Coordination failure cycle

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December 2005

COORDINATION FAILURE CYCLE

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March, 2012

Abstract

This paper proposes a theory of endogenous fluctuations, grounded on a repeated game with strategic complementarity under incomplete information. The equilibrium is characterized by a persistent regime of high activity, where aggregate output tends to expand, followed by a persistent contractionary phase in a recurring cycle. The regime persistence is driven by belief hysteresis, where learning in active regime fuels optimism, propelling an expansion. After an inevitable regime switch, rational persistent pessimism ensues, leading to a prolonged contraction. The equilibrium cycle is unique, stochastic, and converges to a stationary distribution, which characterizes the nature of fluctuations in equilibrium.

Keywords: endogenous cycle, coordination game, learning, global games, hysteresis

JEL Classification: C73, D83, E32

*E-mail: prungcharoenkitkul@imf.org and phurichr@bot.or.th. This paper is based on a chapter of my D.Phil thesis. I am grateful to Kevin Roberts, Godfrey Keller, and Nicolas Melissas for their helpful comments. All errors are mine.

1 Introduction

The dominant macroeconomic paradigm views aggregate fluctuations as a propagation of exogenous aggregate shocks to fundamentals, but falls short of explaining 'how' business cycles arise in the first place. A competing theory may be found in models with multiple rational-expectations equilibria, where changes in expectations can lead to equilibrium shift in a self-fulfilling way, laying the basis for fluctuations unrelated to exogenous shocks.[1] However, shifts in expectations still remain unexplained under this approach, leaving multiple equilibria at best a caricature of fluctuations. Moreover, subsequent research has found the existence of multiplicity to be sensitive to small perturbation to payoffs and information structure, which can serve as an equilibrium selection device.[2] When a unique equilibrium is selected by such device, strategic complementarity typically ends up playing only a shock propagating role in increasing aggregate volatility, and again there is no room for fluctuations to emerge endogenously.[3]

This paper proposes a theory of endogenous fluctuations where strategic complementarity plays a central role. The model envisages an economy of large population, whose collective action determines the dynamics of the economy: if a large enough fraction of agents is active, the aggregate output is likely to expand, otherwise fall. An expansion benefits everyone, but particularly those who have been active thus introducing strategic complementarity. With heterogeneous costs of action and incomplete information about others' costs, each agent must form belief and continuously update her belief as the economy evolves. Type heterogeneity and incomplete information enable a unique strategic equilibrium to be selected, and the equilibrium dynamics is uniquely determined at any given time.

The equilibrium is characterized by a persistent regime of high activity, where the aggregate output keeps rising until some threshold is reached, after which the economy enters a persistent contractionary phase, a regime of low activity. Fluctuations arise as the economy endogenously and perpetually cycles between the two regimes. Regimes are persistent because of belief hysteresis: the event of regime being high tomorrow commands a higher posterior probability if a high regime is observed today, as agents learn that fundamentals must be good enough to justify today's expansion. Similarly, in a

[1] Cooper and John (1988) provide a generalization of the role of strategic complementarity and multiple equilibria in macroeconomic context, the idea of which dated back to at least Goodwin (1951) and Keynes' discussion of animal spirit.

[2] Carlsson and van Damme (1993) are a seminal work on equilibrium selection in a coordination game via an introduction of incomplete information about the game payoffs. The approach proves useful in many macroeconomic applications, and has been extensively explored and generalized. See Morris and Shin (2003).

[3] This volatility view of strategic complementarity is generally held by most recent works; for example Angeletos and Pavan (2007) investigate the sensitivity of unique equilibrium to public information. Angeletos and La'O (2012) discuss the propagation of shocks to higher-order beliefs, giving rise to extrinsic-shocks driven fluctuations.

contractionary regime, belief in a low activity equilibrium is reinforced by past observation of low regime. This belief hysteresis results in a persistent play in the selected equilibrium, where an attack on the existing regime never occurs until the expansion has continued for an overly extended period. In the limit, every agent with a nontrivial strategic problem chooses the same action, driving the expansion in a high regime, and the contraction in a low regime.

A number of antecedents are related to this paper. In a seminal work on regime shifts, Chamley (1999) studies a repeated coordination game where a unique equilibrium is also selected and characterized by most players choosing the same action, with occasional regime switches where the players simultaneously switch to the opposite action. The model's dynamics is driven by an aggregate shock, and regime switches occur when this exogenous shock crosses some thresholds. Chamley's model therefore views strategic complementarity as a propagation mechanism. In contrast, our model produces endogenous cycles in equilibrium, and a qualitatively different dynamic outcome. A key source of departure is that in our model it is the dynamics of aggregate activity that is subject to strategic complementarity in agents' actions. The aggregate dynamic process has an inherent stabilizing mechanism, as the game payoffs prohibit the aggregate variable from following an explosive path. However, the endogenous dynamics introduces an additional dimension to agents' learning problem, allowing agents to test their belief about the fundamentals as the state evolves. Under rational learning, this belief moves only with inertia, leading to a lack of steady state. In a high regime, a successively higher output is interpreted as reaffirming the fundamentals to be favorable, encourages agents to continue being active and pushes output higher still, even if the system may in fact edge nearer to its inevitable crash. The reinforcing interactions between learning from past history and the higher-order beliefs allow the effect of regime persistence to dominate, in agents' calculation, the possibility of an abrupt regime change. The failure to coordinate an early regime switch implies that an expansion can continue prolongedly, forcing a crash only late in a cycle, and similarly for recovery. An endogenous cycle emerges as a result, even if the fundamentals in fact never change.

Our model's learning dynamics generate a *delay* in a strategy switch, a property highlighted in many existing works. In the model of bubble in Abreu and Brunnermeier (2003), rational players find it optimal to ride a bubble for a while, even if they know it must eventually burst, because the incentives to delay selling and time the market make it difficult to synchronize an attack on the bubble until late. In our model, strategic delay to switching actions is also partly incentivized by a short-term gain to production contribution, but crucially the extent of this delay is reinforced by belief hysteresis every time there is an expansion without triggering a crash. The sustained belief hysteresis leads to an outcome of maximal delay, and provides the main impetus in driving the cycle.[4] In

[4]Belief hysteresis also exists in Chamley (1999) immediately after a regime switch, since agents know

Caplin and Leahy (1994) and Chamley and Gale (1994), strategic delay results from the unobservability of other private signals, with an abrupt macro adjustment taking place only when private signals are strong enough for a sufficiently large number of individuals to act, which then causes an information cascade. A form of information cascade also features in our model, as a publicly observed regime switch reveals information about private signals to all players, although in the presence of already maximal delay there is no abrupt change in agents' optimal strategy after such a switch.

Cycle theory has a long history in economics, but the paradigm is largely regarded as the periphery rather than the core of macroeconomics.[5] Firstly, the often deterministic nature of the cycle model gives an excessively sharp prediction, casting doubt on its ability to completely capture phenomenon as complex as the business cycle. Secondly, it is unclear how the model may be empirically tested or fitted to the data. Finally, in many models of this type, cycle is but one possible outcome coexisting along side many others including a steady date, and is therefore far from being a generic description of fluctuations.

Our model's equilibrium cycle is free of these criticisms. The equilibrium is uniquely selected, predicting fluctuations as a generic phenomenon. At the same time, the equilibrium cycle is modeled as a Markov process with asymmetric transition probabilities on a closed loop, making the cycle noisy rather than deterministic. The equilibrium outcome converges to a stationary distribution, which, we propose, is a natural equilibrium concept for a model of recurring fluctuations. The equilibrium distribution allows quantitative assessments of welfare and open doors for empirical investigation. In terms of policy implications, we argue that 'thinking cyclically' in this way yields new insights that are otherwise not available, while consistent with existing policy wisdom.

The model is set out in Section 2. Section 3 discusses the model's basic properties and the belief evolution dynamics. The key result of hysteresis and recurring regime switching is derived in Section 4, which implies the existence of cycle. Section 5 provides the conditions ensuring the equilibrium uniqueness. Section 6 explores the quantitative welfare and policy implications of the model, using a simulation exercise, before Section 7 concludes.

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a regime switch cannot be immediately followed by another switch. However, this hysteresis effect wanes over time, as the information about the fundamentals become increasingly diffuse in the absence of a regime switch. In our model, agents gain information from the evolution of aggregate variable, which works to sustain the existing regime.

[5]The cycle approach counts among the early contributors such names as Richard Goodwin, John Hicks and Nicholas Kaldor. Influential studies on financial crises such as Kindleberger and Aliber (2005) and Minsky (2008) also describe crisis developments as following a recurring cycle pattern. Boldrin and Woodford (1990) survey the cycle literature and discuss why the approach seems to have fallen out of fashion.

2 The Model

2.1 Production technology

Time t is discrete. There is a large population of size normalised to n, and the population set is given by $[z_t, z_t + n]$, where z_t is a stochastic process. There is a single consumption good, the stock of which is measured in discrete units $y_t \in Y = \{0, 1, ..., N\}$. At the beginning of period t, agent $i \in [z_t, z_t + n]$ decides whether to contribute to the production of y_t, by choosing action $x_{it} \in \{0, 1\}$ denoting inactivity and investment respectively. The evolution of y_t is determined by joint investment efforts of all agents, $\int_{i\in[z_t,z_t+n]} x_{it}di$, and is modelled as a discrete-time birth-and-death process on Y. Specifically, the transition probabilities for y_t are given by

$$\Pr\left(y_{t+1} = y \middle| y_t, \int_{i\in[z_t,z_t+n]} x_{it}di\right) = \begin{cases} b_t & \text{for } y = y_t + 1, \\ \delta & \text{for } y = y_t, \\ 1 - b_t - \delta & \text{for } y = y_t - 1, \end{cases}$$

where $\delta > 0$ is a constant, and b_t is an increasing step function

$$b_t = \begin{cases} b_h & \text{for } \int_{i\in[z_t,z_t+n]} x_{it}di > n - \kappa, \\ b_l & \text{for } \int_{i\in[z_t,z_t+n]} x_{it}di < n - \kappa, \end{cases} \tag{2.1}$$

where $(1 - \delta) > b_h > \frac{1-\delta}{2} > b_l > 0$, so that if the number of investing agents exceeds the threshold $n - \kappa$, then the production of an additional unit of y_t is more likely to succeed than not. It is therefore natural to think of b_h and b_l as representing the expansionary and contractionary regimes respectively (high and low regimes, in short). At boundary states N or 0, the probability of $y_{t+1} = y_t$ is δ and that of reflection is $1 - \delta$ irrespective of aggregate investment. Any tie-breaking rule for the case when $\int_{i\in[z_t,z_t+n]} x_{it}di = n - \kappa$ can be chosen without affecting the key results.

2.2 Agents' objective function

y_t is a public good providing equal utility $u(y_t)$ to all agents for free in each period t. On the other hand, contributing to the production of y_t is costly. The investment costs differ across agents, and agent $i \in [z_t, z_t + n]$ incurs an investment cost $c_i(y_t)$ if she decides to invest in period t. In return, each agent who invests at time t will earn an extra private lump-sum gain f at time $t+1$ if $y_{t+1} = y_t + 1$, i.e. if the production at time t is successful in creating an extra unit of the good. Therefore, choosing to invest amounts to betting that y_t will go up next period. The return f provides private incentive to the production of y_t.

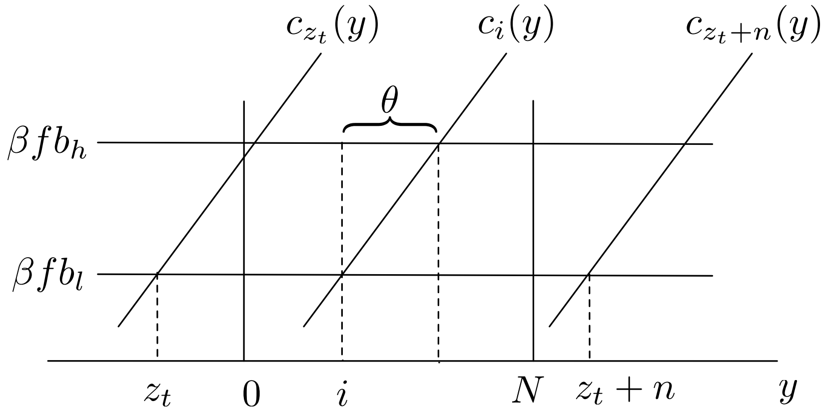


Figure 1: Cost functions $c_i\left(y\right)$

The objective function of agent i at time t is the lifetime discounted payoff

$$U_{it} = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[u\left(y_s\right) - x_{is} c_i\left(y_s\right) + f_{is}\right]$$

where β is the discount factor and

$$f_{it} = \begin{cases} f & \text{if } y_t - y_{t-1} = 1, \text{ and } x_{it-1} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Any individual agent's decision whether to invest clearly depends on the prevailing regime, which in turn depends on other agents' investment decisions. As will become clear, agents are effectively playing a coordination game.

The cost functions $c_i\left(y_t\right)$ are assumed to take the following linear form

$$c_i\left(y_t\right) = \frac{\beta f\left(b_h - b_l\right)}{\theta}\left(y_t - i\right) + \beta f b_l, \qquad i \in \left[z_t, z_t + n\right] \tag{2.2}$$

where $\theta > 0$. The identity index i serves to identify the cost ranking of agents ($i = z_t$ being the highest-cost agent). The variable z_t determines the aggregate investment cost of the population, and is therefore a measure of underlying fundamentals. Because the cost functions only depend on the difference $y_t - i$, they satisfy the quasilinearity property, i.e. any two functions are a horizontal parallel displacement of each other. The fact that $i \in \left[z_t, z_t + n\right]$ also implies a specific correlation structure between the agents' costs, amenable to an equilibrium selection mechanism that ensures a unique equilibrium always obtains. Figure 1 plots the cost functions $c_i\left(y\right)$.

2.3 Timing

The game proceeds in three recurring stages as follow:

1. At time t agents hold some identical belief about $z_t^* \equiv z_t + \kappa$, expressed as a probability density function h_t on a compact interval Z_t (h_t being zero on $\mathbb{R} \backslash Z_t$). The state y_t is observed by everyone.

2. Given such beliefs, all agents choose actions simultaneously to maximise their expected utility. When the equilibrium is unique, the resulting regime is publicly observed. Bayesian rational agents are allowed to revise their beliefs about z_t^*. The updated common belief is denoted by a density $\tilde{h}_t$ on an interval $\tilde{Z}_t$. A new state y_{t+1} is selected according to the transition probabilities defined by the regime.

3. After the regime is determined, z_{t+1} is randomly drawn from a commonly known uniform transition density $p\left(z_{t+1} \middle| z_t\right)$, where

$$p\left(z_{t+1} \middle| z_t\right) = \begin{cases} \frac{1}{2\varepsilon} \text{ if } z_{t+1} \in [z_t - \varepsilon, z_t + \varepsilon] \\ 0 \text{ otherwise.} \end{cases} \tag{2.3}$$

In other words, z_t is a random walk. A new set of beliefs about z_{t+1} is generated from this *shock transition rule* and the game returns to stage 1, continuing forever thereafter.

The parameters θ, κ, n and ε are taken to be large relative to 1, so that the dynamics of beliefs can be analysed without being subject to the constraint set by the discreteness of the state space Y. For instance, the assumption implies that $y_t - \theta \approx y_{t-1} - \theta$. Furthermore, the case of special interest is where ε is small relative to θ, thus limiting the role of exogenous shocks.

As the analysis will show, movements in z_t are *not* the proximate cause of fluctuations in y_t. The randomness of z_t keeps the posterior belief of z_t diffuse and non-degenerated, which ensures a unique equilibrium selection. Fluctuations in y_t are instead driven by persistence and hysteresis in the agents' belief, which will be shown to cycle over time. Therefore, fluctuations in this model arise from an endogenous cycle mechanism, and not an amplification of exogenous shocks.

3 Preliminary Analysis

In this section, we first show that the lifetime discounted payoff defined by the stock of goods y_t is subject to strategic complementarity, and hence our model is essentially a series of coordination games with learning over time. There would be multiple equilibria in the complete information case, leading to indeterminate dynamics of y_t for some range of initial conditions. With incomplete information about z_t, Bayesian rational agents form beliefs about the value of z_t using all available data, and under some conditions about these beliefs and their evolution, there will exist a unique equilibrium that survives iterated deletion of dominated strategies, characterized by a 'switching strategy'. We intuitively describe the stages of Bayesian learning in this environment, the resulting belief evolution, and the conditions under which a unique switching strategy equilibrium obtains. We then highlight salient features of this belief distribution, central for describing the equilibrium properties in the next section.

3.1 Strategic complementarity

Agent i uses all information available at time t, denoted $\mathcal{I}_t^i$, to form a belief about the value of b_t, expressed as a perceived probability over $\{b_l, b_h\}$. In view of this, let us define

$$\pi_t^i = \Pr\left(b_t = b_h | \mathcal{I}_t^i\right) \tag{3.1}$$

and

$$E_t^i(b_t) = \pi_t^i b_h + \left(1 - \pi_t^i\right) b_l.$$

The value of being agent i at time t, $V\left(\mathcal{I}_t^i\right)$, is given by

$$V\left(\mathcal{I}_t^i\right) = \max_{x \in \{0,1\}} \left\{ u(y_t) - x c_i(y_t) + \beta \left[E_t^i(b_t) x f + \tilde{V} \right] \right\}. \tag{3.2}$$

To understand equation 3.2, note that $u(y_t) - x c_i(y_t)$ is simply the period-t payoff conditional on investment decision x. It is convenient to use $x \in \{0, 1\}$ as an indicator function, so that the private return in period $t+1$, conditional on $y_{t+1} = y_t + 1$ and x, is simply given by xf. Thus, the expected private return in period $t+1$ as of time t conditional on x is given by $E_t^i(b_t) xf$. $\tilde{V}$ is a summation of terms involving $\pi_t^i, b_h, b_l, \delta$ and the corresponding $V\left(\mathcal{I}_{t+1}^i\right)$. When optimising, each agent takes $\tilde{V}$ as a constant, as no single agent can unilaterally affect the evolution of the aggregate stock.

It follows from the functional representation 3.2 then that investment ($x = 1$) is optimal for agent i at time t if

$$\beta f E_t^i(b_t) > c_i(y_t) \tag{3.3}$$

which can be reduced to

$$\pi_t^i = \frac{y_t - i}{\theta} \tag{3.4}$$

That is, agent i will invest at time t if she believes the probability of a high regime is sufficiently high. But this probability π_t^i is higher if and only if the underlying set of strategy profiles assigns a larger number of agents to the investment strategy, as agents know the rule determining the regime (equation 2.1). The payoff in each period is therefore subject to strategic complementarity; the return to investment is increasing in the number of opponents who are investing.

Not all agents' decisions are subject to strategic considerations; for any agent i, a dominant strategy may exist for sufficiently extreme values of y_t. In particular, $x = 1$ is a dominant strategy for agent i if $c_i(y_t) < \beta f b_l$, i.e. if $y_t - i < 0$. Similarly $x = 0$ is a dominant strategy if $c_i(y_t) > \beta f b_h$, i.e. if $y_t - i > \theta$. Therefore any agent $i \in [z_t, z_t + n]$ does not have a dominant strategy if and only if $y_t \in [i, i + \theta]$. Equivalently, for any fixed y_t, the set of agents without a dominant strategy is given by $[y_t - \theta, y_t]$.

If agent $i = z_t^* \equiv z_t + \kappa$ has a dominant strategy to invest, all other agents with lower costs also invest, and the regime is guaranteed to be high. When agent z_t^* does not have a dominant strategy, iterated elimination of dominated strategies can identify a unique switching equilibrium, under which there exists a threshold agent i^* such that all $i > i^*$ invest, while the rest remain inactive. In this equilibrium, a high regime is guaranteed if $z_t^* > i^*$, thus agent z_t^* may be referred to as the *decisive* agent.

A *delay* to a regime switch occurs when a large fraction of agents in the strategically nontrivial set $[y_t - \theta, y_t]$ remains invested in equilibrium as long as the regime is high (and inactive when the regime is low). Thus, there is a greater delay in high regime if i^* lies closer to the lower bound of the interval, $y_t - \theta$ (and in low regime if i^* is closer to y_t). Longer delay means the prevailing regime has a reinforcing effect to sustain itself in equilibrium, creating a history-dependent strategy, a property one may call *hysteresis*.

3.2 Learning and belief evolution

Each stage of the game can be described in more details as follow:

3.2.1 Stage 1: Initial beliefs

Since the same law of evolution is used by all agents for updating purposes, the class of homogeneous beliefs is stationary. In solving the model, it therefore proves convenient to specify some common initial belief. An obvious choice is the shock transition rule, because it is exactly the belief held immediately after a regime switch, which reveals momentarily the true value of z_t. Thus, if the value of z_0^* is publicly known at time 1, then the common belief is that

$$z_1^* \text{ is uniformly distributed on } \left[z_0^* - \varepsilon, z_0^* + \varepsilon\right].$$

This belief serves as a convenient starting point to compute the general stationary class of homogeneous beliefs in equilibrium.[6]

[6] The class of homogeneous beliefs may additionally be dynamically stable under some technical conditions, as successive learning gradually eliminates any initial belief heterogeneity. Provided that agents' beliefs of z^* have the same mean and are distributed over the same support, their belief densities converge to an identical normal distribution after repeated applications of the shock transition rule, by virtue of the central limit theorem.

3.2.2 Stage 2: Equilibrium play

Equilibrium uniqueness is an essential prerequisite for our analysis, as the measurability of the strategy profile with respect to z_t^* is needed for the laws of motion for beliefs to be well-defined. The sufficient and necessary condition for there to be a unique Bayesian Nash equilibrium surviving iterated elimination of dominated strategies is established by Carlsson and van Damme (1993) in the context of static global game analysis. By a direct application of the global game argument, there is a unique equilibrium in the present model if and only if for all t there exists a unique solution i_t^* to the equation

$$\Pr\left[z_t^* > i_t^* | \mathcal{I}_t\right] \equiv \bar{H}_t\left(i_t^*\right) = \frac{y_t - i_t^*}{\theta}, \tag{3.5}$$

where $\bar{H}_t\left(i\right)$ is $1-$c.d.f. of belief,

$$\bar{H}_t\left(i\right) \equiv \int_{z=i}^{\sup Z_t} h_t\left(z\right) dz. \tag{3.6}$$

The unique solution, if one exists, corresponds exactly to the unique switching equilibrium, where every agent $i < i_t^*$ chooses $x = 0$, and every $i > i_t^*$ chooses $x = 1$.[7] Note that although there is no heterogeneity in the information set per se, it is the heterogeneous assessments of z_t^* *relative* to i that results in the applicability of iterated dominance argument. The details of conditions ensuring a unique solution to equation 3.5 are deferred to section 5.

The existence of at least one dominance region is critical for iterated dominance argument, and is easy to verify in this case. Suppose that the equilibrium has been unique, and the regime switches from high to low at the end of time t. Since $z_t^* \in [\inf Z_t, i_t^*]$, the lower bound for common belief support at time $t+1$ is given by $\inf Z_t - \varepsilon$. Note that we must have $\inf Z_t < y_t - \theta$ (as $\inf Z_t > y_t - \theta$ would imply $i_t^* = y_t - \theta < \inf Z_t \leq z^*$ and there cannot be a regime switch at time t, a contradiction). It follows that

$$\begin{aligned}
\inf Z_{t+1} &= \inf Z_t - \varepsilon \\
&< y_t - \theta - \varepsilon \\
&\approx y_{t+1} - \theta - \varepsilon \\
&< y_{t+1} - \theta
\end{aligned}$$

so that there exists a dominance region for a low regime; the decisive agent does not invest as a dominant strategy with some probability. As ε is large relative to 1, this region continues to exist throughout the regime. An analogous argument follows when the regime switches from low to high.

[7] For a detailed derivation of this result, see Rungcharoenkitkul (2006).

3.2.3 Stage 3: Learning via truncation and transition

At the end of the second stage, provided there exists a unique equilibrium and no regime switch takes place, agents learn that the value of z_t^* must be consistent with the equilibrium regime and the current state y_t. In particular, learning leads to a one-sided truncation of h_t. As before, since the model is symmetric with respect to the regimes, it suffices to only consider a high regime as the initial condition. The updated distribution is defined by the density $\tilde{h}_t$ where

$$\begin{aligned} \tilde{h}_t(z^*) &= \begin{cases} h_t(z^*) & \text{if } \inf Z_t \geq i_t^*, \\ \frac{h_t(z^*)}{\int_{z=i_t^*}^{\sup Z_t} h_t(z)dz} & \text{otherwise,} \end{cases} \\ &= \frac{h_t(z^*)}{\int_{z=\max\{i_t^*, \inf Z_t\}}^{\sup Z_t} h_t(z)\, dz}, \end{aligned} \tag{3.7}$$

which is non-zero for $z^* \in \tilde{Z}_t \equiv [\max\{i_t^*, \inf Z_t\}, \sup Z_t]$. A truncation always gives rise to a nondegenerate interval $\tilde{Z}$, as $\tilde{Z}$ being empty or a singleton would imply $i_t^* \geq \sup Z_t$, i.e. $i_t^* \geq z_t^*$ which violates the assumption that there has not been a regime switch. On the other hand, if there is a regime switch, say from high to low, at date t then a learning truncation uses the complementary normalisation, i.e.

$$\tilde{h}_t(z^*) = \frac{h_t(z^*)}{\int_{z=\inf Z_t}^{i_t^*} h_t(z)\, dz} \tag{3.8}$$

for $z^* \in \tilde{Z}_t \equiv [\inf Z_t, i_t^*]$.

Following a shock transition in stage 3, the common belief at time $t+1$ is updated using the shock transition probabilities. Specifically, h_{t+1} and Z_{t+1} are given by

$$\begin{aligned} h_{t+1}(z^*) &= \int_{z=z^*-\varepsilon}^{z^*+\varepsilon} p(z^* | z)\, \tilde{h}_t(z)\, dz \\ &= \frac{1}{2\varepsilon} \int_{z=z^*-\varepsilon}^{z^*+\varepsilon} \tilde{h}_t(z)\, dz, \end{aligned} \tag{3.9}$$

which is positive for

$$z^* \in Z_{t+1} = [\max\{i_t^*, \inf Z_t\} - \varepsilon, \sup Z_t + \varepsilon].$$

Stage 1 can then be repeated.

3.3 Properties of belief distribution

This section establishes the core properties of h_t that are needed for the analysis of hysteresis and equilibrium uniqueness. The results are stated in their logical order. Unless stated otherwise, the regime is assumed to be high initially.

First, we define the notion that a density is single-peaked or unimodal, using the following standard definition (the proof for equivalence of each condition is trivial and is omitted). As will be shown, unimodality is an equilibrium property of the belief distribution.

Definition 1. *$h_t(z^*)$ is 'unimodal' if one of the following equivalent conditions holds*

(1) h_t is decreasing in $|z^* - \arg\max h_t(z^*)|$, i.e. it always declines away from its unique peak, or

(2) as z^* increases, the derivative $h_t'(z^*)$ changes sign (from positive to negative) at most once, or

(3) there exists z^0 such that $\bar{H}_t(z^*)$ is concave for all $z^* < z^0$ and convex otherwise.

Agents only learn from the outcome of equilibrium play when the initial information about the fundamental is sufficiently vague. For example, in the few periods immediately after a regime switch, the information about the fundamental is already very precise that an ex post equilibrium outcome is unlikely to convey any useful additional information. The following result states that in any period there can be learning if and only if the equilibrium play is not a corner solution. This result will play a useful role in ensuring equilibrium uniqueness later (see the end of section 5.1).

Claim 1. *Suppose that the regime is high and that there is a unique equilibrium in period t. The equilibrium exhibits a strictly maximum delay in period t, i.e. $i_t^* = y_t - \theta$, if and only if $\inf Z_t \geq y_t - \theta$. Furthermore, there is a maximum delay if and only if there is no learning update (or truncation) in that period.*

Proof. See appendix. □

Let us now investigate the properties of the shock transition rule. Consider these properties in the context of a *pure shock transition*, namely when there is no learning truncation in the same period. Under a pure transition, we have $\tilde{h}_t = h_t$, and hence

$$h_{t+1}(z^*) = \frac{1}{2\varepsilon} \int_{z=z^*-\varepsilon}^{z^*+\varepsilon} h_t(z)\, dz. \tag{3.10}$$

In other words, $h_{t+1}(z^*)$ is an average of h_t on the interval $[z^* - \varepsilon, z^* + \varepsilon]$. That the shock transition rule preserves the symmetry of h_t follows immediately from the symmetry of $p(.|.)$. It is also trivial that the expected value of z_{t+1}^* is the same as that of z_t^*. The first important property of the transition mapping is that it leads to a belief of greater 'riskiness', as made precise in the next result.

Lemma 1. *If h_{t+1} is a shock transition map of h_t, then h_t second-order stochastically dominates h_{t+1}.*

Proof. See appendix. □

A necessary condition for unimodaltity to be an equilibrium property is the following result.

Claim 2. *The unimodality property of a density h_t is preserved by the shock transition rule.*

Proof. See appendix. □

The last result regarding the shock transition rule underpins the uniqueness result to be obtained later.

Lemma 2. *Suppose that h_t is unimodal with a unique mode i_t^m, then under a pure transition mapping,*

(1) $\bar{H}_t$ crosses $\bar{H}_{t+1}$ at some point(s) i_t^c on the interval $[i_t^m - \varepsilon, i_t^m + \varepsilon]$,

(2) if h_t is concave on $[i_t^m - 2\varepsilon, i_t^m + 2\varepsilon]$, then there is a unique crossing point i_t^c.

Proof. See appendix. □

The concavity requirement is only a sufficient condition for there to be a single crossing between $\bar{H}_t$ and $\bar{H}_{t+1}$. The following result establishes a more general link between the single crossing of the cumulative and the density.

Claim 3. *On $Z_{t+1} \backslash \{\inf Z_{t+1}, \sup Z_{t+1}\}$, $\bar{H}_{t+1}$ crosses $\bar{H}_t$ once if and only if h_{t+1} crosses h_t twice.*

Proof. See appendix. □

Next we turn to investigate the effects of truncations on unimodality property.

Claim 4. *If h_t is unimodal, then $\tilde{h}_t$ is also unimodal, regardless of whether there is a regime switch at date t. Furthermore, $\arg\max \tilde{h}_t (z^*) \geq \arg\max h_t (z^*)$, with equality if and only if $i_t^* \leq \arg\max h_t (z^*)$.*

Proof. See appendix. □

Claims 2 and 4 together suggest that if h_t is unimodal then h_{t+1} is also unimodal. This fact implies that if the common belief is initially given by a unimodal distribution, then it remains unimodal in all periods, and for any t, $\bar{H}_t (i)$ is concave for $i < \arg\max h_t (z^*)$, and convex otherwise. To justify the unimodality assumption as an equilibrium property, it is sufficient to assume that agents' belief in the first period of the game is a unimodal distribution. Even if agents initially hold a non-unimodal belief in the first period, a sufficiently large number of pure shock transitions will lead to a convergence to a unimodal distribution by the central limit theorem (see section 4.1 below). Henceforth, we restrict attention to the class of unimodal distributions.

When there is a truncation in period t, equation 3.7 implies that $\tilde{h}_t \geq h_t$ on $\tilde{Z}_t$, with equality only at the points where $h_t = 0$. Note also that $\inf \tilde{Z}_t = i_t^* > \inf Z_t$ and $\sup \tilde{Z}_t = \sup Z_t$. The next result readily follows.

Claim 5. *A left-sidedly truncated distribution first-order stochastically dominates the original version. For example, if the regime is high and $i_t^* > \inf Z_t$, then*

$$\int_{z=z^1}^{\sup Z_t} \tilde{h}_t(z)\, dz \geq \int_{z=z^1}^{\sup Z_t} h_t(z)\, dz \qquad \forall z^1 \in Z_t,$$

with equality if and only if $z^1 \in (-\infty, \inf Z_t] \cup [\sup Z_t, \infty)$.

Claim 5 captures the basic effect of a learning truncation; in the absence of a regime switch, agents update their beliefs that the fundamental is conducive to the prevailing regime. If the fundamental z_t is fixed (but not observed), this truncation effect leads to a greater delay and hysteresis next period (see Rungcharoenkitkul (2006)). Here, agents can choose actions only after the fundamental has randomly moved, but the next result states that the effect of learning truncation in terms of delay still holds.

Lemma 3. *Provided the equilibrium is unique in all periods, a truncation in period t leads to greater delay in period $t + 1$, ceteris paribus.*

Proof. See appendix. □

Lemma 3 essentially says that the hysteresis effect of a learning truncation survives the smoothing effect of transition. An immediate corollary is that a larger truncation would lead to a correspondingly larger delay. We finish with the last result on the effect of truncations on the mode of the distribution.

Claim 6. *Left (right) truncations cannot lower (increase) the mode of the belief density, ceteris paribus. Specifically, if a truncation (say from the left) implies $i_t^* < i_{t+1}^m - \varepsilon$, then the truncation does not affect the mode, i.e. i_{t+1}^m is equal to that yielded from a pure transition of h_t.*

Proof. See appendix. □

4 Regime Switching and Hysteresis

This section focuses on analyzing the equilibrium hysteresis properties, with equilibrium uniqueness assumed as a working hypothesis. The uniqueness conditions and its verification will be the issue of the next section.

Due to stochastic z_t^*, the belief evolution is not perfectly coupled with the y_t process, and the information updating may occur even if y_t does not change over time. As a consequence, the exact degree of hysteresis will depend on both (1) the quantitative impact of truncations (relative to y_t dynamics) on the equilibrium, and (2) the frequency of learning updates. In a high regime for instance, a rising y_t will tend to raise i_t^* if either learning updates are infrequent, or learning truncations only increase $\bar{H}_t(i)$ marginally. Calculating the net hysteresis effect completely would require a computation of an equilibrium outcome in every possible realisation of two dependent time series, y_t and the corresponding beliefs, a seemingly complicated task.

It is helpful to ask a far simpler question; what is the minimum level of hysteresis, taken across all possible realisations of the stochastics, that can be supported without violating the learning procedure? In other words, for a given y_t, what is the largest value of i_t^* that could be justified by some history? We attempt to answer these questions by considering two stages of learning separately.

4.1 The first truncation

In the presence of hysteresis effects, a regime switch from low to high releases a strong public signal to all agents that the fundamental has to be good enough to force such a switch given agents' prior decisions to delay, which collectively favored the old regime. Given a renewed high optimism, agents with a nondominant strategy may choose to invest with probability one for a number of periods, i.e. the equilibrium is a corner solution of maximum hysteresis for a certain length of time. Eventually however, the beliefs will be sufficiently diffuse that the equilibrium becomes an interior solution again. The date at which such an interior solution appears for the first time following a regime switch is called the period of *first truncation*. That is, the periods preceding the first truncation only involve pure shock transitions.

In the first date of truncation, the common belief distribution is exceptionally tractable. The assumption that ε is small relative to θ implies that there is a large number of pure shock transitions between regime switches, the accumulative effect of which can be represented as a sum of independent random variables. Irrespective of the initial belief which may be intractable in general, the central limit theorem can be invoked to obtain a limiting distribution of z_t^*, namely a normal distribution.

The period of first truncation also represents the first moment that a regime switch is deemed by agents to be possible. The question is whether agents will aggressively attack the existing regime in the first period that such an attack can be successful, or act cautiously in that period. The answer of course depends on the belief distribution; if agents believe that the probability of a regime switch is very small, they will act almost as if they are certain that the existing regime will continue (as both sides of equilibrium condition in equation 3.5 is continuous in i). Intuitively one may anticipate that in the period of first truncation, the more uncertain agents are about the exact value of the fundamental (in the sense that the set of potential values of the fundamental is large), the lower the probability that they will assign to a regime switch. Thus if the fundamental has not been observed over a long period of time so that agents are very uncertain about its value, then one may expect agents not to act too aggressively in favour of a regime switch in the first period of truncation. We now seek to confirm this intuition analytically, and thereby characterise the minimum level of hysteresis in the period of first truncation.

Suppose that there is a regime switch from low to high in period $t = -1$, and let all agents hold the same belief in period $t = 0$ that z_0^* is a random variable distributed nondegenerately on the interval $Z_0 \equiv [\inf Z_0, \sup Z_0]$ with finite mean and variance,

$$\begin{aligned} \mu_0 &\equiv \int_{z \in Z_0} z h_0(z)\, dz \\ \sigma_0^2 &\equiv \int_{z \in Z_0} (z - \mu_0)^2 h_0(z)\, dz. \end{aligned}$$

The following conjectures about the initial belief are assumed, and need to be verified as part of the equilibrium properties.

Conjecture 1. $|Z_0| \equiv \sup Z_0 - \inf Z_0$ *is small relative to* θ.

Conjecture 2. $0 \le \frac{\sigma_0^2}{\varepsilon^2} < \infty$.

Conjecture 3. *There exists a constant* $\gamma \in (0, 1]$ *independent of* θ *and* ε *such that* $\inf Z_0 - y_0 + \theta = \gamma\theta$.

Conjectures 1 and 2 place a limit on the fuzziness of the initial belief. Conjecture 3 requires that the equilibrium play in period $t = -1$ must not be degenerate in favour of a regime switch. To see this, note that the existence of γ specified in conjecture 3 implies that $\inf Z_0$ is bounded away from $y_0 - \theta$. Since $\inf Z_0 = i_{-1}^* - \varepsilon$, the conjecture is effectively $i_{-1}^* > y_{-1} - \theta + \varepsilon$. The condition holds in the presence of some hysteresis effects in equilibrium, as then i_{-1}^* would be close to y_{-1}. These conjectures will be verified as being satisfied in equilibrium after proposition 2 is proved below.

After t periods, provided there is no truncation, we have $\inf Z_t = \inf Z_0 - t\varepsilon$. Let the first period of truncation be denoted by t_1, i.e. t_1 is the lowest t such that $\inf Z_t = \inf Z_0 - t\varepsilon < y_t - \theta$. The first result expresses t_1 in terms of the primitive parameters.

Claim 7. *Suppose that conjecture 3 holds. As* ε *becomes large relative to* 1, t_1 *is approximately given by*

$$t_1 \simeq \frac{\gamma\theta}{\varepsilon}.$$

Proof. By definition,

$$t_1 = \inf \{ t \in \{0, 1, ...\} |\, t\varepsilon > \min z_0^* - y_t + \theta \}$$

i.e. there is some small constant $c^1 < \varepsilon$ such that

$$\begin{aligned} t_1\varepsilon &= \inf Z_0 - y_{t_1} + \theta + c^1 \\ t_1\varepsilon &= \inf Z_0 - y_0 + \theta - (y_{t_1} - y_0) + c^1 \end{aligned}$$

Since $y_t - y_0 \in [-t, t]$, let us write $y_t - y_0 = \alpha_1 t$ where $\alpha_1 \in [-1, 1]$. Hence, using conjecture 3, we have

$$\begin{aligned} t_1\varepsilon &= \gamma\theta - \alpha_1 t + c^1 \\ t_1 &= \frac{\gamma\theta + c^1}{\varepsilon + \alpha_1} \\ &\simeq \frac{\gamma\theta}{\varepsilon} \end{aligned}$$

as $\gamma\theta$ is large relative to ε and ε is large relative to 1. □

Consider the effect of pure transitions. Abstracting from truncations, and again viewing z_t^* as a random walk, z_t^* can be written as

$$z_t^* = z_0^* + \sum_{\tau=1}^{t} X_\tau$$

where each X_τ is an i.i.d random variable uniformly distributed on $[-\varepsilon, \varepsilon]$. It follows from the uniform transition density that

$$\begin{aligned} E(X_\tau) &= 0 \\ Var(X_\tau) &= \frac{\varepsilon^2}{3}. \end{aligned}$$

Therefore z_t^* is a sum of independent random variables, with

$$\begin{aligned} E(z_t^*) &= \mu_0 \\ Var(z_t^*) &= \sigma_0^2 + \frac{t\varepsilon^2}{3}. \end{aligned}$$

A limiting case of interest is where there are a large number of pure transitions before there is the first truncation. By the central limit theorem, for large t we have

$$z_t^* \to N\left(\mu_0, \sigma_0^2 + \frac{t\varepsilon^2}{3}\right)$$

so that after a large number of pure transitions,

$$h_t(z^*) \simeq \frac{1}{\sqrt{2\pi\left(\sigma_0^2 + t\varepsilon^2/3\right)}} \exp\left[-\frac{(z^* - \mu_0)^2}{2\left(\sigma_0^2 + t\varepsilon^2/3\right)}\right]. \tag{4.1}$$

Assume for the moment that a unique equilibrium is always ensured. Lemma 1 implies that there exists an interior equilibrium solution for the first time in the period of first truncation, i.e. at date t_1. Our present objective is to determine $i^*_{t_1}$. As $\bar{H}_{t_1}(i)$ is not available in closed form, the strategy is to obtain an asymptotic approximation for $i^*_{t_1}$. Given that the regime is high, the unique interior equilibrium in period t_1 is determined by the intersection between $(y_{t_1} - i)/\theta$ and the concave region of $\bar{H}_{t_1}(i)$. One strategy is then to derive an upper bound for $i^*_{t_1}$ by constructing a tractable auxiliary $\bar{H}^+_{t_1}(i)$ that lies below $\bar{H}_{t_1}(i)$ on the concave region. This procedure is followed in the proof of the next proposition.

Proposition 1. *Let t_1 be the first period of truncation, and assume $\sigma_0^2/\varepsilon^2 < \infty$ as in conjecture 2. Then*

$$\lim_{t_1 \to \infty} i^*_{t_1} = y_{t_1} - \theta.$$

Proof. Consider a point $i_1 = y_{t_1} - \theta + \alpha_2$, for some small $\alpha_2 > 0$. At i_1 and period t_1, we have from equation 4.1,

$$h_{t_1}(i_1) \simeq \frac{1}{\sqrt{2\pi\left(\sigma_0^2 + t_1\varepsilon^2/3\right)}} \exp\left[-\frac{\left(y_{t_1} - \theta + \alpha_2 - \mu_0\right)^2}{2\left(\sigma_0^2 + t_1\varepsilon^2/3\right)}\right] \tag{4.2}$$

Using identical representations used in the proof of claim 7, we can write $y_{t_1} = y_0 + \alpha_1 t_1$ for some $\alpha_1 \in [-1, 1]$, and $\mu_0 - y_0 + \theta \simeq \gamma\theta$, as $\mu_0 - \min z^*_0$ is of order smaller than θ by conjecture 1. Then, using claim 7, we have

$$\begin{aligned} y_{t_1} - \theta + \alpha_2 - \mu_0 &= -\left(\mu_0 - y_0 + \theta\right) + \alpha_1 t_1 + \alpha_2 \\ &= -\gamma\theta + \frac{\alpha_1\gamma\theta}{\varepsilon} + \alpha_2 \\ &\simeq \gamma\theta\left[\frac{\alpha_1}{\varepsilon} - 1\right] \\ &\simeq -\gamma\theta. \end{aligned}$$

Using this equation and claim 7 again, equation 4.2 can be written as

$$\begin{aligned} h_{t_1}(i_1) &\simeq \frac{1}{\sqrt{2\pi\left(\sigma_0^2 + t_1\varepsilon^2/3\right)}} \exp\left[-\frac{\left(-\gamma\theta\right)^2}{2\left(\sigma_0^2 + t_1\varepsilon^2/3\right)}\right] \\ &= \frac{1}{\sqrt{2\pi\left(\sigma_0^2 + \gamma\theta\varepsilon/3\right)}} \exp\left[-\frac{3\left(\frac{\gamma\theta}{\varepsilon}\right)}{2\left(\frac{3\sigma_0^2}{\varepsilon^2}\left(\frac{\varepsilon}{\gamma\theta}\right) + 1\right)}\right]. \end{aligned} \tag{4.3}$$

Thus, since $\sigma_0^2/\varepsilon^2 < \infty$, $h_{t_1}(i_1) \to 0$ exponentially fast as $\gamma\theta/\varepsilon \to \infty$.

Let us define an auxiliary function $h^+_{t_1}$ on $[\inf Z_{t_1}, i_1]$ as

$$h^+_{t_1}(i) = h_{t_1}(i_1), \text{ for all } i \in [\inf Z_{t_1}, i_1].$$

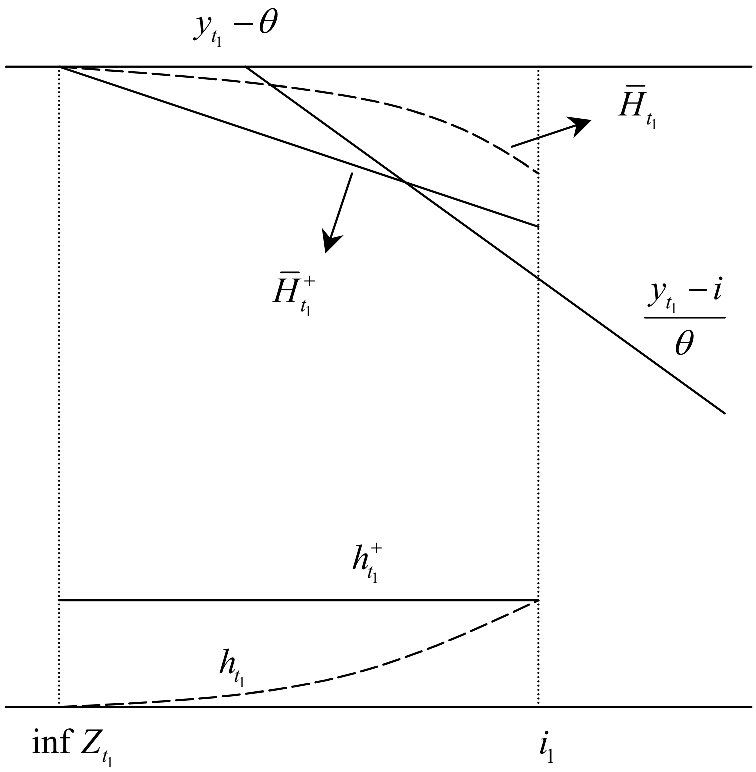


Figure 2: Construction of $\bar{H}^{+}_{t_1}(i)$

The corresponding negative c.d.f is given as usual by,

$$\begin{aligned}\bar{H}^{+}_{t_1}(i) &= 1 - \int_{z=\inf Z_{t_1}}^{i} h^{+}_{t_1}(z)dz \\ &= 1 - h_{t_1}(i_1)\left[i - \inf Z_{t_1}\right]\end{aligned} \tag{4.4}$$

Figure 2 demonstrates the construction of $\bar{H}^{+}_{t_1}(i)$ and consequently the approximation for $i^{*}_{t_1}$.

Let us check the existence. From the diagram, there exists a solution to

$$\bar{H}^{+}_{t_1}(i) = \frac{y_{t_1} - i}{\theta} \tag{4.5}$$

if and only if

$$\begin{aligned}\bar{H}^{+}_{t_1}(i_1) &> \frac{y_{t_1} - i_1}{\theta} \\ &= 1 - \frac{\alpha_2}{\theta}.\end{aligned} \tag{4.6}$$

But from equation 4.4,

$$\begin{aligned}\bar{H}^{+}_{t_1}(i_1) &= 1 - h_{t_1}(i_1)\left[i_1 - \inf Z_{t_1}\right] \\ &\geq 1 - h_{t_1}(i_1)\left[\alpha_2 + \varepsilon\right] \qquad (4.7)\end{aligned}$$

where the last step follows from the definition of i_1 and the fact that $\inf Z_{t_1} \geq y_{t_1} - \theta - \varepsilon$ (as otherwise there would have been a truncation prior to date t_1). According to equation 4.3, as θ becomes large, $h_{t_1}(i_1) \to 0$ exponentially fast. Thus in equation 4.7, the lower bound for $\bar{H}^{+}_{t_1}(i_1)$ and consequently $\bar{H}^{+}_{t_1}(i_1)$ itself also tend to 1 exponentially in θ. Therefore the L.H.S of equation 4.6 tends to 1 at a faster rate than the R.H.S, and a fixed point solution to equation 4.5 always exists for any i_1 given a sufficiently large θ.

Note that as θ/ε (and hence t_1) becomes large, the normal density approximates the tail of $\bar{H}_{t_1}$ more and more accurately, so that by the concavity of $\bar{H}_{t_1}$, $\bar{H}^{+}_{t_1}(i)$ lies everywhere beneath $\bar{H}_{t_1}(i)$ for $i \in [\inf Z_{t_1}, i_1]$ in the limit. Therefore the fixed point solution to equation 4.5 provides a limiting upper bound for the actual $i^*_{t_1}$. Thus, denoting the solution to equation 4.5 by $\sup i^*_{t_1}$, we have from combining equations 4.4 and 4.5,

$$\sup i^*_{t_1}\left[1 - \theta h_{t_1}(i_1)\right] = y_{t_1} - \theta + \theta h_{t_1}(i_1) \inf Z_{t_1}$$

but because $h_{t_1}(i_1)$ tends to zero exponentially in θ, $\lim_{\theta\to\infty} \theta h_{t_1}(i_1) = 0$. Hence

$$\lim_{\theta\to\infty}\left[\sup i^*_{t_1}\right] = y_{t_1} - \theta$$

and the proof is complete. □

Thus, the longer it takes before there is a positive probability of a regime switch being triggered by a dominant strategy, the greater the hysteresis effect at that first period of truncation. Intuitively, a large number of pure shock transitions increases the uncertainty about $z^*_{t_1}$ in such a way that agents assign an increasing probability to $z^*_{t_1}$ being too high to justify a regime switch in the first period of learning. In this case, maximum hysteresis is the limiting outcome at date t_1.

What happens in the periods after the first truncation? Will there continue to be strong hysteresis effect? These questions are addressed next.

4.2 Hysteresis in generic periods

Consider an arbitrary date t where $\bar{H}_t(i)$ gives rise to an interior equilibrium $i^*_t > y_t - \theta$. The objective in this section is to place an upper bound on the level of i^*_t that may be realised in any period t, after some arbitrary sequence of belief truncations and state transitions. In other words, we seek to quantify the minimum level of hysteresis at an arbitrary date t.

In a generic period, it is possible for beliefs to fluctuate. Suppose the common belief is pessimistic and assigns a high probability to z_t being low. A regime switch is believed to be more likely at time t. This common belief would lead to a small delay in period t and a correspondingly large i_t^*, so that an absence of a regime switch would imply a significant learning truncation in that period. Because it is common knowledge that fewer agents are investing and yet the regime still remains high, all agents become relatively more optimistic that the true fundamental is in fact favourable. If such a boost in optimism is sufficiently strong, it can in turn lead to a number of consecutive periods of maximum delay. In other words, pessimistic beliefs are reversed to optimistic ones by the absence of a regime change.

On the other hand, it is not inconceivable that 'bubbles' in the common belief may develop over time, leading agents to believe that a regime switch is just 'around the corner'. That is, they are confident about a high regime today, whilst being convinced about a crash tomorrow. In a high regime, this would imply that the common knowledge density is highly skewed to the right. The skewness of the belief density, once obtained, may be persistent, leading to a similar pattern of play over time. In such a case, there would be large but occasional truncations, and the common knowledge evolution is characterised by spells of optimism (causing maximum delay) interrupted by occasional bursts of pessimism (leading to truncations). Because of this feedback effect, beliefs may cycle between optimistic and pessimistic phases.

The apparent instability of the common belief adds to the complexity of determining h_t explicitly over time. Nonetheless, some important quantitative implications for hysteresis may be assessed without knowing the exact forms of h_t. First, note that the actual timing of a regime switch depends on the speed of y_t evolution relative to that of the common belief dynamics. Suppose that the agents' beliefs fluctuate between a very brief but severe phase of pessimism and a long spell of optimism. If beliefs cycle at a very high speed relative to the dynamics of y_t, then almost every visited state y_t will be tested against the severe pessimism phase. A regime switch therefore takes place as soon as the economy reaches the critical state y_t that cannot be supported by the pessimistic belief. Temporary delay therefore cannot lead to a significant hysteresis when beliefs evolve quickly.

Secondly, the assumption of a unique equilibrium at time $t-1$ clearly implies certain restrictions on h_{t-1}. These restrictions in turn place constraints on the form that h_t may take, and hence on the ensuing degree of delay at (arbitrary) date t. It turns out that the equilibrium uniqueness implies an upper bound on the degree of skewness of the belief density, and hence a lower bound on the degree of delay in any period, leading to the next key result.

Proposition 2. *Suppose that the regime is high, and the equilibrium is unique in periods*

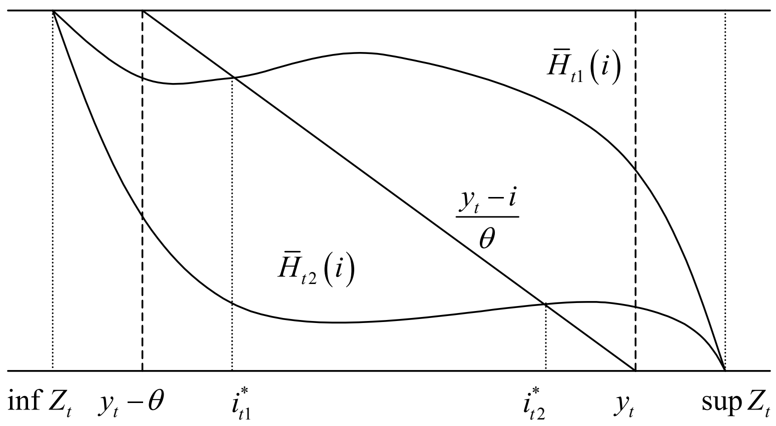


Figure 3: A first-order stochastic dominant distribution leads to greater delay

$t-1$ *and* t. *Then*

$$i_t^* \leq y_t - \theta + \varepsilon.$$

Proof. The objective is to derive an upper bound on i_t^* for any arbitrary date t, where no restriction is imposed on h_t and h_{t-1} except that they must yield a unique equilibrium in period t and $t-1$. For i_t^* to take its maximum value, it clearly has to be an interior solution, i.e. $\inf Z_t < y_t - \theta$. As learning truncations potentially take place prior to date t, agents cannot believe z_t to be too low and we must have $\inf Z_t \geq y_t - \theta - \varepsilon$, thus in sum $\inf Z_t \in [y_t - \theta - \varepsilon, y_t - \theta)$. On the other hand, because equation 5.6 always holds (as a consequence of claim 1; see the discussion following section 5.1) and ε is big relative to one, it follows that $\sup Z_t > y_t$ for any time t after a regime switch. Let us consider an arbitrary fixed pair of $\inf Z_t$ and $\sup Z_t$ that satisfy such constraints. Suppose that two distributions over the fixed support Z_t are rankable in the sense of first-order stochastic dominance. Under the assumption that there is a unique equilibrium in period t, it follows from a simple graph-theoretic argument (see figure 3) that the distribution which first-order stochastically dominates the other leads to a smaller equilibrium i_t^*. Thus, i_t^* is at its greatest if the underlying distribution h_t is first-order stochastically dominated by all other possible distributions within the class of interests. If any distribution h_t is possible, then the upper bound for i_t^* is uninterestingly given by y_t with the underlying distribution being $\bar{H}_{t-1}(i) = 0$, $\forall i$. The next step of our proof is to characterise the class of distributions of interests, namely those which are consistent with a unique equilibrium at each date $t-1$ and t.

For convenience, we reproduce equation 3.9

$$h_t(i) = \frac{1}{2\varepsilon} \int_{z=i-\varepsilon}^{i+\varepsilon} \tilde{h}_{t-1}(z)\, dz,$$

and define

$$\widetilde{H}_{t-1}(i) \equiv \int_{z=i}^{\sup Z_{t-1}} \tilde{h}_{t-1}(z)\,dz. \tag{4.8}$$

As h_t is related to $\tilde{h}_{t-1}$ according to equation 3.9 , the first step is to characterise the class of $\tilde{h}_{t-1}$ consistent with a unique equilibrium in period $t-1$. Recall from equation 3.7 that $\tilde{h}_{t-1}$ is either identical to h_{t-1} or is a left-sidedly truncated version of it, depending on whether $\inf Z_{t-1} \geq i^*_{t-1}$. In terms of the c.d.f, this corresponds to $\widetilde{H}_{t-1}(i) \geq \bar{H}_{t-1}(i)$. Consider two possibilities in turn. First, if $\widetilde{H}_{t-1}(i) = \bar{H}_{t-1}(i)$ then there is no truncation in period $t-1$, implying that the equilibrium is a corner solution of $i^*_{t-1} = y_{t-1} - \theta$. Because there is no other equilibrium, $\widetilde{H}_{t-1}(i)$ and $\bar{H}_{t-1}(i)$ must lie above $(y_{t-1} - i)/\theta$ for all $i \in (y_{t-1} - \theta, y_{t-1}]$. On the other hand, if $\widetilde{H}_{t-1}(i) > \bar{H}_{t-1}(i)$ then there is a truncation in period $t-1$ and $\widetilde{H}_{t-1}(i) > \bar{H}_{t-1}(i) > (y_{t-1} - i)/\theta$ for $i > i^*_{t-1}$ and $\widetilde{H}_{t-1}(i) = 1 > (y_{t-1} - i)/\theta$ for $i \in \left(y_{t-1} - \theta, i^*_{t-1}\right]$. Hence conditional on there being a unique equilibrium in period $t-1$, one can conclude that $\widetilde{H}_{t-1}(i)$ must lie above $(y_{t-1} - i)/\theta$ everywhere except perhaps at the singular point $i = y_{t-1} - \theta$. Let us call the class of distributions $\widetilde{H}_{t-1}(i)$ with such property the *uniqueness class*.

For any fixed support Z_t, there corresponds a unique $\widetilde{Z}_{t-1} \equiv [\inf Z_t + \varepsilon, \sup Z_t - \varepsilon]$. On the support $\widetilde{Z}_{t-1}$, let us define, for a small $e > 0$, a benchmark distribution

$$\tilde{h}^e_{t-1}(i) \equiv \frac{1}{\theta}, \text{ for } i \in [y_{t-1} - \theta + e, y_{t-1} + e],$$

and zero otherwise, with the corresponding c.d.f.

$$\widetilde{H}^e_{t-1}(i) \equiv \frac{y_{t-1} - i}{\theta} + \frac{e}{\theta}.$$

Clearly $\lim_{e \to 0} \widetilde{H}^e_{t-1}(i)$ is first-order stochastically dominated by all other distributions in the uniqueness class. Recall from the proof of 3 that the shock transition rule in equation 3.9 preserves the first-order stochastic dominance ordering. In other words, as $e \to 0$, the benchmark density $\tilde{h}^e_{t-1}$ generates a belief density at time t, say h^e_t, that is stochastically dominated by all other densities h_t which are a shock transition rule of the uniqueness class. Hence, an upper bound on i^*_t is simply the limit as $e \to 0$ of the solution to

$$\bar{H}^e_t(i) \equiv \int_{z=i}^{\sup Z_t} h^e_t(z)\,dz = \frac{y_t - i}{\theta} \tag{4.9}$$

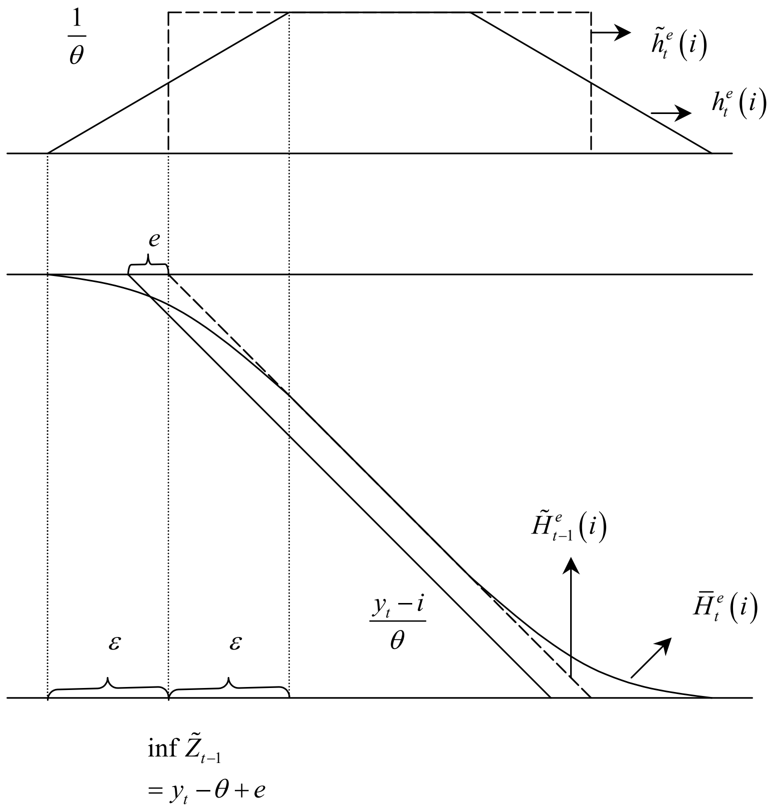


Figure 4: Construction of $\bar{H}_t^e$ and h_t^e

where

$$
\begin{aligned}
h_t^e(i) &= \frac{1}{2\varepsilon}\int_{z=i-\varepsilon}^{i+\varepsilon} \tilde{h}_{t-1}^e(z)\,dz \\
&= \begin{cases} \frac{1}{\theta} & \text{for } i \in [y_t - \theta + e + \varepsilon, y_t + e - \varepsilon] \\ \frac{i-(y_t-\theta+e-\varepsilon)}{2\varepsilon\theta} & \text{for } i \in [y_t - \theta + e - \varepsilon, y_t - \theta + e + \varepsilon] \\ \frac{-i+(y_t+e+\varepsilon)}{2\varepsilon\theta} & \text{for } i \in [y_t + e - \varepsilon, y_t + e + \varepsilon] \end{cases}.
\end{aligned}
$$

Figure 4 illustrates the construction of $\bar{H}_t^e$ and h_t^e.

It can be readily checked that $\bar{H}_t^e(i) = \widetilde{H}_{t-1}^e(i)$ for $i \in [y_t - \theta + e + \varepsilon, y_t + e - \varepsilon]$ and $\bar{H}_t^e(i) > \widetilde{H}_{t-1}^e(i)$ for $i > y_t + e - \varepsilon$, hence the fixed point solution to equation 4.9 must lie in the interval $[y_t - \theta, y_t - \theta + e + \varepsilon]$. Take the limit $e \to 0$, and the proposition is proved. $\square$

Since ε is small relative to θ, proposition 2 states that there is a significant hysteresis in equilibrium. Significant hysteresis in both regimes in turn suggests that the state variable y_t can cycle, even if there is no actual change in the fundamental variable z_t. A high regime persists and y_t tends to rise, until $y_t = z_t^* + \theta$ is reached, inducing a regime switch. A low regime then ensues, y_t declines until $y_t = z_t^*$ is hit, and the regime switches back to high. Even if there is in fact no change in z_t, y_t tends to fluctuate in a unique equilibrium cycle.

Proposition 2 sheds light on the equilibrium properties, allowing us to verify conjectures 1, 2 and 3 made earlier.

Claim 8. *Conjectures 1, 2 and 3 are satisfied in equilibrium.*

Proof. Suppose that there is a regime switch from low to high in period -1, so that $Z_{-1} = \left[i^*_{-1}, \sup Z_{-1}\right]$. Because of the model's symmetry with respect to the regimes, proposition 2 adjusted for a low regime implies that $i^*_{-1} \geq y_{-1} - \varepsilon$. Because of learning truncations, we also have $\sup Z_{-1} \leq y_{-1} + \varepsilon$. Thus

$$\begin{aligned} Z_0 &= \left[i^*_{-1} - \varepsilon, \sup Z_{-1} + \varepsilon\right] \\ &\subseteq \left[y_{-1} - 2\varepsilon, y_{-1} + 2\varepsilon\right] \end{aligned}$$

so that

$$|Z_0| \leq 4\varepsilon \ll \theta$$

validating conjecture 1. Next, note that

$$\frac{\sigma_0^2}{\varepsilon^2} = \int_{z \in Z_0} \left(\frac{z - \mu_0}{\varepsilon}\right)^2 h_0(z)\, dz.$$

Since $\mu_0 \in Z_0$, it follows that for any $z \in Z_0$

$$\frac{z - \mu_0}{\varepsilon} \leq \frac{|Z_0|}{\varepsilon} \leq 4.$$

Thus $\sigma_0^2/\varepsilon^2 < \infty$ and conjecture 2 is verified. Lastly, $i^*_{-1} \geq y_{-1} - \varepsilon > y_{-1} - \theta + \varepsilon$ and hence conjecture 3 is satisfied. □

5 Equilibrium Uniqueness Conditions

We now detail the conditions under which equilibrium uniqueness can be guaranteed, and verify whether these conditions are met.

5.1 Contraction arguments

A sufficient condition for uniqueness can be based on contraction mapping condition:

$$-\frac{\partial}{\partial i}\bar{H}_t(i) < \frac{1}{\theta} \text{ for all } i \in [y_t - \theta, y_t]. \tag{5.1}$$

When this condition holds, it is clear that there is a unique solution to equation 3.5. Using Leibniz's rule on equation 3.6 gives

$$-\frac{\partial}{\partial i}\bar{H}_t(i) = h_t(i), \tag{5.2}$$

so that the uniqueness condition 5.1 is simply

$$\max_{i\in[y_t-\theta,y_t]} h_t(i) < \frac{1}{\theta}. \tag{5.3}$$

In general, uniqueness requires the beliefs about z_t^* to be sufficiently diffuse, and condition 5.3 gives one measure of such diffusion in terms of the sup-norm of the belief posterior.

While the sufficient condition based on sup-norm contraction can be a powerful tool, it will prove to be too strong in the present case. Intuitively, beliefs about z_t^* in the first few periods immediately after a regime switch will be relatively precise, and condition 5.3 may only hold when ε is restricted to be very large relative to θ. When ε is small relative to θ, which is the main case of interest, it will be shown in section 5.3 that even after a large number of belief updates, the condition still cannot be guaranteed although the equilibrium may well be unique. A less demanding condition for uniqueness is therefore required.

Consider an alternative criterion, related to the degree of concavity of $\bar{H}_t(i)$. Note that by an argument that is topologically equivalent to a contraction mapping, if $\bar{H}_t(y_t) > 0$, then the concavity everywhere of $\bar{H}_t(i)$ ensures a unique equilibrium. Similarly, if $\bar{H}_t(y_t-\theta) < 1$, then an everywhere convex $\bar{H}_t(i)$ implies uniqueness. In general, of course, $\bar{H}_t(i)$ is only concave on a certain set and convex otherwise. But suppose that one starts in a high regime and that $\bar{H}_t(y_t) > 0$, then it is easy to see that a unique equilibrium ensues if

$$\text{whenever } \bar{H}_t(i) \text{ is convex, } \bar{H}_t(i) > \frac{y_t - i}{\theta}. \tag{5.4}$$

The condition can also be equivalently expressed as follows. Restrict attention to the range of i for which $\bar{H}_t(i)$ is convex, and define i_t^θ by the solution[8] to

$$-\left.\frac{\partial \bar{H}_t(i)}{\partial i}\right|_{i_t^\theta} = h_t\left(i_t^\theta\right) = \frac{1}{\theta}$$

then there is a unique equilibrium if

$$\bar{H}_t\left(i_t^\theta\right) > \frac{y_t - i_t^\theta}{\theta}. \tag{5.5}$$

[8]Since h_t is always unimodal, there is a unique such solution if one restricts attention to the convex range of $\bar{H}_t$ (i.e. decreasing range of h_t).

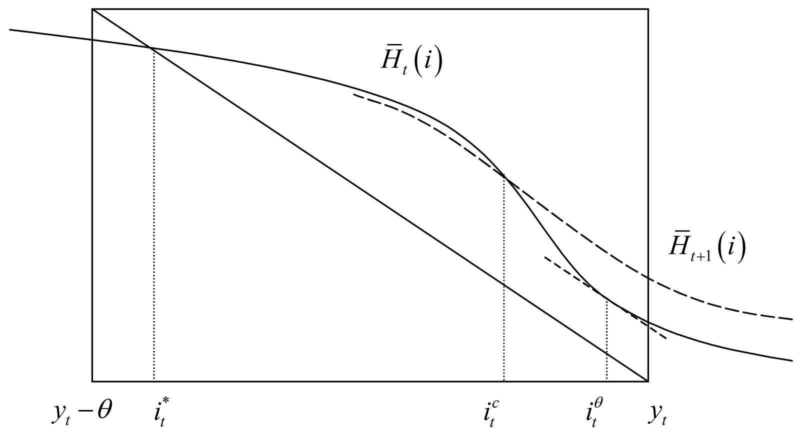


Figure 5: A convexity-based condition for a unique equilibrium, and the first right crossing between $\bar{H}_{t+1}$ and $\bar{H}_t$.

Figure 5 depicts how the 'bounded convexity condition' condition 5.5 implies a unique equilibrium. The condition supplements the bounded sup-norm in equation 5.3. Clearly, if there is no solution i_t^θ to $h_t\left(i_t^\theta\right) = 1/\theta$, then it must follow that $h_t(i) < 1/\theta$ for all i, and the bounded sup-norm condition 5.3 is satisfied.

The prerequisite that $\bar{H}_t(y_t) > 0$ for all t (if one is in a high regime) is intuitively a requirement that agents must always believe that the decisive agent has a dominant strategy with a positive probability. In fact, for $\bar{H}_t(y_t)$ to be larger than 0 for all t in a high regime, it is sufficient to show that $\bar{H}_t(y_t) > 0$ at the starting date of the regime (i.e. in the period after the regime switches from low to high). This is because ε is large relative to 1 and agents continue assigning positive weights to fundamental exceeding y_t as long as the high regime continues. Intuitively, it is immediately after a regime switch and a transition that follows, that the information about the fundamentals is closest to being complete. The applicability of global game technique relies on the shock transition rule to introduce enough uncertainty in this stage. If after a transition, all agents believe with probability one that the true fundamental variable lies within a set defining some coordination game, then the iterated dominance argument fails and there is a multiplicity. The common belief support must therefore be large enough so that the fundamental lies in the dominance region with some probability.

Claim 1 can be used at this point to show that this condition is always satisfied. More concretely, let us suppose that there is a regime switch from high to low at date t.[9] The objective is to prove that $\bar{H}_{t+1}(y_{t+1} - \theta) > 0$, as we now explain briefly. In the beginning of period $t+1$ agents believe that z_{t+1}^* is distributed over $[\inf Z_t - \varepsilon, i_t^* + \varepsilon]$, where both i_t^* and Z_t are common knowledge. If $\inf Z_t - \varepsilon > y_{t+1} - \theta$, then agents believe

[9]We change our convention and choose a low regime as the starting point here so that claim 1 may be applied directly. Recall that the model is symmetric with respect to the regimes.

with probability one that $z^*_{t+1} \in [y_{t+1} - \theta, y_{t+1}]$, so that the decisive group of agents will definitely play a coordination game in period $t+1$ and there must be a multiplicity. Thus a necessary condition for uniqueness in the first period given that the regime just switches to low is that

$$\inf Z_t - \varepsilon < y_{t+1} - \theta \tag{5.6}$$

which is identical to $\bar{H}_{t+1}(y_{t+1} - \theta) > 0$. That this condition is always satisfied can now be verified using claim 1. Consider date t at which there is a regime switch from high to low. The equilibrium at that date is either a corner solution ($i^*_t = y_t - \theta$) or interior ($i^*_t > y_t - \theta$). If it is a corner solution, then the fact that $z^*_t \geq \inf Z_t \geq y_t - \theta$ must imply that $z^*_t = y_t - \theta$ for there to be a regime switch. Thus the true z^*_t becomes common knowledge, i.e. $Z_t = \{z^*_t\}$.[10] It follows that, in period $t+1$, agents believe that $z^*_{t+1} \in [z^*_t - \varepsilon, z^*_t + \varepsilon] = [y_t - \theta - \varepsilon, y_t - \theta + \varepsilon] \approx [y_{t+1} - \theta - \varepsilon, y_{t+1} - \theta + \varepsilon]$, so that $z^*_t - \varepsilon \approx y_{t+1} - \theta - \varepsilon < y_{t+1} - \theta$ and the condition 5.6 is satisfied. On the other hand, suppose more generally that the equilibrium is interior. Then it follows from the claim that $\inf Z_t < y_t - \theta$, and hence $\inf Z_t - \varepsilon < y_t - \theta - \varepsilon \approx y_{t+1} - \theta - \varepsilon < y_{t+1} - \theta$, and again the condition is satisfied.

5.2 A recursive condition

The procedure for checking equilibrium uniqueness can be summarised as follows. If the bounded sup-norm condition in equation 5.3 is satisfied, then there is a unique equilibrium, otherwise we proceed to check if both $\bar{H}_t(y_t) > 0$ and the bounded convexity condition in equation 5.5 hold. When all conditions fail, it is concluded that there is a multiplicity. Analogous conditions can be written down for the case where the initial regime is low.

Given the dynamic nature of the common belief, it must be determined how the equilibrium uniqueness considerations develop over time. Towards this end, it would be useful to have a set of sufficient conditions under which uniqueness can be implied recursively, namely that uniqueness in period t would imply uniqueness in period $t+1$. Let us refer to such a set of sufficient conditions as the *recursive condition* for short. For example, the first-order stochastic dominance is a recursive condition; $\bar{H}_{t+1}$ being greater than $\bar{H}_t$ everywhere is sufficient to ensure that condition 5.4 is recursive, provided that $\bar{H}_{t+1} - \bar{H}_t$ is large relative to $1/\theta$ (which holds since θ is assumed large). However, this particular recursive condition is of limited use, as it obviously cannot be met for all t, e.g. under a pure transition, $\bar{H}_{t+1}$ must cross $\bar{H}_t$ at least once for any $\bar{H}_t$. Intuitively, equilibrium uniqueness is recursive in this case because the first-order stochastic dominance rules out additional equilibria where agents attack the existing regime earlier, by

[10]This is precisely the singular case mentioned earlier in which the information about the fundamentals are complete.

requiring the common belief to become more biased in favour of greater delay.

The essential property of a recursive condition is that the common belief in each period must become sufficiently more diffuse over time, so that there does not exist additional equilibria on which agents can coordinate. Specifically, in view of the uniqueness condition 5.5, the shock transition rule must lead to a 'convexity contraction'[11], so that the validity of iterated dominance can be inferred recursively; namely $\bar{H}_t\left(i_t^\theta\right) > \left(y_t - i_t^\theta\right)/\theta$ implies $\bar{H}_{t+1}\left(i_{t+1}^\theta\right) > \left(y_{t+1} - i_{t+1}^\theta\right)/\theta$.

Let us construct a recursive condition with such property. Suppose that at time t, the regime is high and the bounded convexity condition 5.5 is met. Consider figure 5 again, where both $\bar{H}_t$ and $\bar{H}_{t+1}$ are plotted. As we have not ascertained if $\bar{H}_t$ may cross $\bar{H}_{t+1}$ several times in general, let i_t^c denote the point at which $\bar{H}_{t+1}$ crosses $\bar{H}_t$ from the right for the first time (and necessarily from above), so that $\bar{H}_{t+1}(i) > \bar{H}_t(i)$ for all $i > i_t^c$.[12] Note that uniqueness in period t implies that $\bar{H}_t(i) > (y_t - i)/\theta$ for all $i > i_t^*$. The strategy is to exploit this fact and work out the conditions under which $\bar{H}_t$ can serve as a lower bound for $\bar{H}_{t+1}$ at i_{t+1}^θ, i.e. the point where $-\partial\bar{H}_{t+1}/\partial i = 1/\theta$. In other words, we seek to establish conditions under which we are able to write

$$\begin{aligned}\bar{H}_{t+1}\left(i_{t+1}^\theta\right) &> \bar{H}_t\left(i_{t+1}^\theta\right) &(5.7)\\ &> \frac{y_t - i_{t+1}^\theta}{\theta} &(5.8)\\ &\simeq \frac{y_{t+1} - i_{t+1}^\theta}{\theta}\end{aligned}$$

and hence conclude that there is a unique equilibrium in period $t+1$.

A construction of a recursive condition amounts to obtaining two sufficient conditions that ensure inequalities 5.7 and 5.8 are both satisfied. Since $\bar{H}_{t+1}(i) > \bar{H}_t(i)$ for all $i > i_t^c$, a sufficient condition for inequality 5.7 to hold is that $i_{t+1}^\theta > i_t^c$. On the other hand because there is a unique equilibrium in period t, for inequality 5.8 to hold, it is sufficient and necessary that $i_{t+1}^\theta > i_t^*$. Thus, a recursive condition is simply given by

$$i_{t+1}^\theta > \max\left\{i_t^c, i_t^*\right\}. \tag{5.9}$$

There exists a variety of sufficient conditions that would guarantee that this recursive condition holds. For instance, a condition may be rewritten in terms of the density.

[11]Namely the bounded convexity condition must be recursive, so that when condition 5.5 is met in period t, it is also met in period $t+1$. On the other hand, a shock transition rule always implies a supnorm contraction in the belief density. To see this, recall that a shock transition rule of $h_t(z)$ is merely its average over the interval $[z-\varepsilon, z+\varepsilon]$. Clearly, on any fixed interval, the average cannot exceed the supremum. Hence, the supremum of averages of h_t cannot exceed the supremum of h_t itself.

[12]This is a slight abuse of notation, as i_t^c was defined earlier in lemma 2 as any crossing point. No confusion should arise however, as no other results have relied on that notation.

Because of the unimodality of h_{t+1}, a sufficient condition for $i_{t+1}^{\theta} > \max\{i_t^c, i_t^*\}$ is that

$$h_{t+1}\left(\max\{i_t^c, i_t^*\}\right) > \frac{1}{\theta}. \tag{5.10}$$

Alternatively, a condition based on the mode i_t^m may be constructed. Suppose that we always have $i_t^c > i_t^*$, so that one only needs to show $i_{t+1}^{\theta} > i_t^c$. Since $i_{t+1}^{\theta} > i_{t+1}^m$, a sufficient condition for $i_{t+1}^{\theta} > i_t^c$ comprises two inequalities

$$i_{t+1}^m \geq i_t^m > i_t^c. \tag{5.11}$$

5.3 Verifying uniqueness

Somewhat surprisingly, an implication of the limiting result in equation 4.1 is that the equilibrium uniqueness prior to the first truncation *cannot* be guaranteed by the sup-norm contraction condition 5.3 alone no matter how large is the number of pure shock transitions. As $h_t(z^*)$ is unimodal and attains its maximum at $z^* = E(z_t^*) = \mu_0$, we have

$$\sup h_t \approx \left[2\pi\left(\sigma_0^2 + t\varepsilon^2/3\right)\right]^{-\frac{1}{2}}. \tag{5.12}$$

Thus, a sufficient condition for uniqueness according to equation 5.3 is given by

$$\begin{aligned} \theta &< \sqrt{2\pi\left(\sigma_0^2 + t\varepsilon^2/3\right)} \\ t &> \frac{3\left(\theta^2 - 2\pi\sigma_0^2\right)}{2\pi\varepsilon^2}. \end{aligned} \tag{5.13}$$

The right hand side of equation 5.13 is the minimum number of pure transitions required before uniqueness can be guaranteed by the sup-norm contraction condition in equation 5.3. By claim 7, the expected number of periods before the first truncation is proportional to θ/ε, whilst the right hand side of equation 5.13 is of the order $(\theta/\varepsilon)^2$. In other words, the first truncation takes place before the contraction condition can be satisfied. Therefore, in ensuring uniqueness in the case of pure transitions, the convexity contraction condition must be used.

It is instructive to consider two distinct phases of the learning process separately, and derive uniqueness conditions for each case. The first phase of learning starts in the period immediately after a regime switch. In this phase, subject to the initial belief, agents update their common belief using the shock transition rule alone. After sufficiently many periods of pure shock transition rules, the common belief approximately follows a normal distribution by the central limit theorem. This marks the start of the second phase, which continues until the first learning truncation and subsequently the regime switch. Each phase is now considered in turn.

Suppose that there is a regime switch from low to high in period 0, and that there is a unique equilibrium in that period. Since there is a regime switch in period 0, $\tilde{h}_0$ is defined as in equation 3.8 suitably adjusted for a low regime, i.e.

$$\tilde{h}_0(z^*) = \frac{h_0(z^*)}{\int_{z=i_0^*}^{\sup Z_0} h_0(z)\,dz}.$$

In other words, $\tilde{h}_0$ is a left-sidedly truncated version of h_0, and hence following claim 5 we have

$$\widetilde{H}_0(i) > \bar{H}_0(i), \qquad \text{for } i \in [i_0^*, \sup Z_0].$$

But since there is a unique equilibrium in period 0, we must have

$$\bar{H}_0(i) > \frac{y_0 - i}{\theta} \qquad i > i_0^*,$$

so that we also have

$$\widetilde{H}_0(i) > \frac{y_0 - i}{\theta} \qquad \text{for } i \in [i_0^*, \sup Z_0].$$

In other words, the equilibrium would still remain unique if agents were able to act after the truncation. Thus, there is a unique equilibrium in period 1 if the shock transition rule on $\tilde{h}_0$ satisfies a recursive condition. Using condition 5.10, we have

$$h_1\left(\max\left\{\widetilde{i_0^c}, \frac{y_0 - i}{\theta}\right\}\right) > \frac{1}{\theta} \tag{5.14}$$

where we let $\widetilde{i_0^c}$ denote the first right crossing point between $\bar{H}_1$ and $\widetilde{H}_0$. Note that for sufficiently large θ, the R.H.S tends to 0, whilst the L.H.S tends to $h_1\left(\widetilde{i_0^c}\right)$ which is independent of θ. Furthermore, $h_1\left(\widetilde{i_0^c}\right)$ is large relative to $1/\theta$ if ε is relatively small, i.e. when the smoothing effect takes place slowly. Thus uniqueness carries over for large θ and low ε. Similar arguments can be used for subsequent periods in this phase to infer uniqueness recursively. It may be noted that the sup-norm contraction condition for uniqueness is harder to satisfy for large θ, whereas the opposite is the case for the convexity contraction condition.

Whilst the above argument remains valid in the second phase of the learning process, there exists a far simpler sufficient condition in this case. Once h_t settles down into a normal distribution, the mode becomes static over time and because the density becomes symmetric, we have

$$i_{t+1}^m = i_t^m = i_t^c.$$

In this special case, one needs not have the strict inequality $i_t^m > i_t^c$ to ensure recursive uniqueness. Since equilibrium is a corner solution in this phase, we have $i_t^c > i_t^*$ and it always follows that $i_{t+1}^\theta > i_{t+1}^m = i_t^m = i_t^c$. Thus as long as the distribution remains normal, the equilibrium remains unique.

Consider next the periods after the first truncation. Claim 6 implies that the mode of the belief density will remain static as long as $i_t^* < y_t - \theta + \varepsilon < i_t^m - \varepsilon$, i.e. if

$$y_t - \theta < i_t^m - 2\varepsilon.$$

The belief density ceases to be symmetric here however, thus the argument used in the previous paragraph does not apply. Note that by virtue of lemma 3 which essentially states that the transition mapping perserves first-order stochastic dominance, left-sided truncations imply that in the first period of truncation we must have

$$i_t^c < i_t^m$$

since a pure transition would imply $i_t^c = i_t^m$. Thus, condition 5.11 above is satisfied, and again equilibrium uniqueness is recursive.

Over the subsequent periods, it is less clear-cut whether a recursive condition is satisfied. Nonetheless, recall from lemma 2 that $i_t^c \in [i_t^m - \varepsilon, i_t^m + \varepsilon]$, and consider condition 5.10 again. Suppose that $i_t^c > i_t^*$, then uniqueness is recursive if $h_{t+1}(i_t^c) > 1/\theta$. Because h_t is continuous, for small ε we have approximately $h_{t+1}(i_t^c) \simeq h_{t+1}(i_t^m)$. Then, either $h_{t+1}(i_t^c) \simeq h_{t+1}(i_t^m) > 1/\theta$ in which case uniqueness is recursive due to the convexity contraction, or $h_{t+1}(i_t^c) \simeq h_{t+1}(i_t^m) < 1/\theta$ and the same conclusion follows from the sup-norm contraction.

5.4 Discussion

All aforementioned conditions shares a unifying mechanism central for the uniqueness result. The general mechanism has been identified in the literature on heterogeneity in interactions games (for a survey, see Morris and Shin (2002)) as the invariance of strategic uncertainty with respect to players' types. In our model, the belief about the fundamental is common knowledge, and therefore players can fully work out the strategy of agents with any costs. Invariance of strategic uncertainty here requires that agent's level of optimism about the fundamental *relative* to her tolerance level, i.e. her investment cost, does not vary too much with her cost. Each way of measuring this invariance gives rise to a different uniqueness condition.

The proposed uniqueness conditions are only sufficient conditions, so their violation does not necessarily imply indeterminacy. There can still exist a unique equilibrium even when these conditions are not met. But what happens to the dynamics of y_t when there is a genuine multiplicity? The dynamics of y_t is indeterminate during these periods, and it therefore ceases to convey any information about the underlying fundamental, except that the decisive agent does not have a dominant strategy. Meanwhile, the dynamics for z_t continues independently, and in the absence of learning, lemma 1 implies that the common belief about z_t becomes more diffuse over time. Eventually, by convexity contraction, there exists a unique equilibrium again and the learning resumes. Hysteresis results obtained earlier go through during this phase. The relative lengths of indeterminacy versus uniqueness phases clearly depend on the speed of y_t relative to that of belief evolution. If y_t moves very quickly, then agents quickly learn about the fundamental, and the common belief becomes concentrated rapidly, and multiplicity re-emerges sooner.

6 Cycle and welfare implications

A key positive implication of the model is that y_t can perpetually cycle in a unique strategic equilibrium, even in the absence of any actual change in the fundamental z_t. Holding z_t fixed, the cycle's peak-to-trough length is of the same order of magnitude as θ, due to the maximum hysteresis established by proposition 2. This unique cycle result stands in contrast with limit cycles in many dynamic macroeconomic models, where there is usually an infinite number of equilibrium paths. In this model, cycle is not just a possibility, but is the only well-defined equilibrium outcome.

Another defining feature of the model is that the equilibrium cycle gives rise to a stationary probability distribution over the sets of states Y and regimes, computable using the transition probabilities. Thus, the model goes beyond being a caricature of fluctuations (e.g. by using multiple equilibria as an analogy), and explicitly characterizes fluctuations in terms of a unique well-defined outcome represented by an equilibrium distribution. The equilibrium distribution also allows a quantitative assessment of the average time spent in each state, and offers quantitative predictions that can be fitted to data and potentially rejected in empirical exercises.

The normative implications of the model can also be drawn from the equilibrium distribution, and can give novel insights not available from traditional methods. For example, the conventional macroeconomic theory explains business cycles as arising from shocks being propagated by some mechanism, and roles of stabilization policy is how to optimally offset these shocks. Here, fluctuations in y_t are inevitable, driven by belief hysteresis and persistent phases of high and low optimism. When 'thinking cyclically', the policy's objective is not to stabilize y_t around some fixed level or trend, an impossible task, but in obtaining the second best outcome by influencing the long-term stationary distribution such that the economy spends most of its time, on average, in the active states (high y_t). Influencing transition probabilities in this way can raise welfare, even if there remains a cycle.

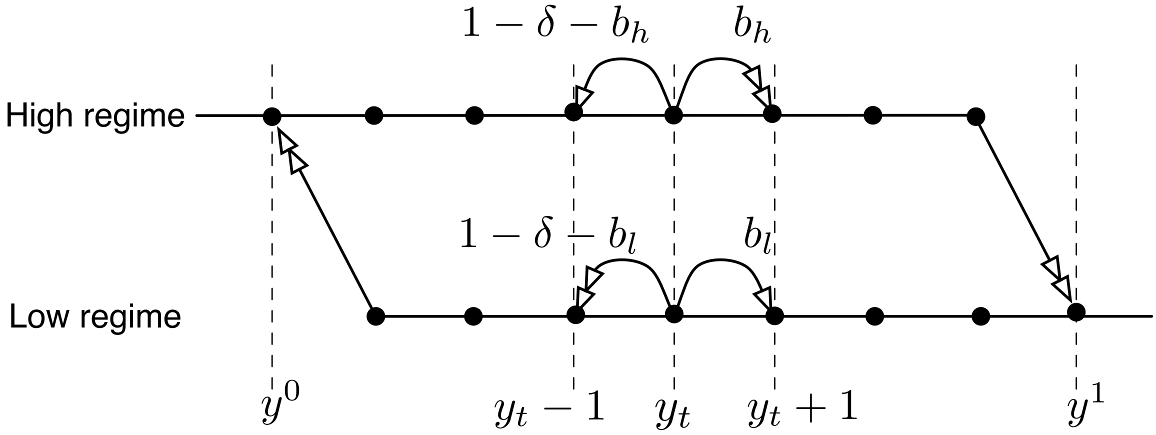


Figure 6: Cycle on a loop with two irreversible transit points.

6.1 Stationary distribution

To focus on the intrinsic fluctuations within the model, we abstract from the influence of aggregate cost shock and consider the case where the random realization of z_t^* is repeatedly equal to its expected value, a constant. In this simple case of *ex post* fixed z_t^* (though perceived to be random *ex ante*), the regime switching points are fixed, and y_t is effectively a discrete-time birth-and-death process on a fixed closed-loop, with two irreversible regime transit points, as depicted in figure 6. Denote the regime-switching states from low to high by y^0 and high to low by y^1, where $z_t^* < y^0 < y^1 < z_t^* + \theta$, and the cycle length $y^1 - y^0$ is no smaller than $\theta - 2\varepsilon$ as implied by proposition 2.

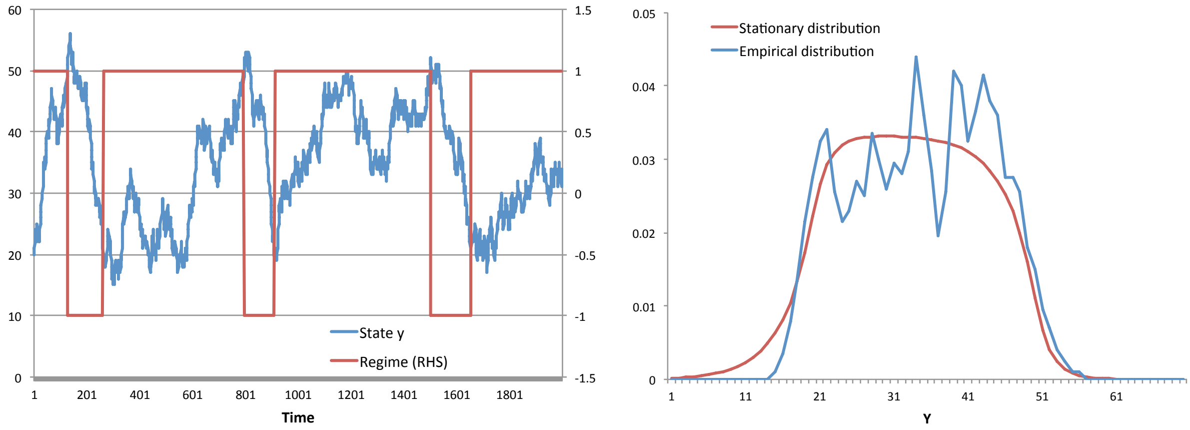
Denote the long-run or equilibrium probability of being in the 2-tuple state $\{y$, high regime$\}$ by $\mu_h(y)$ and in $\{y$, low regime$\}$ by $\mu_l(y)$. Transition probabilities between any pair of 2-tuple state are known and are illustrated in figure 6. Stacking the equilibrium probabilities to define $\mu_s = [\mu_h(0), ..., \mu_h(N), \mu_l(0), ..., \mu_l(N)]'$, and denoting the corresponding transition matrix by P, the equilibrium distribution μ_s is the nonnegative solution to the stationarity condition[13]

$$P\mu_s = \mu_s. \tag{6.1}$$

Although a general explicit solution to this linear system is not tractable, the fact that P defines an irreducible ergodic Markov chain means that μ_s can also be obtained from any column of $\lim_{n\to\infty} P^n$.

The stationary distribution is a natural dynamic equilibrium concept in the presence of perpetual fluctuations, against which welfare can be evaluated. The social welfare function is defined as the sum of individual's value function in equation 3.2. When

[13] An equivalent equilibrium distribution condition is a linear system of full balance conditions (see Kelly (2011)). Note that the process here is not time-reversible, and the detailed balance conditions do not hold in this case.



(a) Evolution of y_t and regime

(b) Empirical and stationary distribution

Figure 7: Numerical simulation results

the aggregate cost parameter z_t is fixed, social welfare is a function of y_t only through the utility function $u(y_t)$. In business cycle application, a monotone increasing $u(y_t)$ is a sensible assumption, as higher states y_t represent higher aggregate activity. Social welfare in this case will depend on the average time spent in higher states relative to lower states, which is a complicated function of the underlying birth and death parameters.

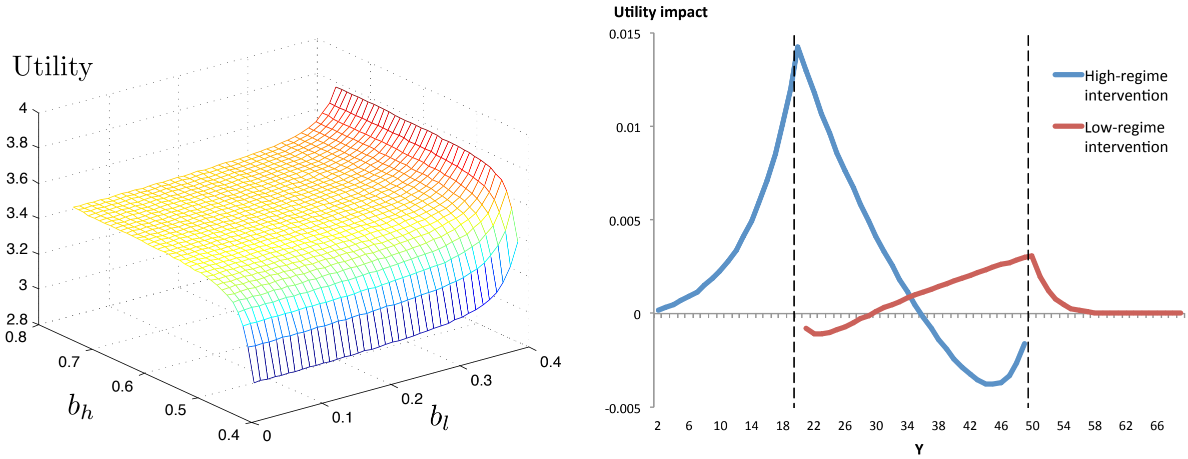
6.2 Simulation

It is straightforward to solve for the stationary distribution numerically. Consider the following configuration:

- Utility function: $u(y) = \log(y)$
- State space parameters: $N = 70$, $y^0 = 20$, and $y^1 = 50$
- Birth and death rates: $b_h = 0.45$, $b_l = 0.3$, and $\delta = 0.2$

Under these set of parameters where $b_h < 1-\delta-b_l$, the cycle has a gradual build-up period in high regime followed by a more abrupt correction in low regime, similar to a typical business cycle or a bubble phenomenon. A random draw of y_t under this configuration is shown in figure 7a, which also plots the corresponding regime (1 means high regime, and -1 low regime). The time-series are visually not very different from typical detrended macro variables, and a simple autoregressive model also fits the series well. However, mistaking the series as a single-regime process will result in incorrect long-run prediction (a steady state rather than a stationary distribution) as well as suboptimal short-run forecast which is regime-dependent.

The empirical distribution of the realized time-series is depicted in figure 7b. In the long-run, the empirical distribution converges to the stationary distribution, which can be



(a) Welfare as a function of b_h and b_l **(b)** Effect of policy intervention

Figure 8: Welfare implications of comparative statics

computed by first deriving μ_s as discussed previously, and summing probabilities across regimes to obtain $\mu(y) = \mu_h(y) + \mu_l(y)$ for $y = 0, 1, ..., 70$. The stationary distribution is shown in figure 7b, which is non-symmetric and is skewed to the right. With a slow buildup and a sharp correction, the economy ends up spending more time in lower states than in higher states. It follows from the model's symmetry and an increasing utility function, that social welfare is lower than the conjugate case of $b_h > 1 - \delta - b_l$, where the correction phase is more gradual than the boom. The relative cycle speeds in different regimes therefore have non-trivial welfare implications.

Influencing the parameters b_h and b_l via policy can potentially raise social welfare. All policy considered here is assumed to take a rule-based form, namely the policy function (of state and regime) remains fixed throughout. For example, policy that leads to a generalized increase in the birth rates should intuitively raise the time spent in higher states. Simulation confirms that social welfare is increasing in both b_h and b_l, as shown in figure 8a, which plots the utility surface as a function of b_h and b_l over the parametric range where a cycle is well-defined. What is perhaps less intuitively obvious, is that raising the high-regime birth rate b_h is subject to diminishing marginal utility, whereas the opposite is true for the low-regime birth rate b_l. The reason for this asymmetry is that, a higher b_h precipitates a regime switch, reducing time spent in high states and introducing an offsetting drag on welfare. On the other hand, a higher b_l merely lowers the downward speed and lessens the severity of the low regime. On balance, this implies that the economy can approach the unconstrained optimal point as b_l approaches the maximum level, even if b_h remains at moderate levels. The policy implication is consistent with the conventional pragmatic wisdom that a policy stimulus should be implemented 'counter-cyclically'.

In practice, policy may be more targeted, designed to influence the transition rates at only certain states. In figure 8b, we plot the additional utility resulting from a state-specific increase in b_h (high-regime intervention) and b_l (low-regime intervention) by 0.05, applied at each different state y.[14] For targeted policy, the marginal benefit is largest when the intervention takes place after the economy has already bottomed out, and the regime has just switched to high. Implementing the targeted policy too late in the high regime, i.e. when y has already risen for some time, is however counterproductive. A targeted policy in low regime raises welfare for a larger set of states, and is most beneficial when the bubble has just burst and the regime has just switched to low. The additional welfare gained is however more limited. Thus, for targeted policy, the optimal plan is to act early; in high regime to speed up the recovery, and in low regime to delay the plunge. Acting too late in a high regime will only quicken a regime switch, which can lower welfare. Acting late in a low regime will delay a welfare-enhancing regime switch. This optimal plan is reminiscent of the policy 'front-loading' wisdom.

6.3 On estimation

When the parameters are unknown, how might they be estimated from the data? Unfortunately, many existing empirical models differ in important ways from our model and are not applicable. The model's structure is distinct from the Markov regime-switching model introduced in Hamilton (1989), since here the probability of a regime switch does not just depend on the existing regime, but also on the state y_t and the aggregate cost z_t. Our model places more restrictions on the probabilistic structure of a regime switch, and while there is still uncertainty about the precise timing of a switch, it is certainly not an event that can take place in any period with an equal probability. Our model, while similar in spirit, is also different from threshold models, since thresholds in our model are history dependent as well as time-irreversible.

Several empirical strategies can however be readily outlined. Because the model predicts conditional distribution of y_t, the maximum likelihood method can certainly be applied, and a regime-specific likelihood function consistent with the transition probabilities of the relevant birth-and-death process can be written down. The stochastic evolution of aggregate cost z_t needs to be treated as a latent variable, necessitating a state space estimation. Inference about z_t helps restrict the set of possible regime switching points and the likelihood, relevant for forecasting exercises. An alternative approach is to minimize the difference between the conditional stationary distribution and observed empirical distribution, summarized as some metric, to obtain parameter estimates.

[14]The theoretical model assumes that b_h and b_l are constant across states, and thus this exercise is only an approximation. However, the assumption is not indispensable, and some generalization follows immediately. For example if b_h and b_l depends on the state, but the difference $b_h - b_l$ is still fixed, then the cost function remains the same as before, and all formal results follow.

7 Conclusion

This paper proposes a theory of endogenous cycle based on a repeated game with strategic complementarity under incomplete information and learning that exhibits strong hysteresis in equilibrium. In booms, it is difficult to shake off optimism, as a regime continuation sends a public signal that reaffirms and adds to confidence, outweighing the impact of private signals whose aggregate is not observed. Similarly in recessions, as long as there is no clear sign of a recovery, pessimism is the only rational belief to hold in equilibrium, resulting in protracted contraction. Put together, the economy cycles perpetually between the two regimes, even without any actual shocks to fundamentals.

The main building blocks of the theory, including strategic complementarity, the cost heterogeneity structure, Bayesian learning and the global game framework, have all been extensively explored in various literature. The equilibrium cycle therefore derives from ideas already familiar in economics, the combined strength of which ensure that the resulting cycle result is free from typical criticisms that have kept models with cycles at the fringe of macroeconomics. Despite its simplicity, the model delivers welfare implications and quantitative predictions that can potentially be evaluated empirically.

The theory of endogenous cycle is perhaps best thought of as complementary to the prevailing shock-and-propagation view, and there are benefits to allowing both mechanisms to coexist in one general model. If the business cycles were in fact driven by both an endogenous cycle mechanism and amplified exogenous shocks, a traditional macroeconomic model would suffer from a misspecification problem, compromising the ability to evaluate the impulse-response of exogenous shocks. On the other hand, any doubt about the role of an endogenous fluctuation mechanism can only be proved or disproved in a model nesting both endogenous cycle and shock propagation mechanisms. But without any model of endogenous fluctuations, economic theory is ultimately void of an explanation for fluctuations phenomena.

Appendix

Proof to claim 1

Proof. Suppose that in period t, the unique equilibrium is characterised by maximum delay, namely $i_t^* = y_t - \theta$, i..e. from the equilibrium condition 3.5,

$$\bar{H}_t (y_t - \theta) = \int_{z=y_t-\theta}^{\sup Z_t} h_t (z) \, dz = 1$$

Since h_t must have no vacuum, it follows that this condition holds if and only if

$$\inf Z_t \geq y_t - \theta = i_t^*.$$

Thus, from equation 3.7, $\tilde{h}_t = h_t$ and $\tilde{Z}_t \equiv [\max \{y_t - \theta, \inf Z_t\}, \sup Z_t] = Z_t$, and there is no learning update. □

Proof to lemma 1

Proof. Note that by viewing z_t^* as a random walk, we can write

$$z_{t+1}^* = z_t^* + X$$

where X is a uniformly distributed random variable over $[-\varepsilon, \varepsilon]$. Thus h_{t+1} is a mean-preserving spread of h_t. The lemma then follows from a well-known fact that the notion of a mean-preserving spread is equivalent to that of second-order stochastic dominance (see Rothschild and Stiglitz (1970)). □

Proof to claim 2

Proof. Suppose that h_t is unimodal. The aim is to show directly that $h'_{t+1} (z^*)$ changes sign at most only once as z^* increases. Differentiating equation 3.10 gives

$$h'_{t+1} (z^*) = \begin{cases} \frac{1}{2\varepsilon} h_t (z^* + \varepsilon) & \text{for } z^* \in [\inf Z_t - \varepsilon, \inf Z_t + \varepsilon] \\ \frac{1}{2\varepsilon} [h_t (z^* + \varepsilon) - h_t (z^* - \varepsilon)] & \text{for } z^* \in [\inf Z_t + \varepsilon, \sup Z_t - \varepsilon] \\ -\frac{1}{2\varepsilon} h_t (z^* - \varepsilon) & \text{for } z^* \in [\sup Z_t - \varepsilon, \sup Z_t + \varepsilon] \end{cases}. \quad (.1)$$

The sign of $h'_{t+1} (z^*)$ is only ambiguous for $z^* \in [\inf Z_t + \varepsilon, \sup Z_t - \varepsilon]$. Given that $h_t (z^*)$ is unimodal, $h_t (z^* + \varepsilon) - h_t (z^* - \varepsilon)$ changes sign at most only once and the claim is proved. □

Proof to lemma 2

Proof. Under a pure transition, we have

$$\begin{aligned}
\bar{H}_{t+1}(i)-\bar{H}_t(i) &= \int_{z=i}^{\sup Z_t+\varepsilon} h_{t+1}(z)\,dz-\int_{z=i}^{\sup Z_t} h_t(z)\,dz \\
&= \frac{1}{2\varepsilon}\int_{z=i}^{\sup Z_t+\varepsilon}\left[\int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'\right]dz-\int_{z=i}^{\sup Z_t} h_t(z)\,dz.
\end{aligned}$$

The double integral on the right hand side is merely a weighted sum, and can be expanded by computing an appropriate weight for each term. Doing so gives

$$\begin{aligned}
\bar{H}_{t+1}(i)-\bar{H}_t(i) &= \int_{z=i+\varepsilon}^{\sup Z_t} h_t(z)\,dz+\int_{z=i-\varepsilon}^{i+\varepsilon}\left(\frac{z-i+\varepsilon}{2\varepsilon}\right)h_t(z)\,dz \\
&\quad -\int_{z=i}^{\sup Z_t} h_t(z)\,dz \\
&= \int_{z=i}^{\sup Z_t} h_t(z)\,dz-\int_{z=i}^{i+\varepsilon} h_t(z)\,dz+\int_{z=i-\varepsilon}^{i+\varepsilon}\left(\frac{z-i+\varepsilon}{2\varepsilon}\right)h_t(z)\,dz \\
&\quad -\int_{z=i}^{\sup Z_t} h_t(z)\,dz \\
&= \int_{z=i-\varepsilon}^{i}\left(\frac{z-i+\varepsilon}{2\varepsilon}\right)h_t(z)\,dz-\int_{z=i}^{i+\varepsilon}\left(\frac{i+\varepsilon-z}{2\varepsilon}\right)h_t(z)\,dz. \qquad (.2)
\end{aligned}$$

Notice that the linear weight on each term of the R.H.S of equation .2 is completely symmetric around i. Thus, it is possible to collect terms with the same weight, and rewrite equation .2 as

$$\bar{H}_{t+1}(i)-\bar{H}_t(i)=\int_{z=0}^{\varepsilon}\left(\frac{\varepsilon-z}{2\varepsilon}\right)\left[h_t(i-z)-h(i+z)\right]dz. \qquad (.3)$$

Any crossing point i_t^c between $\bar{H}_{t+1}$ and $\bar{H}_t$ must solve $\bar{H}_{t+1}(i_t^c)-\bar{H}_t(i_t^c)=0$, thus setting equation .3 to zero, a crossing point i_t^c must be a solution to

$$\int_{z=0}^{\varepsilon}\left(\frac{\varepsilon-z}{2\varepsilon}\right)\left[h_t(i_t^c-z)-h(i_t^c+z)\right]dz=0. \qquad (.4)$$

It follows immediately from equation .4 that $h_t(i_t^c-z)-h(i_t^c+z)$ cannot be positive or negative for all $z\in[0,\varepsilon]$. Equivalently put, h_t cannot be strictly monotonic on $[i_t^c-\varepsilon,i_t^c+\varepsilon]$, and hence we have $i_t^c\in[i_t^m-\varepsilon,i_t^m+\varepsilon]$ since the mode defines the threshold for which h_t changes from being increasing to being decreasing. This proves part (1).

Unimodality of h_t only implies that, for any fixed positive z, $h_t(i-z)-h(i+z)$ changes sign from negative to positive at most only once as i increases. If, in addition, h_t is concave on $[i_t^m-2\varepsilon,i_t^m+2\varepsilon]$, then for $i\in[i_t^m-\varepsilon,i_t^m+\varepsilon]$ and $z\in[0,\varepsilon]$, $h_t(i-z)-h(i+z)$ must be increasing in i, and the R.H.S of equation .3 must also be increasing in

i, and there is a unique solution to equation .4, proving part (2). $\square$

Proof to claim 3

Proof. Since $h_{t+1} > h_t$ on $[\inf Z_{t+1}, \inf Z_t] \cup [\sup Z_t, \sup Z_{t+1}]$, if h_{t+1} crosses h_t twice then Z_{t+1} is a union of three ordered disjoint subintervals, Z_{t+1}^A, Z_{t+1}^B and Z_{t+1}^C (ordered from low to high) such that $h_{t+1}(z) < h_t(z)$ if and only if $z \in Z_{t+1}^B$. On Z_{t+1}^A, $h_{t+1} \geq h_t$ and hence $\bar{H}_t(z) - \bar{H}_{t+1}(z)$ is positive and rises with z (recalling that the derivative of $\bar{H}_t$ is $-h_t$). In other words, as $z \in Z_{t+1}^A$ rises, $\bar{H}_t$ and $\bar{H}_{t+1}$ diverge and hence cannot cross on Z_{t+1}^A. Similarly as $z \in Z_{t+1}^C$ decreases, $\bar{H}_t$ and $\bar{H}_{t+1}$ diverge and thus there cannot be a crossing on Z_{t+1}^C. It follows that on Z_{t+1}^B, we have $\bar{H}_{t+1}\left(\inf Z_{t+1}^B\right) < \bar{H}_t\left(\inf Z_{t+1}^B\right)$ and $\bar{H}_{t+1}\left(\sup Z_{t+1}^B\right) > \bar{H}_t\left(\sup Z_{t+1}^B\right)$, and by continuity of $\bar{H}_t$ and $\bar{H}_{t+1}$ there must be at least one crossing on Z_{t+1}^B (this fact also follows from lemma 1). But on Z_{t+1}^B, $h_{t+1} < h_t$ so that $\bar{H}_{t+1} - \bar{H}_t$ contracts, and hence there is at most one crossing on Z_{t+1}^B and the claim follows. $\square$

Proof to claim 4

Proof. Supposing there is no regime switch at date t, differentiating equation 3.7 gives

$$\tilde{h}_t'(z^*) = \left[\frac{1}{\int_{z=\max\{i_t^*, \inf Z_t\}}^{\sup Z_t} h_t(z)\, dz}\right] h_t'(z^*)$$

so that the signs of h_t' and $\tilde{h}_t'$ are identical and the claim follows. If there is a regime switch at date t, then one differentiates equation 3.8 and the same argument applies. $\square$

Proof to lemma 3

Proof. Following claim 1, we could ignore the case of initial maximum delay. Let the belief at time t, $\bar{H}_t$, be fixed. When there is a truncation at the end of period t, followed by a transition mapping at the beginning of period $t+1$, denote the new belief by $\bar{H}_{t+1}^1$ (with primitive density h_{t+1}^1 on set Z_{t+1}^1). On other hand, if there is no truncation but a transition only, then let the new belief be given by $\bar{H}_{t+1}^2$ (h_{t+1}^2). In terms of these notations, a truncation leads to greater delay, ceteris paribus, if i_{t+1}^* is smaller under belief $\bar{H}_{t+1}^1$ than under $\bar{H}_{t+1}^2$ while the state is kept unchanged, $y_{t+1} = y_t$. Thus, in view of the equilibrium condition 3.5 and on the assumption that equilibrium is always unique, there is a greater delay under truncation if $\bar{H}_{t+1}^1(i) > \bar{H}_{t+1}^2(i)$ for all $i \in [y_{t+1} - \theta, y_{t+1}]$ (namely the primitive distribution underlying $\bar{H}_{t+1}^1$ first-order stochastically dominates that of $\bar{H}_{t+1}^2$). Following claim 5, $\tilde{h}_t$ first-order stochastically dominates h_t if there is a truncation, hence a sufficient condition for $\bar{H}_{t+1}^1 > \bar{H}_{t+1}^2$ is that the shock transition rule preserves the first-order stochastic dominance property. For $\bar{H}^1_{t+1}(i) > \bar{H}^2_{t+1}(i)$ to hold for all i, we must have for each i

$$\begin{aligned}
\int_{z=i}^{\sup Z_{t+1}} h^1_{t+1}(z)\,dz &> \int_{z=i}^{\sup Z_{t+1}} h^2_{t+1}(z)\,dz \\
\int_{z=i}^{\sup Z_{t+1}} \left[\frac{1}{2\varepsilon}\int_{z'=z-\varepsilon}^{z+\varepsilon} \widetilde{h}_t(z')\,dz'\right] dz &> \int_{z=i}^{\sup Z_{t+1}} \left[\frac{1}{2\varepsilon}\int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'\right] dz.
\end{aligned}$$

Using equation 3.7, the inequality becomes

$$\int_{z=i}^{\sup Z_{t+1}} \left[C\int_{z'=z-\varepsilon}^{z+\varepsilon} h^{\#}_t(z')\,dz'\right] dz > \int_{z=i}^{\sup Z_{t+1}} \left[\int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'\right] dz \tag{.5}$$

where $C > 1$ is the truncation normalisation factor, and

$$h^{\#}_t(z) \equiv \begin{cases} h_t(z) & \text{for all } z \in [i^*_t, \sup Z_t] \\ 0 & \text{otherwise.} \end{cases}$$

Note immediately that, for $z = \sup Z_{t+1}$ we have

$$C\int_{z'=z-\varepsilon}^{z+\varepsilon} h^{\#}_t(z')\,dz' = \int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz' = 0,$$

whereas for $z \le i^*_t - \varepsilon$, we have

$$C\int_{z'=z-\varepsilon}^{z+\varepsilon} h^{\#}_t(z')\,dz' = 0 \le \int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'.$$

It follows then that a sufficient condition for inequality .5 to hold is that $C\int_{z'=z-\varepsilon}^{z+\varepsilon} h^{\#}_t(z')\,dz'$ crosses $\int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'$ only once on $(i^*_t - \varepsilon, \sup Z_{t+1})$. To check if this single-crossing condition is satisfied, first let z^0 be an arbitrary crossing point, i.e. any solution to

$$C\int_{z'=z^0-\varepsilon}^{z^0+\varepsilon} h^{\#}_t(z')\,dz' = \int_{z'=z^0-\varepsilon}^{z^0+\varepsilon} h_t(z')\,dz'.$$

Since $C > 1$, it trivially follows that $i^*_t > z^0 - \varepsilon$ (and hence a small change in z^0 does not affect the lower integration limit on the left hand side). Differentiate

$$C\int_{z'=z-\varepsilon}^{z+\varepsilon} h^{\#}_t(z')\,dz' - \int_{z'=z-\varepsilon}^{z+\varepsilon} h_t(z')\,dz'$$

locally around z^0 to get

$$\begin{aligned}
Ch_t\left(z^0+\varepsilon\right) - h_t\left(z^0+\varepsilon\right) + h_t\left(z^0-\varepsilon\right) &= (C-1)\,h_t\left(z^0+\varepsilon\right) + h_t\left(z^0-\varepsilon\right) \\
&> 0
\end{aligned}$$

and hence the single crossing condition is satisfied, and the result follows. □

Proof to claim 6

Proof. Let us consider only left truncations (i.e. when the regime is initially high). By definition,

$$\begin{aligned} i_{t+1}^{m} &\equiv \arg\max_{i} h_{t+1}(i) \\ &= \arg\max_{i} \frac{1}{2\varepsilon}\int_{z=i-\varepsilon}^{i+\varepsilon} \tilde{h}_t(z)\,dz \\ &= \arg\max_{i} \frac{1}{2\varepsilon}\int_{z=\max\{i_t^*,i-\varepsilon,\}}^{i+\varepsilon} \frac{h_t(z)}{C}dz \\ &= \arg\max_{i} \int_{z=\max\{i_t^*,i-\varepsilon\}}^{i+\varepsilon} h_t(z)\,dz \end{aligned}$$

where C is a constant as given in equation 3.7. Clearly, we either have

$$i_{t+1}^{m} = \arg\max_{i} \int_{z=i-\varepsilon}^{i+\varepsilon} h_t(z)\,dz$$

in which case a truncation does not change the mode (from that yielded under a pure transition), or

$$\begin{aligned} i_{t+1}^{m} &= \arg\max_{i} \int_{z=i_t^*}^{i+\varepsilon} h_t(z)\,dz \\ &> \arg\max_{i} \int_{z=i-\varepsilon}^{i+\varepsilon} h_t(z)\,dz \end{aligned}$$

and the claim follows. □

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