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Abstract

This paper uses mixed combinatorial-cum-real particle swarm method to obtain a heuristically optimal order in which the constituent variables can be arranged so as to yield some generalized maximum entropy synthetic indicators that represent the constituent variables in the best information-theoretic sense. It may help resolve the arbitrariness and indeterminacy of Pena's method of construction of a synthetic indicator which at present is very sensitive to the order in which the constituent variables (whose linear aggregation yields the synthetic indicator) are arranged.

JEL Code: C18, C43, C44, C61

Keywords: Synthetic indicators, Composite indices, Pena's distance, Mixed Combinatorial Particle swarm, Sharma-Mittal entropy, Rényi entropy, Tsallis entropy. Kaniadakis entropy, Abe entropy

I. Introduction: A synthetic indicator (or composite index), $Z(n)$, is an n-element array that represents $X(n,m)$, an m-tuple of other n-element arrays (called constituent variables) for $m>1.$ Z is synthetic in the sense that so often it is a linear combination of $x_j \in X;\ j=1,2,...,m$ and, thus, $Z=Xw,$ where w is an m-element array of weights. There are indeed several methods to obtain the weight vector, w, from X, but, at present, we are concerned with Pena's method (Pena, 1977; Somarriba & Pena, 2009) based on his concept of distance (DP2) defined as:

$$DP2_i = \sum_{j=1}^{m}\left[\left(\frac{d_{ij}}{\sigma_j}\right)\left(1-R^2_{j,j-1,...,1}\right)\right]; i=1,2,...,n \qquad ... \qquad (1)$$

where: $i=1,2,...,n$ are cases (e.g. countries); m is the number of constituent variables, X, such that $x_{ij} \in X; i=1,2,...,n; j=1,2,...,m$; $d_{ij}=\left|x_{ij}-x_{rj}\right|; i=1,2,..,n; j=1,2,...,m$; r is the reference case; σ_j is the standard deviation of constituent variable j; $R^2_{j,j-1,...,1}$; $j>1$ is the coefficient of determination in the regression of x_j over $x_{j-1,}x_{j-2},...,x_1$. Moreover, $R_1^2=0$ (Somarriba & Pena, 2009). A synthetic indicator constructed by Pena's method is claimed to have almost all desirable properties (Pena, 1977; Zarazosa, 1996; Somarriba & Pena, 2009; Montero et al., 2010; Garcia et al., 2010; Martína & Fernández, 2011).

Montero et al. (2010) noted and Mishra (2012a) demonstrated that an application of Pena's method of construction of synthetic indicators suffers from indeterminacy since the weight ($w_j = 1 - R^2_{j,j-1,...,1}$) obtained by the j^{th} (standardized) constituent variable, (d_{ij} / σ_j), depends on its position in the order or the value of j. The iterative process suggested by Montero et al. (2010) may not converge as long as the weights continue to be defined as $w_j = 1 - R^2_{j,j-1,...,1}$ and not $w_j^{(t)} = f(w_j^{(t-1)})$, where t and $(t-1)$ stand for the current and the immediately prior iterations and $f(.)$ is a real continuous and bounded function onto itself. Such a condition and thus the convergence is precluded due to the definition of weight as $w_j = 1 - R^2_{j,j-1,...,1}$ which corresponds to a particular configuration (order) belonging to the (discrete) set of all possible $m!$ (m-factorial) configurations (making $m!$ isolated points) conforming to the order in which the constituent variables enter into the formula (eq.1). As a consequence, unless there is some extraneous criterion that determines the order in which the variables enter in eq. 1 above, the synthetic indicator constructed by Pena's method is indeterminate. Mishra (2012b) suggested that maximization of the minimal (absolute) correlation between the synthetic indicator and the constituent variables $(\min_j(|r(Z,d_j)|))$, where Z is Pena's synthetic indicator, may provide such an extraneous criterion.

II. The Objectives of this Paper: Esteban & Morales (1995) provide a comprehensive list of (as many as twenty three) entropy measures. The objectives of this paper are: (i) to use the maximum entropy of Z as the extraneous criterion to obtain Pena's synthetic indicator, and (ii) to gauge into the suitability of a particular measure of entropy from among some well-known measures of entropy.

III. Meaning and Different Measures of Entropy: As Beck (2008) has very lucidly explained, the missing information on the concrete state of a system is related to the entropy of the system and thus 'entropy' is used as a synonym for a possible quantity to measure missing information – the missing information on the actual occurrence of events, given that we only know the probability distribution of the events. To make the point clearer, consider a sample set of K possible events (possible microstates of a system), with the probability of the occurrence of event j being p_j and $\sum_{j=1}^{K} p_j = 1.$ Let the information gain due to the occurrence of a single event j be measured by a function $h(p_j)$, which should be close to zero for p_j close to 1. Then, for a given function $h(.)$ the average information gained during a long sequence of trials is $I(\{p_j\}) = \sum_{j=1}^{K} p_j h(p_j)$ and the entropy is $S = -I.$ Thus, entropy is defined as our missing information on the actual occurrence of events, given that we only know the probability distribution of the events.

Khinchin (1957) formulated four axioms that describe the properties a 'classical' information measure, $I,$ should have. Those are: (i) the information measure, $I,$ must solely depend on the probabilities p_i of the events, or $I = I(p_1, p_2, ..., p_k);$ (ii) the information measure, $I,$ should attain its minimum when $p_j = K^{-1} \; \forall \; j$ or the information content of any probability distribution (other than uniform distribution) must exceed the information content of the uniform distribution; (iii) the information measure should not change if the sample set of events is enlarged by inclusion of an extra event with $p_{k+1} = 0$ and, finally, (iv) the information measure should be independent of the way or the sequence in which the information is collected. The implication of this axiom is that $I(p_{ij})$ factorises into $I(p_i) + I(p_j),$ where p_{ij} is the (joint) conditional probability of occurrence of event j while the event i has already occurred. This is the axiom of additivity of information for independent systems (Beck, 2008). The 'classical' system obeys all the four axioms and has simple formula of expressing the total entropy of a joint system as a simple function of the entropies of the interacting subsystems (called composability property). Variations in defining the different measures of entropy mainly rest on the fulfillment of this axiom of independence. A good measure of entropy should have composability. Additionally, it should have concavity and stability (called Lesche stability) with regard to small perturbations. Concavity means that for the sub-systems U_1 and U_2 belonging to U one has $S(U) \geq \lambda S(U_1) + (1-\lambda)S(U_2)$; $0 \leq \lambda \leq 1$. (Naudts, 2011: p. 43).

Now, we briefly describe the various measures of entropy. We assume that the probability of any event is not zero or, more exactly, $p_j > 0 \ \forall \ j$.

III.1. The Shannon Entropy: This measure of entropy satisfies all the four Khinchin axioms and is measured as $S = -k\sum_{j=1}^{K} p_j \log_e(p_j)$. In the information theoretic context, the constant, k (which has a definite meaning and value for a physical system and is known as the Boltzmann's constant), may be assumed to be unity and, therefore, one may say that Shannon's measure of entropy (S) directly varies with the measure $S = -\sum_{j=1}^{K} p_j \log_e(p_j)$. The Shannon's measure of entropy has the properties of composability, concavity and Lesche-stability.

III.2. The Rényi Entropy: Introduced by Rényi (1970), this measure of entropy has a single parameter, q and is measured as $S_q = \frac{1}{q-1}\log_e(\sum_{j=1}^{K} (p_j)^q$. It satisfies the first three Khinchin axioms, but there is no simple formula of expressing the total Rényi entropy of a joint system as a simple function of the Rényi entropies of the interacting subsystems. This measure of entropy does not have composability, concavity and Lesche-stability (Lesche, 1982). This measure reduces to Shannon's measure of entropy as q approaches unity.

III.3. The Tsallis Entropy: The Tsallis entropy (Tsallis, 1988) is given by $S_q = \frac{1}{q-1}(1-\sum_{j=1}^{K} p_j^q)$ for any real value of q (the entropic index) and in particular, it contains the Shannon entropy in the limiting case as q approaches unity. Tsallis' measure of entropy has composability in a more general sense as shown by Abe (2000). It has concavity for $q > 0$ (convexity for $q < 0$). Lesche-stability of this measure of entropy was shown by Abe (2002), but Lutsko et al. (2009) have argued that if Lesche-stability is properly applied within the usual formalism of non-extensive thermodynamics, the Tsallis entropy is just as unstable as the Rényi entropy.

III.4. The Abe Entropy: The Abe measure of entropy (Abe, 1997) is a symmetric modification of Tsallis measure of entropy, which in symmetric in $q \leftrightarrow q^{-1}$ and q lies in $(0, 1]$. It is given by $S = -\sum_{j=1}^{K} (p_j^q - p_j^{1/q}) / (q - q^{-1})$. Abe measures of entropy is the modifications on Tsallis entropy (Beck, 2008).

III.5. The Kaniadakis Entropy: This measure of entropy was proposed by Kaniadakis (2002) $S = -\sum_{j=1}^{K} (p_j^{1+kappa} - p_j^{1-kappa}) / (2kappa)$. For $kappa = 0$ it gives the Shannon entropy and, therefore, it is a measure of deformation of a statistical distribution suitable to the Shannon entropy.

III.6. The Sharma-Mittal Entropy: This measure of entropy (Sharma & Mittal, 1975) has two parameters, q and r and it may be expressed (Aktürk et al. 2008) as $S = \frac{1}{1-r}\left[\left(\sum_{j=1}^{K} p_j^q\right)^{((1-r)/(1-q))}\right]$. Aktürk et al. argue that the Sharma-Mittal (S-M) measure of entropy must be thought to be a step beyond not both Tsallis and Rényi entropies but rather only as a generalization of Rényi entropy from a thermo-statistical point of view. It also fails to be concave (Masi, 2005), while concavity entails thermodynamic stability. In spite of all these, the Sharma-Mittal measure of entropy incorporates Shannon, Rényi and Tsallis measures of entropy as its special cases. As $r \to 1,$ it gives Rényi's measure; as $r \to q,$ it gives Tsallis measure; as r and q both approach unity, it gives the Shannon measure of entropy. Beck (2008) provides a simpler expression of the Sharma-Mittal measure of entropy, $S = -\sum_{j=1}^{K} p_j^r \left(\frac{p_j^{\alpha} - p_j^{-\alpha}}{2\alpha} \right)$, which gives Tsallis entropy for $r = \alpha$ and $q = 1 - 2\alpha$, Kaniadakis entropy for $r = 0$ and Abe entropy for $\alpha = (q - q^{1/2})/2$ and $r = (q + q^{1/2})/2.$

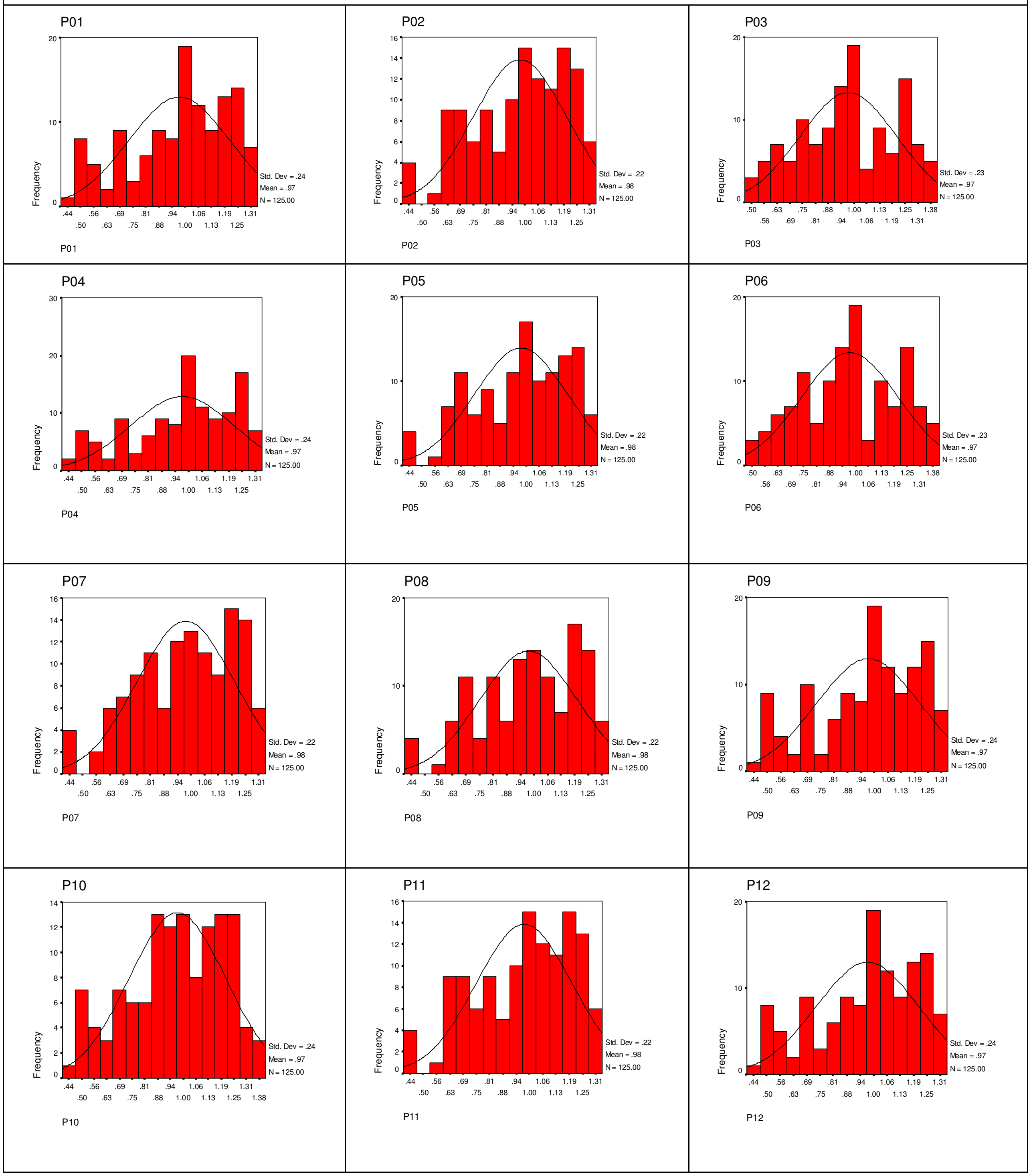
IV. Choice of the Measures of Entropy: For the investigation at hand, we have chosen a few general measures of entropy such as Rényi, Tsallis, Abe, Kaniadakis and Sharma-Mittal measures. It may be noted that in the present context (entropy of Pena's synthetic indicators) we cannot presume stability. More particularly, since the weights assigned by the Pena method depend on the order in which the constituent variables enter into the formula, the weight w_j (associated with d_{ij}) is contingent upon the previously chosen weights (i.e. $w_{j-t} : t = 1,2,... < j$ for d_{ij-t}). Thus, independence is precluded. Additionally, assuming $w_1 = 1$ is arbitrary. In view of these, a regular measure of entropy (such as that of Shannon) may be utterly presumptive and unsuitable.

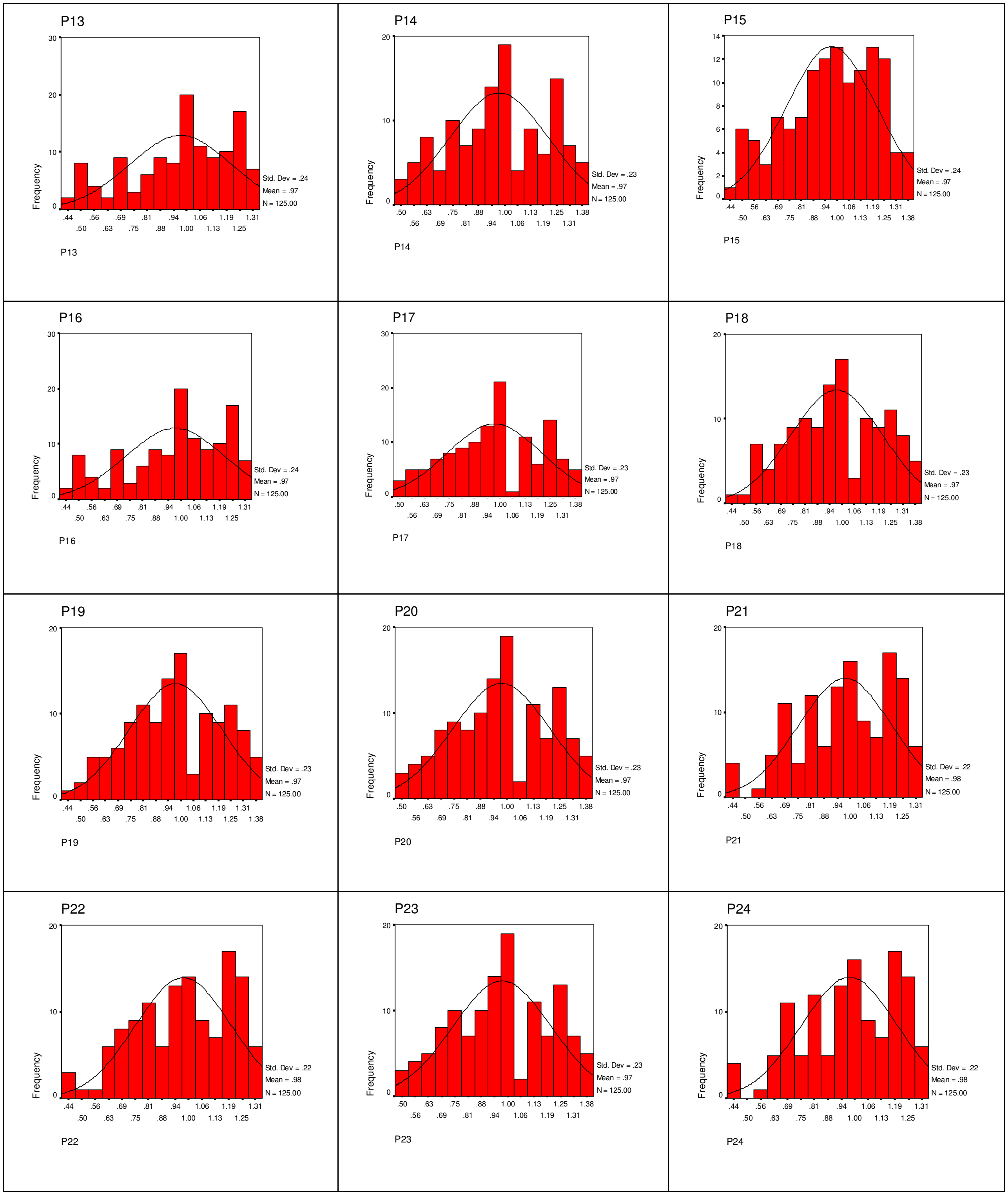
V. Choice of the Method of Optimization: To obtain a maxi-min correlation solution of Pena indicator, Mishra (2012b) chose the discrete particle swarm method of optimization such that the decision variables could take on only those values that conform to a particular permutation (configuration) among the possible m-factorial permutations (configurations) of the decision variables. However, in the present exercise, we must take one or two additional decision variables, depending on the number of parameters in a specific measure of entropy. These additional variables would take on real values. Thus, in the particle swarm method that we use, the parameter space is mixed. Among the total no. of decision variables, the first m will take on only integer values (all permutations of 1, 2, … , m) and the last ones (one or two) will be real. It may also be noted that in this exercise we optimize relative entropy, i.e. $S/S_{\max}$ where $S_{\max}$ is the maximum possible entropy pertaining to the uniform distribution (Rodrigues & Giraldi, 2009) and S is the entropy measure of the relative frequency distribution (approximate probability distribution obtained from the frequency distribution of the Pena Indicator under different permutations of the constituent variables).

VI. The Test Data: The data from Sarker et al. (2007) on Human Development Index (HDI) and its constituents (viz. life expectancy (LE), education (ED) and per capita gross domestic product at the purchasing power parity with the US $ (PCI), used by Mishra (2012a and 2012b)) form the test data to obtain Z (Pena Indicators) that corresponds to particular permutation of constituent variables entering into the Pena's formula in a particular order. In all, twenty-four (4!) permutations are possible.

VI. The Results: First, we may note that different configurations yield different synthetic indicators (Mishra, 2012a) that have different empirical frequency distributions (P_{01} through P_{24}) as depicted in Fig.1. Deviation of the observed frequency distributions from uniform distribution (as well as normal distribution) is clear.

Fig.1: Frequency Distribution of Pena's Synthetic Indicators Obtained from Different Permutations of Constituent Variables





Secondly, among all possible twenty-four Pena's synthetic indicators (configurations presented in Table-1), we have obtained a particular indicator that maximizes a specific (relative) entropy (presented in Table-2). We observe that P_{02} maximizes Shannon, Rényi and Abe entropies; P_{08} maximizes Tsallis entropy; P_{11} maximizes Kaniadakis entropy and P_{21} maximizes Sharma-Mittal entropy. All these indicators have one thing in common: ED is chosen as the leading variable.

Table-1. All Permutations of Constituent Variables of Human Development (LE, ED, PCI and EQ)																		
P	**Order**				**Coded Order**					**P**	**Order**				**Coded Order**			
01	LE	ED	PCI	EQ	1	2	3	4		13	LE	PCI	EQ	ED	1	3	4	2
02	ED	LE	PCI	EQ	2	1	3	4		14	PCI	LE	EQ	ED	3	1	4	2
03	PCI	LE	ED	EQ	3	1	2	4		15	EQ	LE	PCI	ED	4	1	3	2
04	LE	PCI	ED	EQ	1	3	2	4		16	LE	EQ	PCI	ED	1	4	3	2
05	ED	PCI	LE	EQ	2	3	1	4		17	PCI	EQ	LE	ED	3	4	1	2
06	PCI	ED	LE	EQ	3	2	1	4		18	EQ	PCI	LE	ED	4	3	1	2
07	EQ	ED	LE	PCI	4	2	1	3		19	EQ	PCI	ED	LE	4	3	2	1
08	ED	EQ	LE	PCI	2	4	1	3		20	PCI	EQ	ED	LE	3	4	2	1
09	LE	EQ	ED	PCI	1	4	2	3		21	ED	EQ	PCI	LE	2	4	3	1
10	EQ	LE	ED	PCI	4	1	2	3		22	EQ	ED	PCI	LE	4	2	3	1
11	ED	LE	EQ	PCI	2	1	4	3		23	PCI	ED	EQ	LE	3	2	4	1
12	LE	ED	EQ	PCI	1	2	4	3		24	ED	PCI	EQ	LE	2	3	4	1

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Table-2. Identification of Maximum Relative Entropy Pena Indicators of Human Development						
Sl. No.	**P code**	**Variable Order**	**Order Codes**	**Max Relative Entropy**	**Departure from Additivity**	**Entropy Measure**
1	P_{02}	(ED, LE, PCI, EQ)	2, 1, 3, 4	0.920632560	0.0	Shannon Entropy
2	P_{11}	(ED, LE, EQ, PCI)	2, 1, 4, 3	0.910363082	0.390224222	Kaniadakis Entropy
3	P_{08}	(ED, EQ, LE, PCI)	2, 4, 1, 3	0.900989205	0.542937958	Tsallis Entropy
4	P_{02}	(ED, LE, PCI, EQ)	2, 1, 3, 4	0.898986426	0.500000678	Abe Entropy
5	P_{21}	(ED, EQ, PCI, LE)	2, 4, 3, 1	0.896921151	(0.99909739, 0.39081225)	Sharma-Mittal Entropy
6	P_{02}	(ED, LE, PCI, EQ)	2, 1, 3, 4	≃ 0.910360413	(≃ 0.39010978, ≃ 0.00009578)	Rényi Entropy (obtained through S-M)

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Thirdly, it is interesting to observe that none of the entropy-maximizing indicator is a maxi-min correlation solution. It was found that the permutation (P_{10}: 4, 1, 2, 3) was a maxi-min correlation indicator, which is the most inclusive indicator (Mishra, 2007, 2012b). It may also be observed (Table-3) that Shannon, Kaniadakis and Rényi entropy maximizing indicators are relatively more strongly correlated with HDI_1, which is an indicator that maximizes the sum of absolute correlation of the indicator (HDI_1) with the constituent variables, LE, ED, PCI and EQ (Mishra, 2007, 2010a, 2010b).

Table-3. Correlation of HDI_1 & HDI_2 with Pena's Indicators obtained by Permutation of LE, ED, PCI and EQ												
Indi-cators	P(2)	P(8)	P(11)	P(21)	P(1)	P(3)	P(4)	P(5)	P(6)	P(7)	P(9)	P(10)
HDI_1	0.9912	0.9857	0.9912	0.9867	0.9898	0.9927	0.9895	0.9915	0.9930	0.9822	0.9891	0.9767
HDI_2	0.9837	0.9719	0.9836	0.9736	0.9785	0.9849	0.9780	0.9846	0.9856	0.9646	0.9772	0.9550
Indi-cators	P(12)	P(13)	P(14)	P(15)	P(16)	P(17)	P(18)	P(19)	P(20)	P(22)	P(23)	P(24)
HDI_1	0.9897	0.9893	0.9926	0.9763	0.9889	0.9879	0.9832	0.9843	0.9886	0.9835	0.9889	0.9876
HDI_2	0.9783	0.9775	0.9845	0.9543	0.9767	0.9745	0.9652	0.9669	0.9757	0.9666	0.9762	0.9754

VII. Concluding Remarks: If we must choose from among the different entropy measures so as to maximize it for obtaining an entropy-maximizing synthetic indicator (of Pena), we may not favour the Shannon measure. If the Shannon measure would have been suitable, other measures of entropy would have been reduced to that (as the Shannon measure is a special case of all other measures of entropy considered in this exercise). There are deformities as well as the departure from the standard (fourth) Khinchin axiom of independence and conventional additivity (shown in Table-2) that might have generated the patterns reflected in the empirical distribution of the indicators.

However, from among the more general measures of entropy, it would be difficult to suggest as to which one is the best to choose. Aktürk et al. (2008) pointed out that Sharma-Mittal measure of entropy is more akin to Rényi's rather than Tsallis' (which also includes Abe's and Kaniadakis entropies as its special cases). In this study also we have observed this tendency with respect to computation and it appears that the optima of Sharma-Mittal entropies lie on more acute ridges making it difficult to obtain a numerically stable solution. Computation of Rényi's entropy showed instability and we obtained it indirectly through the Sharma-Mittal (S-M) formula, setting r in the neighborhood of unity.

As far as this study suggests, identifying the best Pena indicator by maximization of entropy cannot give us an indubitable and equivocally acceptable solution. The maxi-min solution (Mishra, 2012b) is more determinate, liable to interpretation and clearly suggestive.

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