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7 October 2011

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FIRST DRAFT: OCTOBER 7, 2011
THIS DRAFT: MARCH 23, 2012
LATEST DRAFT: https://sites.google.com/site/marioghossoub

Abstract. In the classical theory of *monotone equimeasurable rearrangements* of functions, "equimeasurability" (i.e. the fact the two functions have the same distribution) is defined relative to a given *additive* probability measure. These rearrangement tools have been successfully used in many problems in economic theory dealing with uncertainty where the monotonicity of a solution is desired. However, in all of these problems, *uncertainty* refers to the classical *Bayesian* understanding of the term, where the idea of *ambiguity* is absent. Arguably, *Knighitan uncertainty*, or *ambiguity* is one of the cornerstones of modern decision theory. It is hence natural to seek an extension of these classical tools of equimeasurable rearrangements to situations of ambiguity. This paper introduces the idea of a monotone equimeasurable rearrangement in the context of *non-additive probabilities*, or *capacities* that satisfy a property that I call *strong nonatomicity*. The latter is a strengthening of the notion of *nonatomicity*, and these two properties coincide for additive measures and for submodular (i.e. concave) capacities. To illustrate the usefulness of these tools in economic theory, I consider an application to a problem arising in the theory of production under uncertainty.

Key words and phrases. Ambiguity, Capacity, Non-Additive Probability, Choquet Integral, Monotone Equimeasurable Rearrangement, Production under Uncertainty.

JEL Classification: C02, C65, D24, D89.

I thank Marc Henry for his comments on earlier drafts, and for his suggestions. I thank John Quiggin, Fabio Maccheroni, Marciano Siniscalchi and Peter Wakker for helpful discussions. Any errors or omissions are entirely mine. Financial support from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

1. Introduction

The theory of monotone equimeasurable rearrangements dates back to the work of Hardy, Littlewood, and Pólya [45]. The theory was then extended by Cambanis et al. [5], Chong and Rice [26], Day [30, 31], Lorentz [51], and Luxemburg [52]. The central result in this classical theory is that for any real-valued function f defined on the real line, there exists a nonincreasing (resp. nondecreasing) function that has the same distribution as the function f for Lebesgue measure. This function is called the nonincreasing (resp. nondecreasing) rearrangement of the function f, and is almost surely unique for Lebesgue measure.

These rearrangement tools have proven to be very fruitful in many areas of economic theory dealing with uncertainty. For instance, in the theory of optimal insurance design, monotonicity of an optimal indemnity schedule is desired since monotone contracts are truthtelling. Rearrangement techniques have been used extensively in the insurance literature [9, 11, 29, 40, 42]. Also, since the seminal work of Landsberger and Meilijson [50], it is well known that in problems of risk sharing between two individuals with preferences that preserve second-order stochastic dominance, Pareto optimal allocations are comonotonic. This is closely related to the idea of a monotone equimeasurable rearrangement; and, indeed, rearrangement techniques have been used in the context of risk-sharing [4, 8, 12, 14, 15, 28]. Equimeasurable rearrangements have also proven to be useful in other ares of economic theory, such as incentive, or agency theory [10, 11], the theory of debt contracting with costly state verification [16], the theory of demand for contingent claims [13, 41], the theory of portfolio choice [46, 48], or even econometrics [23, 24, 25] and the theory of entrepreneurship and innovation [3], for instance[1].

In all of this literature, with the exception of the last cited work, the idea of uncertainty is inherited from the classical theory of choice under uncertainty, as developed by von Neuamnn and Morgenstern [61], De Finetti [32], and Savage [59]. This classical theory follows the Bayesian paradigm, where the uncertainty that a given economic agent faces in a given decision problem is described by a probability measure over a given space of contingencies. When this probability measure is *objective* [61], i.e. independent of the decision maker's (DM) preferences, the problem is typically referred to as a situation of *decision under risk*. When this probability measure is *subjective* [32, 59], i.e. determined from the DM's preferences, the problem is one of *decision under uncertainty*. In both situations, be it a situation of decision under risk or one of decision under uncertainty, a DM has a clear probabilistic assessment of the underlying uncertainty that he faces. However, since the seminal work of Knight [49], there was an implicit discomfort with this Bayesian viewpoint, and the possibility that the DM might not be fully confident in his probabilistic assessment has been alluded to. As Knight writes [49, p. 227],

> *"The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself."*

This suggests that there might be situations of decision under uncertainty where the information available to a DM is too coarse for him to be able to formulate an additive probability measure over the list of contingencies. These occurrences are typically referred to as situations of decision under *Knightian uncertainty*, or *ambiguity*. Yet, this did not penetrate the mainstream theory of choice[2]

[1] In parts of the literature cited above, the term *equimeasruable rearrangement* is not used *per se*, although the mathematical tool used is in fact an equimeasurable rearrangement.

[2] Savage himself was aware of this issue, however. Indeed, he wrote [59, pp. 57-58]: *"[T]here seems to be some probability relations about which we feel relatively "sure" as compared with others. [...] The notion of "sure" and "unsure" introduced here is vague, and my complaint is precisely that neither the theory of personal probability, as it is developed in this book, nor any other device known to me renders the notion less vague."*

until Ellsberg's [36] famous thought experiments, which can be seen as an indication of people's aversion to *unknown unknowns*, or vagueness in beliefs about likelihoods, and as an inconsistency in the classical theory [32, 59]. There is a substantial body of empirical evidence for the pervasiveness of *ambiguity* in situations of choice under uncertainty, and I refer to Camerer [6] for a still timely review.

Ellsberg gave the following example. Consider an urn containing a total of 90 balls, 30 of which are red (R), and the remaining 60 are either black (B) or yellow (Y), with an unknown proportion. Individuals are asked to draw a ball at random from this urn and to consider the following four "gambles":

- *Gamble A*: Win \$100 if you draw a red ball, and win \$0 otherwise;
- *Gamble B*: Win \$100 if you draw a black ball, and win \$0 otherwise;
- *Gamble C*: Win \$100 if you draw a red or yellow ball, and win \$0 otherwise;
- *Gamble D*: Win \$100 if you draw a black or yellow ball, and win \$0 otherwise.

According to the classical Subjective Expected-Utility Theory (SEU) [59], a DM will choose gamble A over gamble B if and only if he believes that the probability of drawing a red ball is higher than that of drawing black ball. Similarly, according to SEU, he will prefer gamble C to gamble D if and only if he believes that drawing a red or yellow ball is more likely than drawing a black or yellow ball. Hence, if you prefer gamble A to gamble B, you will also prefer gamble C to gamble D, assuming that your beliefs are represented by a (unique) subjective probability measure P. In particular, it is the *additivity* of this probability measure P that implies this consistency in choice behavior. Indeed, since $P(R \cup Y) = P(R) + P(Y)$ and $P(B \cup Y) = P(B) + P(Y)$, it follows that

$$P(R) \geqslant P(B) \iff P(R \cup Y) \geqslant P(B \cup Y)$$

In Ellsberg's example, individuals are asked to rank their preferences between gambles A and B on the one hand, and gambles C and D on the other hand. Ellsberg predicted (and his prediction was supported by empirical evidence[3]) that most individuals tend to strictly prefer gamble A to gamble B and gamble D to gamble C, violating the prediction of SEU. This was referred to as the *Ellsberg paradox*, since it is a paradox in the framework of SEU. In essence, the Ellsberg paradox suggests that people prefer *known uncertainties* to *unknown uncertainties*: the probability of winning \$100 in gamble A is exactly $1/3$, whereas the probability of winning \$100 in gamble B is unknown. Similarly, the probability of winning \$100 in gamble D is exactly $2/3$, whereas the probability of winning \$100 in gamble C is unknown.

Largely motivated by the Ellsberg paradox, modern decision theory, also called *neo-Bayesian decision theory* not only distinguishes between (objective) *risk* and (subjective) *uncertainty*, but also between *uncertainty* and *ambiguity*. Neo-Bayesian decision theory is an umbrella term that refers to several models of choice under uncertainty and ambiguity that aim at describing the behavior of an economic agent in the presence of ambiguity, and to accommodate for behavior such as the one described in the three-color urn example above. For example, in Schmeidler [60] ambiguity is represented by a non-additive subjective "probability" measure, called a *capacity*, and preferences are aggregated using an integral defined with respect to capacities: the *Choquet integral*. Schmeidler's [60] seminal work, and his model of decision under ambiguity, which came to be known as *Choquet Expected Utility* (CEU) can be seen as the starting point of decision theoretic investigations of models of choice under ambiguity. It is easy to see how CEU can accommodate for the behavior described

[3] See Camerer and Weber [7] for instance.

in the Ellsberg example. For instance, if κ is a non-additive "probability" measure representing the DM's beliefs in the three-color urn example above, and if $\kappa(R) = \kappa(R \cup B) = \kappa(R \cup Y) = 1/3$, $\kappa(B) = \kappa(Y) = 0$, and $\kappa(B \cup Y) = 2/3$, then the DM will prefer Gamble A to Gamble B *and* Gamble D to Gamble C, as predicted by Ellsberg.

After Schmeidler's work, many axiomatic models of decision under ambiguity were introduced. In Gilboa and Schmeidler [44], ambiguity is described by a set of probability measures, rather than one such measure, and preferences are aggregated using the minimum value of the usual (Lebesgue) integral over this collection. Recently, Ghirardato, Maccheroni, and Marinacci [39] proposed a general model of decision under ambiguity that includes that of [44], and Amarante [2] introduced a model of decision under ambiguity that includes the aforementioned ones. I refer to the recent survey of Gilboa and Marinacci [43] for more on this topic, including other models of decision under ambiguity and applications of these models to several problems in economic theory.

The important contribution of Amarante [2] was to show that Choquet integration is a wide enough aggregation concept for preferences that it can encompass most models of decision under ambiguity, and in particular the most popular ones. Indeed, Amarante [2] shows that most models of decision under ambiguity can be represented as models were the objects of choice (or acts) are evaluated by a Choquet integral with respect to some capacity. The ideas of a capacity, a "distribution" of a function with respect to a capacity, and a Choquet integral are at the core of theory of choice under ambiguity, and the work of Amarante [2] is only a reminder of this. In fact, ever since the idea of a non-additive probability measure entered economic theory, there has been work devoted to extending some of the classical measure-theoretic and probabilistic tools to a setting of non-additive measures, with the purpose of applying these tools to problems in economics where ambiguity prevails. See, for instance, [18, 19, 21, 22, 33, 38, 53, 54, 56, 57], to cite only a few. This paper falls in this line of work. Indeed, the entire literature on equimeasurable rearrangements is confined to the classical measure-theoretic setup where one is given an underlying measure space, and where the equimeasurabiliy of two functions means that they both have the same distribution according to the underlying probability measure. In order to extend the use of these powerful rearrangement techniques to situations of ambiguity, it is imperative to be able to define an equimeasurable rearrangement of a function in the case where equimeasurability is defined relative to an underlying non-additive measure, or a capacity. This is precisely the aim of this paper.

Specifically, let ν be a capacity (Definition 2.1 below) on a given measurable space $(S, \mathcal{G})$, let $X : S \to \mathbb{R}^+$ be a given bounded $\mathcal{G}$-measurable function with $X(S) := [0, M]$, and let $Y = I \circ X$, for some bounded, Borel-measurable map $I : X(S) \to \mathbb{R}^+$. For each bounded function $Y : S \to \mathbb{R}^+$, let $\|Y\|_{sup} := \sup\{Y(s) : s \in S\}$. If ν is continuous (Definition 2.2 below) and verifies a property that will be called *strong nonatomicity with respect to* X (Definition 2.10 below), then there exists a function $\widetilde{Y} : S \to \mathbb{R}^+$ such that:

(1) $\widetilde{Y} = \widetilde{I} \circ X$, for some function $\widetilde{I} : X(S) \to \mathbb{R}^+$ which is nonincreasing, right-continuous, bounded, and Borel-measurable;

(2) $\nu\Big(\{s \in S : Y(s) > t\}\Big) = \nu\Big(\{s \in S : \widetilde{Y}(s) > t\}\Big)$ for each $t \in \mathbb{R}$; and,

(3) $\|\widetilde{Y}\|_{sup} \leqslant \|Y\|_{sup}$.

In particular,

(1) The Choquet integral of Y with respect to ν (Definition 2.5 below) is equal to the Choquet integral of $\widetilde{Y}$ with respect to ν; and,

(2) $\widetilde{Y}$ is *anti-comonotonic* with X (Definition 2.7 below).

$\widetilde{Y}$ will be called a *nonincreasing ν-upper-rearrangement of Y with respect to X*. I show that the property of *strong nonatomicity with respect to* X is closely related to the assumption of nonatomicity of $\nu \circ X^{-1}$, i.e. the assumption that $\nu \circ X^{-1}(\{t\}) = 0$ for each $t \in \mathbb{R}$. Strong nonatomicity and nonatomicity coincide for (additive) measures and for submodular capacities (Definition 2.3 below). Similarly, I consider a nondecreasing rearrangement, and then I examine the special case of nonatomic (additive) measures.

This is a simple, yet powerful result that can be used in situations of ambiguity where monotonicity of a solution is paramount. Just as the classical theory of equimeasurable rearrangements provided a powerful tool in many problems in economic theory, where uncertainty is purely Bayesian, the results of this paper can be seen as a tool for extending these Bayesian analyses in economic theory to situations of ambiguity. As an illustration, I examine a problem of production under uncertainty. Specifically, I consider an economy with a producer and a consumer. The producer faces an uncertain price of an input, and has the possibility of producing several goods, each of which is produced in a random amount that depends on the random price of the input. Uncertainty is represented by a state space, as in the sate-contingent approach to the theory of production under uncertainty [20]. The firm has ambiguous beliefs about the realizations of the uncertain price of the input, and this ambiguity is represented by a capacity on the sate space. The firm's problem is to choose a good to produce so as to minimize the "expected production cost" associated with the (random) amount produced of that good, subject to a minimum production target constraint, as well as some other constraints. This "expected production cost" is a mapping from the collection of all possible outputs to the real line, such as a Choquet integral. In this context, it is natural for the *optimal* good produced to be such that the amount produced of that good is a nonincreasing function of the input's uncertain price. Indeed, given a fixed budget, the more expensive the input the less amount of input can be purchased, and hence the less amount of outputs can be produced, *ceteris paribus*. I show that this desired monotonicity property can be achieved using the ideas developed in this paper.

Outline. The rest of the paper is organized as follows: Section 2 reviews some preliminary definitions and introduces a property of a given capacity that will be called *strong nonatomicity*; Section 3 introduces the idea of a monotone equimeasurable rearrangement in the context of a capacity; Section 4 examines the special case of rearrangements with respect to an additive probability measure and shows how the classical results can be recovered; Section 5 formulates an equimeasurable nonincreasing rearrangement for simple functions; Sections 6 gives an example of the many possible applications of the idea of rearrangement with respect to a capacity; and, Section 7 concludes. Appendix A gives some useful related analysis, and most of the proofs are relegated to Appendix B.

2. Preliminaries

2.1. **Capacities and the Choquet Integral.**

Let $(S, \mathcal{G})$ be a given measurable space. For $C \subseteq S$, denote by $\mathbf{1}_C$ the indicator function of C.

Definition 2.1. A (normalized) *capacity* on $(S, \mathcal{G})$ is a set function $\nu : \mathcal{G} \to [0, 1]$ such that

(1) $\nu(\varnothing) = 0$;

(2) $\nu(S) = 1$; and,

(3) ν is monotone: for any $A, B \in \mathcal{G}$, $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$.

An example of a capacity on a measurable space $(S, \mathcal{G})$ is a set function $\nu := T \circ P$, where P is a probability measure on $(S, \mathcal{G})$ and $T : [0,1] \to [0,1]$ is increasing with $T(0) = 0$ and $T(1) = 1$. Such a function T is usually called a *probability distortion*, and the capacity $T \circ P$ is usually called a *distorted probability measure*.

Definition 2.2. A capacity ν on $(S, \mathcal{G})$ is said to be *continuous from above* if for any sequence $\{A_n\}_n$ in $\mathcal{G}$ such that $A_{n+1} \subseteq A_n$ for each $n \geqslant 1$, we have:

$$\lim_{n \to +\infty} \nu\left(A_n\right) = \nu\left(\bigcap_{n=1}^{+\infty} A_n\right) \tag{2.1}$$

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$$\lim_{n \to +\infty} \nu\left(A_n\right) = \nu\left(\bigcup_{n=1}^{+\infty} A_n\right) \tag{2.2}$$

Finally, a capacity ν on $(S, \mathcal{G})$ is said to be *continuous* if it is both continuous from above and continuous from below.

For instance, if P is a probability measure on $(S, \mathcal{G})$ and $T : [0,1] \to [0,1]$ is increasing and continuous, with $T(0) = 0$ and $T(1) = 1$, then the set function $\nu := T \circ P$ is a capacity on $(S, \mathcal{G})$ which is continuous. This is an immediate consequence of the continuity of the measure P for monotone sequences [27, Prop. 1.2.3] and the continuity of T.

Definition 2.3. A capacity ν on $(S, \mathcal{G})$ is said to be *submodular* (resp. *additive*) if for each $A, B \in \mathcal{G}$,

$$\nu\left(A \cup B\right) + \nu\left(A \cap B\right) \leqslant (\text{resp.} =)\ \nu\left(A\right) + \nu\left(B\right) \tag{2.3}$$

For example, if P is a probability measure on $(S, \mathcal{G})$ and $T : [0,1] \to [0,1]$ is increasing and concave, with $T(0) = 0$ and $T(1) = 1$, then the set function $\nu := T \circ P$ is a capacity on $(S, \mathcal{G})$ which is submodular [33, Ex. 2.1].

Definition 2.4. For a given capacity ν on $(S, \mathcal{G})$ and a given $\psi \in B\left(\mathcal{G}\right)$, the *upper-distribution* of ψ with respect to ν is the function

$$\begin{aligned} G_{\nu,\psi} &: \mathbb{R} \to [0,1] \\ &t \mapsto G_{\nu,\psi}\left(t\right) := \nu\big(\{s \in S : \psi\left(s\right) > t\}\big) \end{aligned} \tag{2.4}$$

and the *lower-distribution* of ψ with respect to ν is the function

$$\begin{aligned} F_{\nu,\psi} &: \mathbb{R} \to [0,1] \\ &t \mapsto F_{\nu,\psi}\left(t\right) := \nu\big(\{s \in S : \psi\left(s\right) \leqslant t\}\big) \end{aligned} \tag{2.5}$$

Then $G_{\nu,\psi}$ is nonincreasing, and if ν is continuous from below then $G_{\nu,\psi}$ is right-continuous [33, p. 46]. Similarly, $F_{\nu,\psi}$ is nondecreasing, and if ν is continuous from above then $F_{\nu,\psi}$ is right-continuous. Clearly, if ν is additive then $F_{\nu,\psi}\left(t\right) = 1 - G_{\nu,\psi}\left(t\right)$, for each $t \in \mathbb{R}$.

If $\nu = T \circ P$, for some probability measure P on $(S, \mathcal{G})$ and some distortion function $T : [0,1] \to [0,1]$, then for any $\phi_1, \phi_2 \in B(\mathcal{G})$, if ϕ_1 and ϕ_2 are identically distributed[4] for P, then they have the same upper-distribution with respect to ν and the same lower-distribution with respect to ν.

For any capacity ν on $(S, \mathcal{G})$ and for any $\phi_1, \phi_2 \in B(\mathcal{G})$, let $\phi_1 \overset{\nu}{\sim} \phi_2$ mean that ϕ_1 and ϕ_2 have the same upper-distribution with respect to ν, and let $\phi_1 \underset{\nu}{\sim} \phi_2$ mean that ϕ_1 and ϕ_2 have the same lower-distribution with respect to ν.

Definition 2.5. For a given capacity ν on $(S, \mathcal{G})$ and a given $\psi \in B^+(\mathcal{G})$, the *Choquet integral* $\oint \psi \, d\nu$ of ψ with respect to ν is defined by

$$\oint \psi \, d\nu := \int_0^{+\infty} \nu\big(\{s \in S : \psi(s) > t\}\big) \, dt = \int_0^{+\infty} G_{\nu,\psi}(t) \, dt \tag{2.6}$$

If $\phi \in B(\mathcal{G})$, then the *Choquet integral* $\oint \phi \, d\nu$ of ϕ with respect to ν is defined by

$$\oint \phi \, d\nu := \int_0^{+\infty} \nu\big(\{s \in S : \phi(s) > t\}\big) \, dt + \int_{-\infty}^{0} \big[\nu\big(\{s \in S : \phi(s) > t\}\big) - 1\big] \, dt \tag{2.7}$$

As a result, if $\phi_1, \phi_2 \in B(\mathcal{G})$ have the same upper-distribution with respect to ν then $\oint \phi_1 \, d\nu = \oint \phi_2 \, d\nu$. This motivates the following definition.

Definition 2.6. Given a capacity ν on $(S, \mathcal{G})$, a mapping $V : B(\mathcal{G}) \to \mathbb{R}$ is said to be *ν-upper-law-invariant* if for any $\phi_1, \phi_2 \in B(\mathcal{G})$,

$$\phi_1 \overset{\nu}{\sim} \phi_2 \implies V(\phi_1) = V(\phi_2)$$

Similarly, V is said to be *ν-lower-law-invariant* if for any $\phi_1, \phi_2 \in B(\mathcal{G})$,

$$\phi_1 \underset{\nu}{\sim} \phi_2 \implies V(\phi_1) = V(\phi_2)$$

For instance, the Choquet integral with respect to a given capacity ν is a ν-upper-law-invariant function on $B(\mathcal{G})$. Moreover, if ν is a *bona fide* countably additive measure then for each $\phi_1, \phi_2 \in B(\mathcal{G})$,

$$\phi_1 \underset{\nu}{\sim} \phi_2 \iff \phi_1 \overset{\nu}{\sim} \phi_2 \iff \nu \circ \phi_1^{-1}(B) = \nu \circ \phi_2^{-1}(B), \forall B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. The last equivalence is a straight-forward application of Dynkin's π-λ theorem (Theorem A.5, p. 20).

The Choquet integral with respect to a measure is simply the usual Lebesgue integral with respect to that measure [55, p. 59]. For any capacity ν on $(S, \mathcal{G})$ and for any $\psi \in B^+(\mathcal{G})$, the following holds [55, Prop. 4.8]:

$$\oint \psi \, d\nu = \int_0^{+\infty} \nu\big(\{s \in S : \psi(s) \geqslant t\}\big) \, dt \tag{2.8}$$

Moreover, for any capacity ν on $(S, \mathcal{G})$ and for any $\phi \in B(\mathcal{G})$, the following holds [55, p. 60]:

[4] That is, $P \circ \phi_1^{-1}(B) = P \circ \phi_2^{-1}(B)$, for any Borel set B.

$$\oint \psi \; d\nu = \int_0^{+\infty} \nu\big(\{s \in S : \phi(s) \geqslant t\}\big) \; dt + \int_{-\infty}^0 \Big[\nu\big(\{s \in S : \phi(s) \geqslant t\}\big) - 1\Big] \; dt \tag{2.9}$$

Finally, as a functional on $B(\mathcal{G})$, the Choquet integral (with respect to some given capacity) is supnorm-continuous, being Lipschitz continuous [55, Prop. 4.11].

Definition 2.7. Two functions $Y_1, Y_2 \in B(\mathcal{G})$ are said to be comonotonic if

$$\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \geqslant 0, \text{ for all } s, s' \in S \tag{2.10}$$

Similarly, two functions $Y_1, Y_2 \in B(\mathcal{G})$ are said to be anti-comonotonic if

$$\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \leqslant 0, \text{ for all } s, s' \in S \tag{2.11}$$

For instance any $Y \in B(\mathcal{G})$ is comonotonic with any $c \in \mathbb{R}$. Moreover, if $Y_1, Y_2 \in B(\mathcal{G})$, and if Y_2 is of the form $Y_2 = I \circ Y_1$, for some Borel-measurable function I, then Y_2 is comonotonic (resp. anti-comonotonic) with Y_1 if and only if the function I is nondecreasing (resp. nonincreasing).

The following proposition gathers some properties of the Choquet integral.

Proposition 2.8. *Let ν be a capacity on $(S, \mathcal{G})$.*

(1) *If $\phi_1, \phi_2 \in B(\mathcal{G})$ are comonotonic, then $\oint (\phi_1 + \phi_2) \; d\nu = \oint \phi_1 \; d\nu + \oint \phi_2 \; d\nu$.*
(2) *If $\phi \in B(\mathcal{G})$ and $c \in \mathbb{R}$, then $\oint (\phi + c) \; d\nu = \oint \phi \; d\nu + c$.*
(3) *If $A \in \mathcal{G}$ then $\oint \mathbf{1}_A \; d\nu = \nu(A)$.*
(4) *If $\phi \in B(\mathcal{G})$ and $a \geqslant 0$, then $\oint a \; \phi \; d\nu = a \; \oint \phi \; d\nu$.*
(5) *If $\phi_1, \phi_2 \in B(\mathcal{G})$ are such that $\phi_1 \leqslant \phi_2$, then $\oint \phi_1 \; d\nu \leqslant \oint \phi_2 \; d\nu$.*
(6) *If ν is submodluar, then for any $\phi_1, \phi_2 \in B(\mathcal{G})$, $\oint (\phi_1 + \phi_2) \; d\nu \leqslant \oint \phi_1 \; d\nu + \oint \phi_2 \; d\nu$.*

Proof. [55, Th. 4.3, Th. 4.6, Prop. 4.11] or [33, Prop. 5.1, Prop. 6.3]. □

For more about capacities and the Choquet integral, I refer to Denneberg [33], Marinacci and Montrucchio [55], or Pap [56].

2.2. **Capacities and Strong Nonatomicity.** Let ν be a given capacity on $(S, \mathcal{G})$, fix some $X \in B(\mathcal{G})$. Then it is easily seen that the set function $\nu \circ X^{-1}$ defined on the Borel σ-algebra of $\mathbb{R}$ is a capacity.

Definition 2.9. The capacity $\nu \circ X^{-1}$ is said to be *nonatomic* if for any $t \in \mathbb{R}$, $\nu \circ X^{-1}(\{t\}) = 0$.

Definition 2.10. The capacity ν is said to be *strongly nonatomic with respect to* X if for any $a, b \in \mathbb{R}$,

$$\nu \circ X^{-1}\Big((a,b)\Big) = \nu \circ X^{-1}\Big([a,b)\Big) = \nu \circ X^{-1}\Big((a,b]\Big) = \nu \circ X^{-1}\Big([a,b]\Big) \tag{2.12}$$

When ν is strongly nonatomic with respect to X, the capacity $\nu \circ X^{-1}$ will be called *strongly nonatomic*.

Intuitively, the requirement that ν be strongly nonatomic with respect to X (i.e. that $\nu \circ X^{-1}$ be strongly nonatomic) is a strengthening of the requirement that $\nu \circ X^{-1}$ be nonatomic. The following proposition formalizes this fact, and its proof is in Appendix B.

Proposition 2.11.

(1) *If ν is strongly nonatomic with respect to X, then $\nu \circ X^{-1}$ is nonatomic.*

(2) *If $\nu \circ X^{-1}$ is nonatomic and if ν is submodular, then ν is strongly nonatomic with respect to X.*

Remark 2.12. If ν is additive, then ν is is strongly nonatomic with respect to X if and only if $\nu \circ X^{-1}$ is nonatomic. That is, when ν is additive, $\nu \circ X^{-1}$ is strongly nonatomic if and only if $\nu \circ X^{-1}$ is nonatomic. This is an immediate consequence of Proposition 2.11, since additivity implies submodularity.

3. Monotone Equimeasurable Rearrangements: The Case of a Capacity

Let ν be a given capacity on a given measurable space $(S, \mathcal{G})$, and let $X \in B^+(\mathcal{G})$ be fixed all throughout. It will be assumed that the function X has a closed range $[0, M]$, where $M := \|X\|_{sup}$. Denote by Σ the σ-algebra $\sigma\{X\}$ of subsets of S generated by X. Then by a classical result [1, Th. 4.41], the elements of $B^+(\Sigma)$ are the functions $Y : S \to \mathbb{R}$ of the form $Y = I \circ X$, for some bounded, nonnegative, and Borel-measurable map $I : X(S) \to \mathbb{R}^+$. For each $I : X(S) \to \mathbb{R}^+$, let $\|I\|_{sup} := \sup\{I(x) : x \in X(S)\}$. Similarly, for each $Y \in B^+(\Sigma)$, let $\|Y\|_{sup} := \sup\{Y(s) : s \in S\}$. Then for any $Y = I \circ X \in B^+(\Sigma)$, $\|Y\|_{sup} = \|I\|_{sup}$.

If $I, I_n : [0, M] \to [0, M]$, for each $n \geqslant 1$, I will write $I_n \uparrow I$ to signify that the sequence $\{I_n\}_n$ is a nondecreasing sequence of functions and that $\lim_{n \to +\infty} I_n(t) = I(t)$, for all $t \in [0, M]$.

Assumption 3.1. *ν is continuous and strongly nonatomic with respect to X.*

Assumption 3.1 implies that $\nu \circ X^{-1}$ is a continuous and nonatomic capacity.

3.1. A Nonincreasing ν-Upper-Equimeasurable Rearrangement.

For a given $Y = I \circ X \in B^+(\Sigma)$, define the map

$$
\begin{aligned}
& G_{\nu,X,I} : \mathbb{R} \to [0,1] \\
& \qquad t \mapsto G_{\nu,X,I}(t) := \nu \circ X^{-1}\big(\{z \in [0, M] : I(z) > t\}\big)
\end{aligned}
\tag{3.1}
$$

to be the upper-distribution of I with respect to $\nu \circ X^{-1}$. Then $G_{\nu,X,I}$ is nonincreasing and right-continuous, due to Assumption 3.1. Moreover,

Proposition 3.2. *If $I, J, I_n : [0, M] \to \mathbb{R}^+$, $n \geqslant 1$, are Borel-measurable then:*

(1) $(I \leqslant J) \Rightarrow (G_{\nu,X,I} \leqslant G_{\nu,X,J})$.

(2) *If ν is also continuous from below, then $(I_n \uparrow I) \Rightarrow (G_{\nu,X,I_n} \uparrow G_{\nu,X,I})$.*

The proof of Proposition 3.2 is in Appendix B. Now, let $Y = I \circ X \in B^+(\Sigma)$, with $I : [0, M] \to \mathbb{R}^+$ bounded and Borel-measurable.

Definition 3.3. Define the function $\widetilde{I}: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\widetilde{I}(t) := \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \tag{3.2}$$

The following proposition gives some properties of the map $\widetilde{I}$. Its proof is in Appendix B.

Proposition 3.4. *Under Assumption 3.1, the following hold:*

1. $\widetilde{I}$ *is nonincreasing and Borel-measurable.*
2. $\widetilde{I}$ *is right-continuous.*
3. *For all* $t \in \mathbb{R}^+$, $G_{\nu,X,I}\left(\widetilde{I}(t)\right) \leqslant \nu \circ X^{-1}\big([0,t]\big)$.
4. *If* $I_1, I_2 : [0,M] \to \mathbb{R}^+$ *are such that* $I_1 \leqslant I_2$, *then* $\widetilde{I}_1 \leqslant \widetilde{I}_2$.
5. I *and* $\widetilde{I}$ *have the same upper-distribution with respect to* $\nu \circ X^{-1}$.
6. *If* $\|I\|_{sup} = N$ $(< +\infty)$, *then* $\|\widetilde{I}\|_{sup} \leqslant N$.
7. *If* $\{I_n\}_n$ *is a sequence of bounded Borel-measurable functions from* $[0,M]$ *into* $\mathbb{R}^+$ *such that* $I_n \uparrow I$, *for some bounded Borel-measurable function* $I : [0,M] \to \mathbb{R}^+$, *then* $\widetilde{I}_n \uparrow \widetilde{I}$.

Definition 3.5. For each $Y = I \circ X \in B^+(\Sigma)$, define the function the function $\widetilde{Y}$ by

$$\widetilde{Y} := \widetilde{I} \circ X \tag{3.3}$$

When Assumption 3.1 holds, Proposition 3.4 implies that the function $\widetilde{Y}$ is bounded, Σ-measurable, anti-comonotonic with X, and has the same upper-distribution as Y with respect to ν. In particular, $\oint \widetilde{Y}\, d\nu = \oint Y\, d\nu$. Moreover, $\|\widetilde{Y}\|_{sup} \leqslant \|Y\|_{sup}$. The function $\widetilde{Y}$ will be called a ***nonincreasing*** $\boldsymbol{\nu}$***-upper-equimeasurable rearrangement of*** Y ***with respect to*** X.

3.2. A Nondecreasing ν-Lower-Equimeasurable Rearrangement.

For a given $Y = I \circ X \in B^+(\Sigma)$, define the map

$$\begin{aligned} F_{\nu,X,I} &: \mathbb{R} \to [0,1] \\ t &\mapsto F_{\nu,X,I}(t) := \nu \circ X^{-1}\big(\{z \in [0,M] : I(z) \leqslant t\}\big) \end{aligned} \tag{3.4}$$

to be the *lower-distribution* of I with respect to $\nu \circ X^{-1}$. Then $F_{\nu,X,I}$ is nondecreasing and right-continuous, due to Assumption 3.1.

Now, let $Y = I \circ X \in B^+(\Sigma)$, with $I : [0,M] \to \mathbb{R}^+$ bounded and Borel-measurable.

Definition 3.6. Define the function $\widetilde{\widetilde{I}}: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\widetilde{\widetilde{I}}(t) := \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \tag{3.5}$$

The following proposition gives some properties of the map $\widetilde{\widetilde{I}}$. Its proof is in Appendix B.

Proposition 3.7. *Under Assumption 3.1, the following hold:*

(1) $\tilde{\tilde{I}}$ *is nondecreasing and Borel-measurable.*

(2) $\tilde{\tilde{I}}$ *is left-continuous.*

(3) *For all* $t \in \mathbb{R}^+$, $F_{\nu,X,I}\left(\tilde{\tilde{I}}(t)\right) \geqslant \nu \circ X^{-1}\big([0,t]\big)$.

(4) *If* $I_1, I_2 : [0,M] \to \mathbb{R}^+$ *are such that* $I_1 \leqslant I_2$, *then* $\tilde{\tilde{I}}_1 \leqslant \tilde{\tilde{I}}_2$.

(5) *If* $Id : [0,M] \to [0,M]$ *denotes the identity function, then* $\widetilde{\widetilde{Id}} \leqslant Id$.

(6) I *and* $\tilde{\tilde{I}}$ *have the same lower-distribution with respect to* $\nu \circ X^{-1}$.

(7) *If* $\|I\|_{sup} = N\ (< +\infty)$, *then* $\|\tilde{\tilde{I}}\|_{sup} \leqslant N$.

Definition 3.8. For each $Y = I \circ X \in B^+(\Sigma)$, define the function the function $\tilde{\tilde{Y}}$ by

$$\tilde{\tilde{Y}} := \tilde{\tilde{I}} \circ X \tag{3.6}$$

When Assumption 3.1 holds, Proposition 3.7 implies that the function $\tilde{\tilde{Y}}$ is bounded, Σ-measurable, comonotonic with X, and has the same lower-distribution as Y with respect to ν. Moreover, $\|\tilde{\tilde{Y}}\|_{sup} \leqslant \|Y\|_{sup}$. The function $\tilde{\tilde{Y}}$ will be called a ***nondecreasing ν-lower-equimeasurable rearrangement of Y with respect to X***.

4. Monotone Equimeasurable Rearrangements: The Case of a Measure

Here I consider the special case of a rearrangement on a nonatomic measure space, and show how the classical results [45, 26, 30, 52] can be obtained as special cases of the results given in the previous section.

Consider the setting of Section 3, and suppose also that P is a given (countably additive) probability measure on (S, Σ). Denote by Ψ_X the probability measure $P \circ X^{-1}$ on the Borel sets, that is, the law of X for the measure P. If $I, I_n : [0,M] \to [0,M]$, for each $n \geqslant 1$, I will write $I_n \downarrow I,\ \Psi_X$-a.s., to signify that the sequence $\{I_n\}_n$ is a nonincreasing sequence of functions and that $\lim\limits_{n\to+\infty} I_n(t) = I(t)$, for Ψ_X-a.a. $t \in [0,M]$. Similarly, I will write $I_n \uparrow I,\ \Psi_X$-a.s., to signify that the sequence $\{I_n\}_n$ is a nondecreasing sequence of functions and that $\lim\limits_{n\to+\infty} I_n(t) = I(t)$, for Ψ_X-a.a. $t \in [0,M]$.

All throughout this section, the following assumption will be made.

Assumption 4.1. Ψ_X *is nonatomic, that is,* X *is a continuous random variable on the probability space* $(S, \mathcal{G}, P)$.

Recall from Remark 2.12 that since P is a *bona fide* measure, nonatomicity and strong nonatomicity of Ψ_X are equivalent.

4.1. A Nondecreasing Rearrangement.

For a given $Y = I \circ X \in B^+(\Sigma)$, define the map

$$\begin{aligned} F_{\Psi_X,I} &: \mathbb{R} \to [0,1] \\ t &\mapsto F_{\Psi_X,I}(t) := \Psi_X\big(\{z \in [0,M] : I(z) \leqslant t\}\big) \end{aligned} \tag{4.1}$$

to be the *distribution function* of I with respect to Ψ_X. Then $F_{\Psi_X,I}$ is nondecreasing and right-continuous, due to Assumption 4.1. The function $t \mapsto 1 - F_{\Psi_X,I}(t)$ is usually called the *survival function* of I with respect to Ψ_X, and for each $t \in \mathbb{R}$,

$$1 - F_{\Psi_X,I}(t) = \Psi_X\big(\{z \in [0,M] : I(z) > t\}\big) := G_{\Psi_X,I}(t)$$

Now, let $Y = I \circ X \in B^+(\Sigma)$, with $I : [0,M] \to \mathbb{R}^+$ bounded and Borel-measurable.

Definition 4.2. Define the function $\tilde{\tilde{I}} : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\tilde{\tilde{I}}(t) := \inf\Big\{z \in \mathbb{R}^+ : F_{\Psi_X,I}(z) \geqslant \Psi_X\big([0,t]\big)\Big\} \tag{4.2}$$

The following proposition gives some properties of the map $\tilde{\tilde{I}}$. Its proof is in Appendix B.

Proposition 4.3. *Let $I : [0,M] \to [0,M]$ be any Borel-measurable map and let $\tilde{\tilde{I}} : [0,M] \to \mathbb{R}$ be defined as in equation (4.2). Then, under Assumption 4.1, the following hold:*

1. *$\tilde{\tilde{I}}$ is left-continuous, nondecreasing, and Borel-measurable;*
2. *For all $t \in \mathbb{R}^+$, $F_{\Psi_X,I}\left(\tilde{\tilde{I}}(t)\right) \geqslant \Psi_X\big([0,t]\big)$*
3. *$\tilde{\tilde{I}}(t) \geqslant 0$, for each $t \in [0,M]$, $\tilde{\tilde{I}}(0) = 0$, and $\tilde{\tilde{I}}(M) \leqslant M$;*
4. *If $I_1, I_2 : [0,M] \to [0,M]$ are such that $I_1 \leqslant I_2$, ϕ-a.s., then $\tilde{\tilde{I}}_1 \leqslant \tilde{\tilde{I}}_2$;*
5. *If $Id : [0,M] \to [0,M]$ denotes the identity function, then $\widetilde{\widetilde{Id}} \leqslant Id$;*
6. *$\tilde{I}$ is Ψ_X-equimeasurable with I, in the sense that for any Borel set B,*

$$\Psi_X\Big(\{t \in [0,M] : I(t) \in B\Big) = \Psi_X\Big(\{t \in [0,M] : \tilde{I}(t) \in B\}\Big) \tag{4.3}$$

7. *If $\overline{I} : [0,M] \to \mathbb{R}^+$ is another nondecreasing, Borel-measurable map which is Ψ_X-equimeasurable with I, then $\overline{I} = \tilde{\tilde{I}}$, Ψ_X-a.s.;*
8. *If $\|I\|_{sup} = N$ $(< +\infty)$, then $\|\tilde{\tilde{I}}\|_{sup} \leqslant N$.*
9. *If $I, I_n : [0,M] \to [0,M]$, for each $n \geqslant 1$, and $I_n \downarrow I$, ϕ-a.s., then $\tilde{\tilde{I}}_n \downarrow \tilde{\tilde{I}}$, Ψ_X-a.s.*

Definition 4.4. For each $Y = I \circ X \in B^+(\Sigma)$, define the function the function $\tilde{\tilde{Y}}$ by

$$\tilde{\tilde{Y}} := \tilde{\tilde{I}} \circ X \tag{4.4}$$

When Assumption 4.1 holds, Proposition 4.3 implies that the function $\tilde{\tilde{Y}}$ is bounded, Σ-measurable, comonotonic with X, has the same distribution as Y with respect to P, and is P-a.s. unique. Moreover, $\|\tilde{\tilde{Y}}\|_{sup} \leqslant \|Y\|_{sup}$. The function $\tilde{\tilde{Y}}$ will be called the ***nondecreasing*** P***-equimeasurable rearrangement of*** Y ***with respect to*** X.

Similarly to the previous construction, for a given a Borel-measurable set $B \subseteq [0, M]$ with $\phi(B) > 0$, there exists a Ψ_X-a.s. unique (on B) nondecreasing, Borel-measurable mapping $\widetilde{\widetilde{I_B}} : B \to [0, M]$ which is Ψ_X-equimeasurable with I on B, in the sense that for any $\alpha \in [0, M]$,

$$\Psi_X\Big(\{t \in B : I(t) \leqslant \alpha\}\Big) = \Psi_X\Big(\{t \in B : \widetilde{\widetilde{I_B}}(t) \leqslant \alpha\}\Big)$$

$\widetilde{\widetilde{I_B}}$ is called a nondecreasing Ψ_X-rearrangement of I on B. Since X is $\mathcal{G}$-measurable, there exists $A \in \mathcal{G}$ such that $A = X^{-1}(B)$, and hence $P(A) > 0$. Now, define $\widetilde{\widetilde{Y_A}} := \widetilde{\widetilde{I_B}} \circ X$. Since both I and $\widetilde{\widetilde{I_B}}$ are bounded Borel-measurable mappings, it follows that $Y, \widetilde{\widetilde{Y_A}} \in B^+(\Sigma)$. Note also that $\widetilde{\widetilde{Y_A}}$ is nondecreasing in X on A, in the sense that if $s_1, s_2 \in A$ are such that $X(s_1) \leqslant X(s_2)$ then $\widetilde{\widetilde{Y_A}}(s_1) \leqslant \widetilde{\widetilde{Y_A}}(s_2)$, and that Y and $\widetilde{\widetilde{Y_A}}$ are P-equimeasurable on A, that is, for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leqslant \alpha\} \cap A) = P\left(\{s \in S : \widetilde{\widetilde{Y_A}}(s) \leqslant \alpha\} \cap A\right)$.

Call $\widetilde{\widetilde{Y_A}}$ the ***nondecreasing P-equimeasurable rearrangement of Y with respect to X on*** A. Note that $\widetilde{Y}_A$ is P-a.s. unique. Note also that if $Y_{1,A}$ and $Y_{2,A}$ are P-equimeasurable on A and if $\int_A Y_{1,A}\, dP < +\infty$, then $\int_A Y_{2,A}\, dP < +\infty$ and $\int_A \psi(Y_{1,A})\, dP = \int_A \psi(Y_{2,A})\, dP$, for any measurable function ψ such that the integrals exist.

Lemma 4.5. *Let $Y \in B^+(\Sigma)$ and let $A \in \mathcal{G}$ be such that $P(A) = 1$ and $X(A)$ is a Borel set*[5]*. Let $\tilde{\tilde{Y}}$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{\widetilde{Y_A}}$ be the nondecreasing P-rearrangement of Y with respect to X on A. Then $\tilde{\tilde{Y}} = \widetilde{\widetilde{Y_A}}$, P-a.s.*

4.2. **A Nonincreasing Rearrangement.** As for the nondecreasing rearrangement, one can define a nonincreasing equimeasurable rearrangement. The construction is similar to that of Section 3.1. However, the difference with the case of a continuous and strongly nonatomic capacity is that in the case of a nonatomic probability measure P, the nonincreasing P-equimeasurable rearrangement will be P-a.s. unique.

5. Nonincreasing Equimeasurable Rearrangements of Simple Functions

As in Section 3, Let $(S, \mathcal{G})$ be a given measurable space, and let $X \in B^+(\mathcal{G})$ be fixed all throughout, such that $X(S) = [0, M]$, where $M := \|X\|_{sup}$. Denote by Σ the σ-algebra $\sigma\{X\}$ of subsets of S generated by X, and let $B^+(\Sigma)$ denote the collection of all bounded, Σ-measurable nonnegative real-valued functions on S. Again, elements of $B^+(\Sigma)$ are the functions $Y : S \to \mathbb{R}$ of the form $Y = I \circ X$, for some bounded, nonnegative, and Borel-measurable map $I : X(S) \to \mathbb{R}^+$.

Let ν be a continuous capacity on (S, Σ) which is strongly nonatomic with respect to X. For any nonnegative, Σ-measurable function Y, there is a sequence $\{Y_n\}_n$ of nonnegative, Σ-measurable simple functions on (S, Σ) that converges monotonically upwards and pointwise to Y [27, Proposition 2.1.7]. Moreover, if Y is bounded (that is, $Y \in B^+(\Sigma)$), then the convergence is uniform [47, Theorem 11.35]. Since for each $n \geqslant 1$, Y_n is Σ-measurable, there is a bounded and Borel-measurable function

[5] Note that if $A \in \Sigma = \sigma\{X\}$ then $X(A)$ is automatically a Borel set, by definition of $\sigma\{X\}$. Indeed, for any $A \in \sigma\{X\}$, there is some Borel set B such that $A = X^{-1}(B)$. Then $X(A) = B \cap X(S)$ [34, p. 7]. Thus $X(A) = B \cap [0, M]$ is a Borel subset of $[0, M]$.

$I_n : [0, M] \to \mathbb{R}^+$ such that $Y_n = I_n \circ X$. Consequently, $I_n \uparrow I$. Thus, by Proposition 3.4, $\widetilde{I}_n \uparrow \widetilde{I}$, where $\widetilde{I}$ and $\widetilde{I}_n$ are defined as in eq. (3.2), for each $n \geqslant 1$. Therefore, $\widetilde{Y}_n \uparrow \widetilde{Y}$, where $\widetilde{Y} := \widetilde{I} \circ X$ is a nonincreasing ν-upper-equimeasurable rearrangement of Y with respect to X, and $\widetilde{Y}_n := \widetilde{I}_n \circ X$ is a nonincreasing ν-upper-equimeasurable rearrangement of Y_n with respect to X, for each $n \geqslant 1$.

Hence, one way to characterize a rearrangement of a nonnegative, Σ-measurable function Y is as a limit of rearrangements of simple functions. In this section, I give a characterization of a nonincreasing ν-upper-equimeasurable rearrangement of a simple function.

5.1. **Nonincreasing ν-Upper-Equimeasurable Rearrangement of a Simple function.** Any Σ-simple function $Y \in B^+(\Sigma)$ can be written as $Y = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{C_i}$, for some $\{\alpha_i\}_{i=1}^{n} \subset \mathbb{R}^+$ and a partition $\{C_i\}_{i=1}^{n}$ of S, where $C_i \in \Sigma$, for each $i \in \{1, \ldots, n\}$. Since $C_i \in \Sigma$, for each $i \in \{1, \ldots, n\}$, and since $\Sigma = \sigma\{X\}$, it follows that

$$Y(s) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{B_i}\big(X(s)\big), \ \forall s \in S \tag{5.1}$$

where B_i is a Borel subset of $X(S) = [0, M]$, for each $i \in \{1, \ldots, n\}$, and $\{B_i\}_i^n$ is a partition of $[0, M]$. in other words, $Y = I \circ X$, where the function I is a simple function on $[0, M]$ of the form

$$I = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{B_i}$$

Without loss of generality, assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n > \alpha_{n+1} := 0$, and recall from eq. (3.2) that

$$\begin{aligned} \widetilde{I}(t) &= \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \\ &= \inf\Big\{z \in \mathbb{R}^+ : \nu \circ X^{-1}([I > z]) \leqslant \nu \circ X^{-1}([0,t])\Big\} \end{aligned}$$

It can be easily verified that

$$\nu \circ X^{-1}([I > z]) = \sum_{i=1}^{n} \nu \circ X^{-1}(B_1 \cup \ldots \cup B_i)\, \mathbf{1}_{[\alpha_{i+1}, \alpha_i)}$$

and that

$$\widetilde{I}(t) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[t_{i-1}, t_i)} \tag{5.2}$$

where

(1) $m_i := \nu \circ X^{-1}(B_1 \cup \ldots \cup B_i)$, for $1 \leqslant i \leqslant n$; and,
(2) $t_0 := 0$ and $t_i := F^{-1}_{\nu \circ X^{-1}}(m_i)$, for $1 \leqslant i \leqslant n$, where $F_{\nu \circ X^{-1}}(t) := \nu \circ X^{-1}([0,t])$.

Note that $m_i = \nu \circ X^{-1}\big([0, t_i]\big) = \nu \circ X^{-1}\big([0, t_i)\big)$, for $1 \leqslant i \leqslant n$. It can be easily checked that I and $\widetilde{I}$ have the same upper-distribution with respect to $\nu \circ X^{-1}$, and, therefore, $Y \overset{\nu}{\sim} \widetilde{Y}$. In particular, Y and $\widetilde{Y}$ have the same Choquet integral with respect to ν.

5.2. **The Choquet Integral of a Simple Function.** For the simple function

$$Y = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{B_i}(X) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{X^{-1}(B_i)}$$

defined in eq. (5.1), with $\alpha_1 > \alpha_2 > \ldots > \alpha_n > \alpha_{n+1} := 0$, the Choquet integral is given by

$$\begin{aligned}\oint Y \, d\nu &:= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, \nu\Big(\bigcup_{j=1}^{i} X^{-1}(B_j) \Big) \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, \nu \circ X^{-1} \left(\bigcup_{j=1}^{i} B_j \right) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, m_i\end{aligned}$$

Since $\widetilde{Y} = \widetilde{I} \circ X = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{X^{-1}([t_{i-1}, t_i))}$ (eq. (5.2)), it then follows that

$$\begin{aligned}\oint \widetilde{Y} \, d\nu &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, \nu\Big(\bigcup_{j=1}^{i} X^{-1}([t_{j-1}, t_j)) \Big) \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, \nu \circ X^{-1} \left(\bigcup_{j=1}^{i} [t_{j-1}, t_j) \right) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, \nu \circ X^{-1}([0, t_i)) \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \, m_i = \oint Y \, d\nu\end{aligned}$$

5.3. **An Example.** Consider a simple function $Y = I \circ X$, with

$$I = \alpha_1 \mathbf{1}_{B_1} + \alpha_2 \mathbf{1}_{B_2} + \alpha_3 \mathbf{1}_{B_3} + \alpha_4 \mathbf{1}_{B_4} \tag{5.3}$$

where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > \alpha_5 := 0$, and B_1, B_2, B_3, B_4 are disjoint Borel subsets of the range of X, and suppose that ν is a strongly nonatomic capacity with respect to X on (S, Σ). Then,

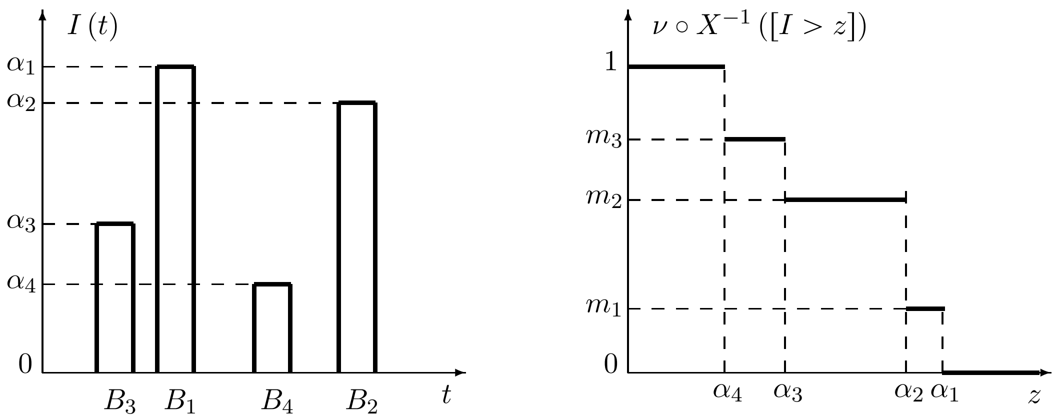
$$\nu \circ X^{-1}([I > z]) = \begin{cases} 1 & \text{if } \ z \in [\alpha_5, \alpha_4) \\ m_3 & \text{if } \ z \in [\alpha_4, \alpha_3) \\ m_2 & \text{if } \ z \in [\alpha_3, \alpha_2) \\ m_1 & \text{if } \ z \in [\alpha_2, \alpha_1) \\ 0 & \text{if } \ z \geqslant \alpha_1 \end{cases}$$

where $m_3 = \nu \circ X^{-1}(B_1 \cup B_2 \cup B_3)$, $m_2 = \nu \circ X^{-1}(B_1 \cup B_2)$, and $m_1 = \nu \circ X^{-1}(B_1)$.

With t_i defined such that $m_i = \nu \circ X^{-1}\big([0, t_i]\big) = \nu \circ X^{-1}\big([0, t_i)\big)$, for $i = 1, 2, 3, 4$, it follows that $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$, and

$$\widetilde{I}(t) = \begin{cases} \alpha_1 & \text{if } \ t \in [0, t_1) \\ \alpha_2 & \text{if } \ t \in [t_1, t_2) \\ \alpha_3 & \text{if } \ t \in [t_2, t_3) \\ \alpha_4 & \text{if } \ t \in [t_3, t_4) \\ 0 & \text{if } \ t \geqslant t_4 \end{cases}$$

Figure 2 below illustrates the shape of the simple function $\widetilde{I}$.



(A) The simple function I (B) The upper-distribution of I

FIGURE 1. A simple function and its upper-distribution.

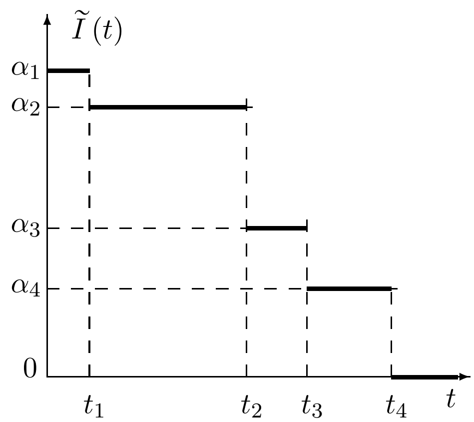


FIGURE 2. A nonincreasing ν-upper equimeasurable rearrangement of I.

6. An Application: Minimizing Production Cost under Ambiguity

Consider an economy with one producer and one consumer, facing an underlying uncertainty represented by a collection S of *states* of the world, as in the sate-contingent approach to the theory of production presented in Chambers and Quiggin [20]. An *event* is a subset of S, and we assume S to be endowed with a σ-algebra of events $\mathcal{G}$.

The producer, or firm, has the possibility of producing a certain range of goods that are seen by the consumer as perfect substitutes, for the sake of simplicity. Denote by $\mathfrak{G}$ the collection of all such goods. Each good $g \in \mathfrak{G}$ can be produced in a (random) amount Y_g which is contingent on the (random) price X of some given resource. X is a random variable on the measurable space $(S, \mathcal{G})$, taken to be bounded and with closed range $X(S) := [0, M]$. Y_g is a function of X, assumed to be nonnegative and bounded in what follows, that is $Y_g \in B^+(\Sigma)$, where Σ denotes the σ-algebra $\sigma\{X\}$. Hence, Y_g can be written in the form $I_g(X)$, for some bounded, nonnegative, and Borel-measurable map $I_g : X(S) \to \mathbb{R}^+$. Any good $g \in \mathfrak{G}$ that the firm can produce can then be identified with the associated function I_g, via the mapping

$$\begin{aligned} \mathcal{I} : \mathfrak{G} &\to B^+(\Sigma) \\ g &\mapsto Y_g = I_g \circ X \end{aligned} \tag{6.1}$$

Assume that $\mathfrak{G}$ is large enough so that the mapping $\mathcal{I}$ can be considered surjective[6]. That is, for any $Z \in B^+(\Sigma)$, there is at least one $g \in \mathfrak{G}$ such that $\mathcal{I}(g) = Z$. We can then identify $\mathfrak{G}$ with $B^+(\Sigma)$. Henceforth, a "good" ($g \in \mathfrak{G}$) will be identified with the "amount" ($\mathcal{I}(g) \in B^+(\Sigma)$) of that good that can be produced, and these terms will be used interchangeably, unless stated otherwise.

Furthermore, assume that there is an upper bound on the (random) amount of any good that can be produced, due to physical limitations, capital limitations, technology limitations, and so forth. Denote this upper bound by N, for some $N \in \mathbb{R}^+$, $N < +\infty$. This upper bound is the same for all production amounts $Y \in B^+(\Sigma)$, since it is assumed to be independent of the specific good produced. In other words, for each $g \in \mathfrak{G}$, and for each state of the world $s \in S$, one has $\mathcal{I}(g)(s) = Y_g(s) \leqslant N < +\infty$.

The firm has ambiguous beliefs about the realizations of X, and hence of Y, for each $Y \in B^+(\Sigma)$. These ambiguous beliefs are represented by a capacity ν on (S, Σ). In producing the random amount Y_g of any given good $g \in \mathfrak{G}$, the firm incurs an "expected cost". This "expected cost" is represented by a mapping $\mathcal{C} : B^+(\Sigma) \to \mathbb{R}^+$ (e.g., a Choquet integral with respect to ν). For a given (random amount of a) good $Y \in B^+(\Sigma)$, the "expected amount" produced by the firm is the quantity $\mathcal{A}(Y)$, where the mapping $\mathcal{A} : B^+(\Sigma) \to \mathbb{R}^+$ is given. The firm seeks to produce a good that will minimize the "expected cost" of production, given a minimum production target and some other constraints. Specifically, the firm's problem is the following:

$$\inf_{Y \in B^+(\Sigma)} \left\{ \mathcal{C}(Y) : Y \leqslant N,\ \mathcal{A}(Y) \geqslant A_0,\ \mathcal{D}(Y) = D_0 \right\} \tag{6.2}$$

where A_0 is a given and fixed minimal production target, and where all other constraints on Y are assumed to be represented by the second constraint, for a given D_0 and a given mapping $\mathcal{D} : B^+(\Sigma) \to \mathbb{R}$.

Proposition 6.1. *If*

(1) *ν is continuous and strongly nonatomic with respect to X;*

(2) *$\mathcal{C}$, $\mathcal{A}$, and $\mathcal{D}$ are ν-upper-law-invariant (e.g. Choquet integrals w.r.t. ν); and,*

(3) *Problem (6.2) admits a solution,*

then there is at least one optimal solution to Problem (6.2) which is anti-comonotonic with X.

Proof. Suppose that Problem (6.2) admits a solution Y^*. Take $\widetilde{Y}^*$ to be a nonincreasing ν-upper-equimeasurable rearrangement of Y^* with respect to X. The rest follows form Definition 2.6 and Proposition 3.4 (5) and (6). □

[6] This assumption can be relaxed to a setting where the set $\mathcal{I}(\mathfrak{G}) := \{I(g) : g \in \mathfrak{G}\} \subseteq B^+(\Sigma)$ is only required to be closed under ν-upper-equimeasurability. This can be seen from a simple examination of the proofs of Proposition 6.1 and Theorem 6.3.

Proposition 6.1 sates that if Problem (6.2) admits a solution, then it admits at least another solution which is anti-comonotonic with X. Nothing guarantees, *a priori*, that the set of optimal solutions for Problem (6.2) is indeed non-empty. Theorem 6.3 below not only guarantees the existence of a solution to Problem (6.2), but also guarantees that that solution is anti-comonotonic with X, under an additional condition on the mappings $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{D}$. This condition can be interpreted as a continuity requirement.

Definition 6.2. A mapping $\rho : B^+(\Sigma) \to \mathbb{R}$ is said to *preserve uniformly bounded pointwise convergence* if for any $Y^* \in B^+(\Sigma)$ and for any sequence $\{Y_n\}_{n\geqslant 1} \subset B^+(\Sigma)$ such that

(1) $\lim_{n\to+\infty} Y_n = Y^*$ (pointwise), and

(2) there is some $N \in (0, +\infty)$ such that $Y_n \leqslant N$, for each $n \geqslant 1$,

the following holds:

$$\lim_{n\to+\infty} \rho(Y_n) = \rho(Y^*)$$

When ρ is defined as a Lebesgue integral with respect to some probability measure P on (S, Σ), i.e. $\rho(Y) = \int Y\, dP$ for each $Y \in B^+(\Sigma)$, then Lebesgue's Dominated Convergence Theorem [27, Th. 2.4.4] implies that ρ preserves uniformly bounded pointwise convergence. More generally, if ρ is a Choquet integral with respect to some continuous capacity ν on (S, Σ), i.e. $\rho(Y) = \oint Y\, d\nu$ for each $Y \in B^+(\Sigma)$, then when seen as an operator on $B^+(\Sigma)$, ρ preserves uniformly bounded pointwise convergence. This is a consequence of [56, Th. 7.16].

Theorem 6.3. *If*

(1) *ν is continuous and strongly nonatomic with respect to X;*

(2) *$\mathcal{C}$, $\mathcal{A}$, and $\mathcal{D}$ are ν-upper-law-invariant (e.g. Choquet integrals w.r.t. ν); and,*

(3) *$\mathcal{C}$, $\mathcal{A}$, and $\mathcal{D}$ preserve uniformly bounded pointwise convergence (e.g. Choquet integrals w.r.t. ν),*

then Problem (6.2) *admits a solution which is anti-comonotonic with X (provided it has a non-empty feasibility set).*

The proof of Theorem 6.3 is given in Appendix B.

7. Conclusion and An Open Question

Classical techniques of monotone equimeasurable rearrangements on a measure space have proven to be very useful and fruitful in several problems in economic theory where uncertainty is present. The formulation of these problems, however, was entirely Bayesian, in the sense that ambiguity was left out of consideration and hence, *de facto*, played no role. On the other hand, largely motivated by the Ellsberg paradox [36], decision theory has developed many models of choice to deal specifically with ambiguity in decision making, starting from the seminal work of Schmeidler [60]. Amarante [2] recently showed that Choquet integration, as an aggregation concept for preferences under ambiguity, is wide enough to cover most of these models. Consequently, to be able to use rearrangement techniques in problems where ambiguity – rather than uncertainty – is present, there ought to be a generalization of the idea of a rearrangement to a context of capacities – rather than additive measures.

In this paper, I defined both a nonincreasing equimeasurable rearrangement and a nondecreasing equimeasurable rearrangement in the context of a capacity that satisfies a property of strong nonatomicity. The latter is a strengthening of the notion of nonatomicity, and both of these properties are equivalent for submodular capacities and measures. Equimeasurability with respect to a capacity is defined in the usual way, as in Denneberg [33]. I also examined the special case of a nonatomic measure, and I showed how the usual properties of a rearrangement on a nonatomic measure space can be obtained as special cases of this paper's main results.

I then considered an application that illustrates the possible use of the notion of a monotone equimeasurable rearrangement in problems where (i) ambiguity is present; and (ii) monotonicity of a solution is a desired property. The problem examined was one of production under ambiguity, where a firm seeks the optimal good to produce so as to minimize the (expected) production cost associated with producing a random amount of that good.

Several issues are left for future research, and my most immediate concern is to extend the Hardy-Littlewood integral inequality to the case of capacities and Choquet integrals. Integral inequalities involving functions and their rearrangements (on a measure space) were first given by Hardy, Littlewood, and Pólya [45] and then generalized by Cambanis et al. [5], Day [31], and Lorentz [51]. All of these results rely heavily on the way in which the Lebesgue integral is constructed. The Choquet integral is a different mathematical object for which the classical techniques used in the aforementioned papers cannot be applied, mainly because of the non-additivity of capacities. A novel approach is required.

Appendix A. Related Analysis

A.1. Two Useful Results.

Lemma A.1. *Let (S, Σ, μ) be a finite nonnegative measure space. If $\{A_n\}_n \subset \Sigma$ is such that $\mu(A_n) = \mu(S)$, for each $n \geqslant 1$, then $\mu\left(\bigcap_{n=1}^{+\infty} A_n\right) = \mu(S)$.*

Proof. Since for each $n \geqslant 1$ one has $\mu(A_n) = \mu(S)$, it follows that $\mu(S \backslash A_n) = 0$, for each $n \geqslant 1$. Therefore, since μ is nonnegative, and by countable subadditivity of countably additive measures [27, Proposition 1.2.2], it follows that $0 \leqslant \mu\left(\bigcup_{n=1}^{+\infty} S \backslash A_n\right) \leqslant \sum_{n=1}^{+\infty} \mu(S \backslash A_n) = 0$. Therefore, $\mu\left(\bigcap_{n=1}^{+\infty} A_n\right) = \mu(S) - \mu\left(\bigcup_{n=1}^{+\infty} S \backslash A_n\right) = \mu(S)$. □

Lemma A.2. *If $(f_n)_n$ is a uniformly bounded sequence of nonincerasing real-valued functions on some closed interval $\mathcal{I}$ in $\mathbb{R}$, with bound N (i.e. $|f_n(x)| \leqslant N$, $\forall x \in \mathcal{I}$, $\forall n \geqslant 1$), then there exists a nonincerasing real-valued bounded function f^* on $\mathcal{I}$, also with bound N, and a subsequence of $(f_n)_n$ that converges pointwise to f^* on $\mathcal{I}$.*

Proof. [17, Lemma 13.15] or [35, pp. 165-166]. □

A.2. Dynkin's π-λ Theorem.

Definition A.3 (π-system)**.** Let S be a nonempty set. A nonempty collection $\mathcal{P}$ of subsets of S is said to be a π-system if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Hence, a π-system is a nonempty collection of subsets of a set, which is closed under finite intersections.

Definition A.4 (λ-system, or Dynkin class)**.** Let S be a nonempty set. A nonempty collection $\mathcal{L}$ of subsets of S is said to be a λ-system if

(1) $S \in \mathcal{L}$;

(2) If $A, B \in \mathcal{L}$ are such that $A \subset B$, then $B \backslash A \in \mathcal{L}$; and,

(3) If $\{A_n\}_n$ is a nondecreasing sequence of elements of $\mathcal{L}$ such that $A_n \uparrow A := \bigcup_{n=1}^{+\infty} A_n$, then $A \in \mathcal{L}$.

Theorem A.5 (Dynkin's π-λ Theorem)**.** *Let S be a nonempty set, $\mathcal{P}$ a π-system in S, and $\mathcal{L}$ a λ-system in S. If $\mathcal{P} \subset \mathcal{L}$ then $\sigma\{\mathcal{P}\} \subset \mathcal{L}$, where $\sigma\{\mathcal{P}\}$ is the σ-algebra of subsets of S generated by $\mathcal{P}$.*

Proof. [1, pp. 135-136]. □

Appendix B. Proofs

B.1. **Proof of Proposition 2.11.**

(1) Fix any $t \in \mathbb{R}$. Then $\nu \circ X^{-1}(\{t\}) = \nu \circ X^{-1}([t,t]) = \nu \circ X^{-1}([t,t)) = \nu \circ X^{-1}(\varnothing) = 0$.

(2) First note that if ν is submodular then so is $\nu \circ X^{-1}$. Fix any $a, b \in \mathbb{R}$. Then, by monotonicity of the capacity $\nu \circ X^{-1}$, $\nu \circ X^{-1}\Big((a,b]\Big) \leqslant \nu \circ X^{-1}\Big([a,b]\Big)$. On the other hand,

$$\begin{aligned}\nu \circ X^{-1}\Big((a,b]\Big) &= \nu \circ X^{-1}\Big((a,b]\Big) + \nu \circ X^{-1}\Big(\{a\}\Big) && \text{(by nonatomicity)}\\ &\geqslant \nu \circ X^{-1}\Big((a,b] \cup \{a\}\Big) + \nu \circ X^{-1}\Big((a,b] \cap \{a\}\Big) && \text{(by submodularity)}\\ &= \nu \circ X^{-1}\Big([a,b]\Big)\end{aligned}$$

and so $\nu \circ X^{-1}\Big((a,b]\Big) = \nu \circ X^{-1}\Big([a,b]\Big)$. Similarly, using the same idea, it is easily shown that $\nu \circ X^{-1}\Big((a,b)\Big) = \nu \circ X^{-1}\Big([a,b)\Big) = \nu \circ X^{-1}\Big((a,b]\Big)$. □

B.2. **Proof of Proposition 3.2.**

(1) Immediate from eq. (3.1) and the monotonicity of $\nu \circ X^{-1}$.

(2) If ν is also continuous from below then the capacity $\nu \circ X^{-1}$ is also continuous from below. The rest then follows from a Monotone Convergence Theorem for the Choquet integral [33, Th. 8.1] and from Proposition 2.8 (3). □

B.3. **Proof of Proposition 3.4.** Suppose that Assumption 3.1 holds. Then:

(1) Let $t_1 \leqslant t_2$. Then by monotonicity of $\nu \circ X^{-1}$, $\nu \circ X^{-1}\big([0,t_1]\big) \leqslant \nu \circ X^{-1}\big([0,t_2]\big)$. Therefore,

$$\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t_1]\big)\Big\} \subseteq \Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t_2]\big)\Big\}$$

Consequently, eq. (3.2) yields $\widetilde{I}(t_1) \geqslant \widetilde{I}(t_2)$. Borel-measurability of $\widetilde{I}$ follows form its monotonicity.

(2) Let $\Upsilon_I(t) := \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant t\Big\}$, so that $\widetilde{I}(t) = \Upsilon_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)$. By Assumption 3.1, to show right-continuity of the function $\widetilde{I}$, it then suffices to show right-continuity of the function Υ_I. First, note that Υ_I is nonincreasing. Let $t_n \downarrow t_0$, let $y_0 := \Upsilon_I(t_0)$, and let $y_n := \Upsilon_I(t_n)$, for each $n \geqslant 1$. Since Υ_I is nonincreasing, $y_n \uparrow x \leqslant y_0$, and so $y_n \leqslant x \leqslant y_0$, for each $n \geqslant 1$. It suffices to show that $x = y_0$. Suppose, *per contra*, that $x < y_0$. By definition of Υ_I, it follows that $y_0 = \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant t_0\Big\}$, and so

$$G_{\nu,X,I}(x) > t_0$$

Now, since $G_{\nu,X,I}$ is nonincreasing, $G_{\nu,X,I}(x) \leqslant \lim_{n\to+\infty} G_{\nu,X,I}(y_n)$. However, since $G_{\nu,X,I}$ is right-continuous (which is a consequence Assumption 3.1),

$$G_{\nu,X,I}(y_n) = G_{\nu,X,I}(\Upsilon_I(t_n)) \leqslant t_n,\ \forall n \geqslant 1$$

Consequently,

$$G_{\nu,X,I}(x) \leqslant \lim_{n\to+\infty} t_n = t_0,$$

a contradiction. Therefore, Υ_I is right-continuous, hence yielding the right-continuity of $\widetilde{I}$.

(3) For all $t \in \mathbb{R}^+$, $G_{\nu,X,I}\Big(\widetilde{I}(t)\Big) = G_{\nu,X,I}\Big(\Upsilon_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)\Big)$. But, as in the proof of (2) above, the right-continuity of $G_{\nu,X,I}$ implies that $G_{\nu,X,I}\Big(\Upsilon_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)\Big) \leqslant \nu \circ X^{-1}\big([0,t]\big)$.

(4) Let $I_1 \leqslant I_2$. Then for each $x \in \mathbb{R}^+$, $\Big\{t : I_1(t) > x\Big\} \subseteq \Big\{t : I_2(t) > x\Big\}$. Hence, by monotonicity of $\nu \circ X^{-1}$, $G_{\nu,X,I_1}(x) \leqslant G_{\nu,X,I_2}(x)$, and so for each $t \in \mathbb{R}^+$

$$\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I_2}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \subseteq \Big\{z \in \mathbb{R}^+ : G_{\nu,X,I_1}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\Big\}$$

By eq. (3.2), this yields $\widetilde{I}_2(t) \geqslant \widetilde{I}_1(t)$.

(5) Fix some $\alpha \geqslant 0$. It suffices to show that

$$\nu \circ X^{-1}\Big(\big\{z \in [0,M] : I(z) > \alpha\big\}\Big) = \nu \circ X^{-1}\Big(\big\{z \in [0,M] : \widetilde{I}(z) > \alpha\big\}\Big)$$

Since $\widetilde{I}$ is nonincreasing, there is some $x_0 \geqslant 0$ such that the set $\big\{z \in [0,M] : \widetilde{I}(z) > \alpha\big\}$ takes the form $[0,x_0)$ or $[0,x_0]$, with $\widetilde{I}(x) \leqslant \alpha$ for each $x > x_0$. Moreover, by right-continuity of $\widetilde{I}$, it follows that $\widetilde{I}(x_0) \leqslant \alpha$. Since $G_{\nu,X,I}$ is nonincreasing,

$$G_{\nu,X,I}(\alpha) \leqslant G_{\nu,X,I}\Big(\widetilde{I}(x_0)\Big) \leqslant \nu \circ X^{-1}\big([0,x_0]\big) = \nu \circ X^{-1}\big([0,x_0)\big),$$

where the last inequality follows from (3) above.

Now, suppose that $G_{\nu,X,I}(\alpha) < \nu \circ X^{-1}\big([0,x_0]\big) = \nu \circ X^{-1}\big([0,x_0)\big)$. Then there is some $z_0 < x_0$ such that $G_{\nu,X,I}(\alpha) = \nu \circ X^{-1}\big([0,z_0]\big) = \nu \circ X^{-1}\big([0,z_0)\big)$. Therefore,

$$\begin{aligned}\widetilde{I}(z_0) &= \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,z_0]\big)\Big\} \\ &= \inf\Big\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant G_{\nu,X,I}(\alpha)\Big\} \\ &\leqslant \alpha\end{aligned}$$

contradicting the fact that $\widetilde{I}(x) > \alpha$ for any $x < x_0$. Therefore,

$$G_{\nu,X,I}(\alpha) = G_{\nu,X,I}\left(\widetilde{I}(x_0)\right) = \nu \circ X^{-1}\big([0,x_0]\big) = G_{\nu,X,\widetilde{I}}(\alpha)$$

(6) Let $N < +\infty$ be such that $\|I\|_{sup} = N$. Then $I(t) \leqslant N$, for each $t \in X(S) = [0,M]$. Since $\widetilde{I}$ is nonincreasing, it suffices to show that $\widetilde{I}(0) \leqslant N$. But

$$\begin{aligned}\widetilde{I}(0) &= \inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,0]\big)\right\}\\ &= \inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant 0\right\} = \inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) = 0\right\}\end{aligned}$$

But since $I(t) \leqslant N$, for each $t \in X(S) = [0,M]$, it follows that $G_{\nu,X,I}(N) = 0$, and so $N \in \left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) = 0\right\}$. Consequently, $\inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) = 0\right\} \leqslant N$.

(7) Suppose that $I_n \uparrow I$. Then, by Proposition 3.2 (2), $G_{\nu,X,I_n} \uparrow G_{\nu,X,I}$. Now, by Proposition 3.4 (4), $\widetilde{I}_n \leqslant \widetilde{I}_{n+1} \leqslant \widetilde{I}$, for each $n \geqslant 1$. Therefore, $\widetilde{I}_n \uparrow \lim_{n\to+\infty} \widetilde{I}_n := \widetilde{K} \leqslant \widetilde{I}$. It then remains to show that $\widetilde{K} \geqslant \widetilde{I}$. Suppose, *per contra*, that there is some $t \geqslant 0$ such that $\widetilde{K}(t) < \widetilde{I}(t)$. Therefore, $\widetilde{I}_n(t) < \widetilde{I}(t)$, for each $n \geqslant 1$. That is, for each $n \geqslant 1$,

$$\inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I_n}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\right\} < \inf\left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\right\} \tag{B.1}$$

Now, let $B(t) := \left\{z \in \mathbb{R}^+ : G_{\nu,X,I}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\right\}$ and let $B_n(t) := \Big\{z \in \mathbb{R}^+ : G_{\nu,X,I_n}(z) \leqslant \nu \circ X^{-1}\big([0,t]\big)\Big\}$, for each $n \geqslant 1$. Since $G_{\nu,X,I_n} \uparrow G_{\nu,X,I}$, it follows that the sequence $\{B_n(t)\}_{n\geqslant 1}$ is nonincreasing, and so $\lim_{n\to+\infty} B_n(t) = \bigcap_{n\geqslant 1} B_n(t) := C(t)$. It also follows that $B(t) \subseteq B_n(t)$, for each $n \geqslant 1$. Therefore,

$$B(t) \subseteq C(t) = \bigcap_{n\geqslant 1} B_n(t) \tag{B.2}$$

Eq. (B.1) and (B.2) then imply that $B(t)$ is a *strict* subset of $B_n(t)$, for each $n \geqslant 1$. Thus, for each $n \geqslant 1$, there is some $z_{0,n} \geqslant 0$ such that $z_{0,n} \in B(t)$ but $z_{0,n} \notin B_n(t)$, i.e.

$$G_{\nu,X,I}(z_{0,n}) \leqslant \nu \circ X^{-1}\big([0,t]\big) \quad \text{and} \quad G_{\nu,X,I_n}(z_{0,n}) > \nu \circ X^{-1}\big([0,t]\big)$$

Hence, $G_{\nu,X,I_m}(z_{0,n}) \geqslant G_{\nu,X,I_n}(z_{0,n}) > \nu \circ X^{-1}\big([0,t]\big)$, for each $m \geqslant n$. Consequently,

$$G_{\nu,X,I}(z_{0,n}) = \lim_{m\to+\infty} G_{\nu,X,I_m}(z_{0,n}) \geqslant G_{\nu,X,I_n}(z_{0,n}) > \nu \circ X^{-1}\big([0,t]\big),$$

contradicting the fact that $G_{\nu,X,I}(z_{0,n}) \leqslant \nu \circ X^{-1}\big([0,t]\big)$. Therefore, $\widetilde{K}(t) \geqslant \widetilde{I}(t)$, and so $\widetilde{K} = \widetilde{I}$. □

B.4. **Proof of Proposition 3.7.** Suppose that Assumption 3.1 holds. Then:

(1) Let $t_1 \leqslant t_2$. Then by monotonicity of $\nu \circ X^{-1}$, $\nu \circ X^{-1}\big([0,t_1]\big) \leqslant \nu \circ X^{-1}\big([0,t_2]\big)$. Therefore,

$$\left\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant \nu \circ X^{-1}\big([0,t_2]\big)\right\} \subseteq \left\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant \nu \circ X^{-1}\big([0,t_1]\big)\right\}$$

Consequently, eq. (3.5) yields $\widetilde{\widetilde{I}}(t_1) \geqslant \widetilde{\widetilde{I}}(t_2)$. Borel-measurability of $\widetilde{\widetilde{I}}$ follows form its monotonicity.

(2) Let $\Delta_I(t) := \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant t\Big\}$, so that $\tilde{\tilde{I}}(t) = \Delta_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)$. By Assumption 3.1, to show left-continuity of the function $\tilde{\tilde{I}}$, it then suffices to show left-continuity of the function Δ_I. First, note that Δ_I is nondecreasing. Let $t_n \uparrow t_0$, let $y_0 := \Delta_I(t_0)$, and let $y_n := \Delta_I(t_n)$, for each $n \geqslant 1$. Since Δ_I is nondecreasing, $y_n \uparrow x \leqslant y_0$, and so $y_n \leqslant x \leqslant y_0$, for each $n \geqslant 1$. It suffices to show that $x = y_0$. Suppose, *per contra*, that $x < y_0$. By definition of Δ_I, it follows that $y_0 = \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant t_0\Big\}$, and so

$$F_{\nu,X,I}(x) < t_0$$

Now, since $F_{\nu,X,I}$ is nondecreasing, $F_{\nu,X,I}(x) \geqslant \lim\limits_{n\to+\infty} F_{\nu,X,I}(y_n)$. However, since $F_{\nu,X,I}$ is right-continuous (which is a consequence Assumption 3.1),

$$F_{\nu,X,I}(y_n) = F_{\nu,X,I}(\Delta_I(t_n)) \geqslant t_n,$$

for each $n \geqslant 1$. Consequently,

$$F_{\nu,X,I}(x) \geqslant \lim_{n\to+\infty} t_n = t_0,$$

a contradiction. Therefore, Δ_I is left-continuous, hence yielding the left-continuity of $\tilde{\tilde{I}}$.

(3) For all $t \in \mathbb{R}^+$, $F_{\nu,X,I}\left(\tilde{\tilde{I}}(t)\right) = F_{\nu,X,I}\Big(\Delta_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)\Big)$. But, as in the proof of (2) above, the right-continuity of $F_{\nu,X,I}$ implies that $F_{\nu,X,I}\Big(\Delta_I\Big(\nu \circ X^{-1}\big([0,t]\big)\Big)\Big) \geqslant \nu \circ X^{-1}\big([0,t]\big)$.

(4) Let $I_1 \leqslant I_2$. Then for each $x \in \mathbb{R}^+$, $\Big\{t : I_2(t) \leqslant x\Big\} \subseteq \Big\{t : I_1(t) \leqslant x\Big\}$. Hence, by monotonicity of $\nu \circ X^{-1}$, $F_{\nu,X,I_2}(x) \leqslant F_{\nu,X,I_1}(x)$, and so for each $t \in \mathbb{R}^+$

$$\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I_2}(z) \geqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \subseteq \Big\{z \in \mathbb{R}^+ : F_{\nu,X,I_1}(z) \geqslant \nu \circ X^{-1}\big([0,t]\big)\Big\}$$

By eq. (3.5), this yields $\tilde{I}_2(t) \geqslant \tilde{I}_1(t)$.

(5) By eq. (3.5), for each $t \in \mathbb{R}^+$,

$$\begin{aligned}\widetilde{\widetilde{Id}}(t) &= \inf\Big\{z \in \mathbb{R}^+ : \nu \circ X^{-1}\big(\{x \in [0,M] : Id(x) \leqslant z\}\big) \geqslant \nu \circ X^{-1}\big([0,t]\big)\Big\} \\ &= \inf\Big\{z \in \mathbb{R}^+ : \nu \circ X^{-1}\big([0,z]\big) \geqslant \nu \circ X^{-1}\big([0,t]\big)\Big\}\end{aligned}$$

Therefore, $\widetilde{\widetilde{Id}}(t) \leqslant t = Id(t)$, for each $t \in [0,M]$.

(6) Fix $\alpha \geqslant 0$. It suffices to show that

$$\nu \circ X^{-1}\Big(\big\{z \in [0,M] : I(z) \leqslant \alpha\big\}\Big) = \nu \circ X^{-1}\Big(\big\{z \in [0,M] : \tilde{\tilde{I}}(z) \leqslant \alpha\big\}\Big)$$

Since $\tilde{\tilde{I}}$ is nondecreasing, there is some $x_0 \in [0,M]$ such that the set $\Big\{x \in [0,M] : \tilde{\tilde{I}}(x) \leqslant \alpha\Big\}$ has the form $[0,x_0)$ or $[0,x_0]$, with $\tilde{\tilde{I}}(x) > \alpha$ for each $x \in (x_0,M]$. Moreover, by left-continuity of $\tilde{\tilde{I}}$, it follows that $\tilde{\tilde{I}}(x_0) \leqslant \alpha$. Since $F_{\nu,X,I}$ is nondecreasing,

$$F_{\nu,X,I}(\alpha) \geqslant F_{\nu,X,I}\left(\tilde{\tilde{I}}(x_0)\right) \geqslant \nu \circ X^{-1}\big([0,x_0]\big) = \nu \circ X^{-1}\big([0,x_0)\big),$$

where the last inequality follows from (3) above.

Now, suppose that $F_{\nu,X,I}(\alpha) > \nu \circ X^{-1}\big([0,x_0]\big) = \nu \circ X^{-1}\big([0,x_0)\big)$. Then there is some $z_0 \in (x_0, M]$ such that $F_{\nu,X,I}(\alpha) = \nu \circ X^{-1}\big([0,z_0]\big) = \nu \circ X^{-1}\big([0,z_0)\big)$. Therefore,

$$\begin{aligned}\tilde{\tilde{I}}(z_0) &= \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant \nu \circ X^{-1}\big([0,z_0]\big)\Big\} \\ &= \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant F_{\nu,X,I}(\alpha)\Big\} \\ &\leqslant \alpha\end{aligned}$$

contradicting the fact that $\tilde{\tilde{I}}(x) > \alpha$ for each $x \in (x_0, M]$. Therefore,

$$F_{\nu,X,I}(\alpha) = F_{\nu,X,I}\left(\tilde{\tilde{I}}(x_0)\right) = \nu \circ X^{-1}\big([0,x_0]\big) = F_{\nu,X,\tilde{\tilde{I}}}(\alpha)$$

(7) Let $N < +\infty$ be such that $\|I\|_{sup} = N$. Then $I(t) \leqslant N$, for each $t \in X(S) = [0,M]$. Since $\tilde{\tilde{I}}$ is nondecreasing, it suffices to show that $\tilde{\tilde{I}}(M) \leqslant N$. But

$$\begin{aligned}\tilde{\tilde{I}}(M) &= \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant \nu \circ X^{-1}\big([0,M]\big)\Big\} \\ &= \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) \geqslant 1\Big\} = \inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) = 1\Big\}\end{aligned}$$

But since $I(t) \leqslant N$, for each $t \in X(S) = [0,M]$, it follows that $F_{\nu,X,I}(N) = 1$, and so $N \in \Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) = 1\Big\}$. Consequently, $\inf\Big\{z \in \mathbb{R}^+ : F_{\nu,X,I}(z) = 1\Big\} \leqslant N$. □

B.5. **Proof of Proposition 4.3.** (1), (2), (5), and (8) follow form Proposition 3.7, since nonatomicity implies strong nonatomicity in this case (Remark 2.12). The other properties are shown below.

(3) By the very definition of $\tilde{\tilde{I}}$ given in equation (4.2), one has $\tilde{\tilde{I}}(t) \geqslant 0$ for each $t \in [0,M]$. Now, $\Psi_X([0,0]) = \Psi_X(\{0\}) = 0$, by nonatomicity of Ψ_X. Therefore, for each $x \geqslant 0$, $\Psi_X\big(\{t \in [0,M] : I(t) \leqslant x\}\big) \geqslant \Psi_X([0,0])$. In particular,

$$\Psi_X\big(\{t \in [0,M] : I(t) \leqslant 0\}\big) = \Psi_X\big(\{t \in [0,M] : I(t) = 0\}\big) \geqslant \Psi_X([0,0])$$

Hence, by equation (4.2), $\tilde{\tilde{I}}(0) \leqslant 0$, and so $\tilde{\tilde{I}}(0) = 0$. Moreover, for each $x \in [0,M]$,

$$1 = \Psi_X([0,M]) \geqslant \Psi_X\big(\{t \in [0,M] : I(t) \leqslant x\}\big)$$

Therefore, $\Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I(x) \leqslant z\}\big) \geqslant \Psi_X\big([0,M]\big)\Big\} = \Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I(x) \leqslant z\}\big) = 1\Big\}$. Since $I(t) \leqslant M$ for each $t \in [0,M]$, it follows that $M \in \Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I(x) \leqslant z\}\big) = 1\Big\}$, and so from equation (4.2) it follows that $\tilde{\tilde{I}}(M) = \inf\Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I(x) \leqslant z\}\big) = 1\Big\} \leqslant M$;

(4) Let $I_1, I_2 : [0,M] \to [0,M]$ be such that $I_1 \leqslant I_2$, Ψ_X-a.s. Then, for each $x \geqslant 0$, $\Psi_X\Big(\{t \in [0,M] : I_1(t) \leqslant x\}\Big) \geqslant \Psi_X\Big(\{t \in [0,M] : I_2(t) \leqslant x\}\Big)$. Therefore, for each $t \in [0,M]$,

$$\Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I_2(x) \leqslant z\}\big) \geqslant \Psi_X\big([0,t]\big)\Big\} \subseteq \Big\{z \in \mathbb{R}^+ : \Psi_X\big(\{x \in [0,M] : I_1(x) \leqslant z\}\big) \geqslant \Psi_X\big([0,t]\big)\Big\}$$

It then follows from equation (4.2) that $\tilde{\tilde{I}}_1 \leqslant \tilde{\tilde{I}}_2$;

(6) To show that $\tilde{\tilde{I}}$ is Ψ_X-equimeasurable with I, one needs to show that for any Borel set B, $\Psi_X\Big(\{t \in [0,M] : I(t) \in B\Big) = \Psi_X\Big(\{t \in [0,M] : \tilde{\tilde{I}}(t) \in B\}\Big)$. First, note that as in Proposition 3.7, one has that for each $\alpha \in [0,M]$,

$$\Psi_X\Big(\{t \in [0,M] : I(t) \leqslant \alpha\}\Big) = \Psi_X\Big(\{t \in [0,M] : \tilde{\tilde{I}}(t) \leqslant \alpha\}\Big)$$

Now, the collection $\big\{[0,\alpha] : \alpha \in [0,M]\big\}$ is a π-system (Definition A.3) that generates the Borel σ-algebra on $[0,M]$ [58, pp. 18-19]. Moreover, the collection of all Borel subsets B of $\mathbb{R}$ such that $\Psi_X \circ I^{-1}(B) = \Psi_X \circ \tilde{\tilde{I}}^{-1}(B)$ is easily seen to be a λ-system (Definition A.4 and [58, Proposition 2.2.3]). Therefore, by Dynkin's π-λ theorem (Theorem A.5), $\Psi_X \circ I^{-1}(C) = \Psi_X \circ \tilde{\tilde{I}}^{-1}(C)$, for each Borel subset C of $[0,M]$. That is, for any Borel set B,

$$\Psi_X\Big(\{t \in [0,M] : I(t) \in B\Big) = \Psi_X\Big(\{t \in [0,M] : \tilde{\tilde{I}}(t) \in B\}\Big)$$

(7) Let $\overline{I} : [0,M] \to \mathbb{R}^+$ be another nondecreasing, Borel-measurable map which is Ψ_X-equimeasurable with I. To show that $\overline{I} = \tilde{\tilde{I}}$, Ψ_X-a.s., it is enough to show that

$$\Psi_X\big(\{x \in [0,M] : \tilde{\tilde{I}}(x) > \overline{I}(x)\}\big) = \Psi_X\big(\{x \in [0,M] : \overline{I}(x) > \tilde{\tilde{I}}(x)\}\big) = 0$$

Let $\mathbb{Q}$ denote the set of all rational numbers. Then

$$\Big\{x \in [0,M] : \overline{I}(x) < \tilde{\tilde{I}}(x)\Big\} = \bigcup_{q \in \mathbb{Q}} \Big(\{x \in [0,M] : \overline{I}(x) < q\} \cap \{x \in [0,M] : q \leqslant \tilde{\tilde{I}}(x)\}\Big)$$

Fix $q \in \mathbb{Q}$ arbitrarily. Since both $\tilde{\tilde{I}}$ and $\overline{I}$ are nondecreasing functions, there are numbers $t_1, t_2 \in [0,M]$ such that $\{x \in [0,M] : \overline{I}(x) < q\} = [0,t_1)$ or $[0,t_1]$, and $\{x \in [0,M] : q \leqslant \tilde{\tilde{I}}(x)\} = (t_2, M]$ or $[t_2, M]$. By nonatomicity of Ψ_X, $\Psi_X([0,t_1)) = \Psi_X([0,t_1])$ and $\Psi_X((t_2,M]) = \Psi_X([t_2,M])$. Thus, since $\overline{I}$ and $\tilde{\tilde{I}}$ are both Ψ_X-equimeasurable with I, one has

$$\Psi_X([0,t_1)) = \Psi_X([0,t_1]) = \Psi_X\big(\{x \in [0,M] : I(x) < q\}\big)$$

and

$$\Psi_X((t_2,M]) = \Psi_X([t_2,M]) = \Psi_X\big(\{x \in [0,M] : q \leqslant I(x)\}\big)$$

Thus, $\Psi_X([0,t_1)) = \Psi_X([0,t_1]) = 1 - \Psi_X([t_2,M]) = \Psi_X([0,t_2)) = \Psi_X([0,t_2])$.

<u>If $t_1 = t_2$</u>, then $[0,t_1) \cap (t_2, M] = [0,t_1) \cap [t_2, M] = [0,t_1] \cap (t_2, M] = \varnothing$, and $[0,t_1] \cap [t_2, M] = \{t_1\}$. Thus,

$$\Psi_X\big([0,t_1) \cap (t_2, M]\big) = \Psi_X\big([0,t_1) \cap [t_2, M]\big) = \Psi_X\big([0,t_1] \cap (t_2, M]\big) = 0$$

and, by nonatomicity of Ψ_X, one has $\Psi_X\big([0,t_1] \cap [t_2, M]\big) = 0$. Therefore,

$$\Psi_X\Big(\{x \in [0,M] : \overline{I}(x) < q\} \ \cap \ \{x \in [0,M] : q \leqslant \tilde{\tilde{I}}(x)\}\Big) = 0$$

<u>If $t_1 > t_2$</u>, then $[0,t_1) = [0,t_2) \cup [t_2,t_1) = [0,t_2] \cup (t_2,t_1)$, and $[0,t_1] = [0,t_2) \cup [t_2,t_1] = [0,t_2] \cup (t_2,t_1]$. Since $\Psi_X([0,t_1)) = \Psi_X([0,t_1]) = \Psi_X([0,t_2)) = \Psi_X([0,t_2])$, it then follows that $\Psi_X((t_2,t_1)) = \Psi_X((t_2,t_1]) = \Psi_X([t_2,t_1)) = \Psi_X([t_2,t_1]) = 0$. Therefore,

$$\begin{cases} \Psi_X\big([0,t_1) \cap (t_2,M]\big) = \Psi_X\big((t_2,t_1)\big) = 0 \\ \Psi_X\big([0,t_1) \cap [t_2,M]\big) = \Psi_X\big([t_2,t_1)\big) = 0 \\ \Psi_X\big([0,t_1] \cap (t_2,M]\big) = \Psi_X\big((t_2,t_1]\big) = 0 \\ \Psi_X\big([0,t_1] \cap [t_2,M]\big) = \Psi_X\big([t_2,t_1]\big) = 0 \end{cases}$$

Thus, $\Psi_X\Big(\{x \in [0,M] : \overline{I}(x) < q\} \ \cap \ \{x \in [0,M] : q \leqslant \tilde{\tilde{I}}(x)\}\Big) = 0$.

Finally, <u>if $t_2 > t_1$</u>, then $[0,t_2) = [0,t_1) \cup [t_1,t_2) = [0,t_1] \cup (t_1,t_2)$, and $[0,t_2] = [0,t_1) \cup [t_1,t_2] = [0,t_1] \cup (t_1,t_2]$. Since $\Psi_X([0,t_2)) = \Psi_X([0,t_2]) = \Psi_X([0,t_1)) = \Psi_X([0,t_1])$, it then follows that $\Psi_X((t_1,t_2)) = \Psi_X((t_1,t_2]) = \Psi_X([t_1,t_2)) = \Psi_X([t_1,t_2]) = 0$. Therefore,

$$\begin{cases} \Psi_X\big([0,t_2) \cap (t_1,M]\big) = \Psi_X\big((t_1,t_2)\big) = 0 \\ \Psi_X\big([0,t_2) \cap [t_1,M]\big) = \Psi_X\big([t_1,t_2)\big) = 0 \\ \Psi_X\big([0,t_2] \cap (t_1,M]\big) = \Psi_X\big((t_1,t_2]\big) = 0 \\ \Psi_X\big([0,t_2] \cap [t_1,M]\big) = \Psi_X\big([t_1,t_2]\big) = 0 \end{cases}$$

Thus, $\Psi_X\Big(\{x \in [0,M] : \overline{I}(x) < q\} \ \cap \ \{x \in [0,M] : q \leqslant \tilde{\tilde{I}}(x)\}\Big) = 0$. Since $q \in \mathbb{Q}$ was chosen arbitrarily, it then follows that $\phi\Big(\big\{x \in [0,M] : \overline{I}(x) < \tilde{\tilde{I}}(x)\big\}\Big) = 0$. Similarly, one can show that $\Psi_X\Big(\big\{x \in [0,M] : \tilde{\tilde{I}}(x) < \overline{I}(x)\big\}\Big) = 0$. Thus, $\tilde{\tilde{I}} = \overline{I}$, Ψ_X-a.s.;

(9) For each Borel-measurable and finite function $\psi : \mathbb{R} \to \mathbb{R}$ define the mapping $\delta(\psi) : \mathbb{R} \to \mathbb{R}$ by

$$\delta(\psi)(t) := \inf\Big\{z \in \mathbb{R} : \Psi_X\big(\{x \in \mathbb{R} : \psi(x) > z\}\big) \leqslant \Psi_X\big((-\infty,t)\big)\Big\}, \ \forall t \in \mathbb{R}$$

Then, as in [37, Proposition 2], $\delta(\psi)$ is nonincreasing and Ψ_X-equimeasurable with ψ. Moreover, by [37, Proposition 2], if $\{f_n\}_n$ is a sequence of Borel-meaurable finite real-valued functions on $\mathbb{R}$ such that $f_n \uparrow f$, Ψ_X-a.s., where f is some Borel-meaurable finite real-valued functions on $\mathbb{R}$, then $\delta(f_n) \uparrow \delta(f)$.

Now, for each Borel-measurable and finite function $\psi : \mathbb{R} \to \mathbb{R}$ define the mapping $\iota(\psi) := -\delta(-\psi)$. Then $\iota(\psi)$ is nondecreasing and Ψ_X-equimeasurable with ψ. Thus, by (7) above, for each Borel-measruable function $I : [0,M] \to [0,M]$, one has $\iota(I) = \tilde{\tilde{I}}$, Ψ_X-a.s., that is, $\Psi_X\Big(\big\{t \in [0,M] : \tilde{\tilde{I}}(t) = \iota(I)(t)\big\}\Big) = 1$. The rest then follows trivially (see also Lemma A.1 on p. 19). □

B.6. **Proof of Lemma 4.5.** Since $P(A) = 1$, one has $P(S \backslash A) = 0$, and so it follows that for all $t \in \mathbb{R}$,

$$
\begin{aligned}
P\left[\{\widetilde{\widetilde{Y_A}} \geqslant t\} \cap A\right] &= P\left[\{Y \geqslant t\} \cap A\right] && \text{(by definition of } \widetilde{\widetilde{Y_A}}\text{)} \\
&= P\left[\{Y \geqslant t\} \cap A\right] + P\left[\{Y \geqslant t\} \cap (S \backslash A)\right] && \text{(since } P(S \backslash A) = 0\text{)} \\
&= P\left[\{Y \geqslant t\}\right] = P\left[\{\tilde{\tilde{Y}} \geqslant t\}\right] && \text{(by definition of } \tilde{\tilde{Y}}\text{)} \\
&= P\left[\{\tilde{\tilde{Y}} \geqslant t\} \cap A\right] + P\left[\{\tilde{\tilde{Y}} \geqslant t\} \cap (S \backslash A)\right] \\
&= P\left[\{\tilde{\tilde{Y}} \geqslant t\} \cap A\right] && \text{(since } P(S \backslash A) = 0\text{)} \\
&= P\left[\{\tilde{\tilde{Y}} \mathbf{1}_A \geqslant t\} \cap A\right]
\end{aligned}
$$

Furthermore, both $\widetilde{\widetilde{Y_A}}$ and $\tilde{\tilde{Y}}$ are nondecreasing in X on A. Hence, by the P-a.s. uniqueness of the nondecreasing P-rearrangement of Y with respect to X on A, it follows that $\tilde{\tilde{Y}} = \widetilde{\widetilde{Y_A}}$, P-a.s. on A, that is, P-a.s. □

B.7. **Proof of Theorem 6.3.** Let $\mathcal{F} := \left\{Y \in B^+(\Sigma) : Y \leqslant N,\ \mathcal{A}(Y) \geqslant A_0,\ \mathcal{D}(Y) = D_0\right\}$ be the feasibility set for Problem (6.2), and assume that $\mathcal{F} \neq \varnothing$. Denote by $\mathcal{F}^{\downarrow}$ the collection of all elements of $\mathcal{F}$ that are anti-comonotonic with X. Then $\mathcal{F}^{\downarrow} \neq \varnothing$, by a proof identical to that of Proposition 6.1. Moreover, by a proof identical to that of Proposition 6.1, for any $Y \in B^+(\Sigma)$ which is feasible for Problem (6.2), there is a $\tilde{Y} \in B^+(\Sigma)$ which is not only feasible for Problem (6.2) and anti-comonotonic with X, but is such that $\mathcal{C}\left(\tilde{Y}\right) = \mathcal{C}(Y)$. Hence, one can choose a maximizing sequence $\{Y_n\}_n$ in $\mathcal{F}^{\downarrow}$ for Problem (6.2). That is,

$$\lim_{n \to +\infty} \mathcal{C}(Y_n) = H := \sup_{Y \in \mathcal{F}} \mathcal{C}(Y)$$

Since $0 \leqslant Y_n \leqslant N$, for each $n \geqslant 1$, the sequence $\{Y_n\}_n$ is uniformly bounded. Moreover, for each $n \geqslant 1$ one has $Y_n = I_n \circ X$. Consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of nonincreasing Borel-measurable functions. Thus, by Lemma A.2, there is a nonincreasing function $I^* : [0, M] \to [0, N]$ and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ such that $\{I_m\}_m$ converges pointwise on $[0, M]$ to I^*. Hence, I^* is also Borel-measurable, and so $Y^* := I^* \circ X \in B^+(\Sigma)$ is such that $0 \leqslant Y^* \leqslant N$, and Y^* is anti-comonotonic with X. Moreover, the sequence $\{Y_m\}_m$, defined by $Y_m := I_m \circ X$, converges pointwise to Y^*. Thus, by the assumption that the mappings $\mathcal{A}$ and $\mathcal{D}$ preserve uniformly bounded pointwise convergence, it follows that $Y^* \in \mathcal{F}^{\downarrow}$. Now, by the assumption that the mapping $\mathcal{C}$ preserves uniformly bounded pointwise convergence, one has

$$\mathcal{C}(Y^*) = \lim_{m \to +\infty} \mathcal{C}(Y_m) = \lim_{n \to +\infty} \mathcal{C}(Y_n) = H$$

Hence Y^* solves Problem (6.2). □

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