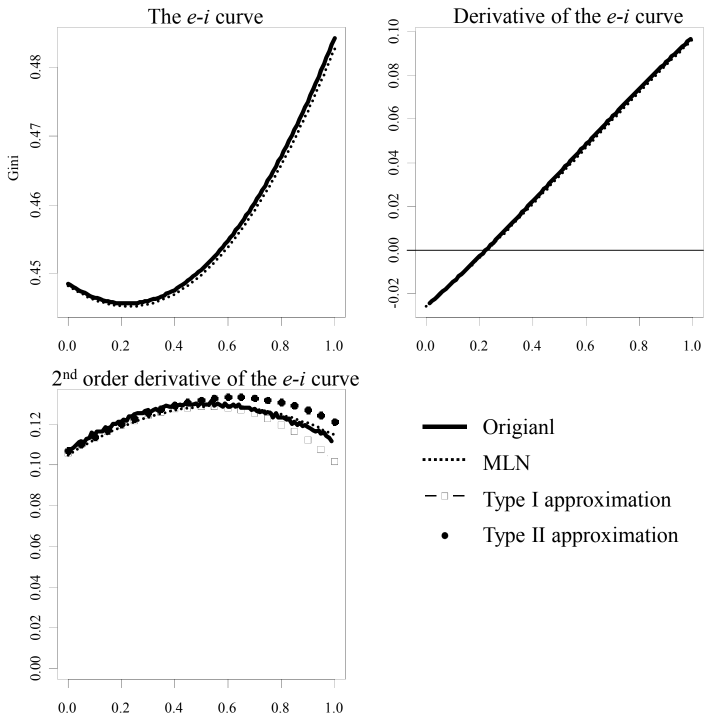


Israel, 2004

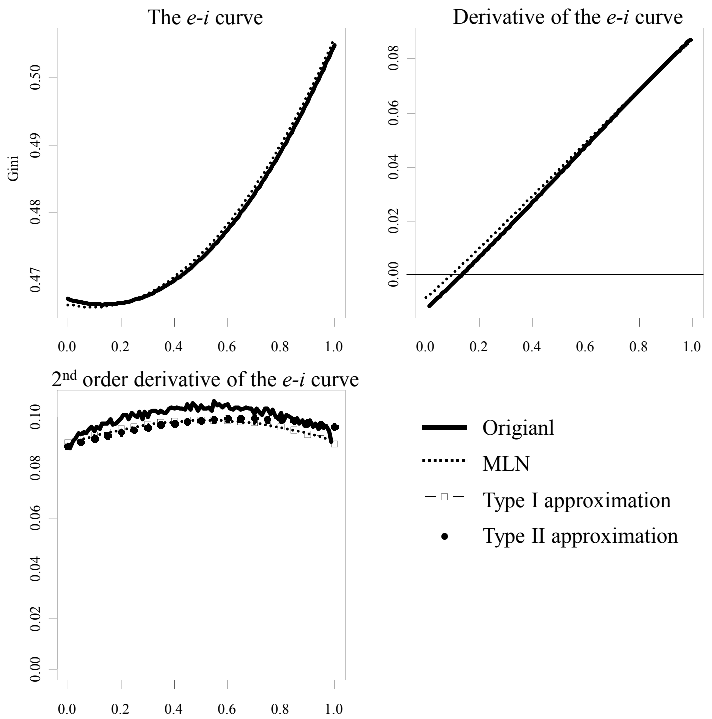


Russia, 2000

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

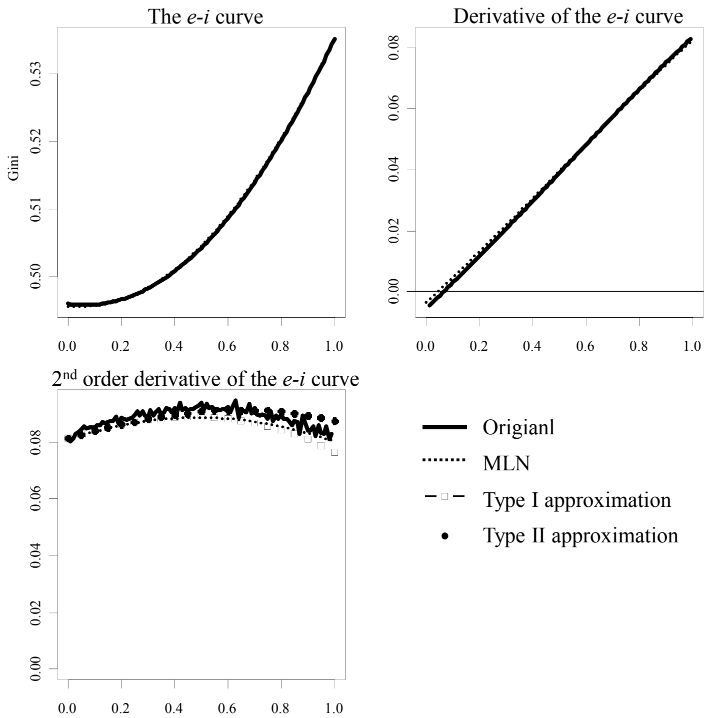


Uruguay, 2004

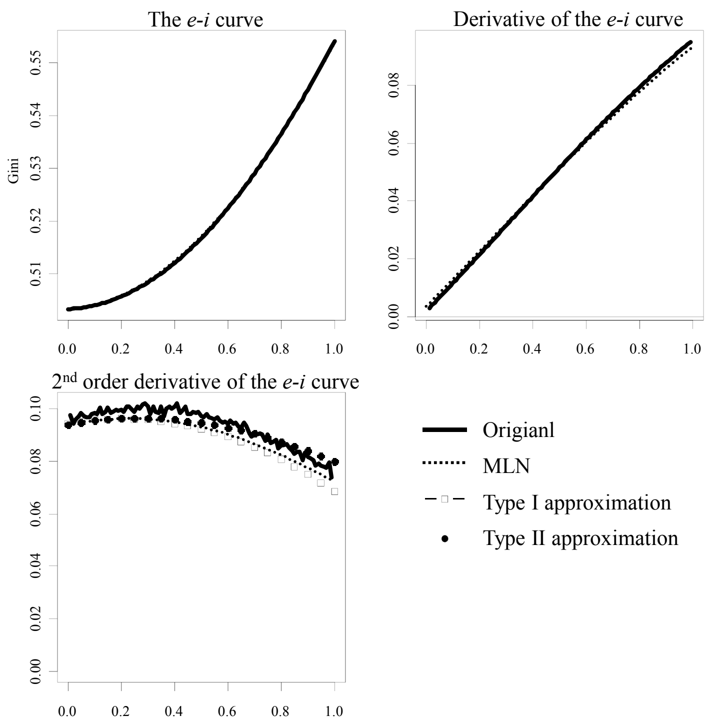


Mexico, 2004

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)

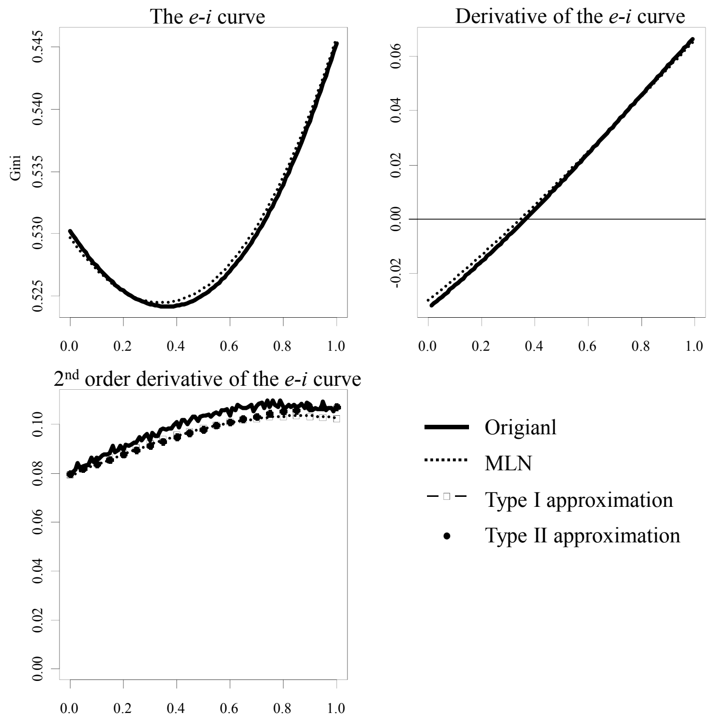


Brazil, 2006

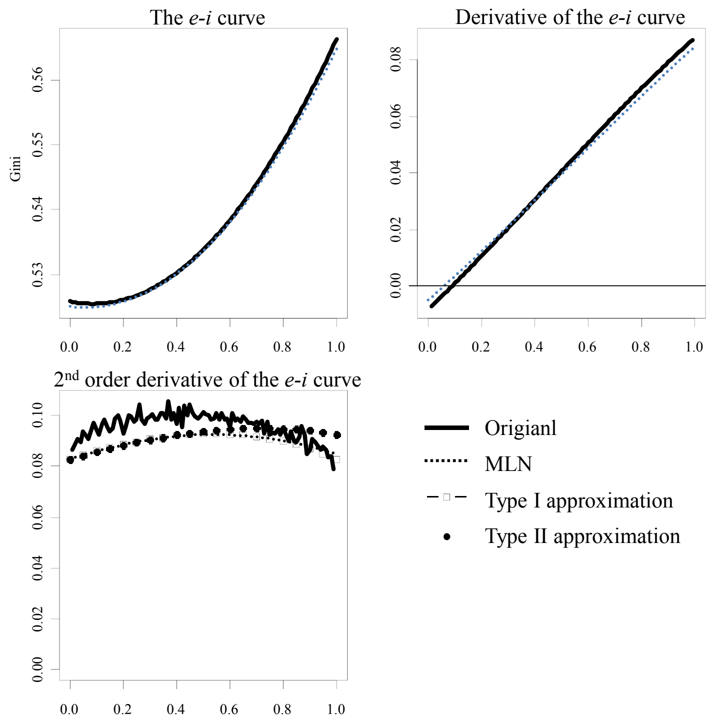


Guatemala, 2006

Annex 3. The Gini Index *e-i* Curve for Individual Equivalised Disposable Income and Its Approximations (*Continued*)



Peru, 2004



Columbia, 2004