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Abstract

Based on the frictional matching framework, the paper provides a theoretical model for a specific type of two-sided platform: The buyer-seller transaction platform. In the model, the number of participants and the source of network externalities are endogenously determined. The platform is shown to exhibit both positive cross-group and negative within-group network externalities. The optimal pricing of the platform depends not only on the cost of providing service and the benefits of the participants, but also on how the marginal entrant (either a buyer or a seller) affects the matching probability. Since the sellers can shift the burden of entry fee to the buyers, the platform never subsidizes the sellers.

1 Introduction

Recent research in two-sided platforms has greatly enhanced our understanding of the factors which might influence the pricing policy of the platforms.[1] For example, early contributions of Rysman (2004), Armstrong (2006) and Rochet and Tirole (2006) all emphasize the importance of externalities in platform's pricing decision, especially its incentives to subsidize the participants who generate large positive externalities to others. Recent contribution by Weyl (2010) also shows that platform pricing can be designed as an insulating tariff to avoid coordination failure in a multi-equilibrium setting, which is common when externalities are present.

The literature on platforms has been based on the unifying insight that profit-maximizing prices charged by the platforms must depend on the degree of externalities. Despite this common denominator, there exists enormous difference between different types of platforms. For example, in certain platforms, there is a clear distinction between between different "sides" (e.g., sellers and buyers in the online auctions, stores and consumers in the credit cards, and female and male in online matching service), while in some other there exists no such distinction (e.g., social networks). Even among platforms in which different sides can be clearly identified, there are some in which buyers and seller can be easily distinguished (e.g., the online auctions)

[1] See Caillaud and Jullien (2003), Rysman (2004), Rochet and Tirole (2006), and Armstrong (2006) for seminal contributions, and the Autumn 2006 symposium issue of *RAND Journal of Economics* and Weyl (2010) for recent developments. Evans and Schmalensee (2007) and Rysman (2009) provide excellent surveys of important issues.

and others in which it cannot (e.g., the online matching service). These differences (and perhaps also others) result in an array of different pricing practices observed in reality.[2] Current literature informs us little beyond the principal that the users who confer greater externalities should be charged less or, when externalities are large enough, even be subsidized. But where do the externalities come from, what determines their size, how they interact with the optimal platform pricing? That different types of platform differ substantially in pricing policy implies that a more detailed investigation of the user's strategic behavior in a platform can further enhance our understanding of the platform's strategic consideration in setting user fees.

In this paper, we set out to answer the above questions in a specific type of two-sided platform: The buyer-seller platform. We explicitly model the price-searching decision of the buyers and price-setting strategy of the sellers, together with the matching outcomes implied by their decision and strategy. A theoretical model which explicitly spells out the details of the participants' interaction within the platforms will have several advantages. First, it can endogenize the size of network externalities. The literature mostly recognizes network externalities in the platform by assuming that the benefit of participating in a platform is a linear function of the number of participants one interacts with.[3] This is a laconic and very useful qualitative approxima-

[2] See Evans and Schmalensee (2007), and especially Table 1 therein, for a thorough but non-exhaustive classification and discussion of pricing strategies in various types of platforms.

[3] An incomplete list of papers using the linear specification is: Armstrong (2006), Armstrong and Wright (2007), Caillaud and Jullien (2001 and 2003), Guthrie and Wright

tion. However, unlike the network products in which the users directly gain utility from the increase of the adopters (see, e.g., Arthur 1989), network externalities in the platforms are usually indirect. Their values critically depend on the rule of transaction and the nature of the participants' interaction, which in turn determine the platform's pricing policy. Second, in a two-sided platform, although the participant enjoy greater positive externalities as the number of participants on the other side of the platform increases, they also suffer a negative externality from participants on the same side.[4,5] This is also an important consideration in the platform's pricing policy, as its incentives to subsidize the participants in order to facilitate positive externalities, a fact much emphasized in the literature, will be checked by the existence of negative externalities. An explicit modeling of interaction within the platform can help our understanding of the inter-play of positive and negative externalities in shaping platform's pricing decision.

Our model incorporate ingredients of both the literature of two-sided platform and frictional price-matching. Specifically, we impose on the traditional model of platforms a frictional matching framework (Burdett et al. 2001) for price-determination. In the framework, a group of sellers (each having one

(2007), Rochet and Tirole (2003, 2006 and 2008) and, Weyl (2010).

[4] Take the online auction platform as an example, although a bidder's (seller's) expected benefit from entering the platform increases with the number of sellers (bidders), his expected benefit also decreases with the number of bidders (sellers).

[5] Belleflamme and Toulenomde (2009), Ellison and Fudenberg (2003) and Ellison, Fudenberg and Mobius (2004) have explicitly considered negative externalities in their model. In the first paper, externalities are exogenous. The latter two are mainly concerned with platform competition, rather than pricing policy. Wely (2010) also considers negative externalities, but only for participants from the *other* side of the platform.

unit of a good) meet a group of buyers (each needing one unit of the good) in a platform. The sellers post prices, and the buyers choose the sellers to buy from. A seller's good is sold (at the price he posts) if and only if at least one buyer visits his store. A buyer, if he is the only visitor of a seller, buys the good with probability one. Otherwise he has an equal chance of buying the good as every other visitor. The platform charges both buyers and sellers for using the platform. Prices set by the platform determine how many buyers and sellers will enter.

We solve for the equilibrium prices of both the platform and the sellers, together with the equilibrium numbers of the sellers and buyers and their utilities. A buyer's utility is shown to be increasing (decreasing) in the number of sellers (buyers). Similarly, a seller's utility is increasing (decreasing) in the number of the buyers (sellers). Moreover, a buyer's or a seller's utility is bounded, regardless of the number of agents on the other side of the platform. The platform's pricing decision is more complicated than in the previous literature. In addition to factors such as service costs and positive externalities considered in the previous literature, it also has to take into consideration its effect on the matching probability and the influence of negative externalities. We therefore provide a model in which externalities, prices, and the number of traders are all endogenously determined. In particular, the presence of negative externalities and the ability of the sellers to pass through their entry fees to consumers are not merely to add a reasonable feature to the platform. It has a strong implication for the platform's pricing policy: the platform never subsidizes the sellers by charging a fee lower than its marginal cost. This provides a theoretical explanation of why, in a buyer-seller platform, it is usually buyers who are subsidized while the sellers seldom are.[6]

Our model is closest to that of Galeotti and Moraga-Gonzalez (2009). Similar to our paper, they also explicitly model the interaction of the buyers and sellers within a platform. In their model, the matching values between the buyers and sellers are (ex post) random, so that it is essentially a product differentiation model. There are two additional features in the paper which are different from our model. First, in their model there is a continuum of buyers whose total mass is restricted to one. Second, the buyers and sellers are ex ante identical, implying that the pricing policy of the platform is either for all the buyers and sellers to enter, or none at all. Given the two features, their paper's main focus is not on how externalities are affected by the numbers of buyers or sellers and, therefore, to show how the platform set fees to balance the tradeoff between entry fees and network externalities, but on the interplay between product variety (in term of the number of sellers) and the buyer's entry fee.

Hagiu (2009) also proposes a model with product differention. The consumer's utility is assumed to be increasing in product variety, which in turn is assumed to be the same as the number of producers. Given the assumptions, the number of producers has a positive network externality for the

[6] As can be seen from Table 1 in Evans and Schmalensee (2007), the sellers are almost always charged by either a usage or an access fee, while the buyers are sometimes free from any charge.

consumers. The paper has not derived the pricing decision of the producer, but it is shown that whether the platform will subsidize the producers or the consumers critically depends on the producer's market power over the consumers, as measured by the ratio of producer's profit to the marginal contribution of an additional producer to consumer's gross surplus.

In our model, the seller's products are identical to the buyers, ex ante or ex post. Therefore, the source of externalities is not product variety, as in the above two papers, but the value of matching probability as determined by the numbers of buyers and sellers and, ultimately, entry fees charged by the platform.

2 Transactions and Frictional Matching

2.1 The Model

Consider the market of a good in which N potential risk-neutral buyers are to trade with N potential risk-neutral sellers on a monopoly platform. Each seller has one unit, and each buyer needs one unit, of the good. The prices charged by the platform determine how many buyers, denoted by N^b, and how many sellers, denoted by N^s, actually enter the platform. Assume that the prices charged by the platform are in the form of entry fees, so that when a seller (buyer) joins the platform, he pays a fee of F^s (F^b).[7] Each

[7] Since in our model the buyers and the sellers transact at most once, the entry fee (properly discounted by matching probability) and transaction fee (commission charged by the platform every time an agent makes transaction) are perfect substitutes.

seller posts a price for the good, which every buyer observes. Based on the observation, every buyer determines the probability that he will visit each seller. A buyer can only buy from the seller he visits. As a result, a seller's commodity might remain unsold if no buyer arrives. This is true even if he is among the lowest-price sellers. If more than one buyer arrives, the good is sold to each visitor with equal probability. This also implies that a buyer might fail to buy the good even if his willingness to pay is greater than the price posted by the seller he visits, as that seller might have more than one visitor.

Under the setup, the role that a platform plays is, on the one hand, to provide price information to the buyers and, on the other hand, to match the buyers and sellers. As mentioned above, a seller might fail to sell his good if no buyer visits him, and a buyer might not be able to buy a good if there are other buyers who visit the same seller. Therefore, this is a price matching model with friction.

The buyers are homogeneous regarding the valuation of the good, but are heterogeneous in the cost of entering and using the platform. Let v^b be each buyer's valuation of the good. Following Armstrong (2006), the heterogeneity in cost is captured by a "cost" or "location" parameter, x^b, which is uniformly distributed on the interval $[0, 1]$. In addition to the entry fee, a buyer with cost parameter x^b will incur an entry cost $t^b x^b$ when he enters the platform, where t^b measures the buyer's sensitivity to the cost. Similarly, the sellers' reservation prices are identical at v^s, and they are heterogeneous in their costs of entering and using the platform, $t^s x^s$, where x^s is also a cost parameter uniformly distributed on $[0,1]$, and t^s is the measure of sensitivity. A buyer's utility function is therefore

$$U^b = \begin{cases} v^b - p - F^b - t^b x^b, & \text{if he visits a seller and buys the good at price } p; \\ -F^b - t^b x^b, & \text{if he visits a seller but fails to buy the good;} \\ 0, & \text{if he does not join the platform.} \end{cases}$$

Similarly, a seller's utility function is

$$U^s = \begin{cases} p - v^s - F^s - t^s x^s, & \text{if he posts a price } p \text{ and sells the good;} \\ -F^s - t^s x^s, & \text{if he posts a price but fails to sell the good;} \\ 0, & \text{if he does not join the platform.} \end{cases}$$

Let the platform's cost of serving a buyer and a seller be c^b and c^s, respectively.[8] We assume that the costs are not very high so that at least two buyers and two sellers enter the platform.[9] The platform's objective is to set the entry fees to maximize its profit:

$$\max_{F^b, F^s} \pi = (F^b - c^b)N^b + (F^s - c^s)N^s.$$

Timing of events is as follow. Stage 1: the platform sets its entry fees (F^s and F^b). Each seller and buyer then decides whether to enter the platform (by incurring entry fees and entry costs). This decision determines the values of N^b and N^s. Stage 2: each seller on the platform posts a price, and each buyer on the platform chooses a probability density function, which determines the

[8] Note that the platform incurs cost c^b (c^s) even if a buyer (seller) fails to trade.

[9] The sufficient condition for this is $c^b \leq \frac{1}{2}(\ln 2)(v^b - v^s) - 4\frac{t^b}{N}$ and $c^s \leq \frac{1}{4}(v^b - v^s) - 4\frac{t^s}{N}$.

buyer's probability of visiting each seller. We call stage 1 as the pricing stage and stage 2 the frictional matching stage. In the following two sections, we will solve for the equilibrium in each stage by backward induction.

2.2 Frictional Matching Stage

In the frictional matching model in Burdett et al. (2001), there is a unique symmetric equilibrium such that every buyer visits each seller with the same probability, and all sellers post the same price. In our model, there is also a symmetric equilibrium:

Proposition 1. (Burdett, et al. 2001) *Given N^b buyers and N^s sellers in the platform, the symmetric equilibrium has every buyer visiting each seller with probability $\frac{1}{N^s}$. Every seller posts the same price*

$$p^* = \frac{v^b[1-(1+\frac{N^b}{N^s-1})(1-\frac{1}{N^s})^{N^b}] + v^s\frac{N^b}{N^s}(1-\frac{1}{N^s})^{N^b}}{1-[1+\frac{N^b}{N^s(N^s-1)}](1-\frac{1}{N^s})^{N^b}}. \tag{1}$$

The expected number of matches is

$$M(N^b, N^s) = N^s[1-(1-\frac{1}{N^s})^{N^b}]. \tag{2}$$

The proof is a simple adaptation of Burdett et al. (2001).[10]

We can rewrite the equilibrium price in (1) as

$$p^* = zv^b + (1-z)v^s, \tag{3}$$

[10] We provide a proof in A1 of the Appendix for the sake of completeness.

where

$$z = \frac{1-(1+\frac{N^b}{N^s-1})(1-\frac{1}{N^s})^{N^b}}{1-[1+\frac{N^b}{N^s(N^s-1)}](1-\frac{1}{N^s})^{N^b}} \in [0,1].$$

For a successful match, the total surplus is $v^b - v^s \equiv v$. Moreover, the benefit for the buyer is $v^b - p^* = (1-z)v$, and that for the seller is $p^* - v^s = zv$. Therefore, the value of z determines the share that the seller gets from the surplus of the transaction. Since z is a function of only N^s and N^b, the buyer's and seller's share of the surplus from transaction is solely determined by their numbers in the platform.

The sellers and the buyers are "complements" in the expected number of matches, $M(\cdot)$, in the sense that $\frac{\partial^2 M}{\partial N^b \partial N^s} > 0$.[11] Moreover, the value of $M(\cdot)$ relative to the number of agents on one side of the platform is a measure of how likely a trader on that side can have a match. The lower its value, the less likely a trader on that side will be successfully matched. Specifically, we measure the degree of friction on side i by $A^i(N^b, N^s) \equiv \frac{M(\cdot)}{N^i}$, $i = b, s$. A^i is also called the arrival rate, and can be shown to be increasing in N^j and decreasing in N^i; $i, j \in \{b, s\}$, $i \neq j$.[12] That is, the arrival rate is increasing in the number of traders on the other side, and decreasing in the number of traders on the same side.

The expected utility functions of a buyer and a seller on the platform can

[11] See A2 in the Appendix.
[12] See A2 in the Appendix.

be rewritten as

$$U^b = [v^b - p^*(N^b, N^s)]A^b(N^b, N^s) - F^b - t^b x^b \equiv u^b - F^b - t^b x^b, \quad (4)$$

$$U^s = [p^*(N^b, N^s) - v^s]A^s(N^b, N^s) - F^s - t^s x^s \equiv u^s - F^s - t^s x^s, \quad (5)$$

where u^b and u^s are willingness-to-pay of the buyer and seller to enter the platform, respectively. We can then investigate how the number of traders affects the equilibrium price and the traders' utilities:[13]

Proposition 2. (a) *The equilibrium price* p^* *is increasing in* N^b *and decreasing in* N^s. (b) *The buyer's and seller's expected utilities exhibit positive cross-group externalities and negative within-group externalities:* $\frac{\partial u^i}{\partial N^j} > (<)\ 0$ *if* $i \neq j$ $(i = j)$, $i, j \in \{s, b\}$.

Proposition 2 shows that in a model in which the matching process and price formation are explicitly spelled out, the platform exhibits not only the well-known positive network externalities in the literature, but also negative externalities as well.

The exogenous specification of linear positive network externalities in the literature implies that the seller's (buyer's) utility is infinite when the number of buyers (sellers) grows without bound. In our matching framework, since the maximum utility a trader gains cannot surpass the surplus of transaction, v, the utility of any trader is necessarily bounded regardless of the number of traders on any side. This is shown in the following corollary.

[13] The proofs of all the propositions are in the Appendix.

Corollary 1. *The willingness of a buyer and a seller to pay to enter the platform, u^b and u^s, is bounded above by v regardless of the number of participants.*

Another important feature of our matching framework is that although positive externalities encourages more agents to enter the platform when there are more agents on the other side, the presence of negative externalities also discourages their entrance. The optimal pricing decision of the platform is therefore more complicated than one with only positive externalities. This issue is discussed in the next section.

3 The Stage of Pricing

Given the equilibrium outcome for the frictional matching stage discussed in the previous section, in this section we will derive the optimal pricing strategy of the platform, together with the equilibrium number of buyers and sellers (N^b and N^s) implied by the optimal strategy.

Since a trader receives zero utility if he does not enter the platform, his expected utility must be at least 0 for him to join the platform willingly. We focus on the interior solution case in which there exists an $\hat{x}^b < 1$ such that $U^b(\hat{x}^b) = 0$, or equivalently $u^b - F^b = t^b\hat{x}^b$.[14] Buyers with expected utilities greater than or equal to 0 (that is, buyers with $x^b \leq \hat{x}^b$) will join the platform. Since x^b is uniformly distributed on $[0, 1]$, the number of buyers

[14] The conditions for having an interior solution on each side are: $t^b > \frac{1}{2}[v(1-\frac{1}{N})^N \ln(1-\frac{1}{N})^{-N} - c^b]$ and $t^s > \frac{1}{2}\{v[1-(1+\frac{N}{N-1})(1-\frac{1}{N})^N - c^s]\}$.

entering the platform, given F^b, is

$$N^b = \Pr(x^b \leq \hat{x}^b)N = x^b N = \frac{u^b - F^b}{t^b}N. \tag{6}$$

The same reasoning applies to the seller's side, so that

$$N^s = \Pr(x^s \leq \hat{x}^s)N = x^s N = \frac{u^s - F^s}{t^s}N. \tag{7}$$

Simultaneously solving for (6), (7), we can write the numbers of buyers and sellers in the platform as the functions of entry fees, $N^b(F^b, F^s)$ and $N^b(F^b, F^s)$.[15] The platform's profit can then be written as

$$\pi = (F^b - c^b)N^b(F^b, F^s) + (F^s - c^s)N^s(F^b, F^s).$$

In the following proposition we characterize the equilibrium fees and the equilibrium number of buyers and sellers in the platform.

Proposition 3. *The profit-maximizing entry fees satisfy*

$$F^b = c^b + \frac{t^b}{N}N^b - (u^s_b N^s + u^b_b N^b), \tag{8}$$

$$F^s = c^s + \frac{t^s}{N}N^s - (u^b_s N^b + u^s_s N^s); \tag{9}$$

where $u^b_i \equiv \frac{\partial u^b}{\partial N^i}$ *and* $u^s_i \equiv \frac{\partial u^s}{\partial N^i}$, $i \in \{b, s\}$. *The equilibrium numbers of participants of buyers and sellers satisfy*

$$vM_b = c^b + 2\frac{t^b}{N}N^{b*}, \text{ and} \tag{10}$$

$$vM_s = c^s + 2\frac{t^s}{N}N^{s*}; \tag{11}$$

[15] Note that N^b and N^s as calculated in (6) and (7) are not necessarily integers. However, the model in Section 2 requires that they be integers. We can take the values of N^b and N^s in Section 2 to be the nearest integers to those defined by (6) and (7) respectively. When N is large, as a meaningful model of two-sided platform should exhibit, this approximation does not the results in the paper.

where $N^{i*} \equiv N^i(F^{b*}, F^{s*})$, $i = b, s$, $M_b \equiv \frac{\partial M}{\partial N^b} > 0$, *and* $M_s \equiv \frac{\partial M}{\partial N^s} > 0$.

It might be helpful to compare the optimal pricing strategy in our model with that in Armstrong (2006) and Rochet and Tirole (2006).[16] In their papers, the equilibrium entry fees are $F^b = c^b + \frac{t^b}{N}N^b - a^s N^s$ and $F^s = c^s + \frac{t^s}{N}N^s - a^b N^b$; where $a^b > 0$ and $a^s > 0$ are the parameters of cross-group positive externalities to buyers and sellers.[17] The effects of the cross-group externalities, $-a^s N^s$ and $-a^b N^b$, help to reduce the the equilibrium fees. We capture the same effects by the terms $-u^s_b N^s$ and $-u^b_s N^b$.[18] However, our model also captures the effects of within-group negative externalities by the terms $-u^b_b N^b$ and $-u^s_s N^s$, which help to raise the equilibrium fees. Consequently, other things being equal, the optimal fees are higher than when only positive externality is considered. In particular, the incentives for the platform to subsidize one side of the platform (by charging a below-cost price), a result much emphasized in the platform-pricing literature, is weaker in our model. For example, in Armstrong (2006) and Rochet and Tirole (2006), if the external effect enjoyed by the buyers, a^b, is large so that $a^b N^b > \frac{t^s}{N}N^s$, then the platform will subsidize the sellers by setting $F^s < c^s$. Note that since a^b is exogenously given and there is no negative externality, if the value of a^b is large, the platform will have an incentive to attract a

[16] Rochet and Tirole (2006) consider the case in which the platform charges not only entry fees but also transaction fees. In order to compare with our model (in which there is only an entry fee), we set the transaction fee to be zero in their model.

[17] In their models, the potential number of users, N, is normalized to 1.

[18] Recall that $u^s_b \equiv \frac{\partial u^s}{\partial N^b}$ and $u^b_s \equiv \frac{\partial u^b}{\partial N^s}$ are the measures of cross-group externalities in our model.

large number of sellers by subsidizing them, and thereby creates enormous network benefit. In that case the platform can charge a very high fee for the buyers. However, this cannot happen in our model. In fact, we will show that the platform never subsidizes the sellers.

Corollary 2. *The total marginal network effect of the seller is negative, i.e.,* $u^b_s N^b + u^s_s N^s < 0$, $\forall N^b \geq 2$ *and* $N^s \geq 2$.

Proof. From the definitions of u^i and A^i, $(i = b, s)$, we know that $u^b N^b + u^s N^s = vM$, implying that $u^b_s N^b + u^s_s N^s = vM_s - u^s$. As a result,

$$\begin{aligned} u^b_s N^b + u^s_s N^s &= vM_s - u^s \\ &= -vz[\frac{N^b}{N^s(N^s-1)}(1-\frac{1}{N^s})^{N^b}] < 0. \end{aligned}$$

□

Corollary 2 and (9) then imply that $F^s > c^s$, i.e., the platform never subsidizes the sellers. However, there are still cases in which the platform charges the buyers a fee lower than the marginal cost.[19] This result is consistent with many pricing strategies in reality, where the buyers (consumers) are usually subsidized while the sellers usually are not.[20] The reason for this result is quite intuitive: since the price of the commodity is set by the sellers, they can shift some of the burden of the entry fee to the buyers. The buyers, on the other hand, can only refrain from joining the platform (in which

[19] For example, $u^s_b N^s + u^b_b N^b \approx -0.02v < 0$ when $N^{b*} = 3$ and $N^{s*} = 2$. $u^s_b N^s + u^b_b N^b \approx 0.06v > 0$ when $N^{b*} = 2$ and $N^{s*} = 2$.

[20] For example, credit card, shopping mall, newspaper and magazine, network TV, online auction et. al.

case the platform loses the revenues from their fees) if they think the fee is too high. In other words, the price elasticity (for entry fee) of the sellers is lower than that of the buyers. Therefore, the platform's cost of raising fees is greater on the buyer's side than on the seller's side.

We can rewrite (8) and (9) as[21]

$$F^{b*} = \frac{1}{2}(u^b + c^b) - \frac{1}{2}(u_b^s N^s + u_b^b N^b), \text{ and} \tag{12}$$

$$F^{s*} = \frac{1}{2}(u^s + c^s) - \frac{1}{2}(u_s^b N^b + u_s^s N^s). \tag{13}$$

As can be seen from (12) and (13), the equilibrium entry fees can be separated into two parts. The first part is the traditional markup pricing formula of the monopolist (without externalities), $\frac{1}{2}(u^i + c^i)$. The second part is the total marginal network effects caused by the agent, $-\frac{1}{2}(u_i^j N^j + u_i^i N^i)$. By our previous discussion, this term is positive for $i = s$, but can be either positive or negative for $i = b$. Therefore, the optimal fee for the sellers is higher than the monopolistic price, but can be either higher or lower for the buyers.

Using (6) and (7), we can also rewrite the platform's profit function as

$$\pi = vM(N^b, N^s) - [c^b N^b + \frac{t^b}{N}(N^b)^2 + c^s N^s + \frac{t^s}{N}(N^s)^2]. \tag{14}$$

The buyers and the sellers can be therefore viewed as two inputs to produce successful matchings as output, with $vM(N^b, N^s)$ as the production function, and $c^b N^b + \frac{t^b}{N}(N^b)^2 + c^s N^s + \frac{t^s}{N}(N^s)^2$ the cost function. Then equations (10)

[21] By substituting $u^i - F^i = t^i \hat{x}^i$ and $\hat{x}^i = \frac{N^i}{N}$, $i = \{b, s\}$, into (8) and (9).

and (11) simply say that the platform's optimal strategy is to "hire" each input until its marginal product, vM_i, equals its marginal cost, $c^i + 2\frac{t^s}{N}N^s$. A change in fee to one side of the platform affects both the number of agents on this side and (therefore) the externalities enjoyed by agents on the other side. Since the price elasticity for side i is larger when $\frac{t^i}{N}N^i - u_i^i N^i$ is smaller, and the positive network effect which side i brings to side j is larger when $u_i^j N^j$ is larger, the optimal fee F^i is lower when $\frac{t^i}{N}N^i - u_i^i N^i$ is smaller or $u_i^j N^j$ is larger.

4 Some Comparative Static Results

In this section we will perform several comparative statics exercises regarding changes in costs and trading surplus. For each result we only discuss the intuition behind it, and leave its proof to the appendix. Moreover, we only derive results on the seller's side. Those on the buyer's side are symmetric.

If c^s or t^s increases, in order to restore the equilibrium condition $vM_s = c^s + 2\frac{t^s}{N}N^s$, the platform should lower the number of sellers, so that the value of marginal contribution of the sellers increases. Moreover, since the buyers and the sellers are complements (see Section 2.2), when the platform reduces the number of sellers, it also reduces the number of buyers as well. As a result, the number of both sellers and the buyers will decrease in response to an exogenous increase in the costs of serving the seller or the seller's cost of using the platform.

When the surplus from trade, v, increases exogenously, it makes a successful matching more valuable. The platform's best response is to induce more agents to join the platform, so that the marginal contributions of all agents become smaller, in order to recover (10) and (11). Therefore, an increasing in trade surplus leads to the intuitive result that the numbers of both sellers and buyers increase.

The change in the platform's pricing policy in response to parametric change is harder to pin down. However, when the number of users in the platforms is large, as we expect to see in the real world, there will be definite answers, as the following lemma shows.

Lemma 1. *When both N^b and N^s are large, the entry fee is positively related to the matching value v: $\frac{\partial F^i}{\partial v} > 0$. Moreover, it is positively (negatively) related to entry cost of serving users on the same (opposite) side: $\frac{\partial F^i}{\partial c^i} > 0$ and $\frac{\partial F^i}{\partial c^j} < 0$. Finally, an agent's entry fee is positively related to his cost, and negatively related to entry fee of agents on the other side $\frac{\partial F^i}{\partial t^i} > 0$ and $\frac{\partial F^i}{\partial t^j} < 0$.*

When the numbers of buyers and sellers are large, we can also show that the entry fees are substitutes, i.e., $\frac{\partial^2 \pi}{\partial F^b \partial F^s} < 0$.[22] This result is consistent with the famous "seesaw principle" in Rochet and Tirole (2006). The intuition of this result is as follows. If, for example, serving the seller becomes more costly (the profit margin on the seller side is lower), then attracting the buyers is

[22] See A6 for the proof.

more profitable. As a result, the platform will not only raise the fee for the sellers but also lower the fee for the buyers.

In Weyl (2010), the seesaw principle is derived by way of showing the number of participants on the two sides being substitutes. In his model the substitution in the numbers of participants on the two sides is the source of the seesaw effect. However, buyers and sellers being substitutes is not the reason for the seesaw principle in our model. By Weyl's definition, the participation levels on the two sides are substitutes (complements) if $\frac{\partial\pi}{\partial N^b \partial N^s} < 0\,(> 0)$. From (14), we know that $\frac{\partial\pi}{\partial N^b \partial N^s} = M_{bs} > 0$. Therefore, in our model the buyers and sellers are complements, while the fees are substitutes. In other words, the seesaw principle in fees still holds in our model even if the participants in two sides are substitutes.

5 Conclusion

In this paper we provide a theoretical model of the two-sided platform in which the number of buyers and sellers, the seller's prices, and, more importantly, the sources of network externalities are endogenously determined. The platform is shown to exhibit both positive and negative network externalities: A participant's benefit in joining the platform is increasing in the number of participants on the other side of the platform, and decreasing in the number of participants on the same side. Moreover, unlike the case of linear externalities, the benefit of a participant is bounded, even if the number of participants on the other side of the platform goes to infinity. The optimal pricing policy of the platform is shown to depend not only on the costs of providing service and benefit to the participants but, more importantly, also on how a new entrant (either a buyer or a seller) affects the matching probability. Beside providing a microfoundation for how the platforms function, we also derive certain theoretical predictions which differ from past literature. For example, we show that the platform never subsidizes the sellers by charging a fee lower than its marginal cost, but might subsidize the buyers. This result is consistent with the platform pricing policy generally observed in practice.

This paper considers only the monopoly platforms. For future research, it will be interesting to also study the oligopoly case. In particular, since our model provides a microfoundation for the platform, issues that are difficult to tackle in the previous theoretical models such as single- vs. multi-homing choice might be more easily analyzed in the present framework.

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Appendix

A1. The Proof of Proposition 1

Follow Burdett et al. (2001), let $\phi(a)$ be the probability that at least one buyer visits a particular seller when all buyers visit this seller with probability a. Given there are N^b buyers in the platform, $\phi(a) = 1 - (1-a)^{N^b}$. Let Ω be the probability that a given buyer gets served when he visits this seller. Hence,

$$\Omega = \frac{\phi(a)}{N^b a} = \frac{1-(1-a)^{N^b}}{N^b a}.$$

If every seller posts a price p and one contemplates deviating to p^d, the buyer visits the deviant with probability a^d. The probability that he visits each of the nondeviants is $\frac{1-a^d}{N^s-1}$, given there are N^s sellers in the platform. As a result,

$$\Omega^d = \frac{1-(1-a^d)^{N^b}}{N^b a^d},$$

and a buyer who visits a nondeviant gets served with probability

$$\Omega = \frac{1-(1-\frac{1-a^d}{N^s-1})^{N^b}}{N^b(\frac{1-a^d}{N^s-1})}.$$

In the equilibrium,

$$(v^b - p)\Omega = (v^b - p^d)\Omega^d.$$

This condition can be written as

$$\frac{v^b - p}{v^b - p^d} = \frac{(1-a^d)[1-(1-a^d)^{N^b}]}{(N^s-1)a^d[1-(1-\frac{1-a^d}{N^s-1})^{N^b}]}. \tag{15}$$

Because the expected profit of the deviant is $(p^d - v^s)[1-(1-a^d)^{N^b}]$, the first-order condition of the deviant's utility maximize problem is

$$[1-(1-a^d)^{N^b}] + (p^d - v^s)N^b(1-a^d)^{N^b-1}\frac{\partial a^d}{\partial p^d} = 0.$$

If we focus on the interior solution such that $a^d \in (0,1)$, we can differentiate (15) and then insert the symmetric equilibrium conditions $p^d = p$, $a^d = \frac{1}{N^S}$ to derive

$$\frac{\partial a^d}{\partial p^d} = -\frac{(N^s-1)^2[1-(1-\frac{1}{N^s})^{N^b}]}{(N^s)^2[(N^s-1)-(N^s-1+N^b)(1-\frac{1}{N^s})^{N^b}](v^b-p^d)} < 0.$$

Inserting this into the first-order condition, we arrive at

$$p^* = \frac{v^b[1-(1+\frac{N^b}{N^s-1})(1-\frac{1}{N^s})^{N^b}] + v^s\frac{N^b}{N^s}(1-\frac{1}{N^s})^{N^b}}{1-[1+\frac{N^b}{N^s(N^s-1)}](1-\frac{1}{N^s})^{N^b}}.$$

A2. Properties of the Matching Function and the Arrival Rates

We will show that the arrival rate of one side of the platform is increasing (decreasing) in the number of agents on the other (same) side of the platform. To complete the proof, it is necessary to check the properties of the matching function, $M(N^b, N^s)$. We can first show that

$$M_b \equiv \frac{\partial M}{\partial N^b} =(1-\frac{1}{N^s})^{N^b}\ln(1-\frac{1}{N^s})^{-N^s} > 0, \text{ and} \tag{16}$$
$$M_s \equiv \frac{\partial M}{\partial N^s} =1-(1+\frac{N^b}{N^s-1})(1-\frac{1}{N^s})^{N^b} > 0. \tag{17}$$

We can also show that M is concave in both N^b and N^s:

$$M_{bb} \equiv \frac{\partial M_b}{\partial N^b} = -\frac{1}{N^s}(1-\frac{1}{N^s})^{N^b}[\ln(1-\frac{1}{N^s})^{-N^s}]^2 < 0, \tag{18}$$

$$M_{ss} \equiv \frac{\partial M_s}{\partial N^s} = -\frac{N^b(N^b-1)}{N^s(N^s-1)^2}(1-\frac{1}{N^s})^{N^b} < 0. \tag{19}$$

Also,

$$M_{bs} \equiv \frac{\partial M_b}{\partial N^s} = -\frac{N^s-(N^s-1+N^b)\ln(1-\frac{1}{N^s})^{-N^s}}{N^s-1}(1-\frac{1}{N^s})^{N^b} > 0. \tag{20}$$

Finally,

$$\begin{aligned}\frac{\partial A^i}{\partial N^i} &= \frac{1}{N^i}(M_i - \frac{M}{N^i}),\\ \frac{\partial A^i}{\partial N^j} &= \frac{M_l}{N^k}, \forall i, j \in \{b, s\}, i \neq j.\end{aligned}$$

Since $M_i - \frac{M}{N^i} < 0$ by the concavity of M in N^i, we know that $\frac{\partial A^i}{\partial N^i} < 0$. Also, $\frac{\partial A^i}{\partial N^j} > 0$ since $M_j > 0$.

A3. The Proof of Proposition 2

First note that

$$\begin{aligned}\frac{\partial U^b}{\partial N^s} &= -\frac{\partial p^*}{\partial N^s}A^b + (v^b - p^*)\frac{\partial A^b}{\partial N^s},\\ \frac{\partial U^s}{\partial N^b} &= \frac{\partial p^*}{\partial N^b}A^s + (p^* - v^s)\frac{\partial A^s}{\partial N^b},\\ \frac{\partial U^b}{\partial N^b} &= -\frac{\partial p^*}{\partial N^b}A^b + (v^b - p^*)\frac{\partial A^b}{\partial N^b},\\ \frac{\partial U^s}{\partial N^s} &= \frac{\partial p^*}{\partial N^s}A^s + (p^* - v^s)\frac{\partial A^s}{\partial N^s}.\end{aligned}$$

To prove this proposition, it suffices to show that (i) the sign of $\frac{\partial p^*}{\partial N^i}$ is positive if $i = b$ and negative if $i = s$; and (ii) the sign of $\frac{\partial A^i}{\partial N^j}$ is positive if

$i \neq j$ and is negative if $i = j$. As (ii) is already proved in A2, we only need to prove (i). Since $p^* = v^s + (v^b - v^s)z$, the sign of $\frac{\partial p}{\partial N^i}$ is the same as the sign of $\frac{\partial z}{\partial N^i}$. Furthermore, let $\alpha = 1 - (1 + \frac{N^b}{N^s - 1})(1 - \frac{1}{N^s})^{N^b}$ and $\beta = \frac{N^b}{N^s}(1 - \frac{1}{N^s})^{N^b}$, then

$$z = \frac{\alpha}{\alpha + \beta} = \frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta} + 1}.$$

It's easy to show that

$$\begin{aligned}\frac{\partial z}{\partial N^b} &= \frac{\partial z}{\partial \frac{\alpha}{\beta}} \frac{\partial \frac{\alpha}{\beta}}{\partial N^b}, \\ \frac{\partial z}{\partial N^s} &= \frac{\partial z}{\partial \frac{\alpha}{\beta}} \frac{\partial \frac{\alpha}{\beta}}{\partial N^s};\end{aligned}$$

where $\frac{\partial z}{\partial \frac{\alpha}{\beta}} = (\frac{\alpha}{\beta} + 1)^{-2} > 0$. Therefore, we know that $\frac{\partial z}{\partial N^b} > 0$ if and only if $\frac{\partial \frac{\alpha}{\beta}}{\partial N^b} > 0$; and $\frac{\partial z}{\partial N^s} < 0$ if and only if $\frac{\partial \frac{\alpha}{\beta}}{\partial N^s} < 0$. It can be shown that

$$\frac{\partial \frac{\alpha}{\beta}}{\partial N^b} = \frac{1}{(N^b)^2}(1 - \frac{1}{N^s})^{-N^b}[N^b \ln(1 - \frac{1}{N^s})^{-N^s} - M] > 0, \tag{21}$$

$$\frac{\partial \frac{\alpha}{\beta}}{\partial N^s} = [\frac{1}{N^b(N^s - 1)}(1 - \frac{1}{N^s})^{-N^b}](M - N^b - M_s) < 0.^{23} \tag{22}$$

As a result, $\frac{\partial z}{\partial N^b} > 0$ and $\frac{\partial z}{\partial N^s} < 0$.

A4. The Proof of Corollary 1

To prove this proposition, we will show that A^b converges to 1 and p^* converges to v^s as N^s goes to infinity; and A^s converges to 1 and p^* converges to v^b as N^b goes to infinity. By the definitions of A^b and A^s, we know that

[23] To verify these, first note that $N^b \geq 2$ and $N^s \geq 2$ ensures that $\ln(1 - \frac{1}{N^s})^{-N^s} > 1$ and $(1 - \frac{1}{N^s})^{N^b} \in (0, 1)$. Second, the term $M \leq \min\{N^b, N^s\}$.

A^b converges to 1 as long as M converges to N^b; and A^s converges to 1 as long as M converges to N^s. What we have to show is that the limit of M is N^b when N^s goes to infinity, and is N^s when N^b goes to infinity. Since $1-\frac{1}{N^s}<1$ implies that $(1-\frac{1}{N^s})^{N^b}$ approaching 0 when N^b is large, it is obvious that M converges to N^s as N^b goes to infinity. To show that M converges to N^b as N^s goes to infinity, we use the L'Hôpital's rule:

$$\lim_{N^s\to\infty} N^s[1-(1-\frac{1}{N^s})^{N^b}] = \lim_{N^s\to\infty} \frac{-N^b(1-\dfrac{1}{N^s})^{N^b-1}\dfrac{1}{(N^s)^2}}{-\dfrac{1}{(N^s)^2}} = N^b.$$

Next we will find the limits of p^* when N^b or N^s grows to infinity. Recall that $p^* = zv^b + (1-z)v^s$, and that

$$z = \frac{M - N^b(1-\frac{1}{N^s})^{N^b-1}}{M - \frac{N^b}{N^s}(1-\frac{1}{N^s})^{N^b-1}}.$$

Since $\lim_{N^s\to\infty} M = N^b$, we have $\lim_{N^s\to\infty} z = 0$. By L'Hôpital's rule, $\lim_{N^b\to\infty} N^b(1-\frac{1}{N^s})^{N^b-1} = 0$, implying that $\lim_{N^b\to\infty} z = 1$. We therefore show that p^* converges to v^b when N^b is large, and converges to v^s when N^s is large.

A5. The Proof of Proposition 3

Totally differentiating N^b and N^s, we have

$$\begin{bmatrix} \frac{t^b}{N} - u^b_b & -u^b_s \\ -u^s_b & \frac{t^s}{N} - u^s_s \end{bmatrix} \begin{bmatrix} dN^b \\ dN^s \end{bmatrix} = \begin{bmatrix} -dF^b \\ -dF^s \end{bmatrix}.$$

Solving for this equation system, we can derive the following:

$$\frac{\partial N^b}{\partial F^b} = \frac{-(\frac{t^s}{N} - u^s_s)}{\Delta},$$
$$\frac{\partial N^s}{\partial F^b} = \frac{-u^s_b}{\Delta},$$
$$\frac{\partial N^b}{\partial F^s} = \frac{-u^b_s}{\Delta},$$
$$\frac{\partial N^s}{\partial F^s} = \frac{-(\frac{t^b}{N} - u^b_b)}{\Delta};$$

where $\Delta = (\frac{t^b}{N} - u^b_b)(\frac{t^s}{N} - u^s_s) - (u^s_b)(u^b_s)$.

The two first-order conditions of the platform's profit maximizing problem are

$$\frac{\partial \pi}{\partial F^b} = N^b + (F^b - c^b)\frac{\partial N^b}{\partial F^b} + (F^s - c^s)\frac{\partial N^s}{\partial F^b} = 0, \text{ and} \tag{23}$$
$$\frac{\partial \pi}{\partial F^s} = (F^b - c^b)\frac{\partial N^b}{\partial F^s} + N^s + (F^s - c^s)\frac{\partial N^s}{\partial F^s} = 0. \tag{24}$$

Solving for this equation system, we arrive at

$$F^b = c^b + \frac{t^b}{N}N^b - u^s_b N^s - u^b_b N^b,$$
$$F^s = c^s + \frac{t^s}{N}N^s - u^b_s N^b - u^s_s N^s.$$

By the fact that $F^i = u^i - \frac{t^i}{N}N^i$, the first-order conditions can be written as $u^b + u^b_b N^b + u^s_b N^s = c^b + 2\frac{t^b}{N}N^s$ and $u^s + u^b_s N^b + u^s_s N^s = c^s + 2\frac{t^s}{N}N^s$. By the definitions of M and u^i we know that $u^b N^b + u^s N^s = vM$. Therefore, $u^b_b N^b + u^b + u^s_b N^s = vM_b$ and $u^b_s N^b + u^s + u^s_s N^s = vM_s$. We therefore have

$$vM_b = c^b + 2\frac{t^b}{N}N^b,$$
$$vM_s = c^s + 2\frac{N^s}{N}N^s.$$

A6. The Proof of Comparative Static Results

Firstly, we investigate the effects of the change in the exogenous parameters on the number of users. We already know that $M_{bb} < 0$, $M_{ss} < 0$ and $M_{bs} = M_{sb} > 0$ from A2. Furthermore, the Hessian matrix associated with π is

$$H = \begin{bmatrix} \frac{\partial^2 \pi}{(\partial F^b)^2} & \frac{\partial^2 \pi}{(\partial F^b)(\partial F^s)} \\ \frac{\partial^2 \pi}{(\partial F^b)(\partial F^s)} & \frac{\partial^2 \pi}{(\partial F^s)^2} \end{bmatrix},$$

with H being negative definite if and only if $|H_1| < 0$ and $|H| > 0$. However, $|H| = \frac{\phi}{\Delta^2}$ where $\phi \equiv (2\frac{t^b}{N} - vM_{bb})(2\frac{t^s}{N} - vM_{ss}) - (vM_{bs})^2$. Therefore, the second-order condition, $|H| > 0$, implies that $\phi > 0$.

Totally differentiating (10) and (11), we have

$$\begin{bmatrix} 2\frac{t^b}{N} - vM_{bb} & -vM_{sb} \\ -vM_{bs} & 2\frac{t^s}{N} - vM_{ss} \end{bmatrix} \begin{bmatrix} \mathrm{d}N^b \\ \mathrm{d}N^s \end{bmatrix} = \begin{bmatrix} M_b \mathrm{d}v - \mathrm{d}c^b - 2\frac{N^b}{N}\mathrm{d}t^b \\ M_s \mathrm{d}v - \mathrm{d}c^s - 2\frac{N^s}{N}\mathrm{d}t^s \end{bmatrix}.$$

Therefore,

$$\frac{\partial N^b}{\partial c^b} = -\frac{1}{\phi}(2\frac{t^s}{N} - vM_{ss}) < 0,$$
$$\frac{\partial N^s}{\partial c^b} = -\frac{1}{\phi}(vM_{bs}) < 0,$$

$$\frac{\partial N^s}{\partial c^s} = -\frac{1}{\phi}(2\frac{t^b}{N} - vM_{bb}) < 0,$$
$$\frac{\partial N^b}{\partial c^s} = -\frac{1}{\phi}(vM_{bs}) < 0,$$

$$\frac{\partial N^b}{\partial t^b} = -\frac{2\frac{N^b}{N}}{\phi}(2\frac{t^s}{N} - vM_{ss}) < 0,$$
$$\frac{\partial N^s}{\partial t^b} = -\frac{2\frac{N^b}{N}}{\phi}(vM_{bs}) < 0,$$

$$
\begin{aligned}
\frac{\partial N^s}{\partial t^s} &= -\frac{2\frac{N^s}{N}}{\phi}(2\frac{t^b}{N} - vM_{bb}) < 0, \\
\frac{\partial N^b}{\partial t^s} &= -\frac{2\frac{N^s}{N}}{\phi}(vM_{bs}) < 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial N^b}{\partial v} &= -\frac{1}{\phi}[M_b(2\frac{t^s}{N} - vM_{ss}) + M_s vM_{bs}] > 0, \\
\frac{\partial N^s}{\partial v} &= -\frac{1}{\phi}[M_s(2\frac{t^b}{N} - vM_{bb}) + M_b vM_{bs}] > 0.
\end{aligned}
$$

Next, we investigate the platform's pricing policy in response to parametric changes. To do so, we differentiate (6) and (7) with respect to all parameters concerned, respectively. Then the partial derivatives can be written as the general formula:

$$
\frac{\partial F^i}{\partial y} =(u_i^i - \frac{t^i}{N})\frac{\partial N^i}{\partial y} + u_j^i\frac{\partial N^j}{\partial y}, \tag{25}
$$

where $y = c^i$, t^i or v for all $i, j \in \{b, s\}$. When the numbers of buyers and sellers are large enough, $\ln(1-\frac{1}{N^s})^{-N^s}$ is approximately equal to 1, and N^i-1 is approximately equal to N^i, $i \in \{b, s\}$. Substitute these into (16) to (22) and z, we have the following approximations:

$$
\begin{aligned}
M_b &\approx(1 - \frac{1}{N^s})^{N^b}, \\
M_s &\approx1 - (1 + \frac{N^b}{N^s})(1 - \frac{1}{N^s})^{N^b} > 0,
\end{aligned}
$$

$$
\begin{aligned}
M_{bb} &\approx -\frac{1}{N^s}(1 - \frac{1}{N^s})^{N^b}, \\
M_{ss} &\approx -\frac{(N^b)^2}{(N^s)^3}(1 - \frac{1}{N^s})^{N^b}, \\
M_{bs} &\approx\frac{N^b}{(N^s)^2}(1 - \frac{1}{N^s})^{N^b},
\end{aligned}
$$

$$\begin{aligned}
\alpha \approx& 1-(1+\frac{N^b}{N^s})(1-\frac{1}{N^s})^{N^b},\\
\beta \approx& \frac{N^b}{N^s}(1-\frac{1}{N^s})^{N^b},\\
\frac{\partial \frac{\alpha}{\beta}}{\partial N^b} \approx& \frac{1}{(N^b)^2}(1-\frac{1}{N^s})^{-N^b}(N^b-M),\\
\frac{\partial \frac{\alpha}{\beta}}{\partial N^s} \approx& [\frac{1}{N^bN^s}(1-\frac{1}{N^s})^{-N^b}](M-N^b-M_s).
\end{aligned}$$

We therefore have

$$\begin{aligned}
u_b^b \approx& -\frac{1}{N^s}(1-\frac{1}{N^s})^{N^b},\\
u_s^b \approx& \frac{N^b}{(N^s)^2}(1-\frac{1}{N^s})^{N^b},\\
u_b^s \approx& \frac{N^b}{(N^s)^2}(1-\frac{1}{N^s})^{N^b},\\
u_s^s \approx& -\frac{(N^b)^2}{(N^s)^3}(1-\frac{1}{N^s})^{N^b}.
\end{aligned}$$

Putting these into (25), it is straightforward to obtain the comparative static results: $\frac{\partial F^i}{\partial v} > 0$, $\frac{\partial F^i}{\partial t^i} > 0$, $\frac{\partial F^i}{\partial c^i} > 0$, $\frac{\partial F^i}{\partial t^j} < 0$ and $\frac{\partial F^i}{\partial c_j} < 0$ for all $i, j \in \{b, s\}$, $i \neq j$.

In addition, twice differentiating the profit function with respect to the fee on both sides, we have

$$\frac{\partial^2 \pi}{\partial F^b \partial F^s} = (\pi_{bb}\frac{\partial N^b}{\partial F^s} + \pi_{bs}\frac{\partial N^s}{\partial F^s})\frac{\partial N^b}{\partial F^b} + (\pi_{bs}\frac{\partial N^b}{\partial F^s} + \pi_{ss}\frac{\partial N^s}{\partial F^s})\frac{\partial N^s}{\partial F^b}, \tag{26}$$

where $\pi_{ij} \equiv \frac{\partial^2 \pi}{\partial N^i \partial N^j}$. Substituting the approximation values above into (26), we can easily show that $\frac{\partial^2 \pi}{\partial F^b \partial F^s} < 0$.