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Abstract

This paper outlines a dynamic model with three levels of government: federal, state and local in the Stackelberg game structure with the superor government as the leader and all its subordinate governments the followers.It studies the optimal design of block grants and matching grants from both the federal government and the state governments to their numerous subordinate governments respectively as well as the optimal public expendtures and public capital stocks of different levels of government in the long run. Using specific form of utility function, we find that the optimal intergovernmental grants are very different between the level of federal government and state governments.

Keywords: Block grants; Matching grants; Public spending; Public capital stocks; Public investment

1 Introduction

Designs of federal grants to localities have recently received great attention in both theory and practice. Practically intergovernmental grants are very important because in many transitional economies in China, Eastern Europe and Russia, the national governments are faced the problem of rationalize the scheme of intergovernmental grants so as to achieve continuing economic growth as well as fiscal equity between developed areas and backward areas. For example,the Chinese government is now implementing a national program called *the great exploitation of western areas* for the purpose of bridging the gap in economic development that has become wider since the reform and opening in late 1970′s between backward western areas such as Tibet and developed eastern areas such as Shanghai.Other examples concern some Latin American countries such as Argentina and Brazil which have been reforming their existing systems of grant allocation since the early 1980′s.

In theory, one prominent feature is that all studies about intergovernmental grants modeled only two levels of government, to my knowledge, by uniting all subnational governments including state, metropolitan,county and town as the level of local government. This is obviously a serious limitation not only because such kind of government structure is very scarce in the real world (perhaps with the exception of Taiwan), but more importantly, it cannot shed light on the possible different policies of intergovernmental grants adopted by different levels of government, for example, the possible difference between the federal grants from federal government to state governments and the state grants from state governments to their subordinate governments in the United States. Another limitation in theory is that most studies only considered a static utility maximization framework, but in the real world all levels of governments invest and formulate capitals and, as we all know, matching grants for public investment from superior governments to their subordinate governments are important forms of intergovernmental grants. On the other hand, in the few papers using a dynamic approach[1], although more than one level of government is considered, the dynamic optimization is constrained to the lowest level. As a result, these papers obtained only a partial macro-equillibrium, leaving both matching and nonmatching grants as exogenously given.

Motivated by the above considerations, this paper discusses the problem of intergovernmental grants by considering the optimal choices of three levels of government: federal government, state governments and all the other governments subordinate to state governments which we take as local governments. The model is within the Stackelberg game structure among different levels of government with both local governments and state governments accumulating capitals. For simplicity, we do not consider federal public capital stocks. The approach taken in this paper, partly from the optimal local fiscal theory developed in Arnott and Grieson (1981), Starrett (1980) and Gordon (1983), focuses directly on the relation between the federal government and numerous state

[1] For example, see Zou [10], Barro [2003], Brueckner [2000], Solow [2003], Yin (2008) and Zhang and Xu (2011).

governments as well as between each state government and its numerous local governments in choosing the optimal matching grants and block grants while ignoring the effects of taxes imposed by different levels of governments on the private sector.

This paper is organized as follows. In Section 2,we set up the general framework for the dynamic Stackelberg (leader-follower) game: (i) between state governments (the leaders) and their numerous local governments (the followers) respectivly, and (ii) between the federal government (the leader) on one side and numerous state governments (the followers) and local governments (the followers) on the other. Some preliminary results are derived in this general, abstract form. In Section 3, hrough a concrete example we see how the optimal choices of intergovernmental grants, public spending and public capital stocks of different levels of governments can be computed. In Section 4, we give some detailed analysis and policy implications of the results we derived in Section 3. Finally in Section 5, we conclude the paper.

2 Basic model

In this paper,we assume there are one federal government and m state governments in the economy. A typical state government i $(i = 1, 2, ...m)$ has n_i local (subordinate) governments, where a typical locality ij $(j = 1, 2, ...n_i)$ has a preference defined on federal public spending f, state i public spending s_i, state i public capital stock k_i, its own public spending l_{ij} and its own public capital stock k_{ij}. Thus locality ij's utility function can be written as:

$$u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}),\ \ i = 1, 2, ...m, j = 1, 2, ...n_i \tag{1}$$

We assume the utility function is twice differentiable and satisfies:

$$\frac{\partial u^{ij}}{\partial f} > 0, \frac{\partial u^{ij}}{\partial s_i} > 0, \frac{\partial u^{ij}}{\partial l_{ij}} > 0, \frac{\partial u^{ij}}{\partial k_i} > 0, \frac{\partial u^{ij}}{\partial k_{ij}} > 0$$

$$\frac{\partial^2 u^{ij}}{\partial f^2} < 0, \frac{\partial^2 u^{ij}}{\partial s_i^2} < 0, \frac{\partial^2 u^{ij}}{\partial l_{ij}^2} < 0, \frac{\partial^2 u^{ij}}{\partial k_i^2} < 0, \frac{\partial^2 u^{ij}}{\partial k_{ij}^2} < 0$$

and Inada condition:

$$\lim_{f\to 0}\frac{\partial u^{ij}}{\partial f} = \infty, \lim_{s_i\to 0}\frac{\partial u^{ij}}{\partial s_i} = \infty, \lim_{l_{ij}\to 0}\frac{\partial u^{ij}}{\partial l_{ij}} = \infty, \lim_{k_i\to 0}\frac{\partial u^{ij}}{\partial k_i} = \infty, \lim_{k_{ij}\to 0}\frac{\partial u^{ij}}{\partial k_{ij}} = \infty$$

$$\lim_{f\to \infty}\frac{\partial u^{ij}}{\partial f} = 0, \lim_{s_i\to \infty}\frac{\partial u^{ij}}{\partial s_i} = 0, \lim_{l_{ij}\to \infty}\frac{\partial u^{ij}}{\partial l_{ij}} = 0, \lim_{k_i\to \infty}\frac{\partial u^{ij}}{\partial k_i} = 0, \lim_{k_{ij}\to \infty}\frac{\partial u^{ij}}{\partial k_{ij}} = 0$$

To focus on the optimal design of intergovernmental grants, we bypass the problem of optimal taxation for all levels of government and assume each locality and each state has fixed fiscal revenues T_{ij} and T_i respectively. Locality ij receives the following grants from state i: a nonmatching grant G_{ij}, a matching grant for local public investment $\alpha_{ij}\dot{k}_{ij}$ and a matching grant for local public spending $g_{ij}l_{ij}$ with α_{ij} and g_{ij} the matching rates repectively. Thus the budget constraint for locality ij is:

$$\dot{k}_{ij} = T_{ij} + g_{ij}l_{ij} + \alpha_{ij}\dot{k}_{ij} + G_{ij} - l_{ij} \tag{2}$$

or

$$\dot{k}_{ij} = \frac{1}{1-\alpha_{ij}}(T_{ij} + g_{ij}l_{ij} + G_{ij} - l_{ij}) \tag{3}$$

where $\dot{k}_{ij}$ represents locality ij's public investment.

Similarly, state government i receives the following grants from federal government: a nonmatching grant G_i,a matching grant for state public investment $\alpha_i\dot{k}_i$ and a matching grant for state public spending g_is_i with α_i and g_i the matching rates repectively. On the other hand, it transfers grants to all its n_i localities. Thus the budget constraint for state goverment i is:

$$\dot{k}_i = T_i + g_is_i + \alpha_i\dot{k}_i + G_i - s_i - \sum_{j=1}^{n_i}(g_{ij}l_{ij} + \alpha_{ij}\dot{k}_{ij} + G_{ij}) \tag{4}$$

Substitute Eq. (3) into Eq. (4),we can rewrite the Eq. (4) as:

$$\dot{k}_i = \frac{1}{1-\alpha_i}(T_i + g_is_i + G_i - s_i - \sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}(g_{ij}l_{ij} + G_{ij}) - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}}(T_{ij} - l_{ij})) \tag{5}$$

where $\dot{k}_i$ represents state i's public investment.

For simplicity, we assume federal government does not own capital stock. Let T_f denote the tax revenue collected by the federal government. The federal government uses it to finance its own public spending as well as all the federal grants to m states. Thus the budget constraint for the federal government is:

$$T_f = f + \sum_{i=1}^{m}(g_is_i + \alpha_i\dot{k}_i + G_i) \tag{6}$$

Substitute Eq. (5) into Eq. (6),we can rewrite the Eq. (6) as:

$$\begin{aligned} T_f \quad = \quad & f + \sum_{i=1}^{m}\frac{1}{1-\alpha_i}(g_is_i + G_i) + \sum_{i=1}^{m}\frac{\alpha_i}{1-\alpha_i}(T_i - s_i - \\ & \sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}(g_{ij}l_{ij} + G_{ij}) - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}}(T_{ij} - l_{ij})) \end{aligned} \tag{7}$$

2.1 Local government′s optimization

Given federal public spending, state public spending , state public capital stocks, and all the other local public spending and local public capital stocks, locality ij ($i = 1$, 2, ...m, $j = 1$, 2, ...n_i) chooses its own public spending l_{ij} and capital stock k_{ij} to maximize a discouted utility over an infinite time horizon:

$$\max_{l_{ij},k_{ij}} \int_0^{\infty} u^{ij}(f, s, l_{ij}, k_i, k_{ij})e^{-\rho t}dt$$

subject to its budget constraint:

$$\dot{k}_{ij} = \frac{1}{1-\alpha_{ij}}(T_{ij} + g_{ij}l_{ij} + G_{ij} - l_{ij}) \tag{3}$$

where ρ [2]is the time discount factor.

Define the Hamiltonian function as:

$$H_{ij} = u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) + \frac{\lambda_{ij}}{1-\alpha_{ij}}(T_{ij} + g_{ij}l_{ij} + G_{ij} - l_{ij})$$

where λ_{ij} is the Hamiltonian multiplier representing the private marginal value of locality ij′s public capital stock.

The first-order conditions are given by Eq. (3) and the follows:

$$\frac{\partial u^{ij}}{\partial l_{ij}} + \frac{\lambda_{ij}}{1-\alpha_{ij}}(g_{ij} - 1) = 0 \tag{8}$$

$$\dot{\lambda}_{ij} = \rho\lambda_{ij} - \frac{\partial u^{ij}}{\partial k_{ij}} \tag{9}$$

plus the transversity condition:

$$\lim_{t\to\infty} \lambda_{ij}k_{ij}e^{-\rho t} = 0 \tag{10}$$

2.2 State government′s optimization

In each state, the state government and its n_i localities play the Stackelberg game with the state government as the leader and its localities the followers. That is, given federal public spending and all federal grants, each state government maximizes the weighted welfare of its localities by fully incorporating all the localities′ first-order conditions in Section 2.1 into its own maximization.

Specifically, state government i ($i = 1, 2, ..m$) chooses its own public spending s_i, public capital stocks k_i, block grants G_{ij}, rates of state matching grants g_{ij} and α_{ij} as well as all its localites′ public spending l_{ij}, capital stocks k_{ij} and Hamiltonian multipliers λ_{ij} to maximize the weighted welfare of its localities:

[2] Here we implicitly assume that the time discount is uniform for all localities. In Section 4, we will privide a simple approach to test this assumption.

$$\max_{l_{ij},k_{ij},\lambda_{ij},s_i,k_i,g_{ij},\alpha_{ij},G_{ij}} \int_0^\infty \sum_{j=1}^{n_i} \xi_{ij} u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) e^{-\rho t} dt$$

where ξ_{ij} is the weight assigned to locality ij $(j = 1, 2, ...n_i)$.
Define the Hamiltonian function as:

$$\begin{aligned} H_i &= \sum_{j=1}^{n_i} \xi_{ij} u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) + \sum_{j=1}^{n_i} \chi_1^{ij} (\frac{\partial u^{ij}}{\partial l_{ij}} + \frac{\lambda_{ij}}{1-\alpha_{ij}}(g_{ij}-1)) \\ &+ \sum_{j=1}^{n_i} \chi_2^{ij}(\rho\lambda_{ij} - \frac{\partial u^{ij}}{\partial k_{ij}}) + \sum_{j=1}^{n_i} \frac{\chi_3^{ij}}{1-\alpha_{ij}}(T_{ij} + g_{ij} l_{ij} + G_{ij} - l_{ij}) \\ &+ \sum_{j=1}^{n_i} \frac{\chi_4^i}{1-\alpha_i}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i} \frac{1}{1-\alpha_{ij}}(g_{ij} l_{ij} + G_{ij}) \\ &- \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{1-\alpha_{ij}}(T_{ij} - l_{ij})) \end{aligned}$$

where χ_1^{ij}, χ_3^{ij} are the Hamiltonian multipliers associated with Eq. (3), (9) respectively, χ_4^i is the Hamiltonian multiplier associated with state government i's budget constraint Eq. (5), χ_2^{ij} is the Lagrange multipler associated with Eq. (8)

Now the first-order conditions are given by Eq. (3),(5),(8),(9) and the follows:

$$\frac{\partial H_i}{\partial l_{ij}} = \xi_{ij} \frac{\partial u^{ij}}{\partial l_{ij}} + \chi_1^{ij} \frac{\partial^2 u^{ij}}{\partial l_{ij}^2} - \chi_2^{ij} \frac{\partial^2 u^{ij}}{\partial k_{ij} \partial l_{ij}} + \frac{\chi_3^{ij}}{1-\alpha_{ij}}(g_{ij}-1) + \frac{\chi_4^i}{1-\alpha_i} \frac{\alpha_{ij} - g_{ij}}{1-\alpha_{ij}} = 0 \tag{11}$$

$$\frac{\partial H_i}{\partial \lambda_{ij}} = \frac{\chi_1^{ij}}{1-\alpha_{ij}}(g_{ij}-1) + \rho\chi_2^{ij} = \rho\chi_2^{ij} - \dot{\chi}_2^{ij} \tag{12}$$

$$\frac{\partial H_i}{\partial k_{ij}} = \xi_{ij} \frac{\partial u^{ij}}{\partial k_{ij}} + \chi_1^{ij} \frac{\partial^2 u^{ij}}{\partial l_{ij} \partial k_{ij}} - \chi_2^{ij} \frac{\partial^2 u^{ij}}{\partial k_{ij}^2} = \rho\chi_3^{ij} - \dot{\chi}_3^{ij} \tag{13}$$

$$\frac{\partial H_i}{\partial s_i} = \sum_{j=1}^{n_i} \xi_{ij} \frac{\partial u^{ij}}{\partial s_i} + \sum_{j=1}^{n_i} \chi_1^{ij} \frac{\partial^2 u^{ij}}{\partial l_{ij} \partial s_i} - \sum_{j=1}^{n_i} \chi_2^{ij} \frac{\partial^2 u^{ij}}{\partial k_{ij} \partial s_i} + \frac{\chi_4^i}{1-\alpha_i}(g_i - 1) = 0 \tag{14}$$

$$\frac{\partial H_i}{\partial k_i} = \sum_{j=1}^{n_i} \xi_{ij} \frac{\partial u^{ij}}{\partial k_i} + \sum_{j=1}^{n_i} \chi_1^{ij} \frac{\partial^2 u^{ij}}{\partial l_{ij} \partial k_i} - \sum_{j=1}^{n_i} \chi_2^{ij} \frac{\partial^2 u^{ij}}{\partial k_{ij} \partial k_i} = \rho\dot{\chi}_4^i - \chi_4^i \tag{15}$$

$$\frac{\partial H_i}{\partial g_{ij}} = \frac{\chi_1^{ij}\lambda_{ij}}{1-\alpha_{ij}} + (\chi_3^{ij} - \frac{\chi_4^i}{1-\alpha_i})\frac{l_{ij}}{1-\alpha_{ij}} = 0 \tag{16}$$

$$\frac{\partial H_i}{\partial \alpha_{ij}} = \frac{\chi_1^{ij}\lambda_{ij}}{(1-\alpha_{ij})^2}(g_{ij}-1) + (\chi_3^{ij} - \frac{\chi_4^i}{1-\alpha_i})\frac{1}{(1-\alpha_{ij})^2}(T_{ij}+g_{ij}l_{ij}+G_{ij}-l_{ij}) = 0 \tag{17}$$

$$\frac{\partial H_i}{\partial G_{ij}} = \frac{\chi_3^{ij}}{1-\alpha_{ij}} - \frac{\chi_4^i}{1-\alpha_i}\frac{1}{1-\alpha_{ij}} = 0 \tag{18}$$

plus transversality conditions:

$$\lim_{t\to\infty} \chi_2^{ij}\lambda_{ij}e^{-\rho t} = 0 \tag{19}$$

$$\lim_{t\to\infty} \chi_3^{ij}k_{ij}e^{-\rho t} = 0 \tag{20}$$

$$\lim_{t\to\infty} \chi_4^{i}k_{i}e^{-\rho t} = 0 \tag{21}$$

Proposition 1 *: Eq. (11)∼(18) and be simplified to as the follows:*

$$\xi_{ij}\frac{\partial u^{ij}}{\partial l_{ij}} - \chi_3^{ij} - \chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial l_{ij}} = 0 \tag{22}$$

$$\dot{\chi}_3^{ij} = \rho\chi_3^{ij} + \chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}^2} - \xi_{ij}\frac{\partial u^{ij}}{\partial k_{ij}} \tag{23}$$

$$\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial u^{ij}}{\partial s_i} - \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial s_i} + \frac{\chi_4^i}{1-\alpha_i}(g_i-1) = 0 \tag{24}$$

$$\dot{\chi}_4^{i} = \rho\chi_4^{i} - \sum_{j=1}^{n_i}\xi_{ij}\frac{\partial u^{ij}}{\partial k_i} + \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial k_i} \tag{25}$$

$$\chi_3^{ij} = \frac{\chi_4^i}{1-\alpha_i} \tag{26}$$

Proof. From Eq. (18), we have Eq. (26)

From Eq. (8) and our assumption $\frac{\partial u_{ij}}{\partial l_{ij}} > 0$, we have: $\lambda_{ij} \neq 0$. Thus by substituting Eq. (26) into Eq.(16), we have: $\chi_1^{ij} = 0$. At the same time, Eq. (17) is automatically satisfied.

Substiute $\chi_1^{ij} = 0$ into Eq. (12), we have: $\dot{\chi}_2^{ij} = 0$, thus $\chi_2^{ij} =$ constant.

Substitute Eq. (26) and $\chi_1^{ij} = 0$ into Eq. (11), we have Eq. (22).

Substitute $\chi_1^{ij} = 0$ into Eq. (13)∼(15), we have Eq. (23)∼ (25) respectively.

■

Note: during the simplification, χ_1^{ij}, χ_2^{ij} are both eliminated.

2.3 Federal government′s optimization

Given the optimal choices of all state governments and local governments, the federal government as the leader in its Stackelberg game with all states and localities the followers chooses its own public spending f, block grants G_i, rate of federal matching grants g_i and α_i ($i = 1, 2, ..m$) as well as l_{ij}, k_{ij}, λ_{ij}, s_i, k_i, g_{ij}, α_{ij}, G_{ij}, χ_3^{ij}, χ_4^i to maximize the the whole social welfare, with ξ_i as the weight assigned to state i, i.e.

$$\max_{f,g_i\alpha_i,G_i,l_{ij},k_{ij},\lambda_{ij},s_i,k_i,g_{ij},\alpha_{ij},G_{ij},\chi_3^{ij},\chi_4^i} \int_0^{\infty} \sum_{i=1}^{m} \xi_i \sum_{j=1}^{n_i} \xi_{ij} u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) e^{-\rho t} dt$$

subject to first-order conditions for all states and localities given by Eq. (3), (5), (8), (9), (22)$\sim$(26) and its own budget constraint Eq. (7)

Define the Hamiltonian function as:

$$\begin{aligned} H \quad = \quad & \sum_{i=1}^{m} \xi_i \sum_{j=1}^{n_i} \xi_{ij} u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_1^{ij}(\rho\lambda_{ij} - \frac{\partial u^{ij}}{\partial k_{ij}}) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_2^{ij} \\ & \times(\rho\chi_3^{ij} + \chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}^2} - \xi_{ij}\frac{\partial u^{ij}}{\partial k_{ij}}) + \sum_{i=1}^{m} q_3^i(\rho\chi_4^i - \sum_{j=1}^{n_i} \xi_{ij}\frac{\partial u^{ij}}{\partial k_i} \end{aligned}$$

$$\begin{aligned} & + \sum_{j=1}^{n_i} \chi_2^{ij}\frac{\partial u^{ij}}{\partial k \partial k_i}) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{q_4^{ij}}{1-\alpha_{ij}}(T_{ij} + g_{ij}l_{ij} + \alpha_{ij}k_{ij} + G_{ij} - l_{ij}) + \\ & \sum_{i=1}^{m} \frac{q_5^i}{1-\alpha_i}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}(g_{ij}l_{ij} + G_{ij}) - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}} \\ & \times(T_{ij} - l_{ij})) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_6^{ij}(\frac{\partial u^{ij}}{\partial l_{ij}} + \frac{\lambda_{ij}}{1-\alpha_{ij}}(g_{ij}-1)) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_7^{ij}(\xi_{ij}\frac{\partial u^{ij}}{\partial l_{ij}} \\ & - \chi_3^{ij} - \chi_2^{ij}\frac{\partial^2 u}{\partial k \partial l}) + \sum_{i=1}^{m} q_8^i(\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial u^{ij}}{\partial s_i} - \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial s_i} + \frac{\chi_4^i}{1-\alpha_i}(g_i - 1)) \\ & + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_9^{ij}(\chi_3^{ij} - \frac{\chi_4^i}{1-\alpha_i}) + q_{10}(T_f - f - \sum_{i=1}^{m}\frac{1}{1-\alpha_i}(g_i s_i + G_i) \end{aligned}$$

$$- \sum_{i=1}^{m}\frac{\alpha_i}{1-\alpha_i}(T_i - s_i - \sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}(g_{ij}l_{ij} + G_{ij}) - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}}(T_{ij} - l_{ij}))$$

where q_1^{ij}, q_2^{ij}, q_3^i, q_4^{ij}, q_5^i are Hamiltonian multipliers associated with Eq. (9), (23), (25), (3), (5) respectively,q_6^{ij}, q_7^{ij}, q_8^i, q_9^{ij}, q_{10} are Lagrange multipliers associated with Eq. (8), (22), (24), (26), (7) respectively.

The first-order conditions are given by Eq. (3), (5), (8), (9), (22)~(26) and the follows:

$$\frac{\partial H}{\partial \lambda_{ij}} = \rho q_1^{ij} + \frac{q_6^{ij}}{1-\alpha_{ij}}(g_{ij}-1) = \rho q_1^{ij} - \dot{q}_1^{ij} \tag{27}$$

$$\frac{\partial H}{\partial \chi_3^{ij}} = \rho q_2^{ij} - q_7^{ij} + q_9^{ij} = \rho q_2^{ij} - \dot{q}_2^{ij} \tag{28}$$

$$\frac{\partial H}{\partial \chi_4^{i}} = \rho q_3^{i} + \frac{q_8^{i}}{1-\alpha_{i}}(g_{i}-1) - \frac{q_9^{ij}}{1-\alpha_i} = \rho q_3^{i} - \dot{q}_3^{i} \tag{29}$$

$$\begin{aligned}\frac{\partial H}{\partial k_{ij}} &= \xi_i\xi_{ij}\frac{\partial u^{ij}}{\partial k_{ij}} - q_1^{ij}\frac{\partial^2 u^{ij}}{\partial k^2} + q_2^{ij}(\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^3} - \xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}^2}) + q_3^i(-\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_i \partial k_{ij}} \\ &+\chi_2^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}^2\partial k_i}) + q_6^{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial k_{ij}} + q_7^{ij}(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial k_{ij}} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial l_{ij}}) \\ &+q_8^i(\xi_{ij}\frac{\partial^2 u_{ij}}{\partial s_i\partial k_{ij}} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial s_i}) \ = \ \rho q_4^{ij} - \dot{q}_4^{ij}\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial H}{\partial k_i} &= \xi_i\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial u^{ij}}{\partial k_i} - \sum_{j=1}^{n_i} q_1^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial k_i} + \sum_{j=1}^{n_i} q_2^{ij}(\chi_2^{ij}\frac{\partial^3 u^{ij}}{\partial k_{ij}^2\partial k_i} \\ &-\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial k_i}) + q_3^i(-\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_i^2} + \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial k_i^2}) \\ &+\sum_{j=1}^{n_i} q_6^{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial k_i} + \sum_{j=1}^{n_i} q_7^{ij}(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial k_i} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial l_{ij}\partial k_i}) \\ &+q_8^i(\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial^2 u^{ij}}{\partial s_i\partial k_i} - \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial s_i\partial k_i}) \ = \ \rho\dot{q}_5^i - \dot{q}_5^i\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial H}{\partial f} &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}\frac{\partial u^{ij}}{\partial f} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_1^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial f} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_2^{ij} \\ &\times(\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial f} - \xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial f}) + \sum_{i=1}^{m} q_3^i(-\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_i\partial f} + \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial k_i\partial f}) \\ &+\sum_{i=1}^{m}\sum_{j=1}^{n_i} q_6^{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial f} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} q_7^{ij}(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial f} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial l_{ij}\partial f}) \\ &+\sum_{i=1}^{m} q_8^i(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial s_i\partial f} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial s_i\partial f}) - q_{10} \ = \ 0\end{aligned} \tag{32}$$

$$\frac{\partial H}{\partial g_i} = \frac{q_5^i}{1-\alpha_i}s_i + \frac{q_8^i}{1-\alpha_i}\chi_4^i - \frac{q_{10}}{1-\alpha_i}s_i = 0 \tag{33}$$

$$\begin{aligned}\frac{\partial H}{\partial \alpha_i} \quad = \quad & (q_5^i - q_{10})\frac{1}{(1-\alpha_i)^2}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}(g_{ij}l_{ij} + G_{ij}) \\ & - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}}(T_{ij} - l_{ij})) + \frac{q_8^i}{(1-\alpha_i)^2}\chi_4^i(g_i - 1) - q_9^{ij}\sum_{j=1}^{n_i}\frac{\chi_4^i}{(1-\alpha_i)^2}\end{aligned} \tag{34}$$

$$\frac{\partial H}{\partial G_i} = (q_5^i - q_{10})\frac{1}{1-\alpha_i} = 0 \tag{35}$$

$$\begin{aligned}\frac{\partial H}{\partial l_{ij}} \quad = \quad & \xi_i\xi_{ij}\frac{\partial u^{ij}}{\partial l_{ij}} - q_1^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial l_{ij}} + q_2^{ij}(\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial l_{ij}} - \xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial l_{ij}}) + q_3^i(-\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_i\partial l_{ij}} \\ & + \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial k_i\partial l_{ij}}) - \frac{q_4^{ij}}{1-\alpha_{ij}} + \frac{q_5^i}{1-\alpha_i}\frac{\alpha_{ij} - g_{ij}}{1-\alpha_{ij}} + q_6^{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}^2} + q_7^{ij} \\ & \times(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}^2} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial l_{ij}^2}) + q_8^i(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial s_i\partial l_{ij}} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial s_i\partial l_{ij}}) \; = \; 0\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial H}{\partial s_i} \quad = \quad & \xi_i\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial u^{ij}}{\partial s_i} - \sum_{j=1}^{n_i}q_1^{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial s_i} + \sum_{j=1}^{n_i}q_2^{ij}(\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial s_i} - \xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_{ij}\partial s_i}) \\ & + q_3^i(-\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial^2 u^{ij}}{\partial k_i\partial s_i} + \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial k_i\partial s_i}) - \frac{q_5^i}{1-\alpha_i} + \sum_{j=1}^{n_i}q_6^{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial s_i} \\ & + \sum_{j=1}^{n_i}q_7^{ij}(\xi_{ij}\frac{\partial^2 u^{ij}}{\partial l_{ij}\partial s_i} - \chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}\partial l_{ij}\partial s_i}) + q_8^i(\sum_{j=1}^{n_i}\xi_{ij}\frac{\partial^2 u^{ij}}{\partial s_i^2} \\ & - \sum_{j=1}^{n_i}\chi_2^{ij}\frac{\partial^3 u_{ij}}{\partial k_{ij}^2\partial s_i^2}) + q_{10}\frac{\alpha_i - g_i}{1-\alpha_i} \; = \; 0\end{aligned} \tag{37}$$

$$\frac{\partial H}{\partial g_{ij}} = \frac{q_4^{ij}}{1-\alpha_{ij}}l_{ij} - \frac{q_5^i}{1-\alpha_i}\frac{l_{ij}}{1-\alpha_{ij}} + \frac{q_6^{ij}\lambda_{ij}}{1-\alpha_{ij}} + q_{10}\frac{\alpha_i}{1-\alpha_i}\frac{l_{ij}}{1-\alpha_{ij}} = 0 \tag{38}$$

$$\frac{\partial H}{\partial \alpha_{ij}} = (q_4^{ij} - \frac{q_5^i}{1-\alpha_i} + q_{10}\frac{\alpha_i}{1-\alpha_i})\frac{1}{(1-\alpha_{ij})^2}(T_{ij} + g_{ij}l_{ij} + G_{ij} - l_{ij}) + \frac{q_6^{ij}\lambda_{ij}(g_{ij} - 1)}{(1-\alpha_{ij})^2} \tag{39}$$

$$\frac{\partial H}{\partial G_{ij}} = \frac{q_4^{ij}}{1-\alpha_{ij}} - \frac{q_5^i}{1-\alpha_i}\frac{1}{1-\alpha_{ij}} + q_{10}\frac{\alpha_i}{1-\alpha_i}\frac{1}{1-\alpha_{ij}} = 0 \tag{40}$$

plus transversality conditions:

$$\lim_{t\to\infty} q_1^{ij}\lambda_{ij}e^{-\rho t} = 0 \tag{41}$$

$$\lim_{t\to\infty} q_2^{ij}\chi_3^{ij}e^{-\rho t} = 0 \tag{42}$$

$$\lim_{t\to\infty} q_3^{i}\chi_4^{i}e^{-\rho t} = 0 \tag{43}$$

$$\lim_{t\to\infty} q_4^{ij}k_{ij}e^{-\rho t} = 0 \tag{44}$$

$$\lim_{t\to\infty} q_5^{i}k_{i}e^{-\rho t} = 0 \tag{45}$$

Proposition 2 *The social marginal utilities of public capital stocks of all localities and states equal the social marginal utility of federal tax income.*

Proof. *:* From our definitions, q_4^{ij}, q_5^i, q_{10} are the social marginal utilities of locality ij's capital stocks, state i' capital stocks ($i = 1,\ 2,\ ...m,\ j = 1,\ 2,\ ...n_i$) and federal tax income respectively.

From Eq. (35), we have: $q_5^i = q_{10.}$

Substitute $q_5^i = q_{10.}$ into Eq. (40),we have: $q_4^{ij} = q_{10}$ ■

Remark: from the above proposition, we can see that all local and state public capital stocks are equivalent in regard to their marginal contributions to social welfare. Perhaps a little surprising, raising federal taxes has the same welfare effect as the accumulation of capital stocks by subnational governments (i.e. state and locality). The reason is that we assume the federal government balances its budget in every period. As a result, more federal taxes means more federal public spending and more federal grants, which contribute directly and indirectly to the social welfare.

3 An explicit example

In Section2, we have set up a general model to discuss the optimal design of intergovernmental grants, but the first-order conditions are too complex to derive some interesting results. In this Section, we will specify the form of utility function to derive an explicit solution to our model. Suppose utility function for locality ij ($i = 1,\ 2,\ ...m,\ j = 1,\ 2,\ ...n_i$) are:

$$u^{ij}(f, s_i, l_{ij}, k_i, k_{ij}) = \theta_1^{ij}\ln f + \theta_2^{ij}\ln s_i + \theta_3^{ij}\ln l_{ij} + \theta_4^{ij}\ln k_i + \theta_5^{ij}\ln k_{ij} \tag{46}$$

where $\theta_1^{ij},\ \theta_2^{ij},\ \theta_3^{ij},\ \theta_4^{ij},\ \theta_5^{ij} > 0$

Obviously all our assumptions in Section 2 concerning utility function are satisfied.

3.1 Locality ij $(i = 1, 2, ...m, j = 1, 2, ...n_i)$

The first-order conditions (8), (9) for locality ij can now be rewritten as:

$$l_{ij} = \frac{1-\alpha_{ij}}{1-g_{ij}} \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{47}$$

$$\dot{\lambda}_{ij} = \rho\lambda_{ij} - \frac{\theta_5^{ij}}{k_{ij}} \tag{48}$$

Substitute Eq. (4) into Eq. (3):

$$\dot{k}_{ij} = \frac{1}{1-\alpha_{ij}}(T_{ij} + G_{ij}) - \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{49}$$

3.2 State i $(i = 1, 2, ...m)$

Under the optimal choices of its n_i localities, state government i chooses its own public spending, public capital stocks, state block grants and state matching grants to maximize the welfare in state i, i.e.

$$\max_{\substack{\lambda_{ij},k_{ij},k_i,s_i,g_{ij},\alpha_{ij},\\ G_{ij},j=1,\ldots n_i}} \int_0^\infty \sum_{j=1}^{n_i} \xi_{ij}(\theta_1^{ij}\ln f + \theta_2^{ij}\ln s_i + \theta_3^{ij}\ln l_{ij} + \theta_4^{ij}\ln k_i + \theta_5^{ij}\ln k_{ij})e^{-\rho t}dt$$

$$\text{s.t. } \dot{\lambda}_{ij} = \rho\lambda_{ij} - \frac{\theta_5^{ij}}{k_{ij}} \tag{48}$$

$$\dot{k}_{ij} = \frac{1}{1-\alpha_{ij}}(T_{ij} + G_{ij}) - \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{49}$$

$$\dot{k}_i = T_i + g_i s_i + \alpha_i \dot{k}_i + G_i - s_i - \sum_{j=1}^{n_i}(g_{ij}l_{ij} + \alpha_{ij}\dot{k}_{ij} + G_{ij}) \tag{4}$$

where $\sum_{j=1}^{n_i} \xi_{ij} = 1$

Combine Eq. (47), (49) and (4), we can rewrite Eq. (4) as:

$$\begin{aligned} \dot{k}_i \quad &= \quad \frac{1}{1-\alpha_i}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i}\frac{\alpha_{ij}}{1-\alpha_{ij}}T_{ij} \\ &\quad -\sum_{j=1}^{n_i}\frac{1}{1-\alpha_{ij}}G_{ij} + \sum_{j=1}^{n_i}\frac{\theta_3^{ij}}{\lambda_{ij}}\frac{1}{1-g_{ij}}(\alpha_{ij} - g_{ij})) \end{aligned} \tag{50}$$

Define the Hamiltonian function as:

$$
\begin{aligned}
H_i \quad = \quad & \sum_{j=1}^{n_i} \xi_{ij}(\theta_1^{ij} \ln f + \theta_2^{ij} \ln s_i + \theta_3^{ij} \ln l_{ij} + \theta_4^{ij} \ln k_i + \theta_5^{ij} \ln k_{ij}) + \sum_{j=1}^{n_i} \chi_1^{ij}(\rho\lambda_{ij} \\
& -\frac{\theta_5^{ij}}{k_{ij}}) + \sum_{j=1}^{n_i} \chi_2^{ij}(\frac{1}{1-\alpha_{ij}}(T_{ij}+G_{ij}) - \frac{\theta_3^{ij}}{\lambda_{ij}}) + \frac{\chi_3^i}{1-\alpha_i}(T_i + g_i s_i + G_i \\
& -s_i - \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{1-\alpha_{ij}} T_{ij} - \sum_{j=1}^{n_i} \frac{1}{1-\alpha_{ij}} G_{ij} + \sum_{j=1}^{n_i} \frac{\theta_3^{ij}}{\lambda_{ij}} \frac{1}{1-g_{ij}}(\alpha_{ij} - g_{ij}))
\end{aligned}
$$

where $\chi_1^{ij}, \chi_2^{ij}, \chi_3^i$ are the Hamiltonian multipliers associated with Eq. (48)∼(50) respectively.

The first-order conditions are given by Eq. (48)∼(50) and the follows:[3]

$$\frac{\partial H_i}{\partial \lambda_{ij}} = \rho\chi_1^{ij} + \chi_2^{ij}\frac{\theta_3^{ij}}{\lambda_{ij}^2} - \frac{\chi_3^i}{1-\alpha_i}\frac{\theta_3^{ij}}{\lambda_{ij}^2}\frac{1}{1-g_{ij}}(\alpha_{ij}-g_{ij}) + \xi_{ij}\frac{\theta_3^{ij}}{l_{ij}}\frac{\partial l_{ij}}{\partial \lambda_{ij}} = \rho\chi_1^{ij} - \dot{\chi}_1^{ij} \tag{51}$$

$$\frac{\partial H_i}{\partial k_{ij}} = \xi_{ij}\frac{\theta_5^{ij}}{k_{ij}} + \chi_1^{ij}\frac{\theta_5^{ij}}{k_{ij}^2} = \rho\chi_2^{ij} - \dot{\chi}_2^{ij} \tag{52}$$

$$\frac{\partial H_i}{\partial k_i} = \sum_{j=1}^{n_i} \xi_{ij}\frac{\theta_4^{ij}}{k_i} = \rho\chi_3^i - \dot{\chi}_3^i \tag{53}$$

$$\frac{\partial H_i}{\partial s_i} = \frac{\sum_{j=1}^{n_i}\xi_{ij}\theta_2^{ij}}{s_i} + \frac{\chi_3^i}{1-\alpha_i}(g_i - 1) = 0 \tag{54}$$

$$\frac{\partial H_i}{\partial g_{ij}} = \frac{\chi_3^i}{1-\alpha_i}\frac{\theta_3^{ij}}{\lambda_{ij}}\frac{2g_{ij}-1-\alpha_{ij}}{(1-g_{ij})^2} + \xi_{ij}\frac{\theta_3^{ij}}{l_{ij}}\frac{\partial l_{ij}}{\partial g_{ij}} = 0 \tag{55}$$

$$\frac{\partial H_i}{\partial \alpha_{ij}} = \frac{\chi_2^{ij}(T_{ij}+G_{ij})}{(1-\alpha_{ij})^2} + \frac{\chi_3^i}{1-\alpha_i}(-\frac{T_{ij}+G_{ij}}{(1-\alpha_{ij})^2} + \frac{\theta_3^{ij}}{\lambda_{ij}}\frac{1}{1-g_{ij}}) + \xi_{ij}\frac{\theta_3^{ij}}{l_{ij}}\frac{\partial l_{ij}}{\partial \alpha_{ij}} = 0 \tag{56}$$

$$\frac{\partial H_i}{\partial G_{ij}} = \frac{\chi_2^{ij}}{1-\alpha_{ij}} - \frac{\chi_3^i}{1-\alpha_i}\frac{1}{1-\alpha_{ij}} = 0 \tag{57}$$

plus transversality conditions:.

$$\lim_{t\to\infty} \chi_1^{ij}\lambda_{ij}e^{-\rho t} = 0 \tag{58}$$

$$\lim_{t\to\infty} \chi_2^{ij}k_{ij}e^{-\rho t} = 0 \tag{59}$$

[3] We will use the relation: $l_{ij} = l_{ij}(\lambda_{ij}, \alpha_{ij}, g_{ij})$ From Eq. (47)

$$\lim_{t\to\infty} \chi_3^i k_i e^{-\rho t} = 0 \tag{60}$$

Proposition 3 : $\alpha_{ij} = g_{ij}$

Proof. From Eq. (47), we have the following relations:

$$\frac{\partial l_{ij}}{\partial g_{ij}} = \frac{l_{ij}}{1 - g_{ij}} \tag{61}$$

$$\frac{\partial l_{ij}}{\partial \alpha_{ij}} = -\frac{l_{ij}}{1 - \alpha_{ij}} \tag{62}$$

From Eq. (57):

$$\chi_2^{ij} = \frac{\chi_3^i}{1 - \alpha_i} \tag{63}$$

In Eq. (56), eliminate χ_2^{ij} from Eq. (63) and use the relation Eq. (62):

$$\lambda_{ij} = \frac{1 - \alpha_{ij}}{1 - g_{ij}} \frac{1}{(1 - \alpha_i)\xi_{ij}} \chi_3^i \tag{64}$$

Combine Eq. (55) and (64) and use the relation (61), we can derive the desired result. ■

Proposition 3 states that the state government i ($i = 1, 2, ..m$) should set the rates of the two kinds of state matching grants for each of its localities to be equal with one aother. This is surprising since these two kinds of matching grants serve different purposes: one is to subside local public spending, the other is to encourage local public investment, and in practice are considered to be uncorrelated with each other. However, from our model, this common practice is obviously not the optimal choice.

Using proposion 3, we can rewrite Eq. (47), (49), (50), (64) as :

$$l_{ij} = \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{65}$$

$$k_{ij} = \frac{1}{1 - g_{ij}}(T_{ij} + G_{ij}) - \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{66}$$

$$\dot{k}_i = \frac{1}{1 - \alpha_i}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i} \frac{g_{ij}}{1 - g_{ij}} T_{ij} - \sum_{j=1}^{n_i} \frac{1}{1 - g_{ij}} G_{ij}) \tag{67}$$

$$\lambda_{ij} = \frac{1}{(1 - \alpha_i)\xi_{ij}} \chi_3^i \tag{68}$$

Combine Eq. (62), (68):

$$\chi_2^{ij} = \xi_{ij}\lambda_{ij} \tag{69}$$

Since ξ_{ij} is exogenously given and $\theta_5^{ij} > 0$, combine Eq. (48), (52), (69),we have:.

$$\chi_1^{ij} = 0 \tag{70}$$

From Eq. (54),we have:

$$s_i = \frac{\sum_{j=1}^{n_i} \xi_{ij}\theta_2^{ij}}{\chi_3^i}\frac{1-\alpha_i}{1-g_i} \tag{71}$$

Thus Eq. (51)∼(57) are reduced to Eq. (52), (68)∼(71) together with proposition 3.

3.3 federal government

Under the optimal choices of all the local and state governments, the federal government chooses its public spending, optimal federal block grants and federal matching grants to maximize the whole social welfare, i.e.

$$\max_{\substack{\lambda_{ij},k_{ij},k_i,\chi_3^i,g_{ij},\\ G_{ij},f,g_i,\alpha_i,G_i}} \int_0^\infty \sum_{i=1}^m \xi_i \sum_{j=1}^{n_i} \xi_{ij}(\theta_1^{ij}\ln f + \theta_2^{ij}\ln s_i + \theta_3^{ij}\ln l_{ij} + \theta_4^{ij}\ln k_i + \theta_5^{ij}\ln k_{ij})e^{-\rho t}dt$$

$$\text{s.t. } \dot{\lambda}_{ij} = \rho\lambda_{ij} - \frac{\theta_5^{ij}}{k_{ij}} \tag{48}$$

$$\dot{k}_{ij} = \frac{1}{1-g_{ij}}(T_{ij} + G_{ij}) - \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{66}$$

$$\dot{k}_i = \frac{1}{1-\alpha_i}(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i}\frac{g_{ij}}{1-g_{ij}}T_{ij} - \sum_{j=1}^{n_i}\frac{1}{1-g_{ij}}G_{ij}) \tag{67}$$

$$\dot{\chi}_3^i = \rho\chi_3^i - \sum_{j=1}^{n_i}\xi_{ij}\frac{\theta_4^{ij}}{k_i} \tag{53}$$

$$\lambda_{ij} = \frac{1}{(1-\alpha_i)\xi_{ij}}\chi_3^i \tag{68}$$

and its own budget constraint:

$$T_f = f + \sum_{i=1}^m (g_i s_i + \alpha_i \dot{k}_i + G_i) \tag{6}$$

where $\sum_{i=1}^{m}\xi_i = 1$

Substitute Eq. (67) into Eq. (6), we can rewrite Eq. (6) as:

$$T_f = f+\sum_{i=1}^{m}\frac{1}{1-\alpha_i}(g_i s_i+G_i)+\sum_{i=1}^{m}\frac{\alpha_i}{1-\alpha_i}(T_i-s_i-\sum_{j=1}^{n_i}\frac{g_{ij}}{1-g_{ij}}T_{ij}-\sum_{j=1}^{n_i}\frac{1}{1-g_{ij}}G_{ij}) \tag{72}$$

Define the Hamiltonian function as:

$$\begin{aligned} H \quad = \quad & \sum_{i=1}^{m}\xi_i\sum_{j=1}^{n_i}\xi_{ij}(\theta_1^{ij}\ln f+\theta_2^{ij}\ln s_i+\theta_3^{ij}\ln l_{ij}+\theta_4^{ij}\ln k_i+\theta_5^{ij}\ln k_{ij})+\sum_{i=1}^{m}\sum_{j=1}^{n_i}q_1^{ij} \\ & \times(\rho\lambda_{ij}-\frac{\theta_5^{ij}}{k_{ij}})+\sum_{i=1}^{m}\sum_{j=1}^{n_i}q_2^{ij}(\frac{1}{1-g_{ij}}(T_{ij}+G_{ij})-\frac{\theta_3^{ij}}{\lambda_{ij}})+\sum_{i=1}^{m}\frac{q_3^i}{1-\alpha_i}(T_i \\ & +g_i s_i+G_i-s_i-\sum_{j=1}^{n_i}\frac{g_{ij}}{1-g_{ij}}T_{ij}-\sum_{j=1}^{n_i}\frac{1}{1-g_{ij}}G_{ij})+\sum_{i=1}^{m}q_4^i(\rho\chi_3^i-\sum_{j=1}^{n_i}\xi_{ij}\frac{\theta_4^{ij}}{k_i}) \\ & +\sum_{i=1}^{m}\sum_{j=1}^{n_i}q_5^{ij}(\lambda_{ij}-\frac{1}{(1-\alpha_i)\xi_{ij}}\chi_3^i)+q_6(T_f-f-\sum_{i=1}^{m}\frac{1}{1-\alpha_i} \\ & \times(g_i s_i+G_i)-\sum_{i=1}^{m}\frac{\alpha_i}{1-\alpha_i}(T_i-s_i-\sum_{j=1}^{n_i}\frac{g_{ij}}{1-g_{ij}}T_{ij}-\sum_{j=1}^{n_i}\frac{1}{1-g_{ij}}G_{ij}) \end{aligned}$$

where q_1^{ij}, q_2^{ij}, q_3^i, q_4^i are Hamiltonian multipliers associated with Eq. (48), (66), (67), (53) respectively, q_5^{ij}, q_6 are Lagrange multipliers associated with Eq. (68), (6) respectively.

The first-order conditions are given by Eq. (58), (53), (66)∼(68) and the follows:[4]

$$\frac{\partial H}{\partial\lambda_{ij}} = \rho q_1^{ij}+q_2^{ij}\frac{\theta_3^{ij}}{\lambda_{ij}^2}+q_5^i+\frac{\xi_i\xi_{ij}\theta_3^{ij}}{l_{ij}}\frac{\partial l_{ij}}{\partial\lambda_{ij}} = \rho q_1^{ij}-\dot{q}_1^{ij} \tag{73}$$

$$\frac{\partial H}{\partial k_{ij}} = \frac{\xi_i\xi_{ij}\theta_5^{ij}}{k_{ij}}+q_1^{ij}\frac{\theta_5^{ij}}{k_{ij}^2} = \rho q_2^{ij}-\dot{q}_2^{ij} \tag{74}$$

$$\frac{\partial H}{\partial k_i} = \frac{\xi_i\sum_{j=1}^{n_i}\xi_{ij}\theta_4^{ij}}{k_i}+q_4^i\frac{\sum_{j=1}^{n_i}\xi_{ij}\theta_4^{ij}}{k_i^2} = \rho q_3^i-\dot{q}_3^i \tag{75}$$

$$\begin{aligned} \frac{\partial H}{\partial\chi_3^i} \quad = \quad & \rho q_4^i-\sum_{j=1}^{n_i}\frac{q_5^i}{\xi_{ij}}\frac{1}{1-\alpha_i}+(\frac{\xi_i\sum_{j=1}^{n_i}\xi_{ij}\theta_2^{ij}}{s_i}+\frac{q_3^i}{1-\alpha_i}(g_i-1) \\ & -\frac{q_6}{1-\alpha_i}(g_i-\alpha_i))\frac{\partial s}{\partial\chi_3^i} \;=\; \rho q_4^i-\dot{q}_4^i \end{aligned} \tag{76}$$

[4] We will use the relation: $l_{ij} = l_{ij}(\lambda_{ij})$, $s_i = s_i(\chi_3^i,\alpha_i,g_i)$ From Eq. (65), (71) respectively.

$$\frac{\partial H}{\partial g_{ij}} = (q_2^{ij} - \frac{q_3^i}{1-\alpha_i} + q_6 \frac{\alpha_i}{1-\alpha_i}) \frac{T_{ij} + G_{ij}}{(1-g_{ij})^2} = 0 \tag{77}$$

$$\frac{\partial H}{\partial G_{ij}} = (q_2^{ij} - \frac{q_3^i}{1-\alpha_i} + q_6 \frac{\alpha_i}{1-\alpha_i}) \frac{1}{1-g_{ij}} = 0 \tag{78}$$

$$\frac{\partial H}{\partial f} = \frac{\sum_{i=1}^m \sum_{j=1}^{n_i} \xi_i \xi_{ij} \theta_1^{ij}}{f} - q_6 = 0 \tag{79}$$

$$\begin{aligned} \frac{\partial H}{\partial g_i} &= (q_3^i - q_6) \frac{s_i}{1-\alpha_i} + (\frac{\xi_i \sum_{j=1}^{n_i} \xi_{ij} \theta_2^{ij}}{s_i} + \frac{q_3^i}{1-\alpha_i}(g_i - 1) \\ &\quad - \frac{q_6}{1-\alpha_i}(g_i - \alpha_i)) \frac{\partial s_i}{\partial g_i} = 0 \end{aligned} \tag{80}$$

$$\begin{aligned} \frac{\partial H}{\partial \alpha_i} &= \frac{1}{(1-\alpha_i)^2}(q_3^i - q_6)(T_i + g_i s_i + G_i - s_i - \sum_{j=1}^{n_i} \frac{g_{ij}}{1-g_{ij}} T_{ij} - \sum_{j=1}^{n_i} \frac{1}{1-g_{ij}} G_{ij}) \\ &\quad - \sum_{j=1}^{n_i} \frac{q_5^i}{\xi_{ij}} \chi_3^i + (\frac{\xi_i \sum_{j=1}^{n_i} \xi_{ij} \theta_2^{ij}}{s_i} + \frac{q_3^i}{1-\alpha_i}(g_i - 1) - \frac{q_6}{1-\alpha_i}(g_i - \alpha_i)) \frac{\partial s_i}{\partial \alpha_i} \end{aligned} \tag{81}$$

$$\frac{\partial H}{\partial G_i} = (q_3^i - q_6) \frac{1}{1-\alpha_i} = 0 \tag{82}$$

plus transversality conditions:

$$\lim_{t\to\infty} q_1^{ij} \lambda_{ij} e^{-\rho t} = 0 \tag{83}$$

$$\lim_{t\to\infty} q_2^{ij} k_{ij} e^{-\rho t} = 0 \tag{84}$$

$$\lim_{t\to\infty} q_3^i k_i e^{-\rho t} = 0 \tag{85}$$

$$\lim_{t\to\infty} q_4^i \chi_3^i e^{-\rho t} = 0 \tag{86}$$

Proposition 4 *: In steady state in the long run,* $g_i = 0$ $(i = 1, 2, ...m)$

Proof. : From Eq. (82):

$$q_3^i = q_6 \tag{87}$$

Substitute (87) into (78):

$$q_2^{ij} = q_6 \tag{88}$$

[5]From Eq. (65), (71), we have the following relations:

$$\frac{\partial l_{ij}}{\partial \lambda_{ij}} = -\frac{l_{ij}}{\lambda_{ij}} \tag{89}$$

$$\frac{\partial s_i}{\partial g_i} = \frac{s_i}{1-g_i} \tag{90}$$

From Eq. (90): $\frac{\partial s_i}{\partial g_i} \neq 0$.Thus by substituting Eq. (87) into Eq. (80),we have:

$$s_i = \frac{\xi_i \sum_{j=1}^{n_i} \xi_{ij}\theta_2^{ij}}{q_6} \tag{91}$$

Substitute Eq. (87), (91) into Eq. (81):

$$\sum_{j=1}^{n_i} \frac{q_5^i}{\xi_{ij}} = 0 \tag{92}$$

In steady state, $\dot{q}_1^{ij} = 0$,thus by substituting Eq. (88) into Eq. (73) and using Eq. (89), we have:

$$q_6 \frac{\theta_3^{ij}}{\lambda_{ij}^2} + q_5^{ij} - \frac{\xi_i \xi_{ij} \theta_3^{ij}}{\lambda_{ij}} = 0 \tag{93}$$

Divide both sides of Eq. (93) by ξ_{ij}, sum up subscript j from 1 to n_i and use Eq. (92):

$$\xi_i \sum_{j=1}^{n_i} \frac{\theta_3^{ij}}{\lambda_{ij}} = q_6 \sum_{j=1}^{n_i} \frac{\theta_3^{ij}}{\lambda_{ij}^2 \xi_{ij}} = 0 \tag{94}$$

From Eq. (68), $\frac{1}{\xi_{ij}\lambda_{ij}}$does not contain the subscript j and can be removed out of the symbol $\sum$. Thus from Eq. (94), we have:

$$\lambda_{ij} = \frac{q_6}{\xi_i \xi_{ij}} \tag{95}$$

Combine Eq. (68), (95):

$$q_6 = \frac{\xi_i}{1-\alpha_i} \chi_3^i \tag{96}$$

Combine Eq. (71), (91), (96),we can derive our desired result. ■

From proposition 4, federal government should grant nothing to state governments to subside their public spending. This result stands in sharp contrast to the the real world where state governments usually receive positive matching grants for their public spending from federal government.[6]

[5]In face, by applying Proposition 2, we can directly get Eq. (87), (88).

[6]Also see Gong and Zou [8], for some detailed discussion about a kind of reverse intergovernmental transfer from the local government to the federal government.

In steady state in the long run, $\dot{\lambda}_{ij} = \dot{k}_{ij} = \dot{k}_i = \dot{\chi}_3^i = 0$. Apply proposition 4 when necessary, we can rewrite Eq. (48), (66), (67), (53), (6) as:

$$k_{ij} = \frac{\theta_5^{ij}}{\rho \lambda_{ij}} \tag{97}$$

$$G_{ij} = \frac{\theta_3^{ij}(1 - g_{ij})}{\lambda_{ij}} - T_{ij} \tag{98}$$

$$T_i + G_i - s_i - \sum_{j=1}^{n_i} \frac{g_{ij}}{1 - g_{ij}} T_{ij} - \sum_{j=1}^{n_i} \frac{1}{1 - g_{ij}} G_{ij} = 0 \tag{99}$$

$$k_i = \frac{\sum_{j=1}^{n_i} \xi_{ij} \theta_4^{ij}}{\rho \chi_3^i} \tag{100}$$

$$\sum_{i=1}^{m} G_i = T_f - f \tag{101}$$

In Eq. (99), sum up subscript i from 1 to m and substitute Eq. (98), (101) into Eq. (99):

$$T_f + \sum_{i=1}^{m} T_i + \sum_{i=1}^{m} \sum_{j=1}^{n_i} T_{ij} = f + \sum_{i=1}^{m} s_i + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\theta_3^{ij}}{\lambda_{ij}} \tag{102}$$

From Eq. (74):

$$f = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \xi_i \xi_{ij} \theta_1^{ij}}{q_6} \tag{103}$$

Substitute Eq. (91), (95), (103) into Eq. (102), we can derive q_6 :

$$q_6 = -\frac{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \xi_i \xi_{ij} (\theta_1^{ij} + \theta_2^{ij} + \theta_3^{ij})}{T_f + \sum_{i=1}^{m} T_i + \sum_{i=1}^{m} \sum_{j=1}^{n_i} T_{ij}} \tag{104}$$

Proposition 5 *: In the long run, the optimal local public spending, state public spending, federal public spending, local public capital stock, state public capital stock, state block grant and federal block grant are as follows:*

$$l_{ij} = \frac{\xi_i \xi_{ij} \theta_3^{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \xi_i \xi_{ij} (\theta_1^{ij} + \theta_2^{ij} + \theta_3^{ij})} (T_f + \sum_{i=1}^{m} T_i + \sum_{i=1}^{m} \sum_{j=1}^{n_i} T_{ij}) \tag{105}$$

$$s_i = \frac{\xi_i \sum_{j=1}^{n_i} \xi_{ij} \theta_2^{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \xi_i \xi_{ij} (\theta_1^{ij} + \theta_2^{ij} + \theta_3^{ij})} (T_f + \sum_{i=1}^{m} T_i + \sum_{i=1}^{m} \sum_{j=1}^{n_i} T_{ij}) \tag{106}$$

$$f = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}\theta_1^{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}(\theta_1^{ij}+\theta_2^{ij}+\theta_3^{ij})}(T_f+\sum_{i=1}^{m}T_i+\sum_{i=1}^{m}\sum_{j=1}^{n_i}T_{ij}) \quad (107)$$

$$k_{ij} = \frac{\xi_i\xi_{ij}\theta_5^{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}(\theta_1^{ij}+\theta_2^{ij}+\theta_3^{ij})}\frac{1}{\rho}(T_f+\sum_{i=1}^{m}T_i+\sum_{i=1}^{m}\sum_{j=1}^{n_i}T_{ij}) \quad (108)$$

$$k_i = \frac{\xi_i\sum_{j=1}^{n_i}\xi_{ij}\theta_4^{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}(\theta_1^{ij}+\theta_2^{ij}+\theta_3^{ij})}\frac{1}{\rho(1-\alpha_i)}(T_f+\sum_{i=1}^{m}T_i+\sum_{i=1}^{m}\sum_{j=1}^{n_i}T_{ij}) \quad (109)$$

$$G_{ij} = \frac{\xi_i\xi_{ij}\theta_3^{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}(\theta_1^{ij}+\theta_2^{ij}+\theta_3^{ij})}(1-g_{ij})(T_f+\sum_{i=1}^{m}T_i+\sum_{i=1}^{m}\sum_{j=1}^{n_i}T_{ij})-T_{ij} \quad (110)$$

$$G_i = \frac{\xi_i\sum_{j=1}^{n_i}\xi_{ij}(\theta_2^{ij}+\theta_3^{ij})}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_i\xi_{ij}(\theta_1^{ij}+\theta_2^{ij}+\theta_3^{ij})}(T_f+\sum_{i=1}^{m}T_i+\sum_{i=1}^{m}\sum_{j=1}^{n_i}T_{ij})-\sum_{j=1}^{n_i}T_{ij}-T_i \quad (111)$$

Proof. : Substitute Eq. (104) into Eq. (91), (103), we have Eq. (106), (107) respectively.

Substitute Eq. (95), (104) into Eq. (65), (97), (98), we have Eq. (105),(108),(110) respectively.

Subsititute Eq. (96), (104) into Eq. (100), we have Eq. (109)

Substitute Eq. (106), (110) into Eq. (99), we have Eq. (111). ■

4 Analysis and policy implications

Proposition 6 *: A rise in tax collected by any level of government will increase both public spending and public capital stocks of all levels of government in the long run, i.e.*

$$\frac{\partial y}{\partial x} > 0, x = T_f, T_{i\prime}, T_{i\prime j\prime},\ y = l_{ij}, s_i, f, k_{ij}, k_i$$

for any i, $i\prime = 1, 2, ... m$, $j = 1, 2, ...n_i$, $j\prime = 1, 2, ...n_{i\prime}$.

The proof is simple from Eq. (105)∼(109). This proposition, however, is strong because it shows that taxes collected by any level of government have complete externalities. If we interprete a rich government as the one with more public spending, more public capital stocks and thus more collected taxes (vice versa), this proposition supports a well-known policy set by Deng Xiaoping in China, that is *Let some districts and some people become rich in advance, afterwards they will bring along the backward districts and people to grow rich together.* Perhaps the districts which become rich in advance do not intentionally seek to help the other districts, but the economy equillibrium in the long run requires them to.

Proposition 7 *: (i) An increase in the preference for any level of public spending increases the public spending of the corresponding level of government, but reduces the public spending of all the other governments and the public capital stocks of all governments.*

(ii) An increase in the preference for any level of public capital stocks increases the public capital stocks of the corresponding level of government, but has no effect on the public capital stocks of all the other governments and the public spending of all governments.

Mathematically, let M be the set defined as follows:

$$M = \{f, s_i, l, k_i, k, i = 1, 2...m,\ j = 1, 2, ...n_i\}$$

then:

$$\frac{\partial y}{\partial \theta_1^{ij}} \begin{cases} > 0 & y = f \\ < 0 & y \in M | \{f\} \end{cases}, \frac{\partial y}{\partial \theta_2^{ij}} \begin{cases} > 0 & y = s_i \\ < 0 & y \in M | \{s_i\} \end{cases}, \frac{\partial y}{\partial \theta_3^{ij}} \begin{cases} > 0 & y = l_{ij} \\ < 0 & y \in M | \{l_{ij}\} \end{cases}$$

$$\frac{\partial y}{\partial \theta_4^{ij}} \begin{cases} > 0 & y = k_i \\ = 0 & y \in M | \{k_i\} \end{cases}, \frac{\partial y}{\partial \theta_5^{ij}} \begin{cases} > 0 & y = k_{ij} \\ = 0 & y \in M | \{k_{ij}\} \end{cases}$$

Again the proof is simple from Eq. (105)∼(109). Let us take locality ij as an example to make clear one important policy implication of this proposition.

If local government ij depends more on the public spending of its superior govern- ments, i.e. state government i or (and) federal government, which is demonstrated by an increase in θ_1^{ij} or (and) θ_2^{ij}, both its public spending l_{ij} and public capital stocks k_{ij} will decrease in the long run; in comparison, depending more on the public capital stocks of its superior government, i.e. the state i government, will have no negative effect on either locality ij′s public spending or its public capital stocks. This implies that state governments should pay more attention to infrastructure investment, which helps form state public capitals, than to direct purchases from their localities.

In regard to public spending, we can say something more. Following Davoodi and Zou [4], the level of fiscal decentralization is defined as the spending by subnational governments as a fraction of the total government spending. In our model, we can compute the long run level of fiscal decentralization. From Eq. (105)∼ (107) as follows:

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i} l_{ij} + \sum_{i=1}^{m} s_i}{\sum_{i=1}^{m}\sum_{j=1}^{n_i} l_{ij} + \sum_{i=1}^{m} s_i + f} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i} \xi_i \xi_{ij} (\theta_2^{ij} + \theta_3^{ij})}{\sum_{i=1}^{m}\sum_{j=1}^{n_i} \xi_i \xi_{ij} (\theta_1^{ij} + \theta_2^{ij} + \theta_3^{ij})} \qquad (112)$$

Eq. (112) tells us that the long run level of fiscal decentralization is just the sum of the weights all localities assign to subnational (local and state) public spending as a fraction of the total sum of weights. This can help us understand the different levels of fiscal decentralization in the real world. For example, in the United States, the state and local governments are more free and more independent to make their own public policies and have more influences on the people. As a result, more social welfare weights are assigned to local and state public spending, thus the greater level of fiscal decentralization in the long run. In China, the case is reversed.

Proposition 8 *: (i) The more emphasis local governments put on current public expenditure and investment, the smaller both local and state capital stocks in the long run;*

(ii) In the long run, the federal matching grant for state public investment is very effective to increase the corresponding state capital stocks, but the state matching grant for local public investment has no effect on the corresponding local public capital formation.

Proof. *:* From Eq. (108), (109) we have: (i) $k_i, k_{ij} \propto \frac{1}{\rho}$ where ρ is the uniform time discount factor, (ii) $k_i \propto \frac{1}{1-\alpha_i}$ while k_{ij} is not the function of α_{ij}, which proves our proposition. ■

In this propositon, (i) and the former part of (ii) is intuitive, since excessive emphasis on current expenditures and investment is a kind of short sight, which will do harm to the long run capital formation; on the other hand,the federal matching grant for state public invest- ment is intended to help form state capitals.[7]. However, the latter part of (ii) is quite counter-intuitive. In general,we expect that a rise in the state matching grant for local public investment would lead the corresponding localities to divert more local resourses from public consumption to public investment, thus an increase in state public capital stocks in the long run. To put it into a policy context, we may question the effectiveness of many state incentive programs for local welfare.

This proposition also sheds light on some interesting findings: we can estimate the capital stocks in a certain locality in the long run. Conceptually it is satisfactory to think of local governments as doing no production and owning no capital. Then local governments just buy a flow of output from their private sectors. These purchased services, which local governments make available to their local households, correspond to the input that matters for private production.[8] As long as local governments and their private sectors have the same production function, the results would be the same whether local governments do their own production to accumulate local public capitals or purchase final output from private sectors. In a word, from the perspective of social welfare, it is reasonble to think that local households have the same preference for local

[7] Note in our model, the matching rate α_i is undetermined, meaning any value of α_i between 0 and 1 is optimal. In the real world, the federal government has to consider many political and historical factors so as to arrive at a realistic choice of α_i.

[8] we do not consider private capitals and private production in our model

public spending and local public capital stocks, i.e.: $\theta_3^{ij} = \theta_5^{ij}$, at least approximately. Thus from Eq. (105), (108), the public capital stock k_{ij} in locality ij is just $\frac{1}{\rho}$ times the local public spending l_{ij}.If we choose $\rho = 0.05$,[9]this is 20 times. Conversely, if locality $ij\prime$s public capital stocks can be derived from other approaches, we can estimate the time discount factor ρ in locality ij, and even test our implicit assumption that ρ is uniform for all localities in the economy.

Proposition 9 *: (i) In the long run, an increase in the tax collected by locality ij implies less block grants from state i government to locality ij and from federal government to state i, but more any other intergovernmental block grants; similarly, an increase in the tax collected by state i implies less block grant from federal government to state i, but more any other intergovernmentl block grants.*

(ii) In state i in the long run, more block grants from state government to its localities imply less matching grants for the public spending of the corresponding localities, and vice versa.

Proof. from Eq. (110),(111) we have: (i) $\frac{\partial G_{ij}}{\partial T_{ij}}$, $\frac{\partial G_i}{\partial T_{ij}} < 0$, $\frac{\partial x}{\partial T_{ij}} > 0$ where $x = G_{i\prime j\prime}$ ($i\prime \neq i$ or $j\prime \neq j$) or $G_{i\prime}$ ($i\prime \neq i$); $\frac{\partial G_i}{\partial T_i} < 0$, $\frac{\partial G_{i\prime}}{\partial T_i} > 0 (i \neq i\prime)$, $\frac{\partial G_{i\prime j\prime}}{\partial T_i} > 0$ (for any $i\prime, j\prime$). (ii) $G_{ij} \propto 1 - g_{ij}$, which proves our proposition. ■

This proposition has important policy implications. Firstly, from (i), if one district becomes rich and can collect more taxes, the federal (national) government should reduce the federal block grant to this district and increase federal block grants for all the other districts. In China,the eastern areas such as Shanghai is very developed in comparison with western areas such as Tibet and can collect much more taxes. From our proposition, the Chinese government should divert more national block grants from the east to the west. This is just part of the national program called *the great exploitation of the western areas* in China. Secondly and somewhat surprisingly, part (ii) tells us that there is a tradeoff in the level of state government when it chooses the policies of state grants for its localities. People may think that in order to give more help to the development of some backward localities, their superior state governments should increase both block grants and matching grants, including the matching grant for local public spending. But from Proposition 9, this common idea is not optimal in regard to social welfare. In comparison, federal government is not subject to this kind of tradeoff, partly because it simply sets the rate of matching grants for state public spending to zero (see Propositon 4). Also note from Proposition 3, the two kinds of state matching rates received by local governments are equal with one another but both undetermined, leaving for the state governments to choose by considering some non-economic factors.

5 Conclusion

In a dynamic model of multiple levels of government, this paper examined the optimal design of intergovernmental grants, optimal public spending and public

[9]Here we follow Gong and Zou [6]

capital stocks. By considering three levels of government in the Stackelberg game structure, this paper shed light for the first time on the difference in the optimal intergovernmental grants adopted by different levels of government, i.e. federal government and state governments in our paper. We summarize the main findings here. (i) In the long run, the state governments should set the rates of two kinds of state matching grant: one is for local public investment, the other is for local public spending, to be equal with one another; the federal government, while not subject to this constraint, should simply set the rate of federal matching grants for state public spending to zero. (ii) The federal matching grants for state public investment is very effective in the sense that when the federal matching rate α_i approaches one, the capital stocks of the corresponding state government i approaches infinity in the long run (See Eq. (109)); however, the state matching grant for local public investment has no effect on the corresponding localitie′s long run capital formation. In addition, since in our model, the rates of the two kinds of state matching grant are both undetermined, it does not matter for the state government i ($i = 1, 2, ...m$) to simply set the matching rate for local public investment α_{ij} ($j = 1, 2, ...n_i$) to zero. Then from proposition 3, the state matching rate for local public spending g_{ij} is also zero. In other words, it is optimal for state governments to transfer only block grants to their localities. (of course, there are numerous other optimal choices) (iii) While the optimal choices of block grants and matching grants have nothing to do with one aother on the level of federal government, there is a clear tradeoff between the state block grants and state matching grants for local public spending. That is, it is not optimal for state governments to increase both block grants and matching grants for local public spending for their localities at the same time.

Even though our paper has extended the usual static approach within the framework of two levels of government, it still suffers from several limitations: we did not model the private sector and took taxes collected by all levels of governments as exogenously given; we did not consider federal public capital accumulation, which may influence some of our results; and we did not discuss the possibly different kinds of local and state public spending.[10]I hope that future reseach could extend our model by considering the above limitations so as to come closer to the real world.

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[10]see Gong and Zou [8] for some detailed discussion about three kinds of local public spending in the framework of two levels of government

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