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Abstract

We give an explicit solution to the perpetual American capped power put option pricing problem in the Black-Scholes-Merton Model. The approach is mainly based on free-boundary formulation and verification. For completeness we also give an explicit solution to the perpetual American standard power (≥ 1) option pricing problem.

Key words: The perpetual American capped power put option, geometric Brownian motion, free-boundary.

1 Introduction

A standard American power put option is a financial contract that allows the holder to sell an asset for a prescribed amount at any time. The price of this asset is raised to some power. The case of power being one corresponds to the usual American put option.

For the European power put option, the value of this option is given (see [2], [3], [4] , [7]). For the perpetual American power (≥ 1) put option, for completeness, we give the value of this option and the optimal stopping time.

A capped power option is a power option whose maximum payoff is set to a prescribed level. For the European capped power put option , the value of this option is given (see [2], [3]). For the perpetual American capped power put option, we give the value of this option and the optimal stopping time.

Throughout this note, the approach is mainly based by free-boundary formulation and verification. Only one exception is Theorem 2.2.

This note is organized as follows. In Section 2, we explicitly solve the perpetual American power put option pricing problem. In Section 3, we explicitly solve the perpetual American capped power put option pricing problem.

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2 The perpetual American power put option

The arbitrage-free price of the perpetual American power put option is given by

$$V_*(x) = \sup_\tau E_x(e^{-r\tau}(K - (X_\tau)^i)^+) \tag{1}$$

where K is the strike price, τ is a stopping time, i is a positive constant greater than or equal 1, and $x > 0$ is the initial value of the stock price process $X = (X_t)_{t\geq 0}$. In (1), the supremum is taken over all stopping times τ of the process X started at x. The stock price process $X = (X_t)_{t\geq 0}$ is assumed to be a geometric Brownian motion. That is,

$$dX_t = rX_t dt + \sigma X_t dB_t \tag{2}$$

where $r, \sigma > 0$. The infinitesimal generator of X is given by

$$L_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}. \tag{3}$$

As in the case of standard perpetual American put option, we suppose that there exists a point $b \in (0, K^{\frac{1}{i}})$ such that

$$\tau_b = \inf\{t \geq 0 : X_t \leq b\}. \tag{4}$$

is optimal in (1). Then we solve the following free-boundary problem for unknown V and b. Here $0 < b^i < K$.

$$\begin{aligned}
L_X V &= rV & x > b, && (5)\\
V(x) &= (K - x^i)^+ & x = b, && (6)\\
V'(x) &= -ix^{i-1} & x = b, && (7)\\
V(x) &> (K - x^i)^+ & x > b, && (8)\\
V(x) &= (K - x^i)^+ & 0 < x < b. && (9)
\end{aligned}$$

The steps to solve this free-boundary problem is same as in the case of the standard American put option ($i = 1$) (see [5]), so we only outline the steps. Since $V(x) \leq K$, the equation (5) implies

$$V(x) = cx^{-r/D} \tag{10}$$

where $D = \frac{\sigma^2}{2}$ and c is an undetermined constant. Using (10), we solve two equations (6) and (7) to give

$$\begin{aligned}
b &= \left(\frac{K}{1 + Di/r}\right)^{1/i}, && (11)\\
c &= \frac{Di}{r}\left(\frac{K}{1 + Di/r}\right)^{1+\frac{r}{Di}}. && (12)
\end{aligned}$$

Thus $V(x)$ is written as

$$V(x) = \begin{cases} \frac{Di}{r}\left(\frac{K}{1+Di/r}\right)^{1+\frac{r}{Di}} x^{-r/D} & \text{if } x \in [b,\infty), \\ K - x^i & \text{if } x \in (0,b]. \end{cases} \tag{13}$$

Now we have the following theorem.

Theorem 1.1 $V(x)$ coincides with $V_*(x)$, and the optimal stopping time is given by τ_b.

The steps to prove this theorem is same as in the case of the standard American put option ($i = 1$) (see [5]), so we omit.

It should be noted that for $i < 1$, $\lim_{x\to 0} |V'(x)| = \infty$, so we assume that $i \geq 1$ in this section.

3 The perpetual American capped power put option

The arbitrage-free price of the perpetual American capped power put option is given by

$$V_*(x) = \sup_{\tau} E_x\left(e^{-r\tau}\min\left[(K - (X_\tau)^i)^+, \overline{C}\right]\right) \tag{14}$$

where $\overline{C}(< K)$ and i are positive constants. First we suppose there exists a point $b \in (h, K^{\frac{1}{i}})$ such that

$$\tau_b = \inf\{t \geq 0 : X_t \leq b\} \tag{15}$$

is optimal in (14). Here h satisfies $K - h^i = \overline{C}$. Then we solve the following free-boundary problem for unknown V and b.

$$\begin{aligned} L_X V &= rV & & x > b, & (16)\\ V(x) &= (K - x^i)^+ & & x = b, & (17)\\ V'(x) &= -ix^{i-1} & & x = b, & (18)\\ V(x) &> (K - x^i)^+ & & x > b, & (19)\\ V(x) &= (K - x^i)^+ & & h < x \leq b, & (20)\\ V(x) &= \overline{C} & & 0 < x \leq h. & (21) \end{aligned}$$

It is clear that for $0 < x \leq h$, the arbitrage-free price from (14) is given by (21). Thus, $0 < x \leq h$ is the stopping region.

Since we suppose that $b \in (h, K^{\frac{1}{i}})$, b is same as in the case of the American power put option. Thus the free-boundary b is given by

$$b = \left(\frac{K}{1 + Di/r}\right)^{1/i}. \tag{22}$$

$V(x)$ is written as

$$V(x) = \begin{cases} \frac{Di}{r}\left(\frac{K}{1+Di/r}\right)^{1+\frac{r}{Di}} x^{-r/D} & \text{if } x \in [b, \infty), \\ K - x^i & \text{if } x \in (h, b], \\ \overline{C} & \text{if } x \in (0, h]. \end{cases} \tag{23}$$

Note that $b > h$ is equivalent to $\overline{C} > K \cdot \frac{Di/r}{1+Di/r}$.

Theorem 2.1 Suppose that $\overline{C} > K \cdot \frac{Di/r}{1+Di/r}$, then $V(x)$ coincides with $V_*(x)$, and the optimal stopping time is given by τ_b.

Proof. From our earlier consideration we can suppose that $x \in (h, \infty)$. Since $P(X_s = h) = 0$ and $P(X_s = b) = 0$, the change-of-variable formula (see Remark 2.3 in [5]) with the smooth-fit condition (18) gives

$$\begin{aligned} V(X_t) = V(X_0) \quad &+ \int_0^t e^{-rs}(L_X V - rV)(X_s) I(X_s \neq h, X_s \neq b) ds \\ &+ \int_0^t V_x(X_s)\sigma X_s I(X_s \neq h, X_s \neq b) dB_s \\ &+ \frac{1}{2}\int_0^t (V_x(X_s+) - V_x(X_s-)) I(X_s = h) dl_s^c(X). \end{aligned} \tag{24}$$

When $V(x) = (K - x^i)$, we see that $(L_X V - rV)(x) = x^i(1-i)(r+\frac{\sigma^2}{2}i) - rK < 0$ for $h < x \leq b$. For $i \geq 1$, it clearly holds. For $i < 1$, from $x^i < h$ it is easily seen to hold. Thus

$$e^{-rt}(K - {X_t}^i)^+ \leq e^{-rt}V(X_t) \leq V(x) + M_t$$

where $M = (M_t)_{t \geq 0}$ defined by

$$M_t = \int_0^t V_x(X_s)\sigma X_s I(X_s \neq h, X_s \neq b) dB_s$$

is a continuous martingale (because $|V'(x)|$ is bounded for all $x > 0$). Thus it is easily verified by standard means (using the localization of M and Fatou's lemma) that we get that

$$V_*(x) \leq V(x)$$

for all $x \in (h, \infty)$.

Next we set $t = \tau_b \wedge \tau_n$ in (24). Here $(\tau_n)_{n \geq 1}$ is a localization sequence of bounded stopping times for M. For $s \leq \tau_b$, $X_s \geq b > h$. Hence the fourth term in the right-hand side of this equation is zero. Moreover using (16), we find the second term in the right-hand side is also zero. Finally using the optional sampling theorem we get

$$E_x(e^{-r(\tau_b \wedge \tau_n)} V(X_{\tau_b \wedge \tau_n})) = V(x). \tag{25}$$

Letting n go to infinity and using the dominated convergence and (17), we get that

$$V_*(x) \geq V(x)$$

for all $x \in (h, \infty)$. The proof is completed.

Theorem 2.2 Suppose that $\overline{C} \leq K \cdot \frac{Di/r}{1+Di/r}$, then $V(x)$ is written as

$$V(x) = \begin{cases} \bar{C}(K-\bar{C})^{\frac{r}{Di}} \cdot x^{-\frac{r}{D}} & \text{if } x \in (h, \infty), \\ \overline{C} & \text{if } x \in (0, h]. \end{cases} \tag{26}$$

The optimal stopping time is given by τ_h.

Proof. We set $\tau_h = \inf\{t \geq 0 : X_t \leq h\}$. Then

$$E_x(e^{-r\tau_h}(K - (X_{\tau_h})^i)^+) \tag{27}$$
$$= E_x(e^{-r\tau_h}(K - h^i))$$
$$= (K - h^i) E_x(e^{-r\tau_h})$$
$$= \bar{C}(K-\bar{C})^{\frac{r}{Di}} \cdot x^{-\frac{r}{D}} \tag{28}$$

where the final equality follows by the formula for the expected first hitting time for a geometric Brownian motion (see e.g. [1]).

Now we show for $K^{\frac{1}{i}} > x > h$,

$$\bar{C}(K-\bar{C})^{\frac{r}{Di}} \cdot x^{-\frac{r}{D}} > K - x^i. \tag{29}$$

We set $f(x)$ to be equal to the left-hand side minus the right-hand side in (29). Clearly $f(h) = 0$. To show that $f'(x) > 0$, it suffices to show

$$\bar{C} x^{\frac{r}{D}} (-\frac{r}{D}) x^{-\frac{r}{D}-1} + i(K - \overline{C}) x^{-1} \geq 0 \tag{30}$$

because $x > h$. Since $\overline{C} \leq K \cdot \frac{Di/r}{1+Di/r}$, (30) holds. Since $V(x) > 0$, (29) and (30) imply that (h, ∞) is a continuous region. On the other hand, $(0, h]$ is a stopping region. Thus τ_h is the optimal stopping time and $V(x)$ is given by (26). The proof is completed.

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