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Distressed Sales and Liquidity in OTC Markets

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ABSTRACT: We present a stylized model of the over-the-counter (OTC) markets in the tradition of Duffie, Gârleanu, and Pedersen [10] with three distinctive features: (i) Buyers' willingness to pay is private information. (ii) Dividends depend on the state of the macro economy. (iii) Sellers become financially distressed if they cannot sell for too long. Unlike the existing body of work in this literature the probability of trade is endogenous, which in turn opens the door to many interesting results, such as liquidation sales, predation and boom and bust cycles.

Keywords: fire sales, predation, liquidity, boom and bust cycles

JEL: D8, G1

1 Introduction

Participants in over the counter (OTC) markets face two types of frictions. One is searching for a counterpart and the other is, once the counterpart is found, whether the parties can agree on the terms of trade. A recent literature, spurred by the influential work of Duffie, Gârleanu, and Pedersen [10]¹ captures the former friction, but not the second. In these models trade materializes as soon as agents contact each other. In reality, however, it is not uncommon at all for parties to walk away without trading; disagreement, in fact, is the more likely outcome.

With the advancing communication technology the former friction (searching for a counterpart) is a secondary concern. For instance nowadays sellers can package the necessary documentation of a contract, such as a term sheet, detailed long form confirmation, an Excel pricing sheet, into an iPhone or Android app enabling potential buyers to download and browse the content using their smart phones. If interested, the app allows them to contact the seller for further details.² The key friction, therefore, is the latter—that is, whether the buyer wants to purchase or not. This in turn depends on whether the asset is indeed what the buyer is looking for.

Many investors, for instance, participate in OTC derivatives markets for hedging purposes. Investors face different types and levels of risk and therefore need different types of contracts

¹See Afonso [1], Lagos and Rocheteau [16], Rocheteau and Weill [20] or Vayanos and Wang [22] among others.

²See <http://derivatrust.com/index.html>. Electronic trading platforms, such as OTCBB or OTC Link coupled with instant messaging capabilities also make it easy to contact sellers.

to cover their exposure. The idiosyncrasy in buyers' needs implies that an asset that turns out to be a good fit for a particular buyer may be a poor fit for another. The buyer finds out these details only after meeting the seller and discussing the underlying structure of the asset. To capture this idea we assume that the dividend of an asset consists of two parts; a deterministic and aggregate component that is the same across all assets plus an idiosyncratic component that determines how good a fit the asset is for the potential buyer's needs. The buyer realizes the quality of the fit after linking up with the seller and the realization is his private information.

The idiosyncrasy in dividends implies that, unlike the existing literature, the probability of trade is endogenous.³ In equilibrium buyers follow a threshold rule. If the quality of fit is sufficiently high then the deal goes through, otherwise buyers walk away. The aggregate yield, too, affects the probability of trade. As it grows large the deal becomes more lucrative and buyers pay less attention to the goodness of fit because the opportunity cost of not buying kicks in. When the aggregate part grows beyond a smaller threshold distressed trades materialize for sure (they are cheap) and if it exceeds a higher threshold all trades, distressed or regular, go through with probability 1 whether the fit is good or bad.

As hinted above, the likelihood of trade as well as other equilibrium objects depend on whether sellers are distressed or not, which brings us to the second component of the model—the fact that sellers may become desperate if they cannot sell for too long. Anecdotal evidence suggests that sellers may become financially distressed due to, for instance, pressing debt obligations, nearing margin calls, hedging motives, being caught in a short squeeze etc. This notion is captured by an adverse shock, which, if hits, causes sellers to grow impatient and makes them more eager to off-load their holdings. We show that in equilibrium financially distressed sellers pursue *liquidation sales* (or fire sales): they quote prices that are substantially below fundamental values and consequently trade faster.

Liquidation sales are associated with significant profit losses, but more importantly they open the door for predation. We show that during periods where sellers are more likely to encounter financial distress (e.g. crises or recessions) the followings occur. First, the number of fire sales rises. Second, all sellers, regular and distressed, quote lower prices. And most importantly, third, customers exhibit what we call *predatory buying*: they become more selective and hold off purchasing despite the abundance of distressed sales and lower prices. By doing so customers strategically slow down the speed of trade causing the percentage of desperate sellers to grow further. This, in turn, exerts more pressure on sellers forcing them for further price cuts. This cycle dries up liquidity and increases the cost of liquidation for distressed sellers. Indeed, from a distressed seller's point of view, liquidity disappears when it is mostly needed.

Though it lacks an agreed upon definition in the literature, predation is a prevalent feature of financial markets. Anecdotal evidence is abound documenting numerous forms of predatory trading.⁴ Based on these observations, a recent body of theoretical work explores various mechanisms through which predation takes place. For instance in Attari et al. [2] predators lend to the financially fragile preys in an effort to obtain higher profits by trading against them for a prolonged time. In Brunnermeier and Pedersen [5] if a distressed trader is forced to

³In the aforementioned body of work all assets yield identical returns. As long as there are gains from trade (there always is) the deal goes through, which is why trade is automatic contingent on the meeting.

⁴E.g. see Brunnermeier and Pedersen [5] (pp. 1853-4) or Carlin et al.[8].

liquidate, other strategic traders initially sell in the same direction driving down the price even faster and then buy back at dirt cheap prices. Carlin et al. [8] describe an equilibrium where cooperation among traders breaks down leading to predatory trading and episodic illiquidity.⁵ Our model contributes to this literature by providing yet another mechanism through which predation manifests itself: Investors deliberately hold off purchasing, which raises the percentage of distressed sellers in the market, which in turn leads to further price cuts.

The state of the macro-economy, too, affects the equilibrium objects. The aggregate yield is the same across all assets and takes two values; x_h if the economy is in the high state and x_l if it is in the low state. A high state is a period where fundamentals are strong, so all assets yield greater returns. A low state corresponds to the opposite scenario. We analytically show that the market goes through boom and bust cycles as the macro economy switches between high and low states. In the high state prices rise *and* trade speeds (boom) up while in the low state prices fall *and* trade slows down (bust). The reason is this. Assuming states are sufficiently persistent, purchasing the asset in the high state is more lucrative than purchasing it in the low state. This is why buyers accept higher prices in the high state. On the other hand, sellers have no means of transferring the extra value across states, so they have strong incentives to trade while the asset is still valuable. Therefore they limit the price rise to a modest amount making sure that trade indeed speeds up in the high state.

There is an interesting interplay between the changing market conditions and the demographics of agents. We show that during boom episodes the number of owners rises while the number of sellers shrinks. The percentage of distressed sellers, too, shrinks during booms. The opposite happens during busts. The change in demographics is a natural outcome of the previous result. The increased speed of trade in a boom means that more sellers trade and become buyers while at the same time more buyers purchase and become owners; hence the pendulum tilts towards owners and away from sellers. In addition if trade speeds up then sellers quickly trade and exit before becoming distressed, which is why the percentage of distressed sellers falls.

The model naturally suggests two proxies of liquidity, the first of which is the probability of trade and the second is the profit loss in a liquidation sale. We discuss how these measures respond to the parameters of interest. Curiously, though, the proxies almost always point to the opposite directions and disagree whether liquidity improves or worsens when a parameter changes. This is, perhaps, not too surprising since these proxies, by construction, quantify different aspects of liquidity; however it is clear that one cannot rely on a single measure to fully apprehend liquidity. The contradictory nature of the proxies may also explain why in the literature there seems to be no definition of liquidity that is generally agreed upon (Lagos [13]).

2 Model

The model specification is a variation of Duffie, Gârleanu, and Pedersen [10]. We consider a continuous-time economy with a fixed supply $a > 0$ of indivisible assets that yield a flow of dividends q .⁶ Investors are risk neutral and divided into four categories; buyers, non-trading

⁵See also Coval and Stafford [9], Morris and Shin [17], Pritsker [18], Pulvino [19], Shleifer and Vishny [21] among others.

⁶For a model where the asset is divisible and agents' holdings are unrestricted see Lagos and Rocheteau [16].

owners, regular sellers and distressed sellers. Similar to [10] we have a ‘closed loop’ setting where no agent leaves the market and there is no entry from outside; specifically the total measure of agents η is fixed and exceeds a . Each buyer wants to purchase one unit of the asset to consume its dividends. After trading buyers become owners and remain so until they are hit by a liquidity shock which turns them into regular sellers. The shock arrives with a Poisson rate σ and reduces the flow value of dividends from q to zero, which is why sellers wish to trade and liquidate their holdings.⁷ Once the asset has been sold, the seller comes back to the market as a buyer (see the flowchart in Figure 1).

If regular sellers cannot trade for too long then they may become distressed. This notion is captured by another idiosyncratic adverse shock, which, too, arrives at an exogenous Poisson rate $\mu > 0$. The shock is similar in nature to the liquidity shock above and may be associated with factors such as pressing debt obligations, margin calls from other positions etc. Such difficulties are more likely to arise during financial crises or recessions, so it is sensible to think that μ rises during such periods. Buyers and regular sellers have the same discount factor $\delta > 0$ whereas distressed sellers are more impatient and discount future utility with $\bar{\delta} > \delta$. A larger value of $\bar{\delta}$ implies a more severe shock.

The dividend $q = x_s + v$ consists of an aggregate component x_s plus an idiosyncratic component v . The aggregate component x_s is same across all assets and takes two values; x_h if the economy is in the high state and x_l if it is in the low state, where $x_h > x_l$. In the high state fundamentals are strong and all assets yield greater returns. The opposite is true in the low state. The transition is according to a first order Markov process, where

$$\Pr(x_{t+1} = x_s | x_t = x_s) = \lambda, \text{ where } s = h, l.$$

The parameter λ governs persistence and we assume that $\lambda > 1/2$ i.e. if the economy is in state s today, then it is likely to remain in the same state tomorrow. All agents know x_h , x_l and λ . The difference between x_h and x_l filters its way into buyers’ and sellers’ value functions and it is the main reason behind the boom and bust cycle result in Section 5.

The idiosyncratic component $v \in [0, 1]$ is a random draw from the unit interval via the cdf F .⁸ As pointed out in the introduction buyers differ in terms of their tastes and preferences, so the realization of v determines how good a fit the asset is for the buyer’s liking. A high value of v indicates a good fit and a low value indicates a poor fit. We assume that v is independent across buyers, so the same asset may be liked by one investor and disliked by another. From a buyer’s perspective the search process amounts to finding a high enough v . Unlike the aggregate component x_s the value of v does not change over time; once an asset is purchased the buyer enjoys the same v forever. At this point it may seem that the aggregate component x_s plays no role in determining the probability of trade, however this is not true. As it turns out, if x_s is sufficiently large then buyers do not pay any attention to the goodness of fit; all meetings result in trade even if v turns out to be zero. We impose the following assumption on F .⁹

⁷The liquidity shock in the literature is typically associated with hedging needs arising from a position in another market; see, for instance, [10], [16] or [22].

⁸This is a standard technique to accommodate preference heterogeneity among buyers; see for instance Jovanovic [12], Wolinsky [24], among others.

⁹Log-concavity is a mild assumption. Many distributions including Uniform, Normal, Exponential, χ^2 satisfy

- **Assumption 1.** *The survival function $\Phi = 1 - F$ is log-concave, i.e.*

$$f^2(v) + f'(v)\Phi(v) > 0, \forall v.$$

The valuation v is a buyer's private information. The seller cannot observe v ; he only knows the cdf F and the state of the economy. He cannot tailor the price for each customer; so he must quote the same take-it-or-leave-it price p for all customers. The pricing mechanism in the aforementioned papers, and mostly in reality, is bargaining.¹⁰ Modelling bargaining in a complete information setting is straightforward; however with private information this becomes a non-trivial task as disagreement, delay, multiple or a continuum of equilibria are common in such models; see Kennan and Wilson [14]. To analytically characterize the equilibrium we assume that the transaction necessarily takes place at the initially quoted price.

The market is characterized by trading frictions and operates via search and matching. There are two sources of frictions in the model:

1. Locating and meeting a trading partner. Trading partners are matched over time bilaterally. We assume that agents meet each other according to a Poisson process with fixed search intensity $\alpha > 0$. The arrival rate of a trading partner is proportional to the measure of the partner's group. Specifically, a buyer meets a distressed seller at rate $\alpha m_{s,d}$ and regular sellers at rate $\alpha m_{s,r}$ where $m_{s,d}$ and $m_{s,r}$ denote the steady state measures of distressed and regular sellers in state $s = h, l$. A seller, on the other hand meets buyers at rate $\alpha m_{s,b}$, where $m_{s,b}$ is the measure of buyers.¹¹
2. The second friction is, whether the asset, once located, turns out to be a good fit for the potential buyer. The model is based on private information, so unlike models of complete information, meetings do not necessarily result in trade. The probability of trade $\Phi_{s,j}$ is endogenous and depends on the seller's type j as well as the state of the economy s . With some abuse of notation, we denote the probability of trade as well as the survival function with Φ , because, as it turns out, the probability of trade $\Phi_{s,j}$ equals to $\Phi(\bar{v}_{s,j})$ where $\bar{v}_{s,j}$ is an threshold below which no trade takes place (see below)

As pointed out in the introduction, the key friction is the latter and it is behind most of the results in the paper.

this property; see Bagnoli and Bergstrom [3].

¹⁰Inter-dealer trading systems allow dealers to post and disseminate their prices to the market place. The two major systems are OTC Link and FINRA's OTCBB. The former system is equipped with electronic messaging and allows dealers to negotiate but OTCBB is a quotation only system.

¹¹Duffie and Sun [11] present a formal proof of this argument. See also Vayanos and Wang [22].

3 Analysis

3.1 Steady State Measures

The asset is in fixed supply a , so the measures of agents in possession of the asset (owners + regular sellers + distressed sellers) add up to a , that is

$$m_{s,o} + m_{s,r} + m_{s,d} = a. \quad (1)$$

The total measure of agents η is also fixed and exceeds a . It follows that the steady state measure of buyers, too, is fixed and equals to

$$m_{s,b} = \eta - a > 0.$$

Without loss in generality fix $m_{s,b} = 1$ so that η equals to $1 + a$. Remaining measures $m_{s,o}$, $m_{s,r}$ and $m_{s,d}$ are endogenous and are determined by the fact that in steady state the inflow into a group of investors equals to the outflow from it. Similar to Duffie et al. [10], we have a *closed loop* setup in the sense that no agent ever leaves the market and there is no inflow from outside. Unlike [10], though, buyers in our model are not subject to the liquidity shock.

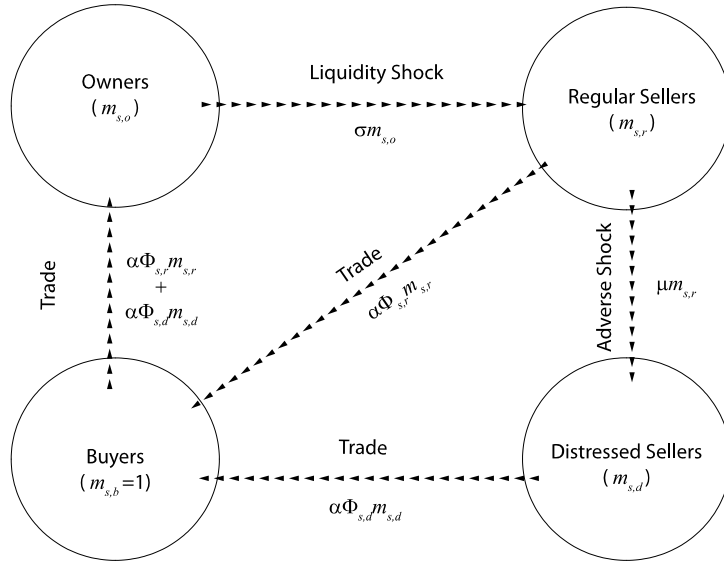


Fig 1 - Flow Chart

Consider desperate sellers. The inflow $\mu m_{s,r}$ consists of regular sellers hit by the adverse shock. The outflow $\alpha m_{s,d} \Phi_{s,d}$ comprises of sellers who trade and become buyers.¹² Setting inflow equal to outflow yields

$$\alpha m_{s,d} \Phi_{s,d} = \mu m_{s,r}. \quad (2)$$

¹²A distressed seller meets a buyer at rate $\alpha m_{s,b}$ and trades with probability $\Phi_{s,d}$ hence the total flow is $\alpha m_{s,b} m_{s,d} \Phi_{s,d}$. Recall that $m_{s,b} = 1$.

Now consider regular sellers. The inflow $\sigma m_{s,o}$ consists of owners hit by the liquidity shock. The outflow has two components: $\alpha m_{s,r} \Phi_{s,r}$ which are regular sellers who trade and become buyers plus $\mu m_{s,r}$ which are regular sellers who become desperate. Therefore

$$\sigma m_{s,o} = \alpha m_{s,r} \Phi_{s,r} + \mu m_{s,r}. \quad (3)$$

Proposition 1 *Equations (1), (2) and (3) pin down the steady state measures $m_{s,o}$, $m_{s,d}$ and $m_{s,r}$ as follows:*

$$\begin{aligned} m_{s,d} &= a \left\{ \frac{\alpha}{\sigma} \Phi_{s,d} \left\{ 1 + \frac{\alpha}{\mu} \Phi_{s,r} \right\} + \frac{\alpha}{\mu} \Phi_{s,d} + 1 \right\}^{-1}, \\ m_{s,o} &= m_{s,d} \times \frac{\alpha}{\sigma} \Phi_{s,d} \left\{ 1 + \frac{\alpha}{\mu} \Phi_{s,r} \right\}, \\ m_{s,r} &= m_{s,d} \times \frac{\alpha}{\mu} \Phi_{s,d}. \end{aligned} \quad (4)$$

The measures depend on exogenous parameters α , a , μ and σ as well as the probabilities of trade $\Phi_{s,j}$ which are endogenous and controlled by buyers.¹³ As we show later, distressed sellers trade at lower prices; so ceteris paribus, buyers wish to encounter such sellers more often. Since buyers control the probabilities they can make this happen. To see how, focus on the fraction of distressed sellers in the market

$$\theta_s \doteq \frac{m_{s,d}}{m_{s,d} + m_{s,r}} = \frac{1}{1 + \frac{\alpha}{\mu} \Phi_{s,d}}. \quad (5)$$

Note that θ increases if the probability of trade $\Phi_{s,d}$ falls. Indeed if buyers squeeze $\Phi_{s,d}$ then distressed sellers cannot trade fast enough; their lingering presence in the market slows down the outflow from the ‘pool of distressed’ and increases θ . As it turns out, the rising θ intensifies the competition among distressed sellers forcing them for further price cuts. This is the basic mechanism behind predation result in section 4.1.

3.2 Owners

Letting Γ_s denote the value function of an owner in state s , we have

$$\delta \Gamma_s = v + \mathbb{E}x_s + \sigma \{ \mathbb{E}\Pi_{s,r} - \Gamma_s \},$$

where

$$\mathbb{E}x_s = \lambda x_s + (1 - \lambda) x_{\bar{s}} \quad \text{and} \quad \mathbb{E}\Pi_{s,r} = \lambda \Pi_{s,r} + (1 - \lambda) \Pi_{\bar{s},r}. \quad (6)$$

¹³The following table summarizes the signs of the partial derivatives of the measures with respect to the parameters of interest (the algebra is skipped):

	σ	μ	$\Phi_{s,d}$	$\Phi_{s,r}$
$m_{s,d}$	+	+	-	-
$m_{s,r}$	+	-	+	-
$m_{s,o}$	-	+	+	+

A rise in the arrival rate of the liquidity shock σ turns more owners into sellers, so $m_{s,d}$ and $m_{s,r}$ rise while $m_{s,o}$ falls. Similarly a rise in the arrival rate of the adverse shock μ causes more relaxed sellers to become distressed; hence $m_{s,r}$ falls while $m_{s,d}$ goes up. The effect of μ on the measure of owners $m_{s,o}$ is more subtle. The rising μ increases the fraction of distressed sellers, and, as we show later, distressed sellers trade faster than regular sellers; so trade speeds up. This, in turn, means that more buyers become owners, hence $m_{s,o}$ goes up. Using similar arguments, and the flowchart, one can explain the signs wrt $\Phi_{s,j}$.

An owner keeps enjoying the idiosyncratic dividend v plus the expected value of the aggregate dividend $\mathbb{E}x_s$ until he is hit by the liquidity shock σ , which turns him into a regular seller, whose value function in state s is denoted by $\Pi_{s,r}$. Rearranging yields

$$\Gamma_s = \frac{v + \mathbb{E}x_s + \sigma \mathbb{E}\Pi_{s,r}}{\sigma + \delta}, \quad (7)$$

Note that the expectations are conditional on the state. The expression $\mathbb{E}x_h$ is the expected value of the aggregate yield contingent on purchasing the asset in the high state; other expectations are likewise. One can show that $\mathbb{E}x_h > \mathbb{E}x_l$ if $\lambda > 1/2$. In addition, below we show that $\mathbb{E}\Pi_{h,r}$ is greater than $\mathbb{E}\Pi_{l,r}$. So, controlling for the idiosyncratic dividend v , the function Γ_h exceeds Γ_l , that is, being an owner is more valuable in the high state than it is in the low state. As pointed out earlier, this difference is the main reason behind the boom and bust cycles.

3.3 Buyers

Now turn to buyers. Letting Ω_s denote their value function in state s we have

$$\begin{aligned} \Omega_s &= \frac{\lambda}{1 + \delta} \times \{\alpha m_{s,r} I_{s,r} + \alpha m_{s,d} I_{s,d} + (1 - \alpha m_{s,r} - \alpha m_{s,d}) \Omega_s\} \\ &\quad + \frac{1 - \lambda}{1 + \delta} \times \{\alpha m_{s,r} I_{\tilde{s},r} + \alpha m_{s,d} I_{\tilde{s},d} + (1 - \alpha m_{s,r} - \alpha m_{s,d}) \Omega_{\tilde{s}}\}, \end{aligned}$$

where

$$I_{s,j} = \int_0^1 \max\{\Gamma_s(v) - p_{s,j}, \Omega_s\} dF(v).$$

The expression $I_{s,j}$ is the expected surplus to a buyer contingent on having met a type j seller in state s . As long as the difference $\Gamma_s(v) - p_{s,j}$ exceeds the opportunity cost Ω_s the buyer purchases, otherwise he walks away. With this information the interpretation of Ω_s is straightforward: With probability λ the current state s persists; at rate $\alpha m_{s,r}$ the buyer meets a regular seller and obtains $I_{s,r}$ and similarly at rate $\alpha m_{s,d}$ he meets a distressed seller and obtains $I_{s,d}$. In case the buyer does not meet a trading partner, he continues to enjoy Ω_s . With probability $1 - \lambda$ the state switches to \tilde{s} and the remainder of the expression can be interpreted similarly. Observe, however, that at the time the state switches from s to \tilde{s} , the measures are still $m_{s,r}$ and $m_{s,d}$.

For any given price $p_{s,j}$ we conjecture an associated threshold (or ‘reservation value’) $\bar{v}_{s,j}$ satisfying

$$p_{s,j} + \Omega_s = \Gamma_s(\bar{v}_{s,j})$$

i.e. leaving the buyer indifferent between buying and searching. After substituting for Γ_s the indifference condition becomes

$$p_{s,j} + \Omega_s = \frac{\bar{v}_{s,j} + \mathbb{E}x_s + \sigma \mathbb{E}\Pi_{s,r}}{\sigma + \delta}. \quad (8)$$

Buyers’ decision is simple: purchase if $v \geq \bar{v}_{s,j}$ and keep searching otherwise. Clearly the

probability of trade $\Phi_{s,j}$ is endogenous and equals to

$$\Phi_{s,j} = \Pr(v \geq \bar{v}_{s,j}) = \Phi(\bar{v}_{s,j}),$$

where $\Phi = 1 - F$ is the survival function. As mentioned earlier, not all meetings result in trade; for trade to occur the asset has to be a good match for the buyer. Substitute Γ_s from (7) into $I_{s,j}$ and use the indifference condition (8) to obtain

$$I_{s,j} = \int_{\bar{v}_{s,j}}^1 \frac{v - \bar{v}_{s,r}}{\sigma + \delta} dF(v) = \int_{\bar{v}_{s,j}}^1 \frac{\Phi(v) dv}{\sigma + \delta}.$$

In the second step we have used integration by parts. Now focus on the expression for Ω_s . Straightforward algebra yields

$$\delta\Omega_s = k\alpha m_{s,r}\mathbb{E}I_{s,r} + k\alpha m_{s,d}\mathbb{E}I_{s,d} + (1-k)\alpha m_{\bar{s},r}\mathbb{E}I_{\bar{s},r} + (1-k)\alpha m_{\bar{s},d}\mathbb{E}I_{\bar{s},d}, \quad (9)$$

where

$$k = \frac{1 - \lambda + \delta}{2 - 2\lambda + \delta} \quad \text{and} \quad \mathbb{E}I_{s,j} = \lambda I_{s,j} + (1 - \lambda) I_{\bar{s},j}. \quad (10)$$

Note that k is a constant between 0.5 and 1, so the function Ω_s is a weighted average of the expected consumer surpluses in both states. The weight k exceeds 0.5 and rises in λ ; i.e. the current state s has a greater weight in Ω_s and its weight gets bigger as the state becomes more persistent.

3.4 Sellers

Sellers' value functions, denoted by $\Pi_{s,j}$, are given by

$$\begin{aligned} \bar{\delta}\Pi_{s,d} &= \lambda X_{s,d} + (1 - \lambda) X_{\bar{s},d} \\ \delta\Pi_{s,r} &= \lambda [X_{s,r} + \mu(\Pi_{s,d} - \Pi_{s,r})] + (1 - \lambda) [X_{\bar{s},r} + \mu(\Pi_{\bar{s},d} - \Pi_{\bar{s},r})], \end{aligned} \quad (11)$$

where

$$X_{s,j} = \alpha\Phi(\bar{v}_{s,j})(p_{s,j} + \Omega_s - \Pi_{s,j}). \quad (12)$$

The expression $X_{s,j}$ is the expected net surplus to a type j seller in state s . The seller encounters a buyer at rate α and the buyer purchases with probability $\Phi(\bar{v}_{s,j})$. If trade occurs the seller obtains price $p_{s,j}$ plus Ω_s (he becomes a buyer now) minus $\Pi_{s,j}$ (he is no longer a seller). With this information it is easy to interpret $\Pi_{s,d}$ and $\Pi_{s,r}$. Note that a regular seller keeps track of the state of the economy as well as the possibility of becoming distressed in each state, whereas a distressed seller worries only about the state of the economy because he is already distressed and will remain so until he sells.

The function $\Pi_{s,r}$ is linked to several contingencies; straightforward algebra yields:

$$(\delta + \mu)\Pi_{s,r} = c_1 X_{s,r} + (1 - c_1) X_{\bar{s},r} + \frac{\mu c_2}{\bar{\delta}} X_{s,d} + \frac{\mu(1 - c_2)}{\bar{\delta}} X_{\bar{s},d}, \quad (13)$$

where

$$c_1 = \frac{\delta\lambda + \mu(2\lambda - 1)}{\delta + \mu(2\lambda - 1)} \quad \text{and} \quad c_2 = 1 - \lambda + c_1(2\lambda - 1)$$

are constants in the unit interval (given that $\lambda > 1/2$).

A type j seller in state s solves

$$\max_{p_{s,j} \in \mathbb{R}_+} \Pi_{s,j} \quad \text{s.t.} \quad \bar{v}_{s,j} = (\sigma + \delta)(p_{s,j} + \Omega_s) - \mathbb{E}x_s - \sigma \mathbb{E}\Pi_{s,r}$$

taking Ω_s as given.¹⁴ The function $\Pi_{s,j}$ is a weighted average of $X_{s,j}$ at different nodes; so, the optimal price $p_{s,j}$ must, by the Bellman principle, maximize the net surplus $X_{s,j}$. The FOC, thus, is given by¹⁵

$$p_{s,j} + \Omega_s - \Pi_{s,j} = \frac{\Phi(\bar{v}_{s,j})}{(\sigma + \delta)f(\bar{v}_{s,j})}, \quad \forall s, j. \quad (14)$$

It is easy to verify the second order condition;¹⁶ thus the solution above corresponds a maximum. Inserting the FOC into (12) yields

$$X_{s,j} = \frac{\alpha\Phi^2(\bar{v}_{s,j})}{(\sigma + \delta)f(\bar{v}_{s,j})}, \quad \forall s, j.$$

Substituting this into (11) and (13) yields closed form expressions of the value functions $\Pi_{s,d}$ and $\Pi_{s,r}$ when sellers optimize. Now we can define the equilibrium.

Definition 2 *A steady-state symmetric equilibrium is characterized by value functions Γ_s , Ω_s , $\Pi_{s,d}$, $\Pi_{s,r}$ given by (7), (9), (11), (13) and the pair $\mathbf{v}^* = \{\bar{v}_{s,j}^*\} \in [0, 1]^4$ and $\mathbf{p}^* = \{\bar{p}_{s,j}^*\} \in \mathbb{R}_+^4$ satisfying indifference (8) and profit maximization (14). The steady state measures $m_{s,d}^*$, $m_{s,r}^*$ and $m_{s,o}^*$, also implicitly part of the equilibrium, can be recovered from (4) by substituting $\Phi_{s,j} = \Phi(\bar{v}_{s,j}^*)$.*

Combine the indifference conditions in (8) with FOCs (14) to obtain

$$\Delta_{s,j} \doteq \Phi(\bar{v}_{s,j})/f(\bar{v}_{s,j}) + (\sigma + \delta)\Pi_{s,j} - \mathbb{E}x_s - \sigma \mathbb{E}\Pi_{s,r} - \bar{v}_{s,j} = 0 \quad \text{for all } s, j. \quad (15)$$

Existence of an equilibrium amounts to showing that there exists some $\mathbf{v}^* \in [0, 1]^4$ satisfying (15). However there are four equilibrium conditions all of which are non linear in \mathbf{v}^* , so it is not practical to attempt to prove existence and analytically characterize the equilibrium for the full-fledged model. Instead we take the following route.

¹⁴From the seller's point of view, cutting the price directly improves the buyer's willingness to trade, but the seller fails to take into account how this drop changes equilibrium prices and the buyer's value of search; see [7].

¹⁵Alternatively one can directly differentiate the value functions $\Pi_{s,d}$ and $\Pi_{s,r}$ wrt the prices. One still gets the same FOCs, but inevitably the algebra is more cumbersome.

¹⁶We have

$$X_{s,j}'' = -\alpha(\sigma + \delta) \times \{f'(\bar{v}_{s,j})(\sigma + \delta)\{p_{s,j} + \Omega_s - \Pi_{s,j}\} + 2f(\bar{v}_{s,j})\}.$$

Substitute the FOC (and omit the argument $\bar{v}_{s,j}$) to obtain

$$X_{s,j}'' = -\alpha(\sigma + \delta) \times \{f'\Phi + 2f^2\}/f.$$

The expression is negative because of log concavity.

The model has two major components: (i) sellers become desperate as they are unable to sell and (ii) the economy switches between high and low states. In the next section we ignore the second component and assume that the economy remains in the same state forever. Then, in section 5 we take the opposite approach; we focus on the economy's transition between high and low states assuming that no seller becomes distressed. These simplifications allow us to obtain several key results analytically. In section 6 we return to the full fledged model.

4 Liquidation Sales and Predation

Let $\lambda = 1$ so that the economy remains in the same state s forever (absorbing Markov state). Equations in (15) simplify to

$$\Delta_r(\bar{v}_r, \bar{v}_d) = \Phi(\bar{v}_r) / f(\bar{v}_r) + \delta\Pi_r - x - \bar{v}_r = 0 \quad \text{and} \quad (16)$$

$$\Delta_d(\bar{v}_r, \bar{v}_d) = \Phi(\bar{v}_d) / f(\bar{v}_d) + (\sigma + \delta)\Pi_d - \sigma\Pi_r - x - \bar{v}_d = 0, \quad (17)$$

Now, we have two, instead of four, equations to analyze and note that throughout this section we dispense with the state index s .

Proposition 3 *The equilibrium exists and it is unique. In equilibrium distressed sellers pursue 'liquidation sales'; they accept to trade at lower prices and consequently sell faster, i.e. $p_d^* < p_r^*$ and $\Phi_d^* > \Phi_r^*$.*

In the proof we show that the locus of $\Delta_r = 0$ is downward sloping (wrt \bar{v}_r) whereas the locus of $\Delta_d = 0$ is upward sloping; so, they intersect once in the $\bar{v}_r - \bar{v}_d$ space (as seen in panel 2c), which implies that there exists a unique \mathbf{v}^* satisfying (16) and (17).

More importantly, the equilibrium is characterized by liquidation sales (or fire sales). After being hit by the adverse shock a distressed seller grows impatient and quotes a lower price in an effort to quickly exit from his position. The price-cut produces the desired outcome. The inequality $\Phi_d^* > \Phi_r^*$ says that distressed trades materialize faster than regular trades.

Attempting a liquidation sale, of course, is costly. Had the seller not become distressed he would have traded at p_r^* (the "fundamental value") but the shock forces him to trade at the lower price p_d^* ; so the difference is the forgone profits incurred in the liquidation process. The ratio $\xi = (p_r^* - p_d^*) / p_r^*$ —the profit loss as a percentage of the fundamental value—therefore is a natural proxy for liquidity. The higher ξ , the more costly the liquidation, and therefore the lower the liquidity. In section 7 we discuss how ξ responds to the key parameters of the model.

Liquidation sales are prevalent in financial markets (e.g. see Coval and Stafford [9] or Shleifer and Vishny [21]) and occur for a variety of reasons including paying regulatory fines, meeting margin calls or other pressing debt obligations—all of which are summarized in the adverse shock in our model.¹⁷ Furthermore liquidation sales typically come with spill-over effects onto regular sales (Shleifer and Vishny [21]) and may trigger predatory buying. We touch upon these issues below.

¹⁷Real asset markets, too, exhibit fire sales. Pulvino [19] finds that commercial airplanes sold by distressed airlines brings 10 to 20 percent lower prices when compared to planes sold by undistressed regular airlines. See also Campbell et al. [6] for a discussion about foreclosures and fire sales in the real estate market.

4.1 Predation

Proposition 4 *If the adverse shock arrives more often, i.e. if μ rises, then the equilibrium price p_d^* falls yet the probability of trade Φ_d^* decreases. Buyers deliberately delay purchasing from desperate sellers despite the falling prices. This behavior (labelled as ‘predation’) further increases the percentage of desperate sellers.*

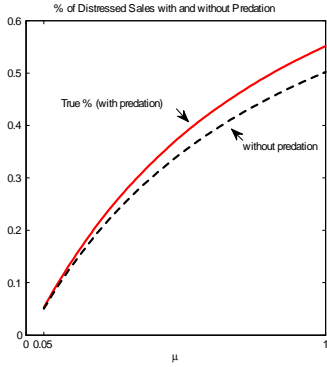


Fig 2a

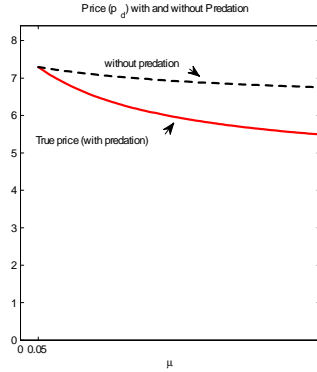


Fig 2b

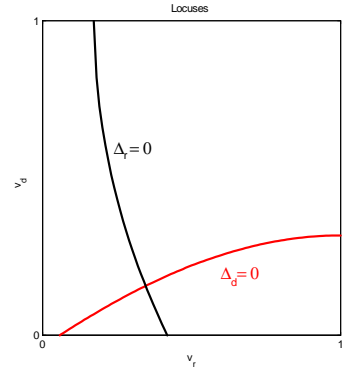


Fig 2c

It is sensible to think that μ increases during financial crises or recessions where an increasing number of sellers encounter financial distress. The proposition says that during such times distressed sellers offer further price cuts, yet buyers are reluctant to purchase. The mechanism behind the result is this. An increase in μ causes sellers’ and buyers’ value functions to move in opposite directions; sellers are worse off and buyers are better off. Specifically, the fraction of desperate sellers, θ , rises with μ , and intensifies the competition for distressed sellers. Realizing that many other sellers are in the same dire situation, distressed sellers are forced to cut their already low prices. The question is whether price cuts generate the desired outcome and the answer is no. The probability of trade Φ_d^* falls, instead of rising. To understand why note that distressed sales come with greater consumer surplus; so the rising θ boosts buyers’ value of search. Realizing that there are plenty of good deals in the market buyers hold off purchasing and search longer, i.e. they lower Φ_d^* . This response has the following feedback effect, which is what we label as *predation*. By lowering Φ_d^* buyers strategically slow down the speed of trade and cause θ to grow further. The growing θ , in turn, puts additional downward pressure on prices and so on.

The arguments can be seen in Figure 2. The solid lines in panels 2a and 2b are the true values of θ and p_d^* , whereas the dashed lines are what they would have been had the probabilities of trade remained intact, so that the change in θ and p_d^* would be purely for exogenous reasons; namely the rising μ .¹⁸ In both figures the difference between the two lines is due to predation.

Anecdotal evidence is abound documenting numerous forms of predatory behavior in financial markets¹⁹. Inspired by these observations, a recent body of theoretical work explores mechanisms through which predation takes place e.g. see Attari et al. [2], Brunnermeier and

¹⁸The dashed lines are obtained by fixing $\Phi_d^* = 0.89$ and $\Phi_r^* = 0.59$ which are the equilibrium values when $\mu = 0.05$. This is why in both simulations the solid and dashed lines start at the same point when $\mu = 0.05$.

¹⁹See for instance Brunnermeier and Pedersen [5] (pp. 1853-4), Carlin et al.[8] or Shleifer and Vishny [21].

Pedersen [5] or Carlin et al. [8] (in the introduction we briefly discuss these papers). Our model contributes to this literature by illustrating a new scheme through which predation manifests itself.

4.2 Probability of Trade and the Aggregate Yield

We now explore the link between the aggregate yield x and the probability of trade. As seen above, from a buyer's point of view the search process amounts to finding a high enough v since all assets yield the same deterministic x . So, it may appear that the aggregate yield x plays no role in determining the probability of trade; however this is not true. As we show next buyers pay little or no attention to v if x is large enough.

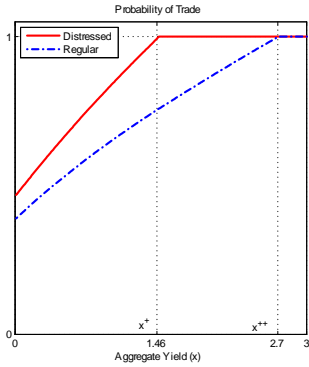
Proposition 5 *Both Φ_r^* and Φ_d^* rise in x . Furthermore*

$$\Phi_r^* < \Phi_d^* < 1 \quad \text{if } 0 < x < x^+$$

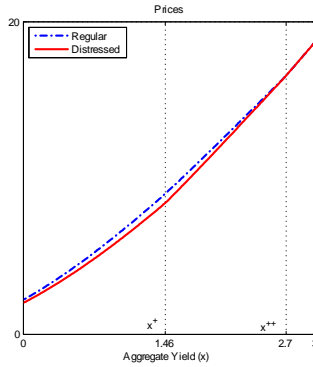
$$\Phi_r^* < \Phi_d^* = 1 \quad \text{if } x^+ \leq x < x^{++}$$

$$\Phi_r^* = \Phi_d^* = 1 \quad \text{if } x^{++} \leq x,$$

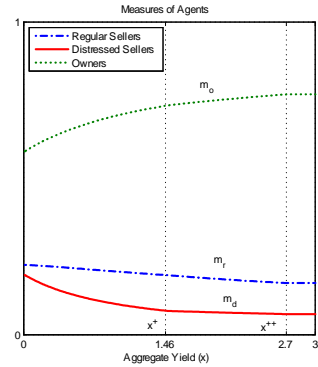
where x^+ and x^{++} are thresholds given by (22) and (27).



3a - Probabilities



3b - Prices



3c - Measures

If x is small then the goodness of fit v has considerable weight in determining whether the deal goes through or not. Indeed if $x < x^+$ then no meeting automatically results in trade; even distressed sellers face some uncertainty about whether or not the transaction will take place. However as x grows large buyers pay less attention to v as the opportunity cost of not buying starts to weigh in, which is why both Φ_r^* and Φ_d^* rise in x . When x grows beyond x^+ distressed trades materialize for sure (they are cheap) and when it exceeds x^{++} all trades, distressed or otherwise, go through for sure. See Figure 3a for an illustration.

Figure 3b is also easy to interpret. Realizing buyers' eagerness to purchase, sellers reflect the rise in x onto their prices, which is why both p_r and p_d are upward sloping. Once x grows beyond x^+ the probability Φ_d^* hits 1 and distressed sellers' FOC no longer holds with equality (prices coming out of the FOC produce probabilities in excess of 1) so they set prices simply

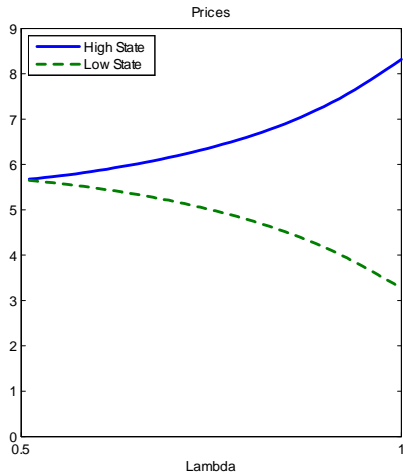
to satisfy $\Phi_d^* = 1$. This is why p_d^* start to grow faster and catches up with p_r^* after x^+ . Once x goes above x^{++} , regular sellers, too, resort to the corner outcome and set prices to implement $\Phi_r^* = 1$.

Figure 3c says that the measure of owners m_o grows large while the measures of sellers, m_r and m_d , shrink with x . This stems from the rising probabilities. Indeed if trade speeds up, then a large number of sellers trade and become buyers, while at the same time the same number of buyers trade and become owners; hence the outcome.

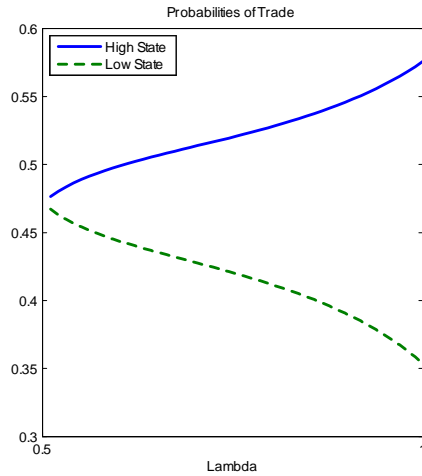
5 Boom and Bust Cycles

In this section we shut down the channel whereby sellers become distressed (specifically we let $\mu = 0$) and assume that all sellers are regular. The key feature that we want to highlight, instead, is the economy's transition between high and low states and how that transition affects the equilibrium objects.

Proposition 6 *Fix $\mu = 0$. The equilibrium prices and the probabilities of sale are high in the high state and low in the low state i.e. $p_h^* > p_l^*$ and $\Phi_h^* > \Phi_l^*$.*



4a - Prices



4b - Probabilities

The proposition says that the market booms in the high state (prices are high, expected time to sale is short and the trade volume is high) and goes bust in the low state (prices are low, expected time to sale is long and the trade volume is low). The intuition is this. The asset generates higher returns in the high state, so controlling for the probability of trade, buyers are ready to pay more. Alternatively, controlling for prices, they are more eager to purchase. Realizing this, sellers increase prices in the high state; however they limit the price rise to a modest amount. This is because they cannot transfer the additional value across states, so they have strong incentives to trade while the asset is still valuable. By limiting the price increase to a small amount they ensure that buyers are indeed more likely to purchase in the high state.

For an illustration see panels 4a and 4b, where equilibrium objects are plotted against the persistence parameter λ . Clearly we have $p_h^* > p_l^*$ and $\Phi_h^* > \Phi_l^*$. Note that a rising λ exacerbates the price and probability gaps across states; so, transitional fluctuations are most severe if $\lambda \approx 1$ and less pronounced if $\lambda \approx 1/2$.

6 Simulations

Now we turn to the full fledged model and provide some sensitivity analysis via numerical simulations. We assume that the asset is in total supply $a = 1$. It pays $x_h = 1$ in the high state and $x_l = 0$ in the low state while the idiosyncratic dividend v is uniformly distributed in $[0, 1]$. The persistence parameter λ is set to 0.8, so a state, on average, persists for 5 periods. The arrival rate of the liquidity shock σ equals to 0.3; hence the expected duration of ownership is 3.33 periods. The adverse shock arrives at rate $\mu = 0.4$. All agents, except distressed sellers, have time preferences with discount rate $\delta = 0.1$, whereas distressed sellers are more impatient and have $\bar{\delta} = 0.3$. Finally the search intensity α equals to 1.

$v \sim U(0, 1)$	$x_h = 1$	$x_l = 0$	$\lambda = 0.8$	$a = 1$
$\sigma = 0.3$	$\mu = 0.4$	$\delta = 0.1$	$\bar{\delta} = 0.3$	$\alpha = 1$

Figures 5, 6 and 7 plot equilibrium objects against the parameters of interest μ , $\bar{\delta}$ and σ .

- **Observation 1.** *The results about liquidation sales and boom and bust cycles obtain in the full fledged model.*

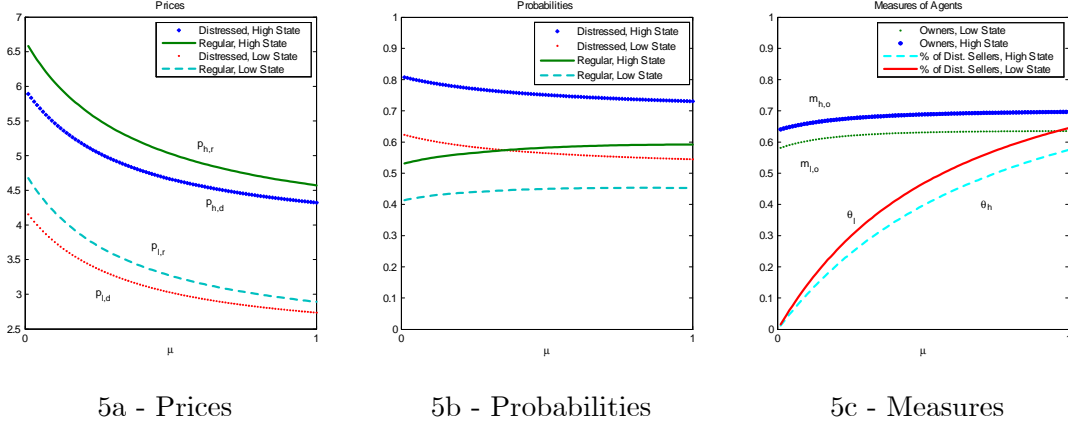
Panels *a* and *b* in Figures 5, 6 and 7 show that $p_{*,r}^* > p_{*,d}^*$ and $\Phi_{*,r}^* > \Phi_{*,d}^*$, which means that in either state of the economy distressed sellers trade at lower prices and sell faster. The figures also reveal that $p_{h,*}^* > p_{l,*}^*$ and $\Phi_{h,*}^* > \Phi_{l,*}^*$, i.e. controlling for sellers' types, prices and probabilities are high in the high state and low in the low state. Altogether the observations suggest that the analytic results in Propositions 4 and 6 (liquidation sales and boom and bust cycles) go through in the full fledged model.

- **Observation 2.** *In a booming market the number of owners rises while the number of sellers shrinks. The percentage of distressed sellers, too, shrinks in booms. The opposite happens in a market that goes bust.*

Panel *c* in Figures 5, 6 and 7 illustrates the measure of owners $m_{h,o}$ and $m_{l,o}$ in either state of the economy. From this one can deduce the total measure of sellers (regular + distressed) because the asset is in fixed supply $a = 1$; thus owners plus sellers add up to 1. So, if one measure goes up then the other goes down. Instead of plotting sellers' measures $m_{l,r}$, $m_{h,r}$, $m_{l,d}$ and $m_{h,d}$ separately (the graph becomes too crowded) we simply plot the percentage of desperate sellers θ_h and θ_l in either state of the economy.

The simulations in 5c, 6c and 7c reveal that $m_{h,o} > m_{l,o}$ and $\theta_h < \theta_l$, confirming the claim made in the observation. The inequalities follow from the fact that trade is more likely in the high state. The increased speed of trade means that in the high state more sellers trade

and become buyers while at the same time more buyers purchase and become owners; hence $m_{h,o} > m_{l,o}$. In addition when trade speeds up sellers quickly trade and exit before becoming distressed, which is why $\theta_h < \theta_l$.

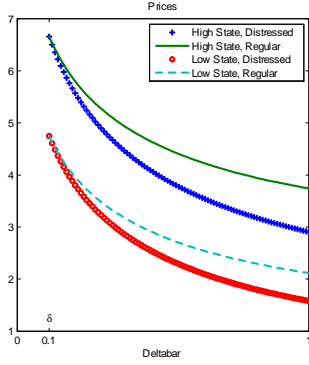


- **Observation 3.** *The predation result obtains in the full model.*

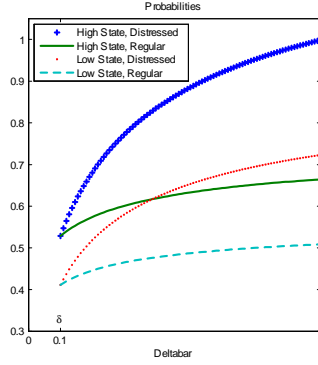
Figure 5 provides a broader picture of the predation result. An increase in μ has three consequences. First, more sellers become distressed and attempt liquidation sales; see the rising percentage of distressed sales in panel 5c. Second, all sellers, regular and distressed, trade at lower prices (panel 5a). Third, customers are reluctant to purchase. Despite the rising number of distressed sales and falling prices, the probability of trade either remains almost flat or, in fact, falls (panel 5b).

We have already discussed the mechanism behind predation, but there is a point to add here. Regular sellers, too, are worse off because of the rising μ . Facing an increasing prospect of becoming distressed in the future, they significantly reduce their prices in an effort to quickly sell before being hit by the shock; see the falling $p_{.,r}^*$ in 5a. This is the *spillover effect* of distressed sales onto the regular sales. The difference between $p_{.,r}^*$ and $p_{.,d}^*$ vanishes as μ grows large, which implies that regular sellers, in fact, cut prices more dramatically than distressed sellers.²⁰

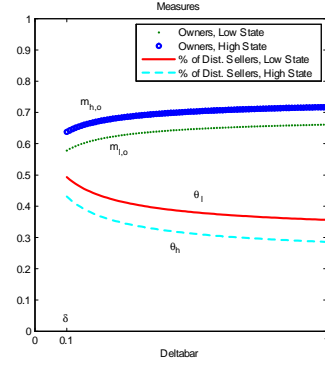
²⁰Distressed sellers are not afraid of becoming distressed anymore; indeed their value function $\Pi_{s,d}$ does not contain μ . Regular sellers, on the other hand, are afraid of becoming distressed (their value function $\Pi_{s,r}$ decreases in μ) hence the effect of μ is more pronounced on regular sellers. This is why $p_{.,r}^*$ falls sharper than $p_{.,d}^*$.



6a - Prices



6b - Probabilities

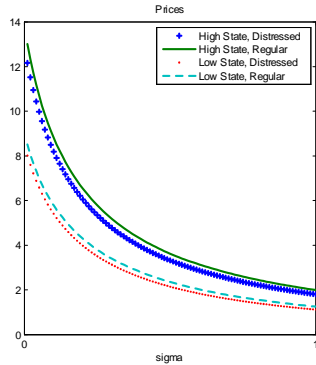


6c - Measures

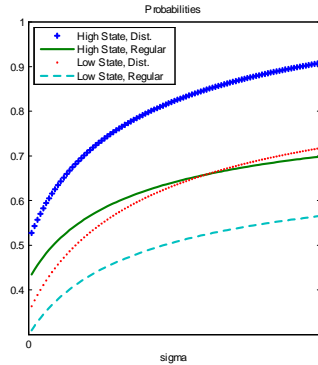
- **Observation 4.** As $\bar{\delta}$ rises, i.e. as the shock becomes more biting, all prices fall, trade speeds up and the percentage of distressed sales falls.

Recall that $\bar{\delta}$, by assumption, must exceed δ , which is why it starts from $\delta = 0.1$ in the simulation. A rise in $\bar{\delta}$ makes desperate sellers even more impatient. Regular sellers, too, are affected by the rising $\bar{\delta}$ as they face a grimmer outlook if they were to become distressed one day. Hence all asset prices drop, but the fall in $p^*_{.d}$ is sharper than the one in $p^*_{.r}$ (observe that the price difference is minimum when $\bar{\delta} \approx \delta$, but gets bigger as $\bar{\delta}$ rises).

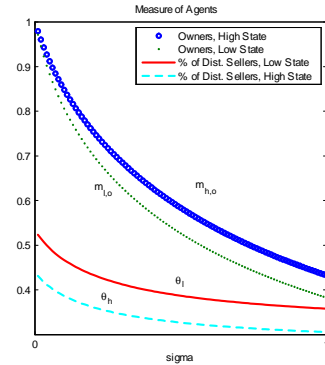
The severity of the shock, unlike its frequency μ , does not change the percentage of distressed sales.²¹ So, from a buyer's perspective, the number of deals stays the same but the deals get sweeter because of the lower prices. Consequently buyers increase the probabilities to catch these deals; as seen in panel 6b. The increased speed of trade raises the number of owners and decreases the percentage of distressed sellers (panel 6c). Note that the rise in $m_{.o}$ or the fall in $\theta_{.}$ is due to the changing probabilities, not due to $\bar{\delta}$ itself.



7a - Prices



7b - Probabilities



7c - Measures

- **Observation 5.** A rise in the arrival rate of the liquidity shock σ increases the number of

²¹Indeed $\partial \theta_s / \partial \bar{\delta} = 0$, i.e. *ceteris paribus* θ is unaffected by $\bar{\delta}$ (one can immediately verify this from (5)).

sellers, which intensifies the competition among them sending prices down and increasing the speed of trade.

When σ rises more owners are hit by the liquidity shock and become sellers (see the falling $m_{\cdot,o}$ in 7c). The increased number of sellers intensifies the competition, which is why prices come down (panel 7a). The sharp fall in prices leads to higher probabilities of trade (panel 7b) hence trade speeds up. The increased speed of trade means that sellers quickly trade and exit before becoming distressed, which is why the percentage of distressed sales θ falls (panel 7c).

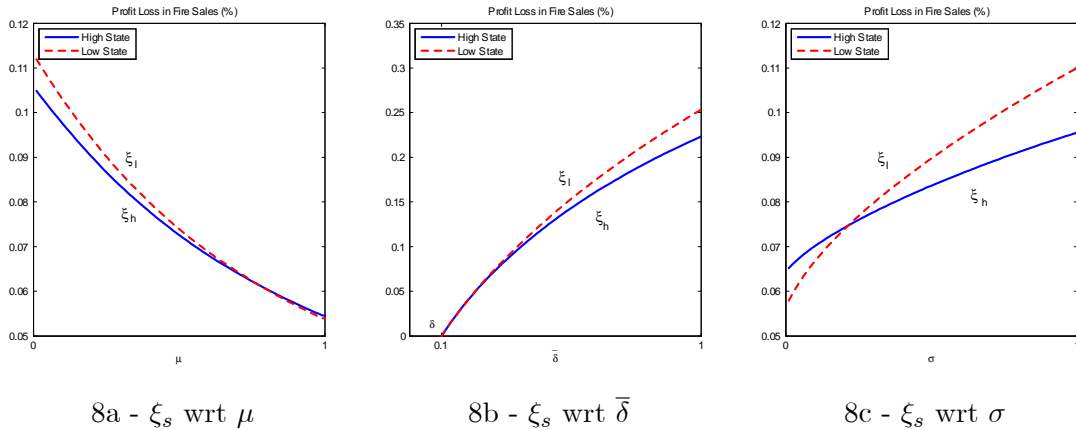
7 Cost of liquidation Sales and Liquidity

As pointed out by Lagos [13], in financial economics a market is considered to be liquid if traders can find a counterpart relatively quickly, and if the cost of trading is relatively small. The model naturally suggests two equilibrium objects measuring these two aspects. The first is the probability of trade $\Phi_{s,j}^*$ which proxies the ease of transacting in the market (this is what Brunnermeier and Pedersen [5] refer as "market liquidity"). The higher the probability, the quicker the trade, the larger the volume, and therefore the higher the liquidity. We have already discussed how $\Phi_{s,j}^*$ responds to the key parameters of the model.

The second proxy

$$\xi_s = \frac{p_{s,r}^* - p_{s,d}^*}{p_{s,r}^*}, \quad s = h, l,$$

is a distressed seller's percentage-wise profit loss. Had the seller not become distressed he would have obtained $p_{s,r}^*$, but in a liquidation he can only obtain $p_{s,d}^*$, so the difference $p_{s,r}^* - p_{s,d}^*$ is the forgone profits. The index ξ_s is, therefore, the percentage-wise loss taking the regular price as a benchmark. Higher values of ξ_s indicate that liquidation sales are costly and therefore the market is illiquid. Below we analyze how ξ_s responds to key parameters of the model.



- **Observation 6.** *In either state ξ_s falls in μ and rises in $\bar{\delta}$ and σ . Attempting a liquidation sale is generally more costly in the low state.*

Panel 8a illustrates the profit loss ξ_s as a declining function of μ . When the shock is infrequent ($\mu \approx 0$) distressed sales take place about 10% below regular sales, however when the

shock is rather frequent ($\mu \approx 2$) the difference is about 5%. This is because, as discussed earlier, regular sellers are more sensitive to μ than distressed sellers are (distressed sellers do not worry about μ as much because they are already distressed). So, although both prices fall, the drop in p_r^* is sharper than the one in p_d^* which is why ξ_s declines. The declining ξ_s , indicates that liquidity improves because distressed sales are less costly. However one has to be careful when interpreting this rather positive-looking result, because the improvement is only relative. In absolute terms all sellers are worse off (all prices fall). Only in relative terms distressed sellers are better off.

Panel 8b shows that ξ_s rises with $\bar{\delta}$. If the shock is mild ($\bar{\delta} \approx \delta$) then there is not much difference between what regular and distressed sellers charge, however as the shock starts to bite ($\bar{\delta} \gg \delta$), then distressed sellers face considerable losses; for instance when $\bar{\delta} \approx 1$ (corresponding to a discount factor 50%) the price difference exceeds 25%. The reason is that distressed sellers are directly affected by $\bar{\delta}$ (it is their own discount factor) whereas regular sellers worry about $\bar{\delta}$ in case they become distressed one day; hence p_d^* falls more sharply than p_r^* , which is why ξ_s goes up. So, from a profit loss perspective liquidity worsens with $\bar{\delta}$.

Finally, panel 8c plots the profit loss against the liquidity shock. A rise in σ turns more owners into sellers which intensifies the competition and brings down the prices. Although both prices fall, p_d^* falls sharper than p_r^* , which is why ξ_s goes up.

- **Observation 7.** *The proxies ξ_s and $\Phi_{s,j}^*$ almost never agree whether liquidity improves or declines.*

The table below summarizes and compares how ξ_s and $\Phi_{s,j}^*$ respond to the key parameters of the model.

<i>Profit Loss ξ_s</i>	<i>vs.</i>	<i>Probability of Trade $\Phi_{s,j}^*$</i>
Falls in μ ; liquidity improves	<i>vs.</i>	Falls in μ (or stays flat); liquidity drops
Rises in $\bar{\delta}$; liquidity worsens	<i>vs.</i>	Rises in $\bar{\delta}$; liquidity improves
Rises in σ ; liquidity drops	<i>vs.</i>	Rises in σ ; liquidity improves

Interestingly the proxies almost always point to opposite directions and never agree whether liquidity improves or worsens. This outcome, perhaps, is not too surprising since the proxies, by construction, capture different aspects of liquidity; however it is clear that one cannot rely on a single measure to fully apprehend liquidity. The contradictory nature of the proxies may be a reason why in the literature there seems to be no definition of liquidity that is generally agreed upon (Lagos [13]).

The proxies agree, though, that liquidity worsens when the economy goes bust. Simulations in Figure 8 show that ξ_l generally exceeds ξ_h indicating that liquidation sales are more costly in the low state. Furthermore, Proposition 6 along with Observation 1 indicate that trade is less likely in the low state. Taken together, both proxies say that liquidity drops as the market transitions into a bust episode.

7.1 Conclusion

This paper contributes to a recent literature, spurred by Duffie et al. [10], studying the OTC markets via search and matching. We complement this literature by assuming that buyers' willingness to pay is private information, that sellers are heterogeneous in terms of their urgency to sell and that asset returns exhibit state dependence. Unlike the existing body of work in this literature the probability of trade is endogenous, which in turn opens the door to many interesting results, such as liquidation sales, predation and boom and bust cycles.

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8 Appendix

Proof of Proposition 3. The proof involves three steps.

Step 0. Preliminaries. To start, substitute $\lambda = 1$ into the expression for $\Pi_{s,d}$ and $\Pi_{s,r}$, given by given by (11) and (13), to obtain

$$\Pi_d = \frac{\alpha}{\bar{\delta}(\sigma+\delta)} \times \frac{\Phi^2(\bar{v}_d)}{f(\bar{v}_d)} \quad \text{and} \quad \Pi_r = \frac{\alpha}{(\sigma+\delta)(\mu+\delta)} \times \left[\frac{\Phi^2(\bar{v}_r)}{f(\bar{v}_r)} + \frac{\mu\Phi^2(\bar{v}_d)}{\bar{\delta}f(\bar{v}_d)} \right]. \quad (18)$$

The following partial derivatives will be useful

$$\begin{aligned} \frac{\partial \Pi_d}{\partial \bar{v}_d} &= -\frac{\alpha\Phi(\bar{v}_d)}{\bar{\delta}(\sigma+\delta)} \times \frac{2f^2(\bar{v}_d)+f'(\bar{v}_d)\Phi(\bar{v}_d)}{f^2(\bar{v}_d)} \quad \text{and} \quad \frac{\partial \Pi_r}{\partial \bar{v}_d} = \frac{\mu}{\mu+\delta} \times \frac{\partial \Pi_d}{\partial \bar{v}_d}, \\ \frac{\partial \Pi_r}{\partial \bar{v}_r} &= -\frac{\alpha\Phi(\bar{v}_r)}{(\sigma+\delta)(\mu+\delta)} \times \frac{2f^2(\bar{v}_r)+f'(\bar{v}_r)\Phi(\bar{v}_r)}{f^2(\bar{v}_r)} \quad \text{and} \quad \frac{\partial \Pi_d}{\partial \bar{v}_r} = 0. \end{aligned}$$

All partial derivatives (except for $\frac{\partial \Pi_d}{\partial \bar{v}_r}$) are negative because of log concavity.

Step 1. Existence-. We will show that the locus of $\Delta_r = 0$ and that of $\Delta_d = 0$ intersect once in $\bar{v}_r - \bar{v}_d$ space, where Δ_r and Δ_d are given by (16) and (17). To start, let

$$\kappa_r(\bar{v}_r) \doteq \{\bar{v}_d \in [0, 1] \mid \Delta_r(\bar{v}_r, \bar{v}_d) = 0\}$$

be the locus of $\Delta_r(\bar{v}_r, \bar{v}_d)$. Similarly let $\kappa_d(\bar{v}_r)$ be the locus of Δ_d . We will establish that κ_r is downward sloping whereas κ_d is upward sloping wrt \bar{v}_r . Differentiating (16) and (17) wrt \bar{v}_r and \bar{v}_d we have:

$$\begin{aligned} \frac{\partial \Delta_r}{\partial \bar{v}_r} &= -\frac{f^2(\bar{v}_r)+f'(\bar{v}_r)\Phi(\bar{v}_r)}{f^2(\bar{v}_r)} + \delta \frac{\partial \Pi_r}{\partial \bar{v}_r} - 1 < 0 & \frac{\partial \Delta_r}{\partial \bar{v}_d} &= \delta \frac{\partial \Pi_r}{\partial \bar{v}_d} < 0 \\ \frac{\partial \Delta_d}{\partial \bar{v}_d} &= -\frac{f^2(\bar{v}_d)+f'(\bar{v}_d)\Phi(\bar{v}_d)}{f^2(\bar{v}_d)} + (\sigma + \delta) \frac{\partial \Pi_d}{\partial \bar{v}_d} - \sigma \frac{\partial \Pi_r}{\partial \bar{v}_d} - 1 < 0 & \frac{\partial \Delta_d}{\partial \bar{v}_r} &= -\sigma \frac{\partial \Pi_r}{\partial \bar{v}_r} > 0 \end{aligned} \quad (19)$$

Focus on $\frac{\partial \Delta_r}{\partial \bar{v}_r}$. The first term and $\frac{\partial \Pi_r}{\partial \bar{v}_r}$ are both negative because of log concavity; hence $\frac{\partial \Delta_r}{\partial \bar{v}_r} < 0$. Similarly $\frac{\partial \Delta_r}{\partial \bar{v}_d} < 0$ since $\frac{\partial \Pi_r}{\partial \bar{v}_d}$ is negative. Therefore $\Delta_r(\bar{v}_r, \bar{v}_d) = 0$ defines $\bar{v}_d = \kappa_r(\bar{v}_r)$ as an implicit function of \bar{v}_r (Implicit Function Theorem) with

$$\frac{d\kappa_r}{d\bar{v}_r} = -\frac{\partial \Delta_r / \partial \bar{v}_r}{\partial \Delta_r / \partial \bar{v}_d} < 0,$$

i.e. the locus of $\Delta_r = 0$ is downward sloping wrt \bar{v}_r . Similarly one can verify that $\frac{\partial \Delta_d}{\partial \bar{v}_d} < 0$ and $\frac{\partial \Delta_d}{\partial \bar{v}_r} > 0$; therefore

$$\frac{d\kappa_d}{d\bar{v}_r} = -\frac{\partial \Delta_d / \partial \bar{v}_r}{\partial \Delta_d / \partial \bar{v}_d} > 0,$$

which means that the locus of $\Delta_d = 0$ is upward sloping.

Now we prove that $\kappa_r(0) > \kappa_d(0)$ and $\kappa_r(1) < \kappa_d(1)$. Start by substituting $(\bar{v}_r, \bar{v}_d) = (0, 0)$ into Δ_r and Δ_d and observe that $\Delta_r(0, 0) > \Delta_d(0, 0)$ because $\bar{\delta} > \delta$. In addition note that $\frac{\partial \Delta_d}{\partial \bar{v}_d} < \frac{\partial \Delta_r}{\partial \bar{v}_d} < 0$ (this follows from log-concavity and that $\frac{\partial \Pi_d}{\partial \bar{v}_d} < \frac{\partial \Pi_r}{\partial \bar{v}_d}$). It follows that $\Delta_r(0, \bar{v}_d) > \Delta_d(0, \bar{v}_d)$ for all $\bar{v}_d > 0$. This, in turn, implies that $\kappa_r(0) > \kappa_d(0)$. Similarly $(\bar{v}_r, \bar{v}_d) = (1, 1)$ into Δ_r and Δ_d and observe that $\Delta_r(1, 1) = \Delta_d(1, 1) = -(1+x)$. Since

$\frac{\partial \Delta_d}{\partial \bar{v}_d} < \frac{\partial \Delta_r}{\partial \bar{v}_d} < 0$ we have $\Delta_r(1, \bar{v}_d) < \Delta_d(1, \bar{v}_d)$ for all $\bar{v}_d < 1$. This inequality implies that $\kappa_r(1) < \kappa_d(1)$.

Since (i) $\frac{d\kappa_r}{d\bar{v}_r} < 0$ and $\frac{d\kappa_d}{d\bar{v}_r} > 0$, (ii) $\kappa_r(0) > \kappa_d(0)$ and (iii) $\kappa_r(1) < \kappa_d(1)$, the Intermediate Value Theorem guarantees existence of a unique $\bar{v}_r^* \in (0, 1)$ such that $\kappa_r(\bar{v}_r^*) = \kappa_d(\bar{v}_r^*) = \bar{v}_d^*$.

Step 2. Liquidation Sales. First we will show that $\bar{v}_d^* < \bar{v}_r^*$, which, in turn, implies that $\Phi(\bar{v}_d^*) > \Phi(\bar{v}_r^*)$. Recall that $\frac{\partial \Delta_r}{\partial \bar{v}_r} < 0$ and $\frac{\partial \Delta_d}{\partial \bar{v}_r} > 0$; hence the difference $\Delta_r - \Delta_d$ decreases in \bar{v}_r . Now, by contradiction suppose that $\bar{v}_r^* = \bar{v}_d^* = v$ and notice that

$$\Delta_r(v, v) - \Delta_d(v, v) = \Pi_r(v, v) - \Pi_d(v, v) = \frac{\alpha \Phi^2(v)(\bar{\delta} - \delta)}{f(v)\bar{\delta}(\mu + \delta)} > 0.$$

The expression is positive because $\bar{\delta} > \delta$. The fact that $\Delta_r(v, v) > \Delta_d(v, v)$ implies that $\bar{v}_r^* \neq \bar{v}_d^*$ because in equilibrium we must have $\Delta_r(\bar{v}_r^*, \bar{v}_d^*) = \Delta_d(\bar{v}_r^*, \bar{v}_d^*)$. The inequality gets worse if $\bar{v}_d^* > \bar{v}_r^*$ because $\Delta_r - \Delta_d$ decreases in \bar{v}_r ; the equilibrium condition can be satisfied only if $\bar{v}_d^* < \bar{v}_r^*$.

The inequality $p_r^* > p_d^*$ is follows from the indifference conditions (8) implying

$$p_r^* - p_d^* = (\bar{v}_r^* - \bar{v}_d^*) / (\sigma + \delta) > 0,$$

which is positive because $\bar{v}_d^* < \bar{v}_r^*$. ■

Proof of Proposition 4. Recall that \bar{v}_r^* and \bar{v}_d^* simultaneously satisfy

$$\Delta_r(\bar{v}_r^*, \bar{v}_d^*) = 0 \text{ and } \Delta_d(\bar{v}_r^*, \bar{v}_d^*) = 0.$$

Omit the superscript * when understood and note that (General Implicit Function Theorem)

$$\frac{d\bar{v}_j}{du} = \frac{\det B_j(u)}{\det A}, \text{ for } u = \mu, x, \bar{\delta}, \sigma \text{ and } j = r, d,$$

where

$$B_r(u) = \begin{bmatrix} -\frac{\partial \Delta_r}{\partial u} & \frac{\partial \Delta_r}{\partial \bar{v}_d} \\ -\frac{\partial \Delta_d}{\partial u} & \frac{\partial \Delta_d}{\partial \bar{v}_d} \end{bmatrix}, \quad B_d(u) = \begin{bmatrix} \frac{\partial \Delta_r}{\partial \bar{v}_r} & -\frac{\partial \Delta_r}{\partial u} \\ \frac{\partial \Delta_d}{\partial \bar{v}_r} & -\frac{\partial \Delta_d}{\partial u} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial \Delta_r}{\partial \bar{v}_r} & \frac{\partial \Delta_r}{\partial \bar{v}_d} \\ \frac{\partial \Delta_d}{\partial \bar{v}_r} & \frac{\partial \Delta_d}{\partial \bar{v}_d} \end{bmatrix}.$$

Note that

$$\det A = \begin{matrix} \frac{\partial \Delta_r}{\partial \bar{v}_r} & \frac{\partial \Delta_d}{\partial \bar{v}_d} \\ (-) & (-) \end{matrix} - \begin{matrix} \frac{\partial \Delta_d}{\partial \bar{v}_r} & \frac{\partial \Delta_r}{\partial \bar{v}_d} \\ (+) & (-) \end{matrix} > 0.$$

The signs of the partial derivatives follow from the proof of the previous proposition; see (19). It follows that

$$\text{sign}(dv_j/du) = \text{sign}(\det B_j(u)),$$

where

$$\det B_r(u) = \frac{\partial \Delta_r}{\partial \bar{v}_d} \frac{\partial \Delta_d}{\partial u} - \frac{\partial \Delta_d}{\partial \bar{v}_d} \frac{\partial \Delta_r}{\partial u} \quad \text{and} \quad \det B_d(u) = \frac{\partial \Delta_d}{\partial \bar{v}_r} \frac{\partial \Delta_r}{\partial u} - \frac{\partial \Delta_r}{\partial \bar{v}_r} \frac{\partial \Delta_d}{\partial u}. \quad (20)$$

The setup is general and it can be used to analyze the signs of the partial derivatives of \bar{v}_r^* and \bar{v}_d^* wrt any one of the parameters $\mu, x, \bar{\delta}, \sigma$; but this proposition is about the sign of \bar{v}_d^* wrt μ ,

so below we focus on $\det B_d(\mu)$. The next proof deals with other scenarios and builds on this setup.

To start, note that

$$\frac{\partial \Delta_d}{\partial \mu} = -\sigma \frac{\partial \Pi_r}{\partial \mu} \quad \text{and} \quad \frac{\partial \Delta_r}{\partial \mu} = \delta \frac{\partial \Pi_r}{\partial \mu},$$

where

$$\frac{\partial \Pi_r}{\partial \mu} = -\frac{\alpha}{(\sigma+\delta)(\mu+\delta)^2} \left[\frac{\Phi^2(\bar{v}_r^*)}{f(\bar{v}_r^*)} - \frac{\delta}{\delta} \frac{\Phi^2(\bar{v}_d^*)}{f(\bar{v}_d^*)} \right]. \quad (+)$$

Note that $\frac{\partial \Pi_r}{\partial \mu}$ is negative, because the expression in the square brackets (call it T_1) is positive.²² Now, substitute $\frac{\partial \Delta_d}{\partial \bar{v}_r}$ and $\frac{\partial \Delta_d}{\partial \bar{v}_d}$, which are given in (19), into $\det B_d(\mu)$ to obtain

$$\det B_d(\mu) = -\frac{\sigma}{\sigma+\delta} \times \frac{\partial \Pi_r}{\partial \mu} \times \frac{2f^2(\bar{v}_r) + f'(\bar{v}_r)\Phi(\bar{v}_r)}{f^2(\bar{v}_r)} > 0. \quad \begin{matrix} (-) \\ (+) \end{matrix}$$

The last expression is positive because of log concavity. We have already established that $\partial \Pi_r / \partial \mu$ is negative; hence $\det B_d(\mu)$ is positive, which implies that $d\bar{v}_d^*/d\mu$ is positive, which in turn implies that the equilibrium probability of sale $\Phi(\bar{v}_d^*)$ falls in μ .

Now we will show that p_d^* , too, falls in μ . Use the FOC (14) and the expression for Π_d , given by (18), to obtain

$$p_d^* + \Omega = \frac{\Phi(\bar{v}_d^*)}{(\sigma+\delta)f(\bar{v}_d^*)} \left[1 + \frac{\alpha}{\delta} \Phi(\bar{v}_d^*) \right].$$

Call the expression on the right hand side T_2 and notice that

$$\frac{dp_d^*}{d\mu} = \frac{\partial T_2}{\partial \bar{v}_d^*} \frac{d\bar{v}_d^*}{d\mu} - \frac{d\Omega}{d\mu}. \quad \begin{matrix} (-) & (+) & (+) \end{matrix}$$

It is easy to verify that $\frac{\partial T_2}{\partial \bar{v}_d^*}$ is negative because of log-concavity; $\frac{d\bar{v}_d^*}{d\mu}$ is positive from above. In addition $\frac{d\Omega}{d\mu} > 0$.²³ Hence $\frac{dp_d^*}{d\mu}$ is negative. ■

Proof of Proposition 5. The first part of the proposition deals with the signs of $\Phi(\bar{v}_r^*)$ and $\Phi(\bar{v}_d^*)$ wrt x . Recall that

$$\text{sign}(d\bar{v}_j^*/du) = \text{sign}(\det B_j(u)) \quad \text{for } j = r, d$$

where $\det(B_r(u))$ and $\det(B_d(u))$ are given by (20). Below we show that $\det B_r(x)$ and

²²To see why combine the FOCs, given by (14), with the value functions Π_d and Π_r , given by (18), to obtain

$$p_r^* - p_d^* = \frac{\Phi(\bar{v}_r^*)}{f(\bar{v}_r^*)(\sigma+\delta)} - \frac{\Phi(\bar{v}_d^*)}{f(\bar{v}_d^*)(\sigma+\delta)} + \frac{\alpha}{(\mu+\delta)(\sigma+\delta)} \times T_1 > 0.$$

This expression is positive since we have established that in equilibrium $p_r^* > p_d^*$. Now focus on the first two terms on the right hand side. The expression $\Phi(v)/f(v)$ falls in v because of log concavity. Since $\bar{v}_r^* > \bar{v}_d^*$ in equilibrium, it follows that the summation of the first two terms is negative. This means that, for $p_r^* > p_d^*$ to hold T_1 must be positive. Hence $\partial \Pi_r / \partial \mu$ is negative.

²³Note that

$$\Omega' \propto m'_r \int_{\bar{v}_r} S(v) dv + m'_d \int_{\bar{v}_d} S(v) dv.$$

In equilibrium $\int_{\bar{v}_r} S(v) dv < \int_{\bar{v}_d} S(v) dv$ since $\bar{v}_d < \bar{v}_r$ (Proposition 3). One can verify that $|m'_d| < m'_r$; hence Ω' is positive.

$\det B_d(x)$ are both negative. Note that

$$\frac{\partial \Delta_d}{\partial x} = \frac{\partial \Delta_r}{\partial x} = -\frac{1}{\sigma + \delta}.$$

It follows that

$$\begin{aligned} \det B_r(x) &= \frac{1}{\sigma + \delta} \left[\frac{\partial \Delta_d}{\partial \bar{v}_d} - \frac{\partial \Delta_r}{\partial \bar{v}_d} \right] < 0 \\ \det B_d(x) &= \frac{1}{\sigma + \delta} \left[\frac{\partial \Delta_r}{\partial \bar{v}_r} - \frac{\partial \Delta_d}{\partial \bar{v}_r} \right] < 0 \end{aligned}$$

In the first line, the expression in square brackets is negative because $\frac{\partial \Delta_d}{\partial \bar{v}_d} < \frac{\partial \Delta_r}{\partial \bar{v}_d} < 0$; see the proof of Proposition 3. The expression in the second line is negative because $\frac{\partial \Delta_r}{\partial \bar{v}_r} < 0$ and $\frac{\partial \Delta_d}{\partial \bar{v}_r} > 0$. The signs of the determinants imply that both \bar{v}_r^* and \bar{v}_d^* fall and therefore $\Phi(\bar{v}_r^*)$ and $\Phi(\bar{v}_d^*)$ rise in x .

Characterization of Corner Solutions. Let \bar{v}_r^+ be the specific value of \bar{v}_r satisfying

$$\frac{\alpha \Phi^2(\bar{v}_r^+)}{(\mu + \delta)f(\bar{v}_r^+)} + \frac{\Phi(\bar{v}_r^+)}{f(\bar{v}_r^+)} - \bar{v}_r^+ - \frac{1}{f(0)} \left[1 + \frac{\alpha \delta}{\delta(\mu + \delta)} \right] = 0. \quad (21)$$

Basic algebra reveals that if $x = x^+$, where

$$x^+ \doteq \frac{\delta(1 + \alpha\delta)}{(\sigma + \delta)f(0)} - \frac{\sigma}{\sigma + \delta} \left[\bar{v}_r^+ - \frac{\Phi(\bar{v}_r^+)}{f(\bar{v}_r^+)} \right], \quad (22)$$

then $\Delta_r(\bar{v}_r^+, 0) = \Delta_d(\bar{v}_r^+, 0) = 0$; hence the the pair $\mathbf{v}^* = (\bar{v}_r^+, 0)$ correspond to an equilibrium.

Recall that \bar{v}_r^* and \bar{v}_d^* both fall in x . So, if $x > x^+$ then \bar{v}_d^* falls below 0, implying that the probability of trade $\Phi(\bar{v}_d^*)$ exceeds 1, which, of course, is impossible. In this parameter region, distressed sellers' FOC no longer holds with equality, since the price satisfying FOC (14) and the indifference condition (8) corresponds to a probability of trade in excess of 1. The concavity of sellers' objective function implies that in this parameter region distressed sellers pick the price p_d satisfying $\bar{v}_d = 0$ (not the FOC) and the indifference condition. More specifically, p_d satisfies

$$\frac{x + \sigma \Pi_r}{\sigma + \delta} = p_d + \Omega, \quad (23)$$

where the equation is obtained by substituting $\bar{v}_d = 0$ and $\lambda = 1$ into the indifference condition (8). Substitute (23) and $\bar{v}_d = 0$, $\lambda = 1$ into distressed sellers value function Π_d , given by (11), to obtain

$$\Pi_d = \frac{\alpha(x + \sigma \Pi_r)}{(\alpha + \delta)(\sigma + \delta)}.$$

Relaxed sellers' problem is still the same. We conjecture that (to be verified below) their FOC

$$p_r + \Omega - \Pi_r = \frac{\Phi(\bar{v}_r)}{(\sigma + \delta)f(\bar{v}_r)} \quad (24)$$

holds with equality. Substitute (24) along with Π_d from above (and $\bar{v}_d = 0$, $\lambda = 1$) into Π_r , given by (13), to obtain

$$\Pi_r = \frac{c_3 \alpha \Phi^2(\bar{v}_r)(\alpha + \delta)}{f(\bar{v}_r)} + c_3 \mu \alpha x \quad (25)$$

where

$$c_3 = [(\delta + \mu)(\alpha + \bar{\delta})(\sigma + \delta) - \mu\alpha\sigma]^{-1} \in (0, 1).$$

Relaxed sellers face the indifference condition

$$\frac{\bar{v}_r + x + \sigma\Pi_r}{\sigma + \delta} = p_r + \Omega. \quad (26)$$

Combine the indifference condition with their FOC (24) above to obtain

$$\Delta_r(v_r) = \frac{\Phi(\bar{v}_r)}{f(\bar{v}_r)} + \delta\Pi_r - \bar{v}_r - x = 0.$$

This function, of course, looks similar to the equilibrium condition in (16), but unlike the former, this one does not depend on \bar{v}_d anymore (now Π_r is a function of \bar{v}_r only). Substitute Π_r from (25) into Δ_r to obtain

$$\Delta_r(v_r) = \frac{\Phi(\bar{v}_r)}{f(\bar{v}_r)} + \frac{\delta c_3 \alpha \Phi^2(\bar{v}_r)(\alpha + \bar{\delta})}{f(\bar{v}_r)} - \bar{v}_r - x c_3 (\sigma + \delta) [\delta(\alpha + \bar{\delta}) + \mu\bar{\delta}] = 0.$$

It is easy to verify that Δ_r falls in \bar{v}_r (assuming log-concavity). In addition $\Delta_r(1) < 0$. So if $\Delta_r(0) > 0$ then there exists an interior $\bar{v}_r^* \in (0, 1)$ satisfying $\Delta_r(v_r) = 0$. Note that

$$\Delta_r(0) > 0 \Leftrightarrow x < x^{++},$$

where

$$x^{++} \doteq \frac{(\alpha + \bar{\delta})\{(\delta + \mu)(\sigma + \delta) + \delta\alpha\} - \mu\alpha\sigma}{f(0)(\sigma + \delta)\{\delta(\alpha + \bar{\delta}) + \mu\bar{\delta}\}}. \quad (27)$$

Therefore if $x < x^{++}$ relaxed sellers' FOC holds with equality and the optimal \bar{v}_r^* is interior;

hence $\Phi(\bar{v}_r^*) < 1$. If, however, $x \geq x^{++}$ then, relaxed sellers, too, set p_r satisfying $\bar{v}_r = 0$ and their indifference condition (26). In this parameter region both $\Phi(\bar{v}_d^*)$ and $\Phi(\bar{v}_r^*)$ are equal to 1. The value functions and other equilibrium objects can be obtained using the steps above. ■

Proof of Proposition 6.

Step 0. Preliminaries. Substitute $\mu = 0$ into (15) to obtain

$$\begin{aligned} \Delta_h(\bar{v}_h, \bar{v}_l) &= \Phi(\bar{v}_h) / f(\bar{v}_h) + (\sigma + \delta)\Pi_h - \bar{v}_h - \mathbb{E}x_h - \sigma\mathbb{E}\Pi_h, \\ \Delta_l(\bar{v}_h, \bar{v}_l) &= \Phi(\bar{v}_l) / f(\bar{v}_l) + (\sigma + \delta)\Pi_l - \bar{v}_l - \mathbb{E}x_l - \sigma\mathbb{E}\Pi_l, \end{aligned}$$

where

$$\Pi_s = \frac{\lambda\alpha\Phi^2(v_s)}{\delta(\sigma + \delta)f(v_s)} + \frac{(1-\lambda)\alpha\Phi^2(v_{\bar{s}})}{\delta(\sigma + \delta)f(v_{\bar{s}})}, \quad s = h, l. \quad (28)$$

Recall that in equilibrium we need $\Delta_h = \Delta_l = 0$. The steady state measures of agents can be obtained from (4) by taking the limit $\mu \rightarrow 0$. We have

$$m_{s,d} = 0, \quad m_{s,o} = \frac{a\alpha\Phi(\bar{v}_s)/\sigma}{\alpha\Phi(\bar{v}_s)/\sigma + 1}, \quad m_{s,r} = \frac{a}{\alpha\Phi(\bar{v}_s)/\sigma + 1}. \quad (29)$$

Buyers' value function Ω_s simplifies to

$$\delta\Omega_s = k\alpha m_{s,r}\mathbb{E}I_{s,r} + (1-k)\alpha m_{\bar{s},r}\mathbb{E}I_{\bar{s},r}, \quad (30)$$

since $m_{s,d} = m_{\bar{s},d} = 0$ (in this version of the model there are no distressed sellers).

Step 1. We show that $\Delta_l - \Delta_h$ increases in \bar{v}_h . We have

$$\Delta_l - \Delta_h = \frac{\Phi(\bar{v}_l)}{f(\bar{v}_l)} - \frac{\Phi(\bar{v}_h)}{f(\bar{v}_h)} + [2\sigma(1-\lambda) + \delta](\Pi_l - \Pi_h) - (2\lambda - 1)(x_l - x_h).$$

Differentiation wrt \bar{v}_h yields:

$$\frac{\partial(\Delta_l - \Delta_h)}{\partial\bar{v}_h} = \frac{f^2(\bar{v}_h) + f'(\bar{v}_h)\Phi(\bar{v}_h)}{f^2(\bar{v}_h)} + [2\sigma(1-\lambda) + \delta] \left(\frac{\partial\Pi_l}{\partial\bar{v}_h} - \frac{\partial\Pi_h}{\partial\bar{v}_h} \right) + 1.$$

The first expression is positive. The second expression, too, is positive if $\partial(\Pi_l - \Pi_h)/\partial\bar{v}_h$ is positive. Note that

$$\Pi_l - \Pi_h = \frac{(2\lambda-1)\alpha\Phi^2(\bar{v}_l)}{\delta(\sigma+\delta)f(\bar{v}_l)} - \frac{(2\lambda-1)\alpha\Phi^2(\bar{v}_h)}{\delta(\sigma+\delta)f(\bar{v}_h)}.$$

Therefore:

$$\frac{\partial(\Pi_l - \Pi_h)}{\partial\bar{v}_h} = \frac{(2\lambda-1)\alpha}{\delta(\sigma+\delta)} \times \Phi(\bar{v}_h) \times \frac{2f^2(\bar{v}_h) + f'(\bar{v}_h)\Phi(\bar{v}_h)}{f^2(\bar{v}_h)}. \quad (31)$$

The expression is positive, again, because of log concavity. Hence $\Delta_l - \Delta_h$ increases in \bar{v}_h .

Step 2. To show $\Phi(\bar{v}_h^*) > \Phi(\bar{v}_l^*)$ we need to demonstrate that $\bar{v}_h^* < \bar{v}_l^*$. By contradiction, suppose that $\bar{v}_h^* = \bar{v}_l^* = \bar{v}$. Then

$$\Delta_l(\bar{v}, \bar{v}) - \Delta_h(\bar{v}, \bar{v}) = (2\lambda - 1)(x_h - x_l).$$

The expression is positive because $x_h > x_l$, which is a contradiction, since in equilibrium $\Delta_l = \Delta_h = 0$. This means that $\bar{v}_h^* \neq \bar{v}_l^*$; so either $\bar{v}_h^* > \bar{v}_l^*$ or $\bar{v}_h^* < \bar{v}_l^*$. Recall that $\Delta_l - \Delta_h$ increases in \bar{v}_h ; thus the inequality gets worse if $\bar{v}_h^* > \bar{v}_l^*$. The equality $\Delta_l = \Delta_h = 0$ is possible only if $\bar{v}_h^* < \bar{v}_l^*$.

Step 3. Now we show that $p_h^* > p_l^*$. A first step is to show that $p_h^* - p_l^*$ decreases in \bar{v}_h . The FOC (14) implies

$$p_h^* - p_l^* = \Pi_h - \Pi_l + \frac{\Phi(\bar{v}_h)}{(\sigma+\delta)f(\bar{v}_h)} - \frac{\Phi(\bar{v}_l)}{(\sigma+\delta)f(\bar{v}_l)} - \Omega_h + \Omega_l. \quad (32)$$

Differentiation wrt \bar{v}_h to yields:

$$\frac{\partial(p_h^* - p_l^*)}{\partial\bar{v}_h} = \underbrace{\frac{\partial(\Pi_h - \Pi_l)}{\partial\bar{v}_h} - \frac{\partial(\Omega_h - \Omega_l)}{\partial\bar{v}_h}}_{G_1} - \frac{f^2(\bar{v}_h) + f'(\bar{v}_h)\Phi(\bar{v}_h)}{f^2(\bar{v}_h)}.$$

The last expression is positive because of log-concavity; so $\partial(p_h^* - p_l^*)/\partial\bar{v}_h$ is negative if G_1 is negative.

The expression $\partial(\Pi_h - \Pi_l)/\partial\bar{v}_h$ is given by (31) and it is negative, but $\partial(\Omega_h - \Omega_l)/\partial\bar{v}_h$

needs some work. We have

$$\begin{aligned} \frac{\delta(\sigma+\delta)}{\alpha(2k-1)} \times (\Omega_h - \Omega_l) &= \{\lambda m_{h,r} - (1-\lambda) m_{l,r}\} \int_{\bar{v}_h}^1 \Phi(v) dv \\ &+ \{(1-\lambda) m_{h,r} - \lambda m_{l,r}\} \int_{\bar{v}_l}^1 \Phi(v) dv \end{aligned}$$

where k is a constant given by (10) and recall that $k > 1/2$. Therefore

$$\frac{\delta(\sigma+\delta)}{\alpha(2k-1)} \times \frac{\partial(\Omega_h - \Omega_l)}{\partial \bar{v}_h} = -\{\lambda m_{h,r} - (1-\lambda) m_{l,r}\} \Phi(\bar{v}_h) + G_2,$$

where

$$G_2 \doteq m'_{h,r} \{\lambda \int_{\bar{v}_h}^1 \Phi(v) dv + (1-\lambda) \int_{\bar{v}_l}^1 \Phi(v) dv\}.$$

Observe that $m_{h,r}$ is given by (29) and its partial derivative wrt \bar{v}_h equals to

$$m'_{h,r} = \frac{\alpha\alpha}{\sigma} f(\bar{v}_h) / [\alpha\Phi(\bar{v}_s) / \sigma + 1]^2 > 0;$$

hence G_2 is positive. Now, substitute $\partial(\Pi_h - \Pi_l) / \partial \bar{v}_h$, given by (31), and the expression of $\partial(\Omega_h - \Omega_l) / \partial \bar{v}_h$ into G_1 and re-arrange to obtain

$$\frac{\delta(\sigma+\delta)}{\alpha} \times G_1 = -\Phi(\bar{v}_h) \times G_3 - (2k-1) G_2,$$

where

$$G_3 \doteq (2\lambda - 1) \left[1 + \frac{f^2(\bar{v}_h) + f'(\bar{v}_h)\Phi(\bar{v}_h)}{f^2(\bar{v}_h)} \right] + (2k-1) \left[\frac{\frac{\alpha}{\sigma}\Phi(\bar{v}_l)+1}{\frac{\alpha}{\sigma}\Phi(\bar{v}_l)+1} - \frac{\frac{\alpha}{\sigma}\Phi(\bar{v}_h)+1}{\frac{\alpha}{\sigma}\Phi(\bar{v}_h)+1} \right].$$

The term G_2 is positive, so G_1 is negative if G_3 is positive, so focus on G_3 . The expression inside the first square brackets in G_3 is positive because of log-concavity. The expression in the second square brackets is positive since $\Phi(\bar{v}_l) < \Phi(\bar{v}_h)$ (from Step 2). The terms $2\lambda - 1$ and $2k - 1$, too, are positive since $\lambda > 1/2$. Hence G_3 is positive, so G_1 is negative and therefore $\partial(p_h^* - p_l^*) / \partial \bar{v}_h$ is negative.

Step 4. Now we can argue that $p_h^* > p_l^*$. Fix \bar{v}_l^* and note that if $\bar{v}_h^* = \bar{v}_l^*$ then $\Pi_h = \Pi_l$ (see 28) and $\Omega_h = \Omega_l$ (see 30) and therefore $p_h^* = p_l^*$ (see 32). Since $p_h^* - p_l^*$ decreases in \bar{v}_h and since $\bar{v}_h^* < \bar{v}_l^*$ in equilibrium (Step 2), it follows that $p_h^* > p_l^*$. ■