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Abstract. Some of the most widely used models in economics are based on variables not yet observed, and their specification depends on future observations; the theory that underpins these delivers the backward/ forward solution. We present a newly unified construction, starting with a more general specification of an ARMA model, yet is capable of delivering in closed form, in both the backward and forward cases, leading to an alternative presentation of causal/non-causal and invertible/non-invertible cases.

A general discussion of the model:

$$\phi_{-p_1} Y_{t+p1} + \ldots + \phi_{-1} Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + \ldots + \phi_{p_2} Y_{t-p2} =$$
$$= \theta_{-q1} X_{t+q1} + \ldots + \theta_{-1} X_{t+1} + \theta_0 X_t + \theta_1 X_{t-1} + \ldots + \theta_{q2} X_{t-q2}$$

is given, first for the case of two stationary random vectors $\{Y_t\},\{X_t\}$, and then for the case in which $\{X_t\}$ is white noise. The case in which future dates are involved using an expectative is also considered.

1 **Introduction.**

Let $L^2(\Omega, F, P, \Re)=\{Y:\Omega \to \Re \mid EY^2 = \int_{\Omega} Y^2(\omega)dP(\omega) < \infty\}$, the Hilbert space of squared integrable real-valued random variables defined on the probability space (Ω, F, P) where F is a sigma-algebra of subsets of Ω and P is a probability measure defined on F, in which the inner product $<Y_1,Y_2> = E(Y_1 \cdot Y_2)$ and norm $\| Y \| = \sqrt{EY^2}$ are both defined. An m-variate time series process is a sequence of column m-vectors $\{Y_t\}$, $Y_t' = (Y_t(1), Y_t(2), ..., Y_t(m))$ formed of elements taken from the space $Y_t(i) \in L^2(\Omega, F, P)$.

The mean of an m-variate process is $\mu_t = E[Y_t] = (E[Y_t(i)]) = (\mu_t(i))$, and the autocovariance is $\Gamma_Y(j,) = E[(Y_{t+j} - \mu_{t+j}) \cdot (Y_t - \mu_t)']$. A process is second order stationary if the mean and covariance do not depend on the integer variable t, which represents time. The case considered herein has a zero mean, hence $E[Y_t] = 0$ and $\Gamma_Y(j) = E[Y_{t+j} \cdot Y_t']$

$\{A_t\}$ is white noise a numerable collection of stationary random variables with mean zero $E[A_t] = 0$, with autocovariance $\Gamma_A(j) = E[A_{t+j} \cdot A_t'] = \Omega$ if $j = 0$ and $\Gamma_A(j) = 0$ if $j \neq 0$.

The m x m matrix Ω is invertible, positive definite and symmetric and is termed the covariance.

The lag operator is defined by $L^k(Y_t(i)) = Y_{t-k}(i)$, where k is an integer. The notation $\sum_{s=0}^{\infty} \| B_s \| < \infty$ means absolute summability and $||B||$ is a matrix norm.

The present study considers an admissible exposition that encompasses causal/non-causal and invertible/non-invertible cases, delivers closed solutions associated with the specification based on a conditional expectative:

$$\phi_{-p_1}E_t\left[Y_{t+p1}\right]+\ldots+\phi_{-1}E_t\left[Y_{t+1}\right]+\phi_0 Y_t+\phi_1 Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}=$$
$$=\theta_{-q1}E_t\left[X_{t+q1}\right]+\ldots+\theta_{-1}E_t\left[X_{t+1}\right]+\theta_0 X_t+\theta_1 X_{t-1}+\ldots+\theta_{q2}X_{t-q2}$$

Starting with the theoretical model:

$$\phi_{-p_1}Y_{t+p1}+\ldots+\phi_{-1}Y_{t+1}+\phi_0 Y_t+\phi_1 Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}=$$
$$=\theta_{-q1}X_{t+q1}+\ldots+\theta_{-1}X_{t+1}+\theta_0 X_t+\theta_1 X_{t-1}+\ldots+\theta_{q2}X_{t-q2}$$

The focus then shifts to the multivariate ARMA case. The remainder of the paper is divided into three sections. In Section 2, a constructive presentation is used to describe a general procedure for dealing with a linear filter. In Section 3, the direct application to the VARMA case is considered, followed by a proposal for dealing with an applied case using a model with an expectative.

2 Linear processes.

Let us take two stationary second-order processes $\{Y_t\}$, $\{X_t\}$. Both are m-variate and have a zero mean, and we now consider a linear model of order (p_1, p_2, q_1, q_2):

$$\phi_{-p_1}Y_{t+p1}+\ldots+\phi_{-1}Y_{t+1}+\phi_0 Y_t+\phi_1 Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}=$$
$$=\theta_{-q1}X_{t+q1}+\ldots+\theta_{-1}X_{t+1}+\theta_0 X_t+\theta_1 X_{t-1}+\ldots+\theta_{q2}X_{t-q2}$$

In this equation $\phi_{-p1}\neq 0$, $\phi_{p2}\neq 0$, $\theta_{-q1}\neq 0$, $\theta_{q2}\neq 0$, all the coefficients are real m x m matrices, ϕ_0 and θ_0 are m x m identity matrices.

The standard notation is the usual $\Phi(L)Y_t=\Theta(L)X_t$

The linear operators:

$$\Phi(L)=\phi_{-p1}L^{-p1}+\ldots+\phi_{-2}L^{-2}+\phi_{-1}L^{-1}+\phi_0+\phi_1 L+\phi_2 L^2+\ldots+\phi_{p2}L^{p2}$$

$$\Theta(L) = \theta_{-q_1} L^{-q1} + ... + \theta_{-1} L^{-1} + \theta_0 + \theta_1 L^1 + ... + \theta_{q_2} L^{q2}$$

in fact have, alternative formulations:

$$\Phi(L) = L^{-p1} \Phi_{p1}(L) = L^{p2} \cdot \Phi_{p2}(L)$$

$$L^{-p1} \cdot \Phi_{p1}(L) = L^{-p1} \cdot (\phi_{-p1} + ... + \phi_0 L^{p1} + \phi_1 L^{p1+1} + ... + \phi_{p2} L^{p1+p2})$$

$$L^{p2} \cdot \Phi_{p2}(L) = L^{p2} \cdot (\phi_{-p1} L^{-p2-p1} + ... + \phi_0 L^{-p2} + \phi_1 L^{-p2+1} + ... + \phi_{p2})$$

$$\Theta(L) = L^{-q1} \Theta_{q1}(L) = L^{q2} \cdot \Theta_{q2}(L)$$

$$L^{-q1} \cdot \Theta_{q1}(L) = L^{-q1} \cdot (\theta_{-q_1} + ... + \theta_0 L^{q1} + \theta_1 L^{q1+1} + ... + \theta_{q_2} L^{q1+q2})$$

$$L^{q2} \cdot \Theta_{q2}(L) = L^{q2} \cdot (\theta_{-q_1} L^{-q1-q2} + ... + \theta_0 L^{-q2} + \theta_1 L^{-q2+1} + ... + \theta_{q_2})$$

The backward and forward solutions respectively require the following two representations:

$$L^{-p1} \Phi_{p1}(L) Y_t = L^{-q1} \Theta_{q1}(L) X_t \text{ and } L^{p2} \Phi_{p2}(L) Y_t = L^{q2} \Theta_{q2}(L) X_t$$

The backward analysis:

$$Y_t = \Phi^{-1}(L) \Theta(L) X_t = L^{p1} \Phi^{-1}_{p1}(L) \cdot L^{-q1} \Theta_{q1}(L) X_t = L^{p1-q1} \Phi^{-1}_{p1}(L) \cdot \Theta_{q1}(L) X_t$$

$$Y_t = \Phi^{-1}(L) \Theta(L) X_t = L^{p1-q1} \cdot [\phi_{-p1} + ... + \phi_0 L^{p1} + .. + \phi_{p2} L^{p1+p2}]^{-1} \cdot [\theta_{-q_1} + ... + \theta_0 L^{q1} + ... + \theta_{q_2} L^{q1+q2}] X_t$$

$$Y_t = \Phi^{-1}(L) \Theta(L) X_t = L^{p1-q1} \cdot [\varphi_0 + \varphi_1 L + \varphi_2 L^2 + ...] \cdot [\theta_{-q_1} + ... + \theta_0 L^{q1} + ... + \theta_{q_2} L^{q1+q2}] X_t$$

If all the roots of the polynomial:

$$\phi_{p1}(z) = \det[\Phi_{p1}(z)] = \det[\phi_{-p1} + ... + \phi_{-1} z^{p1-1} + \phi_0 z^{p1} + \phi_1 z^{p1+1} + \phi_2 z^{p1+2} + ... + \phi_{p2} z^{p1+p2}]$$

lie outside the unit circle, it is known that the inversion is guaranteed and the selection fulfills the condition that the real m x m matrices $\{\varphi_s\}$ are such that $\sum_{s=0}^{\infty} \| \varphi_s \| < \infty$.

It is well known that the product of the matrix series with a matrix polynomial is another well defined matrix series, $\sum_{s=0}^{\infty}\| \psi_s \| < \infty$

$$L^{p1-q1} \cdot \Phi_{p1}^{-1}(L)\Theta_{q1}(L) = L^{p1-q1} \cdot \psi(L) = L^{p1-q1} \cdot [\psi_0 + \psi_1 L + \psi_2 L^2 + ...]$$

Hence, the backward stationary solution is:

$$Y_t = \psi_0 X_{t+p1-q1} + \psi_1 X_{t+p1-q1-1} + \psi_2 X_{t+p1-q1-2} + \psi_3 X_{t+p1-q1-3} + ...$$

The forward analysis:

The rationale for obtaining the forward solution is that the filter $\Phi(L)Y_t=\Theta(L)X_t$ and the dual filter $\Phi(L^{-1})Y_t=\Theta(L^{-1})X_t$ are related, because there is a path travel interchange between traveling forward and going backward. The forward solution is the backward solution in the dual case, but it is pulled back.

Now let us take:

$$L^{p2}\Phi_{p2}(L)Y_t = L^{q2}\Theta_{q2}(L)X_t$$

$$Y_t = \Phi^{-1}(L)\Theta(L)X_t = L^{-p2}\Phi_{p2}^{-1}(L) \cdot L^{q2}\Theta_{q2}(L)X_t = L^{q2-p2}\Phi_{p2}^{-1}(L) \cdot \Theta_{q2}(L)X_t$$

$$\Phi_{p2}^{-1}(L)\Theta_{q2}(L) = (\phi_{-p1}L^{-p2-p1} + ... + \phi_0 L^{-p2} + \phi_1 L^{-p2+1} + ... + \phi_{p2})^{-1} \cdot (\theta_{-q_1}L^{-q1-q2} + ... + \theta_0 L^{-q2} + \theta_1 L^{-q2+1} + ... + \theta_{q_2})$$

and let us apply the transformation L-->L^{-1}

$$\Phi_{p2}^{-1}(L^{-1}) \cdot \Theta_{q2}(L^{-1}) = (\phi_{-p1}L^{p2+p1} + ... + \phi_0 L^{p2} + \phi_1 L^{p2-1} + ... + \phi_{p2})^{-1} \cdot (\theta_{-q_1}L^{q1+q2} + ... + \theta_0 L^{q2} + \theta_1 L^{q2-1} + ... + \theta_{q_2})$$

If det[$\Phi_{p2}(1/z)$] is a polynomial with roots that lie outside the unit circle, we may write

$$\Phi_{p2}^{-1}(L^{-1}) = (\phi_{-p1}L^{p2+p1} + ... + \phi_0 L^{p2} + \phi_1 L^{p2-1} + ... + \phi_{p2})^{-1} = \lambda_0 + \lambda_1 L + \lambda_2 L^2 + ...$$

This selection fulfills the condition that the real m x m matrices $\{\lambda_s\}$ are such that $\sum_{s=0}^{\infty}\|\lambda_s\|<\infty$.

Therefore, $\Phi_{p2}^{-1}(L^{-1})\cdot\Theta_{q2}(L^{-1})=(\lambda_0+\lambda_1L^1+\lambda_2L^2+...)\cdot(\theta_{-q_1}L^{q1+q2}+...+\theta_0L^{q2}+\theta_1L^{q2-1}+...+\theta_{q_2})$

and we again have again a product of a matrix series with a matrix polynomial

$$\Phi_{p2}^{-1}(L^{-1})\cdot\Theta_{q2}(L^{-1})=\psi_0+\psi_1L^1+\psi_2L^2+...$$

Now going backwards by applying the transformation L-->L^{-1}

$$\Phi_{p2}^{-1}(L)\cdot\Theta_{q2}(L)=\psi_0+\psi_1L^{-1}+\psi_2L^{-2}+...$$

It should be noted that det[$\Phi_{p2}(1/z)$] has roots that all lie outside the unit circle, if and only if all the roots of the dual polynomial det[$\Phi_{p2}(z)$] lie inside the unit circle and are non-zero.

It is therefore required that the roots of the polynomial det[$\Phi_{p2}(z)$] all lie inside the unit circle and are not null, in order to ensure the existence of the required convergent matrix series.

$$\Phi^{-1}(L)\Theta(L)=L^{q2-p2}\cdot[\psi_0+\psi_1L^{-1}+\psi_2L^{-2}+...]$$

and the real m x m matrices $\{\psi_s\}$ are such that $\sum_{s=0}^{\infty}\|\psi_s\|<\infty$.

Hence, the forward stationary solution is:

$$Y_t=\psi_0X_{t+q2-p2}+\psi_1X_{t+q2-p2+1}+\psi_2X_{t+q2-p2+2}+\psi_3X_{t+q2-p2+3}+...$$

In summary, the backward case uses:

$L^{-p1}\Phi_{p1}(L)Y_t=L^{-q1}\Theta_{q1}(L)X_t$ and solves

$$Y_t=L^{p1-q1}\Phi_{p1}^{-1}(L)\Theta_{q1}(L)X_t=L^{p1-q1}\Psi(L)X_t=\sum_{s=0}^{+\infty}\psi_sX_{t+p1-q1-s}$$

and the forward model uses $L^{p2}\Phi_{p2}(L)Y_t = L^{q2}\Theta_{q2}(L)X_t$ then

$$Y_t = L^{q2-p2}\Phi_{p2}^{-1}(L)\Theta_{q2}(L)X_t = L^{q2-p2}\Psi(L)X_t = \sum_{s=0}^{+\infty}\psi_s X_{t+q2-p2+s}$$

Collecting the results of the previous analysis we have proved the closed solution of a linear model, as discussed below.

Closed solution of a linear model.

Let us first consider a multivariate backward solution. Let $\{Y_t\}$ and $\{X_t\}$ be two stationary second-order processes that are m-variate, and consider the stochastic equation $\phi(L)Y_t = \theta(L)X_t$ of order (p_1, p2, q1, q2). Let the polynomial det[$\Phi_{p1}(z)$] be such that all its roots lie outside the unit circle, then there exists an integer index given by k= p_1-q_1 and a countable collection of real m x m matrices s $\{\psi_s\}$ with $\sum_{j=0}^{+\infty}\|\psi_s\| < \infty$ such that

$$Y_t = \psi_0 X_{t+k} + \psi_1 X_{t+k-1} + \psi_2 X_{t+k-2} + ... = \sum_{s=0}^{\infty}\psi_s X_{t+k-s}$$ is the stationary solution.

Let us now consider a multivariate forward solution. Let $\{Y_t\}$ and $\{X_t\}$ be two stationary second-order processes that are m-variate, and consider the stochastic equation $\phi(L)Y_t = \theta(L)X_t$ of order (p_1, p_2, q_1, q_2). The polynomial det[$\Phi_{p2}(z)$] is such that all its roots are non zero and lie inside the unit circle; there then exists an integer index given by k=q_2-p_2 , and there exists a countable collection of real m x m matrices $\{\psi_s\}$ with $\sum_{j=0}^{+\infty}\|\psi_s\| < \infty$ such that

$$Y_t = \psi_0 X_{t+k} + \psi_1 X_{t+k+1} + \psi_2 X_{t+k+2} + ... = \sum_{s=0}^{+\infty}\psi_s X_{t+k+s}$$ is the stationary solution.

3 Multivariate ARMA processes.

Let us now assume that $X_t=A_t$ and make the additional assumption that $\{A_t\}$ is white noise .

In this section select $X_t=A_t$ and solve for Y_t or A_t , there are now four cases:.

1. - VMA backward   2. - VAR backward

3. - VMA forward   4. - VAR forward

Take a zero mean stationary process $\{Y_t\}$, this series is a solution of the VARMA(p1,p2,q1,q2) stochastic equation, if it satisfies:

$$\phi_{-p_1}Y_{t+p1}+..+\phi_{-1}Y_{t+1}+\phi_0 Y_t+\phi_1 Y_{t-1}+..+\phi_{p_2}Y_{t-p2}=\theta_{-q1}A_{t+q1}+..+\theta_{-1}A_{t+1}+\theta_0 A_t+\theta_1 A_{t-1}+..+\theta_{q2}A_{t-q2}$$

$\phi_{-p1}\neq 0$, $\phi_{-p2}\neq 0$, $\theta_{-q1}\neq 0$, $\theta_{q2}\neq 0$. $\det[\phi_0]\neq 0$ and $\det[\theta_0]\neq 0$

Corollary 1: VMA Backward. Let $\{Y_t\}$ be a vector stationary process and let $\{A_t\}$ be white noise, and let us consider the stochastic equation $\phi(L)Y_t=\theta(L)A_t$ in the form $L^{-p1}\mathit{\Phi}_{p1}(L)Y_t=L^{-q1}\mathit{\Theta}_{q1}(L)A_t$. If the polynomial $\det[\mathit{\Phi}_{p1}(z)]$ is such that all its roots lie outside the unit circle, there then exists an integer key $k=p_1-q_1$ and a countable collection of real m x m matrices $\{\psi_s\}$ with $\sum_{j=0}^{+\infty}\|\psi_s\|<\infty$, and the solution is given by:

$$Y_t=\psi_0 A_{t+k}+\psi_1 A_{t+k-1}+\psi_2 A_{t+k-2}+\psi_3 A_{t+k-3}+...=\sum_{s=0}^{+\infty}\psi_s A_{t+k-s}\ .$$

Corollary 2: VAR Backward. Let $\{Y_t\}$ be a vector stationary process and let $\{A_t\}$ be white noise, and let us consider the stochastic equation $\phi(L)Y_t=\theta(L)A_t$ in the form $L^{-p1}\mathit{\Phi}_{p1}(L)Y_t=L^{-q1}\mathit{\Theta}_{q1}(L)A_t$. If the polynomial $\det[\mathit{\Theta}_{q1}(z)]$ is such that all its roots lie outside the unit circle, there then exists an integer key $k=q_1-p_1$ and a countable collection of real m x m matrices $\{\pi_s\}$ with $\sum_{j=0}^{+\infty} \|\pi_s\| < \infty$, and the solution is given by:

$$A_t = \pi_0 Y_{t+k} + \pi_1 Y_{t+k-1} + \pi_2 Y_{t+k-2} + \pi_3 Y_{t+k-3} + \ldots = \sum_{s=0}^{+\infty} \pi_s Y_{t+k-s}$$

Corollary 3: VMA Forward . Let $\{Y_t\}$ be a vector stationary process and let $\{A_t\}$ be white noise, and let us consider the stochastic equation $\phi(L)Y_t = \theta(L)A_t$ in the form $L^{p2}\Phi_{p2}(L)Y_t = L^{q2}\Theta_{q2}(L)A_t$.

If the polynomial $\det[\Phi_{p2}(z)]$ is such that all its roots lie inside the unit circle and are not null, there then exists an integer key $k=q_2-p_2$ and a countable collection of real m x m matrices $\{\psi_s\}$ with $\sum_{j=0}^{+\infty} \|\psi_s\| < \infty$, and the solution is given by:

$$Y_t = \psi_0 A_{t+k} + \psi_1 A_{t+k+1} + \psi_2 A_{t+k+2} + \psi_3 A_{t+k+3} + \ldots = \sum_{s=0}^{+\infty} \psi_s A_{t+k+s}$$

Corollary 4: VAR Forward. Let $\{Y_t\}$ be a vector of stationary process and let $\{A_t\}$ be white noise, and let us consider the stochastic equation $\phi(L)Y_t = \theta(L)A_t$ in the form $L^{p2}\Phi_{p2}(L)Y_t = L^{q2}\Theta_{q2}(L)A_t$ If the polynomial $\det[\Theta_{q2}(z)]$ is such that all its roots lie inside the unit circle and are not null, there then exists an integer key $k=p_2-q_2$ and a countable collection of real matrices $\{\pi_s\}$ with $\sum_{j=0}^{+\infty} \|\pi_s\| < \infty$ and

$$A_t = \pi_0 Y_{t+k} + \pi_1 Y_{t+k+1} + \pi_2 Y_{t+k+2} + \pi_3 Y_{t+k+3} + \ldots = \sum_{s=0}^{+\infty} \pi_s Y_{t+k+s}$$

4 Models that depends on future observations.

Let us now consider an application in which the economic agents incorporate expectations in their plans. We may then consider

$$\begin{aligned}&\phi_{-p_1}E_t\left[Y_{t+p1}\right]+\ldots+\phi_{-1}E_t\left[Y_{t+1}\right]+\phi_0Y_t+\phi_1Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}=\\&\qquad=\theta_{-q1}E_t\left[X_{t+q1}\right]+\ldots+\theta_{-1}E_t\left[X_{t+1}\right]+\theta_0X_t+\theta_1X_{t-1}+\ldots+\theta_{q2}X_{t-q2}\end{aligned}$$

where $E_t[Z_t]$ is the conditional expectative of Z_t respect to the sigma field generated by all the past information in the collection { Zt, Z_{t-1}, Z_{t-2},…} and fulfills:

$E_t[Z_{t-j}]=Z_{t-j} \quad j=0,1,...$ .In the case where Z_t=A_t, is given by white noise, the notation will imply the use of the residuals $E_t[A_{t-j}]=\hat{A}_{t-j} \quad j=0,1,...$

The conditional expectative is linear, thus:

$$\begin{aligned}&E_t[\phi_{-p_1}Y_{t+p1}+\ldots+\phi_{-1}Y_{t+1}+\phi_0Y_t+\phi_1Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}]=\\&=E_t[\theta_{-q1}X_{t+q1}+\ldots+\theta_{-1}X_{t+1}+\theta_0X_t+\theta_1X_{t-1}+\ldots+\theta_{q2}X_{t-q2}]\end{aligned}$$

Let us solve for the skeleton:

$$\begin{aligned}&\phi_{-p_1}Y_{t+p1}+\ldots+\phi_{-1}Y_{t+1}+\phi_0Y_t+\phi_1Y_{t-1}+\ldots+\phi_{p_2}Y_{t-p2}=\\&\qquad=\theta_{-q1}X_{t+q1}+\ldots+\theta_{-1}X_{t+1}+\theta_0X_t+\theta_1X_{t-1}+\ldots+\theta_{q2}X_{t-q2}\end{aligned}$$

We now have the relation: $Y_t=\sum_s \psi_s X_{t+k\pm s}$

By applying a conditional expectation and using the concept of limit, it is possible to simplify the expression above according to the rules:

$$\begin{aligned}&E_t[Y_{t-j}]=Y_{t-j} \quad j=0,1,2,3,...\\&E_t[X_{t-s}]=X_{t-s} \quad s=0,1,2,3,...\end{aligned}$$

Either in backward or forward form, we may conclude that $E_t[Y_t]=\sum_s \psi_s E_t[X_{t+k\pm s}]$

It is possible to apply the ideas in two ways: by invoking a learning procedure to obtain an estimate for each expectative, or alternatively by using surveys to fill the expectation terms on the right hand side. However, a sudden change in the information set might alter the anticipated value. The $\{Y_t\}$ path depends not only on past information but also on future expected values. An inherent uncertainty is present, in that the policy maker cannot work in isolation behind a desk; instead he or she must try to gauge public opinion in order to address the uncertainty. It is possible to say that there is time consistency if the sequence of expected values remains constant, otherwise there is inconsistency when the sequence of expected values rapidly changes.

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