Optimal participation in illegitimate market activities: complete analysis of 2-dimensional cases

Carfì, David and Pintaudi, Angelica

University of California at Riverside

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David Carfi
Department of Mathematics
University of California at Riverside
900 Big Springs Road, Surge 231 Riverside
CA 92521-0135, USA.
davidcarfi71@yahoo.it

Angelica Pintaudi
University of Messina
Via dei Verdi 75
98122, Messina, Italy.
angelica.pintaudi@tiscali.it

Abstract. In this paper we consider the **quantitative decision problem** to allocate a certain amount of time upon two possible market activities, specifically a legal one and an illegal one: this problem was considered in literature by Isaac Ehrlich (in his seminal paper "Participation in Illegitimate Activities: A Theoretical and Empirical Investigation", published in The Journal of Political Economy, in 1973) and the mathematical model we propose and use is essentially a formal mathematical translation of the ideas presented by him; but, on the other hand, our approach will allow to apply efficiently and quantitatively the Ehrlich qualitative model. Specifically, in this original paper, we apply the Complete Pareto Analysis of a Differentiable Decision Problem, recently introduced in literature by David Carfi, to examine exhaustively the above Ehrlich-kind decision problem: given by a pair $P = (f, >)$, where the function $f : T \to \mathbf{R}^m$ is a vector payoff function defined upon a compact m-dimensional decision (time) constrain T and with values into the m-dimensional payoff space $\mathbf{R}^m$, for some natural number m (in our paper m is 2). So, the principal aim of this paper is to show how the D. Carfi Pareto Analysis can help to face, **quantitatively**, the decision problems of the Ehrlich-type in some practical cases, also because the computational aspects were not considered by Ehrlich. Our methodologies and approaches permit (in principle), by giving a total quantitative view of the possible payoff space of the Ehrlich-decision problems (and consequently, giving a precise optimal solutions for the decision-maker), to perform quantitative econometric verifications, in order to test the payoff functions chosen in the various Ehrlich models. In particular, we apply our mathematical methodology to determine the topological boundary of the payoff space of a decision problem, for finding optimal strategies in the participation in such legal and illegitimate market activities. The theoretical framework is clarified and applied by an example.

0. Introduction

In this paper we attempt to investigate the quantitative relation between crime activities and legal measurable opportunities by means of an analytical tool introduced in literature by D. Carfi: the Pareto Analysis of a Differentiable Decision Problem. Following Ehrlich we develop a quantitative theoretical contribution to the study of crime in economic terms proposing a formal decision model to evaluate quantitatively the reasonability to commit offenses, which emphasizes the relation between crime and punishment. Following Ehrlich, we shall analyze the interaction between offense and defense, moreover we derive some behavioral implications.

1. The model with two activities

In this section we consider the model presented in Ehrlich [2]. This formal model allows us to present two new generalizations of the economic model itself and a case study.

The context. We assume, in what follows the following points.

Assumption 0. A decision-maker *D* can participate in two different market activities, say 1 and 2; namely, an illegal one, the activity 1, and a legal one, the activity 2.

Assumption 1 (the strategy space). The (universe) strategy space T of the decision maker coincides with the positive cone of the Euclidean space $\mathbf{R}^2$, that is the Cartesian square $(\mathbf{R}_{\geq})^2$ of the real non-negative semi-line $\mathbf{R}_{\geq}$, that is the unbounded real interval $[0, \rightarrow[$.

Interpretation (of the bi-strategy). A strategy t, belonging to the strategy space T, is a pair of non-negative real numbers (t_1, t_2), whose first component t_1 is the time devoted to the first (illegal) activity and the second component t_2 is the time devoted to the second (legal) activity.

Assumption 2 (monotonicity of the returns). We assume that the returns of the two activities are increasing functions W_1 and W_2, both defined on the time semi-line $\mathbf{R}_{\geq}$ and with value into **R.** Specifically, denoted by T_1 and T_2 the first and second Cartesian projections, respectively, of the strategy universe T (both are obviously equal to the semi- line $\mathbf{R}_{\geq}$), we have, for any index i*,* an increasing mapping

$$W_i : T_i \rightarrow \mathbf{R}$$

in such a way that the returns of the activity i depends only upon the time devoted to i itself.

Assumption 3 (uncertainty of illegal returns). We assume that the final returns of the first activity (the illegal one) depend upon the states of the world belonging to the unit interval of the real line [0, 1]. The final return function of the first activity is then a random variable defined on the state space S and with values into the function space $\mathcal{F}(T_1,\mathbf{R})$, that is, an application

$$L : S \rightarrow \mathcal{F}(T_1, \mathbf{R}),$$

mapping each state of the world s of S into a function

$$L_s : T_1 \rightarrow \mathbf{R} : L_s(t_1) = L(s)(t_1).$$

Interpretation (of the states of the world). Our interpretation of the state space S is the following: the state of the world s represents a degree of punishment; namely, 0 represents the state of the world in which the offender is totally safe, on the contrary 1 represents the state of the world in which the offender is fully punished.

Assumption 4 (certainty of legal returns). We assume that the second activity (the legal one) is safe, in the sense that its final returns depend only on the time devoted to it.

Assumption 5 (penalty function). We assume that the function L_s can be defined by the difference

$$L_s (t_1) = W_1 (t_1) - sF_1 (t_1),$$

for any time t_1 in T_1, where the function $F_1 : T_1 \rightarrow \mathbf{R}_{\geq}$ is an increasing function called ***penalty function***.

Definition (payoff random variable). We so can construct a random variable, which we shall call the **payoff random variable of the decision-maker**, on the set S of states of the world and with values in the function space ${}^{T}\mathbf{R}_{\geq}$, namely it is the mapping

$$X : S \rightarrow {}^{T}\mathbf{R}_{\geq},$$

defined by

$$X_s(t) = W_1(t_1) - sF_1(t_1) + W_2(t_2),$$

for each state of the world s in the unit interval S and any pair of times t in the strategy space T.

2. First extension of the model

In this section we extend the classic model proposed in [2] assuming that the returns functions and the penalty function are depending upon the entire strategy pair $t = (t_1, t_2)$.

Assumption 2 bis (monotonicity of the returns). We assume that the returns of the two activities are two increasing functions W_1 and W_2, both defined on the time cone $T = (\mathbf{R}_{\geq})^2$ and with value into the real line $\mathbf{R}$, both functions are increasing with respect to the standard orders on the plane with respective signature (+, -) and (-, +). For example, the standard order on the plane with signature (-, +), also called the cost-benefit order is the order defined by $t >_{cb} t'$ if and only if $t_1 < t'_1$ and $t_2 > t'_2$.

Remark. Denoted by T the strategy cone, we have so, for any index i, an increasing (in the above sense) mapping

$$W_i : T \to \mathbf{R}.$$

Assumption 3 bis (uncertainty of illegal returns). We assume that the final returns of the first activity (the illegal one) depend upon the states of the world belonging to the unit interval of the real line [0, 1]. The return function of the first activity is then a random variable defined on the state space S and with values into the function space ${}^{T}\mathbf{R}$, namely the return function of the first activity is an application

$$L : S \to \mathcal{F}(T, \mathbf{R}),$$

mapping each state of the world $s \in S$ into a function

$$L_s : T \to \mathbf{R}.$$

Interpretation (of the states of the world). Our interpretation of the state space S is the following: the state of the world s represents the degree of punishment, namely 0 represents the state of the world in which the offender is totally safe, on the contrary 1 represents the state of the world in which the offender is fully punished.

Assumption 4 bis (certainty of legal returns). We assume that the second activity (the legal one) is safe, in the sense that its returns depend only on the time devoted to both activities and not upon the states of the world.

Assumption 5 bis (penalty function). We assume that the function L_s can be defined by the difference:

$$W_1 - sF_1,$$

that is by

$$L_s(t) = W_1(t) - sF_1(t),$$

for any pair t in T, where the function

$$F_1 : T \to \mathbf{R}_{\geq}$$

is an increasing penalty function, with respect to the usual order of signature (+, -).

Definition (payoff random variable). We can construct a random variable, which we shall call the payoff random variable of the decision-maker, on the set S of states of the world and with values in the function space ${}^T\mathbf{R}$, namely it is the mapping

$$X : S \to {}^T\mathbf{R}$$

defined by

$$X_s(t) = W_1(t) - sF_1(t) + W_2(t),$$

for each state of the world s in the unit interval S and any pair of times t in the strategy space T.

3. The payoff function associated with a payoff random variable

Definition (the transformation associated with the random variable). Any random variable of the form

$$X : S \to {}^T\mathbf{R}$$

can be read, immediately, as a family of functions in the space ${}^T\mathbf{R}$, indexed by the set S, namely the family

$$X = (X_s)_{s \in S}$$

so as an element of the hypercube $({}^T\mathbf{R})^S$. Moreover, we can associate with the random variable

$$X : S \to {}^T\mathbf{R}$$

the mapping of T into the Cartesian power (the hypercube) $\mathbf{R}^S$ defined by

$$f_X : T \to \mathbf{R}^S : t \to (X_s(t))_{s \in S};$$

we call this mapping the payoff transformation associated with the random variable X.

4. The model with two states of the world

We assume now that the state of the world space S contains only two elements, say 0 and 1. Moreover, we assume that the decision maker has total time at his disposal 1 (in conventional unit).

The strategies of the decision maker are triples of non-negative real numbers $t = (t_1, t_2, t_3)$, with unit total duration:

$$\|t\|_1 := \sum t = 1.$$

The first component t_1, of any possible strategy t, is the time devoted to first activity; the second component t_2 is the time devoted to second activity and the third component t_3 is the remaining time

$$1 - (t_1 + t_2).$$

So that, the decision constraint of the decision maker is the canonical 2-simplex $\Delta_2(\mathbf{R}^3)$ of the 3-space $\mathbf{R}^3$, that is the convex envelope $\mathbf{M}_3$ of the canonical basis e of $\mathbf{R}^3$.

The construction of the decision problem associated with a random variable. The payoff transformation associated with a random variable

$$X : S \to \mathcal{F}(\mathbf{M}_3, \mathbf{R})$$

is the function

$$f_X : \mathbf{M}_3 \to \mathbf{R}^S : f_X(t) = (X_s(t))_{s \in S}.$$

The power $\mathbf{R}^S$ can be identified with the Euclidean plane $\mathbf{R}^2$, since the state space S contains only 2 elements; so that, we can consider the payoff transformation f_X as a vector function of the type

$$f_X : \mathbf{M}_3 \to \mathbf{R}^2 : f_X(t) = (X_0(t), X_1(t)),$$

for each time triple t of the simplex $\mathbf{M}_3$. We obtain, in such a way, a multi-criteria decision problem

$$(f_X, \geq),$$

associated with the random variable X.

Determination of the topological boundary of the problem $(f_X, \geq)$**.** To solve the above decision problem, we consider the function

$$g : \Delta_2 \to \mathbf{M}_3 : g(t_1, t_2) = (t_1, t_2, 1 - t_1 - t_2),$$

g is an affine (bijective) parametrization of the canonical 2-simplex $\mathbf{M}_3$ (of the 3-space $\mathbf{R}^3$) by means of the canonical 2-simplex Δ_2 of the plane $\mathbf{R}^2$ (convex envelope of the canonical basis of the plane and its origin). So that, if

$$f : \mathbf{M}_3 \to \mathbf{R}^2$$

is a vector function defined on $\mathbf{M}_3$, the composite function

$$f \circ g : \Delta_2 \to \mathbf{R}^2$$

is a parametric section of the function f upon the constraint $\mathbf{M}_3$, and moreover, we have

$$(f \circ g)(\Delta_2) = f(g(\Delta_2)) = f(\mathbf{M}_3).$$

Hence, in order to obtain the payoff space of the decision problem $(f, \geq)$, we can apply the Topological Boundary theorem to the section $f \circ g$.

The applicative example. In our specific problem, we shall consider the function

$$h : \Delta_2 \to \mathbf{R}^2$$

defined by

$$h(t) = (X_0(t), X_1(t)) = \\ = (W_1(t_1) + W_2(t_2), W_1(t_1) - F_1(t_1) + W_2(t_2)).$$

And, for the extended model, we shall consider the function

$$h : \Delta_2 \to \mathbf{R}^2,$$

defined by

$$h(t) = (X_0(t), X_1(t)) = \\ = (W_1(t) + W_2(t), W_1(t) - F_1(t) + W_2(t)),$$

for every pair t in the 2-simplex Δ_2.

5. A case study

We face an Ehrlich generalized decision problem, a case study in which an individual has two activities:

1) a legal one (the first one)
2) and an illegal one (the second one)

and in which the conditions are "quite realistic", in the sense that:

- the activities of the individual are in some way connected;
- it is impossible for the decision-maker to avoid the control of the organized crime and consequently he cannot abandon his illegal activity without a "punishment" of this organized crime;
- the Justice takes account of the legal activity when it punishes the decision maker for the illegal one.

5.1 Returns and penalty function

We shall consider the (payoff) function $f : \Delta_2 \to \mathbf{R}^2$, from the canonical 2-simplex of the plane $\mathbf{R}^2$ into $\mathbf{R}^2$, defined by

$$f(t) = (f_1(t), f_2(t)) := (X_1(t), X_0(t)) = (W_1(t) + W_2(t), W_1(t) + W_2(t) - F_2(t)),$$

for every time bi-strategy $t = (x, y)$ in the triangle Δ_2.

Interpretation. If the real pair $t = (x, y)$ is a possible time-strategy of the decision-maker, then:

- the first component x is the time devoted to the legal activity 1;
- the second component y is the time devoted to the illegal activity 2;
- the function W_1 is the return function corresponding with the legal activity 1, and defined by

  $$W_1 (x, y) = \tfrac{1}{4} x^2 - y,$$

  for each pair (x,y) in the triangle Δ_2; note that the time y devoted to the illegal activity affects negatively on W_1.

- The function W_2 is the return function corresponding with the illegal activity 2, defined by

$$W_2(x, y) = xy - \frac{1}{4}x^2 + y + 2,$$

for each pair (x,y) in Δ_2; the time x devoted to the legal activity intervenes negatively on W_2, but a positive amount xy of returns - due to the positive indirect effect of the legal activity 1.

- the function F_2 is the penalty function for the illegal activity 2, defined by

$$F_2(x, y) = 2 + xy - x^2 + \frac{1}{4}y^2,$$

for each pair (x,y) in the 2-simplex Δ_2; the time x devoted to the legal activity reduces the punishment, but the law consider negative the interaction xy between legal and illegal activity so that the term xy is added to the punishment.

5.2 Total Return functions

For what concerns the first component f_1 of the payoff function f, representing the payoff of the decision-maker when he is totally punished, we obtain

$$\begin{aligned} f_1(t) &= X_1(t) = \\ &= W_1(t) + W_2(t) - F_2(t) = \\ &= W_1(x, y) + W_2(x, y) - F_2(x, y) = \\ &= (\tfrac{1}{4}x^2 - y) + (xy - \tfrac{1}{4}x^2 + y + 2) - (2 + xy - x^2 + \tfrac{1}{4}y^2) = \\ &= \tfrac{1}{4}x^2 - \tfrac{1}{4}y^2, \end{aligned}$$

for every (x, y) in Δ_2.

On the other hand, for what concerns the unpunished case, the component f_2 is defined by

$$\begin{aligned} f_2(t) &= X_0(t) = \\ &= W_1(x, y) + W_2(x, y) \\ &= (\tfrac{1}{4}x^2 - y) + (xy - \tfrac{1}{4}x^2 + y + 2) = \\ &= xy + 2, \end{aligned}$$

for every (x, y) in Δ_2.

Remark. Observe that the functions W_2 and F_2 are decreasing with respect to the first argument on the portion T_0, of the simplex Δ_2, formed by the points (x, y) such that $y < 2x$. For what concerns F_2 the interpretation is clear: when the illegal time y is greater than 2 times the legal one, then the punishment is increasing also with respect to the first (legal) argument. On the other hand, on the same subset T_0 the return of illegal activity is increasing also with respect to the first argument (for example, the legal activity supports the illegal one, when the latter is sufficiently organized).

6. Determination of payoff space

This section will be entirely devoted to the determination of the topological boundary of the payoff space $f(\Delta_2)$.

6.1 Payoff function

Thus we have to examine the function f defined by

$$f(x, y) = (\tfrac{1}{4} x^2 - \tfrac{1}{4} y^2, xy + 2)$$
$$= (\tfrac{1}{4} x^2 - \tfrac{1}{4} y^2, xy) + (0,2),$$

for each pair (x, y) of the Euclidean plane, upon the strategy constraint Δ_2.

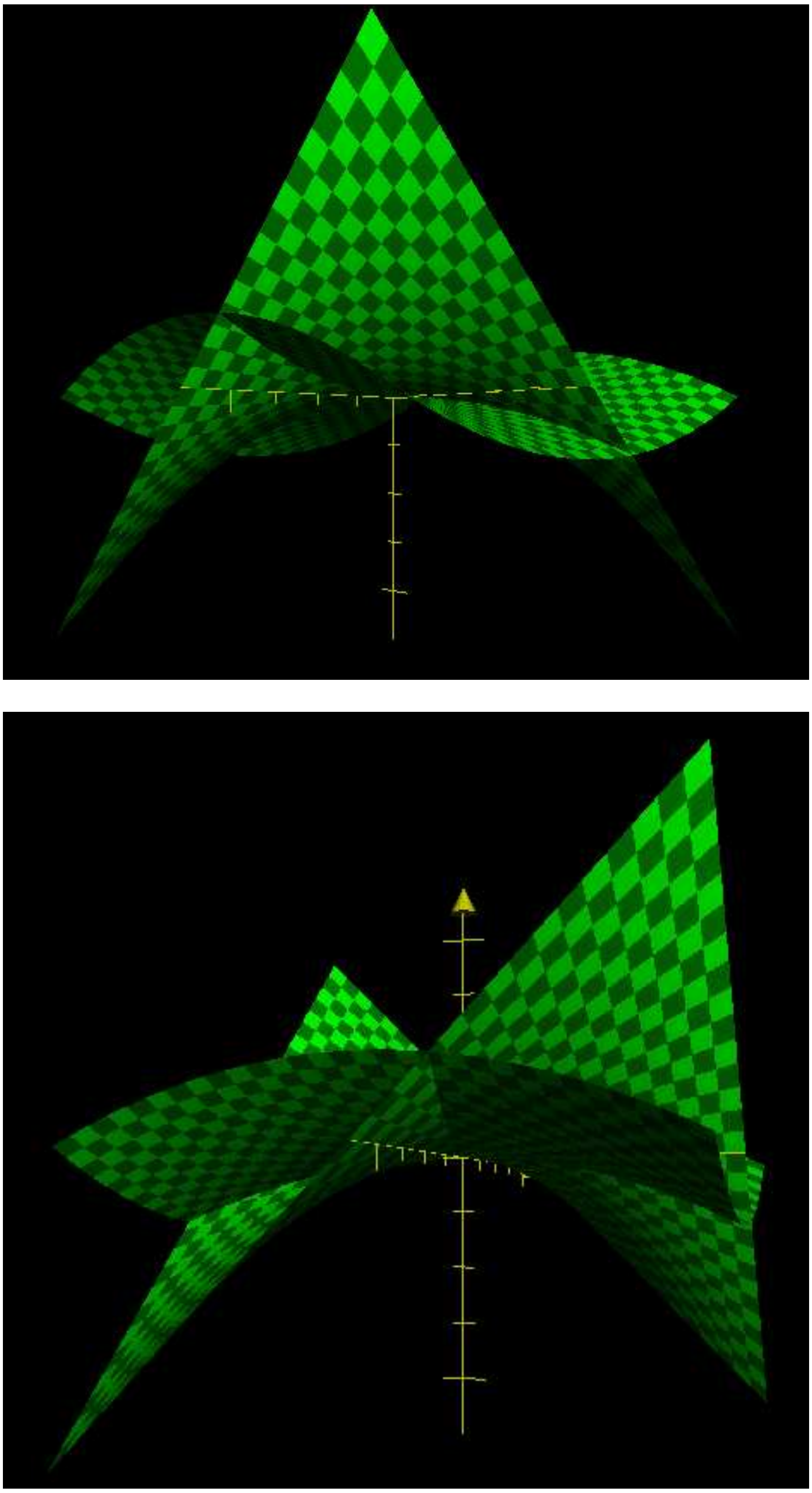

Figure 0. 3D graphical representation of the vector function *f* – (0,2).

6.2 Transformation of the strategy space

First of all we are interested in the determination of the image of the strategic triangle Δ_2. Let 0, e_1 and e_2 be the origin, the first canonical vector and the second canonical vector of the Cartesian strategy plane, respectively. Applying Carfi's methodology, it is sufficient to study the transformation of the boundary of Δ_2 and of the critical zone of f on Δ_2.

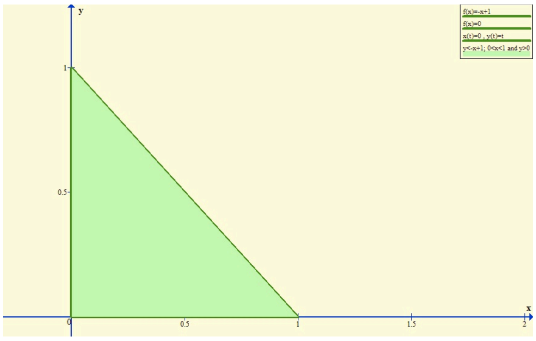

Figure 1. Bi-strategy space, that is, the canonical 2-simplex of the plane.

6.2.1 Transformation of the segment [0, e_1]

We start from the image of the segment [0, e_1], i.e. the image f ([0, e_1]). The segment [0, e_1] is the set of points (x, y) of the Euclidean plane such that the coordinate y equals 0 and the abscissa x lies in the compact interval E := [0, 1]. The value of the payoff function at the generic point (x, 0) of the segment [0, e_1] is the payoff

$$f(x, 0) = (\tfrac{1}{4} x^2, 2),$$

for each time x in the strategy space E.

Note that, for all x in E, the first component x^2 belongs to [0, 1]. Thus the image of the segment [0, e_1] is the segment of the payoff universe with end points (0, 2) and (¼, 2).

In the following figure 2 we show the above transformation. Note that the transformation of the segment [0,e_1] is contained in the payoff universe and not in the strategic universe.

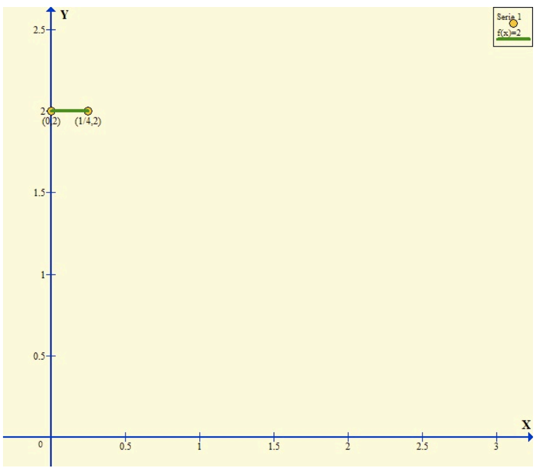

Figure 2. Transformation of the segment [0, e_1] : f([0, e_1]).

6.2.2 Transformation of the segment $[0, e_2]$

The segment $[0, e_2]$ is the set of bi-strategies (x, y) such that the abscissa x is 0 and the second coordinate y lies in the interval $[0, 1]$. The image of the generic point $(0, y)$ of this segment is the payoff

$$f(0, y) = (-\tfrac{1}{4} y^2, 2),$$

for each y in $F := [0, 1]$. The set of all the above transformation is the set of all payoff-points (X, Y), of the payoff universe, with

$$Y = 2$$

and X belonging to the interval $[-\tfrac{1}{4}, 0]$. So the image is the segment with end points $(0, 2)$ and $(-\tfrac{1}{4}, 2)$. As it is shown in the following figure.

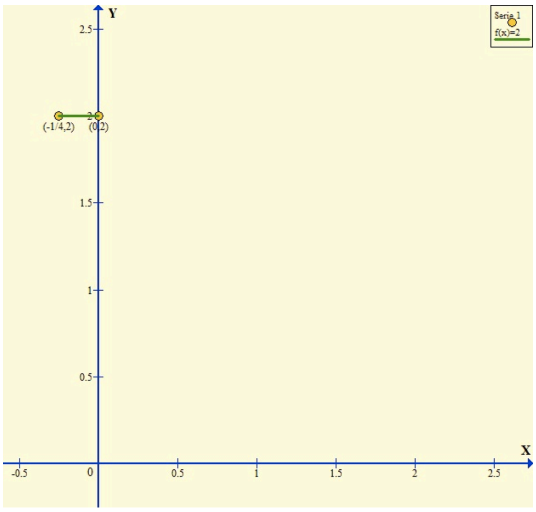

Figure 3. Transformation of the segment $[0, e_2]$: $f([0, e_2])$

6.2.3 Transformation of the segment $[e_1, e_2]$

The segment $[e_1, e_2]$ is the set of all points (x, y) of the plane $\mathbf{R}^2$ such that

$$x + y = 1$$

and whose abscissas x lie in unit interval $[0, 1]$, that is the canonical 1-simplex $\mathbf{M}_2$. The image of the generic bi-strategy (x, y) of this simplex is the payoff

$$f(x, 1-x) = (\tfrac{1}{4} x^2 - \tfrac{1}{4}(1-x)^2, x(1-x) + 2).$$

From here, we get

$$f(x, 1-x) = (\tfrac{1}{2} x, x - x^2) + (-\tfrac{1}{4}, 2).$$

The vector $(-\tfrac{1}{4}, 2)$ represents a translation vector, which we can consider after the study of the non-linear part

$$h(x, 1-x) = (\tfrac{1}{2}x, x - x^2).$$

Let us put

$$X := \tfrac{1}{2}x \text{ and } Y := x - x^2.$$

We deduce that the transformation of the 1-simplex $\mathbf{M}_2$, by the function h, is the set of all points (X, Y) such that

$$Y = -4X^2 + 2X,$$

and X lies in the compact interval $[0, \tfrac{1}{2}] = \tfrac{1}{2}\mathbf{U}$. To trace the graph of this parabolic segment, we note that the vertex is the point $(\tfrac{1}{4}, \tfrac{1}{4})$. Then, the image of our canonical 1-simplex is the set of all payoffs (X, Y) such that

$$Y = 2X - (2X)^2,$$

where X belongs to the interval $[0, \tfrac{1}{2}]$. So the image is the parabolic segment shown in the below figure.

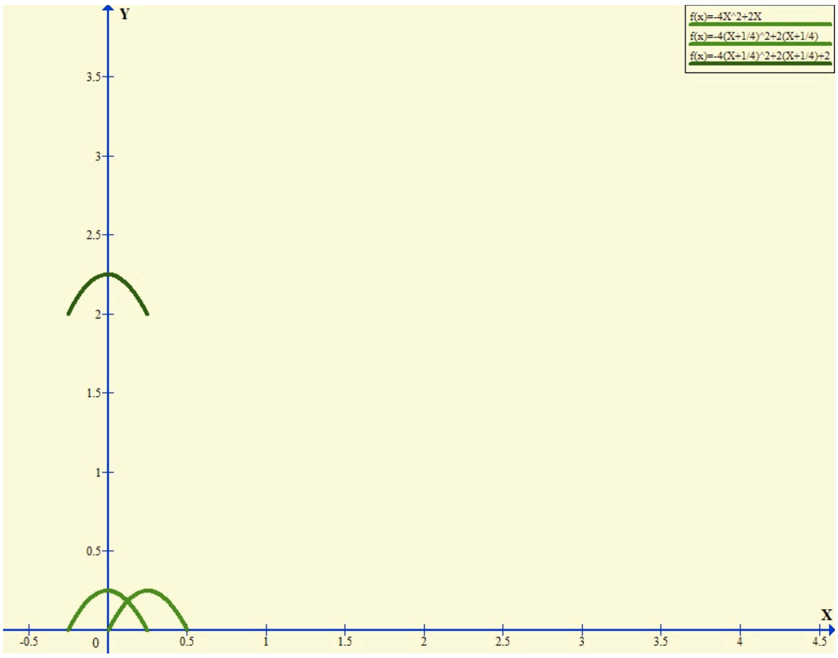

Figure 4. Transformation of the segment $[e_1, e_2]$

6.3 Critical zone

The critical zone of a differentiable game G = (f, >) is the set of all bi-strategies at which the Jacobian determinant of f is 0.

The Jacobian matrix of the function f, at any point (x, y) of the bistrategy space Δ_2, is denoted by $J_f(x, y)$ and it is the matrix having as rows the gradients of the gain functions f_1 and f_2, respectively; the two gradients are defined respectively by

$$\text{grad } f_1(x, y) = (\tfrac{1}{2}x, -\tfrac{1}{2}y),$$

and

$$\text{grad } f_2(x, y) = (y, x),$$

for every bistrategy (x, y) in Δ_2. The Jacobian determinant, at the bistrategy (x, y), is

$$\det J_f(x, y) = \tfrac{1}{2} x^2 + \tfrac{1}{2} y^2,$$

for every pair (x, y) in the bistrategy space Δ_2.

6.4 Critical space

The critical zone is the subset of the bistrategy space Δ_2 of those bistrategies (x, y) at which the Jacobian matrix is not invertible, that is verifying the relation

$$\det J_f(x, y) = 0,$$

i.e., in our case, $x = y = 0$.

In symbols, the critical zone is the singleton

$$C_f = \{(0, 0)\}.$$

6.5 Determination of payoff space of decision problem

Gathering all the preceding information we obtain our payoff space. The image $f([0,1]^2)$ is the set of payoffs (X, Y) such that

$$Y \le -4(X + 1/4)^2 + 2(X + 1/4) + 2$$

with $X \ge -1/4$ and $Y \ge 2$. As it is shown in the following figure.

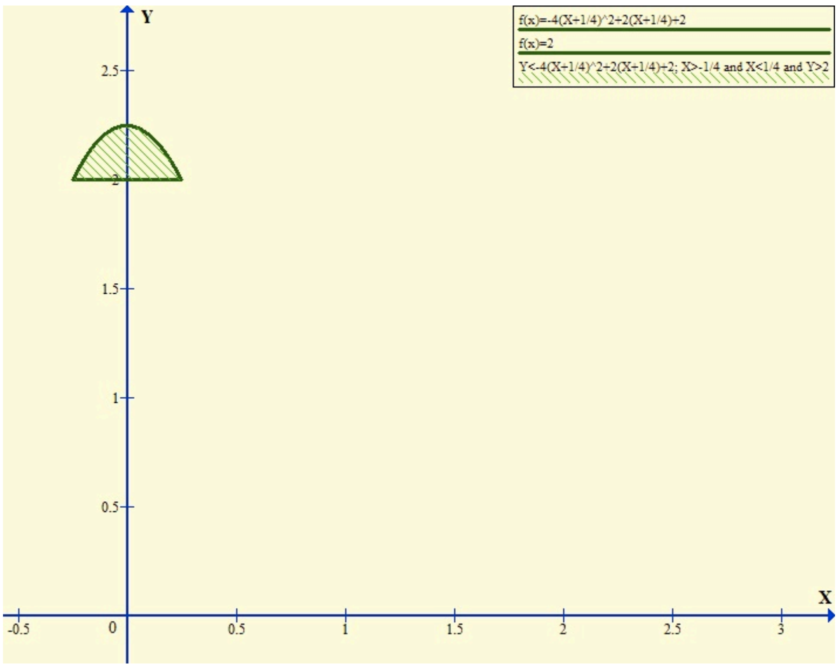

Figure 5. Payoff space of the decision problem.

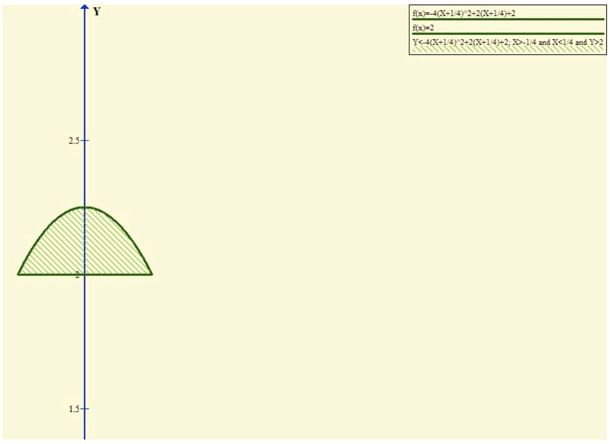

Figure 6. Payoff space of the decision problem (zoom).

7. Solution on the equiprobability assumption for the space S

Now, we are interested in the determination of the optimal solution, when it is known that the states of the world 0 and 1 are equiprobable. In the present case, the optimal solution can be obtained by the maximization of the utility average function

$$\mathbf{E}_{(1/2,\,1/2)} : \mathbf{R}^2 \to \mathbf{R} : (X,Y) \to \tfrac{1}{2} X + \tfrac{1}{2} Y,$$

upon our payoff space $f(\Delta_2)$. Moreover, the unique solution can be obtained also by maximizing the cumulative function g, defined by

$$g(X, Y) = X + Y,$$

upon the payoff constraint $h(\mathbf{M}_2)$ with equation

$$Y = 2X - (2X)^2,$$

where the coordinate X is in the canonical unit interval $\tfrac{1}{2}\mathbf{U}$. That solution belongs to the Pareto maximal boundary of our payoff space $f(\Delta_2)$, translated by the vector $(1/4, -2)$, then, to find the solution upon the payoff space $f(\Delta_2)$, it is sufficient to translate this solution by means of the vector $(-1/4, 2)$.

A section of the function g upon the constraint $h(\mathbf{M}_2)$ is the function

$$s : \tfrac{1}{2}\mathbf{U} \to \mathbf{R}$$

defined by

$$s(X) = g(X, 2X - (2X)^2) = X + 2X - (2X)^2,$$

for every X in $\tfrac{1}{2}\mathbf{U}$. We have

$$s'(X) = 1 + 2 - 8X > 0,$$

which is positive when

$$3 - 8X > 0$$

that is $X < 3/8$. So the optimal solution in the interval $\tfrac{1}{2}\mathbf{U}$ is just the point $3/8$, which determines the optimal solution $(3/8, 3/16)$ for the function g and finally the optimal payoff

$$(3/8, 3/16) + (-\tfrac{1}{4}, 2) = (1/8, 35/16),$$

of the space $f(\Delta_2)$. In order to obtain the corresponding bistrategy (x, y) in the canonical simplex $\mathbf{M}_2$, we recall that

$$h(x, 1-x) = (\tfrac{1}{2} x, x - x^2) = (3/8, 3/16),$$

and thus the corresponding bi-strategy is the pair $(3/4, \tfrac{1}{4})$.

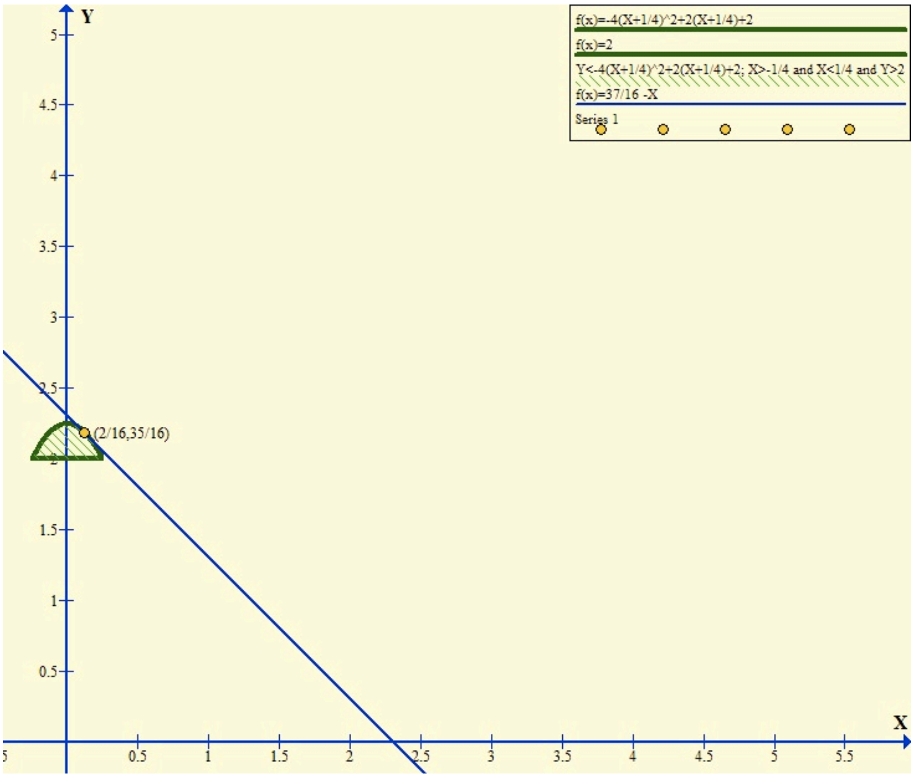

Figure 7. Equiprobability solution.

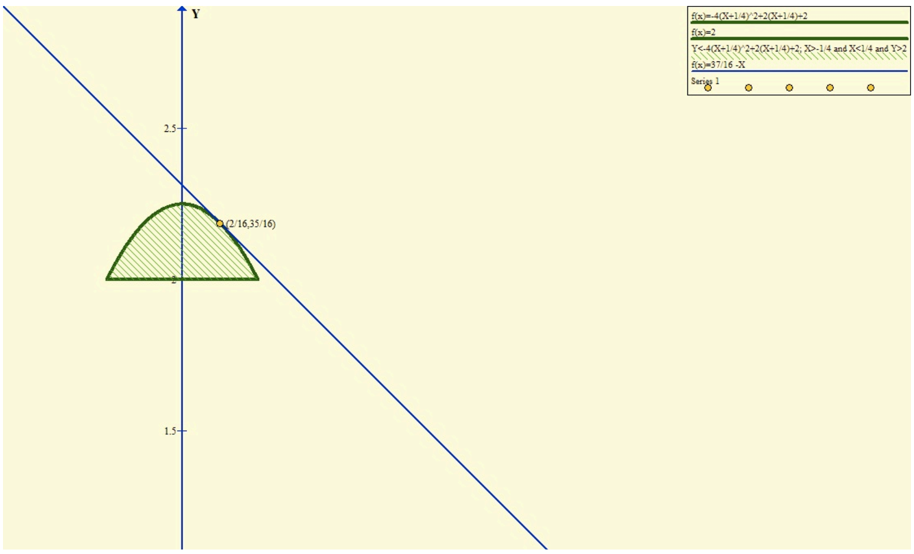

Figure 8. Equiprobability solution (zoom).

8. Solutions on extreme probability assumption for the space S

Starting from the average function $IE_{(p, 1-p)}: IR^2 \rightarrow IR$, which we shall denote here by g, we have

$$g(X, Y) = pX + (1 - p)Y,$$

for every payoff pair (X,Y) in IR^2.

We analyze two extreme cases:

- the case in which the probability of punishment p is maximum, that is p = 1;
- the case where the probability of punishment p equals 0.

We analyze the first case, in which the offender is most likely (*certainly*) to be punished; then the function defined by

$$g(X, Y) = 0X + 1Y,$$

for every payoff pair (X,Y) in IR^2.

Interpretation and remark. We note that the level-sets of the function g will be straight lines parallel to the axis of abscissas (horizontal lines). This level-sets will go to intersect the maximal Pareto boundary parabola at the point (0, 2 + 1/4). In this case, it would be appropriate to devote time equally to the legal and illegal activity (indeed the reciprocal image of our (0, 2+1/4) is (1/2,1/2)), so contrary to the common sense, our decision-maker, as he tends to get the maximum profit possible, will not devote his time only to the legal activity (even if he knows will be certainly punished): he will split his time equally between legal and illegal activities, despite maximal probability of punishment.

In the opposite case the probability of punishment is equal to 0, therefore we obtain the objective function

$$g(X, Y) = 1X + 0Y,$$

whose level sets are vertical straight lines that will go to intersect optimally the payoff constraint at the point (1/4, 2) and the reciprocal image of the point (1/4,2) is the bi-strategy (1,0).

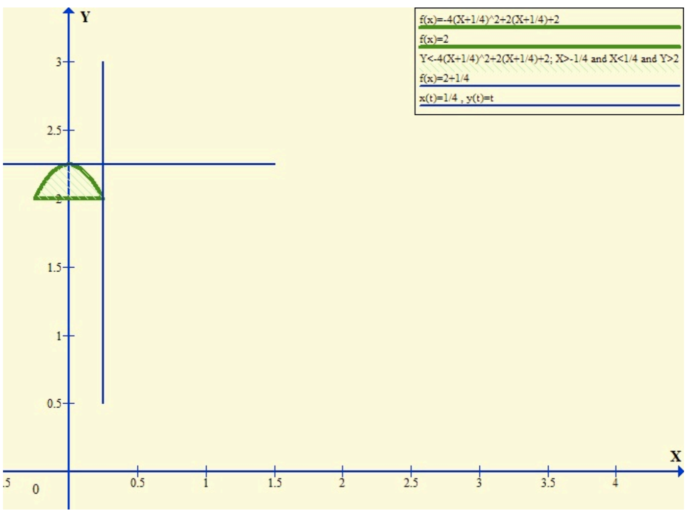

Figure 9. Extreme solutions

7. Conservation and compromises

7.1 Worst punished gain function with respect to illegal activity

By definition, the *worst punished gain function* $f_1^{\#}$ is defined on the legal strategy space E = **U** by

$$f_1^{\#}(x) = \inf_U f_1(x, .),$$

for every legal strategy x in E. The payoff $f_1^{\#}(x)$ is the worst possible payoff of the decision-maker, when it is punished.

Remark. An individual, comparing the gains that he could get by adopting legitimate or illegitimate activities, will notice that a lot of times the profits arising from the illegal ones will be great in comparison to those of legitimate activities. Therefore, he would be brought to infringe. However, he also has to keep in mind of the possible punishments deriving from behaviors against the law. In fact, we can observe three cases:

- the individual devotes his time to illegal activity and he is not punished;
- the individual devotes his time to illegal activity and he is punished;
- the individual devotes his time to legal activity (and paradoxically) he is punished.

You will get:

$$\sup_{x \in E} \inf_{y \in E} f_1(x, y) = v_1^{\#}$$

when the individual has the tendency to maximize in operation some legal activity and to minimize that illegal, since he is afraid of the consequential punishment from the second or he hears again of a scruple of conscience in to break the law.

Instead, in the opposite case:

$$\sup_{y \in F} \inf_{x \in E} f_2(x, y) = v_2^{\#}.$$

The individual will have the tendency to maximize in function of illegal activity and to minimize for that legal; in this case we find us of forehead to an individual that doesn't fear the punishment that could derive from the infringement, rather he could almost say that we find us of forehead to a "inveterate criminal", that has as primary objective that to maximize her profits.

Remark. Further situation, possible but few probable, would be had in the case in which the individual decided to maximize both in comparison to the illegal and legal activity, in other words he would extend to get the maximum possible from both the activities.

From her max-min precedent we can build the conservative bi-value that furnishes us, in this context, a sort of initial status from which to get further for improving his own profits, in fact he could be taken as value of threat in the problems of selection on the Pareto boundaries

$$v^{\#} = (v_1^{\#}, v_2^{\#}).$$

In the case of our study:

$$\begin{aligned} v_1^{\#} &= \tfrac{1}{4} \sup_{x \in [0,1]} \inf_{y \in [0,1]} (x^2 - y^2) = \\ &= \tfrac{1}{4} \sup_{x \in [0,1]} (x^2 - 1) = \\ &= \tfrac{1}{4} (1 - 1) = \end{aligned}$$

$= 0$

and

$v_2^{\#} = \sup_{y \in [0,1]} \inf_{x \in [0,1]} (xy + 2) =$
$= \sup_{y \in [0,1]} (0 + 2) =$
$= 2.$

So that, the conservative bi-value is the pair (0,2).

7.2 Elementary best compromise between punishment-unpunishement

The elementary best compromise bi-gain is the intersection of the segment joining the threat bi-gain $v^{\#}$ with the supremum of the game with the Pareto boundary of the payoff space of the game $f(\Delta_2)$. We calculate this solution on the first parable

$$f(\mathbf{M}_2) + (-2, ¼);$$

this latter point satisfies the system:

$$Y = -4X^2 + 2X \text{ and } Y = X - 1/4$$

leading to the resolvent equation:

$$16X^2 - 4X - 1 = 0,$$

whose acceptable solution is $A = (1 + 5^{1/2})/8$ with corresponding payoff $B = A - 1/4$. So that the Kalai-Smorodinsky payoff solution of our decision problem is the translation of the above pair (A, B) by the vector $(-1/4, 2)$:

$$K' := f(K) = ((1 + 5^{1/2})/ 8, (1 + 5^{1/2})/ 8 - ¼) + (-1/4, 2),$$

the point K is our best compromise in the bi-strategy space, easily determined by retro-image by f.

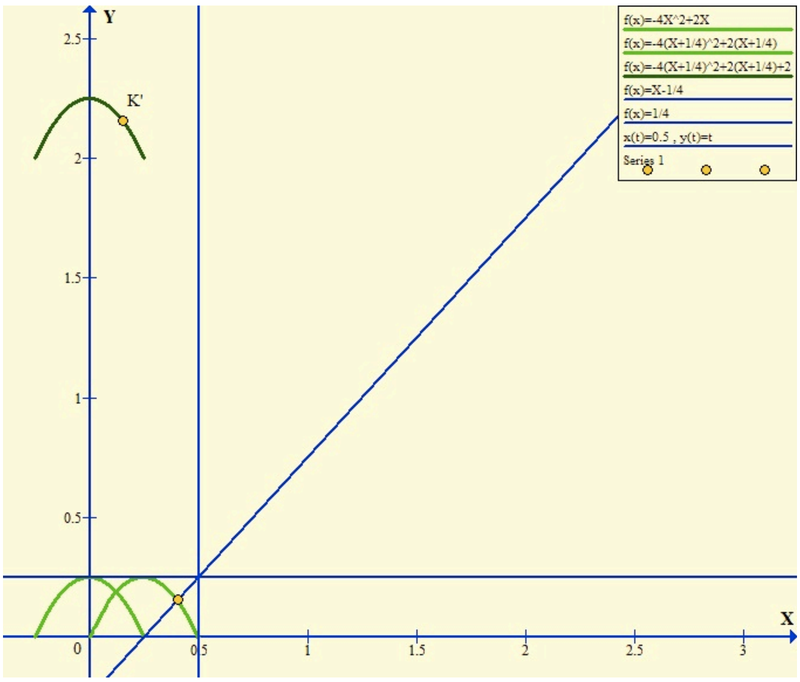

Figure 10. Best compromise solution.

8. Pseudo-Equilibria

8.1. Virtual Nash equilibrium

In our game, the Nash equilibria do not play a major role as we have not precisely a standard game, because of the shape of the strategy space, and moreover, even if we consider the smallest possible game containing our payoff function, it does not fall within our space of strategies; as we shall show below. Let us run out the calculations: we have

$$f(x, y) = ((1/4)(x^2 - y^2), xy + 2).$$

We get:

$$\partial_1 f_1(x, y) = (1/2)\, x > 0$$

within the range [0, 1], showing a increasing trend of f_1. Hence the best reply correspondence of the first virtual player is defined by $B_1(y) = 1$, for any y in F. For the second virtual player, we have two cases; indeed, for any fixed strategy x in E = [0,1], we have $\partial_2 f_2(x, y) = x$, for every y in F = [0,1]; so we have two relevant and exhaustive possibilities: $\partial_2 f_2(x, y) = x > 0$, when we have an increasing trend of the partial gain function $f_2(x, .)$; or $\partial_2 f_2(x, y) = 0$, when we have a constant partial gain function $f_2(x, .)$. Hence, the best reply correspondence of the second virtual player is defined by $B_2(x) = 1$, if $x > 0$, and $B_2(x) = F$, if $x = 0$. In the following figure, we'll show the graph of the multi-function B_2 and the inverse graph of the correspondence B_1 with their intersection: the Nash equilibrium (1,1).

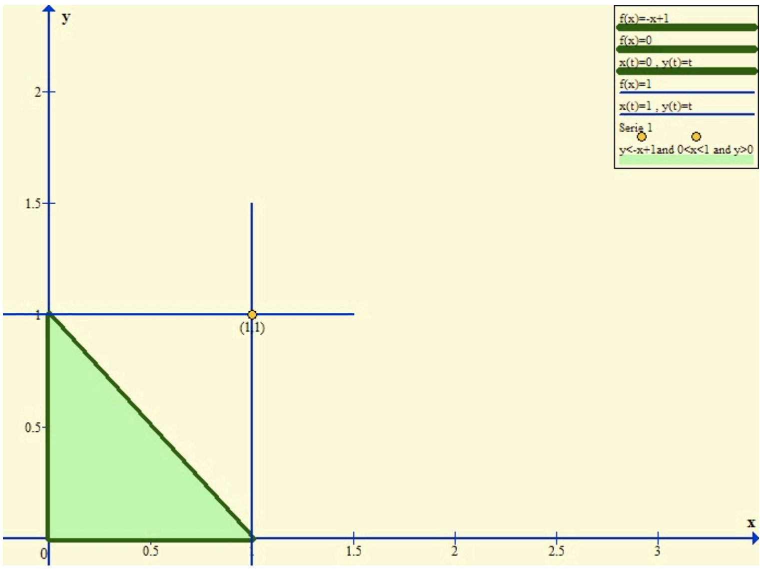

Figure 11. Generalizied Nash equilibrium (Nash equilibrium of the minimal extended game).

8. 2 Generalized Nash solution

However, refining the analysis, i.e. taking into account not the entire strategic area, but the square subspaces with the infimum-vertex at the origin, upon each of them we have a Nash equilibrium. The continuous sequence of Nash equilibria forms a so called *Nash path*. The Nash equilibrium corresponding to the maximum square contained into our strategic triangle falls within our strategy space itself, going to be the best compromise between legal and illegal activity. This extreme equilibrium is what we call the *generalized Nash solution* of our problem.

The below figure shows the Nash path (in yellow) and the generalized Nash solution (½, ½).

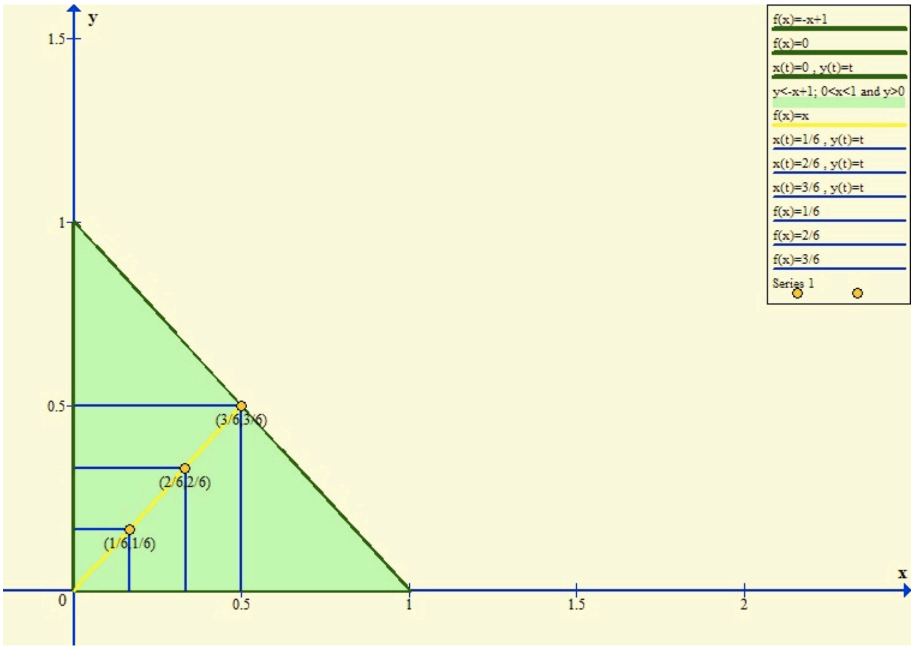

Figure 12. Generalized Nash path solution.

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