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14. May 2007

Online at <http://mpra.ub.uni-muenchen.de/3843/>  
MPRA Paper No. 3843, posted 5. July 2007

May 14, 2007

## THINKING CATEGORICALLY ABOUT OTHERS: A CONJECTURAL EQUILIBRIUM APPROACH

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ABSTRACT. Inspired by the social psychology literature, we study the implications of categorical thinking on decision making in the context of a large normal form game. Every agent has a categorization (partition) of her opponents and can only observe the average behavior in each category. A strategy profile is a *Conjectural Categorical Equilibrium (CCE)* with respect to a given categorization profile if every player's strategy is a best response to some consistent conjecture about the strategies of her opponents.

We show that, for a wide family of games and for a particular categorization profile, every CCE becomes almost Nash as the number of players grows. An equivalence of CCE and Nash equilibrium is achieved in the settings of a non-atomic game. This highlights the advantage of categorization as a simplifying mechanism in complex environments. With much less information in their hands agents behave as if they see the full picture. Some properties of CCE when players categorize 'non-optimally' are also considered.

*JEL classification:* C72, D81, D84.

*Keywords:* Categorization, Conjectural equilibrium, Large games.

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<sup>†</sup>This work is part of a Ph.D. dissertation written at the School of Mathematical Sciences of Tel-Aviv University under the supervision of Prof. Ehud Lehrer. I am grateful to E. Lehrer as well as to P. Jehiel and E. Kalai for their comments.

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## 1. INTRODUCTION

It is commonly accepted in the psychological literature<sup>1</sup> that people represent the world around them with the aid of categories. In particular, information about other people is often being processed with the aid of social categories. As Macrae and Bodenhausen (2000) write

“ Given basic cognitive limitations and a challenging stimulus world, perceivers need some way to simplify and structure the person perception process. This they achieve through the activation and implementation of categorical thinking. Rather than considering individuals in terms of their unique constellations and proclivities, perceivers prefer instead to construe them on the basis of the social categories (e.g. race, gender, age) to which they belong. . .”

The purpose of the current paper is to study some issues related to categorical thinking in the context of decision making. Specifically, our concern here is with equilibrium behavior of agents in a non-cooperative normal form game. Equilibrium is viewed here as a steady state of a recurring interaction between agents with no strategic links among the repetitions. As such, it is highly sensitive to the information (and the way its being processed) that every agent has about the actions of her opponents. It is a key assumption of this paper that categorical thinking affects exactly this information.

To study the implications of categorization in such scenarios we define a solution concept called *Conjectural Categorical Equilibrium* (CCE). This is a special case of Battigalli and Guaitoli’s (1988) conjectural equilibrium. Each player  $i$  is equipped with an exogenously given partition of her opponents. This is the categorization that  $i$  uses in order to facilitate the process of information about the behavior of her opponents. As a consequence,  $i$  is unable to observe the actions taken by each individual player. Instead, she can only tell what is the average behavior within each category in her partition. Thus, when deciding what action to choose, player  $i$  is facing uncertainty as to the actual choices of her opponents. In this case it is natural to assume that  $i$  has some conjecture (which conforms with her information) about the profile of actions that her opponents actually play, and that she plays a best response to her conjecture. When all players behave in this way the resulting strategy profile is a CCE.

When an agent is categorizing her opponents there is a risk that it will lead her to make sub-optimal decisions and to lose utility. Therefore, in order for categorization to be efficient it should have the property that the loss of information incurred by the categorical representation of other agents will not result in choosing the wrong action. In other words, each agent wants to choose the action that she would choose

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<sup>1</sup>See Section 7 for references.

had she known the entire strategy profile of her opponents. If the categorization of every player has this property then every CCE is also a Nash equilibrium. We call such a profile of categorizations *sufficient*. The main result of the paper concerns the existence of non-trivial sufficient categorization profiles.

Our solution concept is plausible only if the number of participating players is large. We therefore consider families of games with an increasing number of players. The result we obtain is asymptotic. It is shown that, with appropriate anonymity and continuity assumptions on the payoff functions, certain categorization profiles become close to being sufficient as the number of players grows to infinity. These categorization profiles are those in which each player lumps together players that have symmetric influence on her payoffs.

The aforementioned result can be interpreted in several ways. First, it highlights the advantage of categorization as a simplifying tool in complex environments. With much less information in their hands agents behave as if they see the full picture. The second interpretation is of normative nature. The result can be seen as a recommendation of how one should categorize others when involved in a game-like situation. Finally, the result also increases the plausibility of Nash equilibrium in large games since it shows that an equilibrium must emerge even if players have limited information about the strategies of their opponents.

The model we use in order to obtain the asymptotic result is adopted from Kalai (2004). There is a finite universal set of actions  $\mathbb{S}$ .  $\Gamma(\mathbb{S})$  is a family of normal form games such that for every game  $G$  in  $\Gamma(\mathbb{S})$  and for every player  $i$  in  $G$  the set of (pure) strategies available to  $i$  is some subset of  $\mathbb{S}$ . With a fixed family  $\Gamma(\mathbb{S})$  in hand, one can very naturally define notions of *uniform* continuity and anonymity in  $\Gamma(\mathbb{S})$ . These are the key assumptions needed in order to obtain the asymptotic existence of a sufficient categorization profile. For a detailed discussion of the relation between our assumptions and results and those of Kalai (2004, 2005) see Section 7.

As noted before, CCE is appealing when the number of players is large. It is therefore natural to study it also in the setting of a game with a continuum of players. Working in the model of Schmeidler (1973), we define CCE for a non-atomic game similarly to its definition in the finite case. A simple sufficient condition for the existence of sufficient categorization profile is provided. We then show that this condition holds for a dense set of non-atomic games. Thus, every non-atomic game can be approximated by a game in which a sufficient categorization profile exists.

The results described so far are of ‘positive’ nature. They emphasize the advantages of categorization as an information processing mechanism. But these advantages may cease to exist if an agent makes use of the ‘wrong’ categorization. To illustrate this point we analyze two examples of non-atomic games in which agents categorize their opponents not as one may think they should. In the first example it is shown that this can lead to a CCE in which all the players get the worst possible

payoff. In the second example there is a CCE which yield a higher *total* payoff for the society than the Nash equilibrium of the game (though the payoffs to some players is lower than their equilibrium payoffs).

The paper is organized as follows. In Section 2 we illustrate our solution concept by means of an example. CCE for both finite normal form games and non-atomic games is formally defined in Section 3. Section 4 contains the main results of the paper about sufficient categorization profiles. The influence that different categorizations may have on social efficiency is exemplified in Section 5. Some remarks about the model are in Section 6. These include a possible refinement of CCE and a result regarding CCE as a purifying device. Related literature is discussed in Section 7. All the proofs are in Section 8.

## 2. CHOOSING UNIVERSITY: A MOTIVATING EXAMPLE

Every year, a large population of school graduates is required to make decisions regarding higher education issues. For concreteness, let us focus on the set of all graduates who have decided to apply to a medical school and are facing the problem of which university to apply to.

There may be many factors influencing the preferences of a candidate. Some of these factors, such as the university's location and reputation, are independent of the choices of other candidates. But other factors are directly affected by the decisions made by other candidates. For instance, the probability of being accepted depends on the profile of other candidates applying to the same university. Also, the financial support that a university provides to a student may be relative to her socioeconomic background in comparison to the socioeconomic background of other students. Every candidate may also have preferences regarding the characteristics of her future fellow students.

The situation can, therefore, be described as a game where candidates are the players and universities are the (common) set of actions available to each player. Notice that a similar game occurs every year but there are no strategic links among the repetitions since every time the set of players is new.

An equilibrium of this recurring game is a list specifying the university that each candidate applies to (or a probability distribution over the set of universities) such that no candidate would like to revise her choice had she known the choices of all other candidates. But the assumption that a candidate knows the decision of any other candidate seems unreasonable in this case. A more realistic assumption is that every candidate has only partial information about the choices of others<sup>2</sup>. This

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<sup>2</sup>We emphasize that the cause for the lack of information is not necessarily that this information is unavailable. It may well be that a candidate uses a simplified representation of the situation due to its complexity. It is not important for our purposes whether the candidate doesn't have the information or chooses not to use it.

information is based on the results of games from previous years. For instance, by visiting a certain university a candidate can get an impression of some characteristics of the students in this institute. And by looking on statistics of acceptance from previous years a candidate can learn the universities to which candidates with high school grades apply.

The CCE solution concept assumes that the information of every candidate about the decisions of others has a particular structure. Namely, each candidate categorizes the rest of the candidates according to some subjective criteria and can only tell the proportion of candidates choosing each university in each category<sup>3</sup>. For example, if the categorization of a certain candidate is based solely on sex (a category of males and a category of females) then this candidate will only know the proportions of the males and of the females which applied to each university. If another candidate categorizes according to school grades (say ‘low’, ‘average’ and ‘high’ categories) then her information will be the distribution of applications among the various universities in each of these 3 sets of candidates. A third candidate may categorize according to both criteria, thus having 6 different categories.

Since the information of a candidate is not complete she can only conjecture what are the true choices of the others. Her conjecture, however, must be consistent with the information she has. If every candidate plays a best response to some consistent conjecture the resulting profile of strategies constitute a CCE. Notice that every Nash equilibrium of the game is a CCE since the true profile of strategies is always a consistent conjecture. But there may be many other CCE’s which are not Nash. However, the main result of this paper shows that in many large games every CCE is close to being Nash if the players are categorizing ‘correctly’.

### 3. DEFINITION OF CCE

**3.1. Finite games.** A game  $G$  in normal form is defined by a triplet  $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ .  $N = \{1, \dots, n\}$  is the set of players. For each  $i \in N$ ,  $S_i$  is the finite set of pure strategies (actions) of player  $i$ . Denote by  $S$  the product  $S = \times_{i \in N} S_i$  and for every player  $i \in N$  let  $S_{-i} = \times_{j \neq i} S_j$ . A typical element of  $S$  ( $s_i, s_{-i}$ ) will be denoted by  $\underline{s}$  ( $s_i, s_{-i}$ ).  $u_i : S \rightarrow \mathbb{R}$  is the utility function of player  $i \in N$ . Each player  $i$  may use a mixed strategy which is a probability distribution over  $S_i$ , usually denoted by<sup>4</sup>  $\sigma_i$ . If  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a profile of strategies then  $\sigma_{-i}$  denotes the strategies of players other than  $i$ . As usual,  $u_i$  will also be used to denote expected utility whenever players use mixed strategies.

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<sup>3</sup>If some candidates play a mixed strategy then the information is only the *expected* proportion of each university in each category.

<sup>4</sup>Thus, for every  $s_i \in S_i$ ,  $\sigma_i(s_i)$  is the probability of player  $i$  choosing the action  $s_i$  according to the mixed strategy  $\sigma_i$ .

Assume that every player  $i \in N$  categorize the rest of the players according to some criteria. Formally, for every  $i \in N$  let  $C_i$  be a partition of the set  $N \setminus \{i\}$ . That is,  $C_i = \{B_1, \dots, B_m\}$  where each  $B_j$  is a non-empty subset of  $N \setminus \{i\}$ ,  $j \neq k$  implies  $B_j \cap B_k = \emptyset$ , and  $\cup_{j=1}^m B_j = N \setminus \{i\}$ . A *categorization profile* is a vector  $C = (C_1, \dots, C_n)$ , where each  $C_i$  is a partition of  $N \setminus \{i\}$ . For two categorization profiles  $C = (C_1, \dots, C_n)$  and  $C' = (C'_1, \dots, C'_n)$ , we say that  $C$  is finer than  $C'$  if<sup>5</sup>  $C_i$  is finer than  $C'_i$  for every  $i \in N$ .

Assume that there is a finite universal set of actions  $\mathbb{S}$  (not to be confused with the product set  $S$ ) such that  $S_i \subseteq \mathbb{S}$  for every  $i \in N$ . Every profile of (possibly mixed) strategies<sup>6</sup>  $\sigma = (\sigma_1, \dots, \sigma_n) \in \times_{i \in N} \Delta(S_i)$  and a non-empty set of players  $B \subseteq N$  induce a probability distribution over  $\mathbb{S}$ , denoted  $\sigma^B$ , which is defined by<sup>7</sup>  $\sigma^B(s) = \frac{1}{|B|} \sum_{i \in B} \sigma_i(s)$  for every  $s \in \mathbb{S}$ . Thus,  $\sigma^B(s)$  is the expected proportion of players choosing  $s$  in the set  $B$  according to the profile of strategies  $\{\sigma_i\}_{i \in B}$ .

Given a player  $i \in N$ , a categorization  $C_i$  of  $N \setminus \{i\}$  and a profile of strategies  $\sigma = (\sigma_1, \dots, \sigma_n)$ , let  $F_{C_i}(\sigma_{-i}) = \{\tau_{-i} : \tau_{-i}^B = \sigma_{-i}^B \text{ for every } B \in C_i\}$  be the set of all strategy profiles of players other than  $i$  which induce the same distribution over  $\mathbb{S}$  like  $\sigma$  in every set  $B \in C_i$ . Elements of  $F_{C_i}(\sigma_{-i})$  are called *consistent conjectures* of player  $i$  when his opponents play according to  $\sigma_{-i}$ .

**Definition 1.**  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a Conjectural Categorical Equilibrium (CCE) w.r.t. the categorization profile  $C = (C_1, \dots, C_n)$  if, for every  $i \in N$ , there exists a profile of strategies  $\tau_{-i} \in F_{C_i}(\sigma_{-i})$  such that  $\sigma_i$  is a best response to  $\tau_{-i}$ .

Assuming that a categorization profile is exogenously given, a profile of strategies constitutes a CCE (w.r.t. the given categorization profile), if every player best responds to *some* conjecture about the strategies of the others. However, the conjecture of every player must be consistent with what she knows about the strategies of others, i.e., within the set  $F_{C_i}(\sigma_{-i})$ .

The set of all CCE in a game  $G$  w.r.t. a given categorization profile  $C$  is denoted by  $CCE_G(C)$ .  $NE_G$  is the set of Nash equilibria of the game  $G$ . The following observation is simple but important (the proof is omitted).

**Lemma 1.** For every game  $G$ ,

- (i) If  $C$  refines  $C'$  then  $CCE_G(C) \subseteq CCE_G(C')$ .
- (ii) If  $C$  is the finest categorization profile in  $G$  (every cell of every categorization contains only one player) then  $CCE_G(C) = NE_G$ .

**Corollary 1.** Every Nash equilibrium is a CCE w.r.t. any categorization profile.

<sup>5</sup>For two partitions  $P$  and  $P'$  of the same set,  $P$  is finer than  $P'$  (or equivalently,  $P'$  is coarser than  $P$ ) if every cell of  $P'$  is a union of cells of  $P$ .

<sup>6</sup>If  $X$  is a finite set then  $\Delta(X)$  denotes the family of all probability measures over  $X$ .

<sup>7</sup> $\sigma_i(s) = 0$  whenever  $s \in \mathbb{S} \setminus S_i$ .

**3.2. A continuum of players.** By its nature, the concept of CCE is more plausible when the number of players is large. It is therefore natural to study this concept in the environment of a non-atomic game. As we shall see below, working in the limit with a continuum of players removes the need for many of the technical details involved in the finite model. As a consequence the results become sharper and clearer.

We follow the model and notation of Schmeidler (1973)<sup>8</sup>. The set of players is identified with the  $T = [0, 1]$  interval equipped with the Lebesgue measure  $\lambda$ . There are  $n$  pure strategies, each of them represented by a vector  $e_i$  from the standard basis of  $\mathbb{R}^n$ . The set of possible mixed strategies of every player is<sup>9</sup>  $P = \text{conv}(\{e_1, \dots, e_n\})$ . A  $T$ -strategy is (the equivalence class of) a measurable function  $\hat{x}$  from  $T$  to  $P$ , specifying the strategy chosen by each of the players.  $\hat{P}$  is the set of all  $T$ -strategies endowed with the  $L_1$  weak topology.

The utility of player  $t_0 \in T$  when she chooses  $e_i$  and almost every player in  $T$  plays according to the  $T$ -strategy  $\hat{x}$  is  $u^i(t_0, \hat{x})$ . Denote  $u(t_0, \hat{x}) = (u^1(t_0, \hat{x}), \dots, u^n(t_0, \hat{x}))$ . The payoff to player  $t_0$  when almost every player in  $T$  plays according to  $\hat{x}$  (and, of course,  $t_0$  is playing  $\hat{x}(t_0)$ ) is the scalar product  $\hat{x}(t_0) \cdot u(t_0, \hat{x})$ . Thus, a game with a continuum of players can be identified with the function  $u : T \times \hat{P} \rightarrow \mathbb{R}^n$ .

We will only consider games  $u$  with the following two properties:

- (1)  $u$  is continuous on  $\hat{P}$  for every  $t \in T$ ; and
- (2)  $u$  is measurable on  $T$  for every  $\hat{x} \in \hat{P}$ .

As in the finite case, assume that every player  $t \in T$  has a finite and measurable partition  $C_t$  of the set  $T$  of players<sup>10</sup>. Define  $F_{C_t}(\hat{x}) = \{\hat{y} \in \hat{P} : \int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda \text{ for every } B \in C_t\}$ . Again, if  $\hat{y} \in F_{C_t}(\hat{x})$  we say that  $\hat{y}$  is a consistent conjecture of player  $t$  when the  $T$ -strategy is  $\hat{x}$ .

**Definition 2.** A  $T$ -strategy  $\hat{x} \in \hat{P}$  is a Conjectural Categorical Equilibrium (CCE) w.r.t. the categorization profile  $C = \{C_t\}_{t \in T}$  if, for  $\lambda$ -almost every  $t \in T$ , there is a  $T$ -strategy  $\hat{y}_t \in F_{C_t}(\hat{x})$  such that  $\hat{x}(t) \cdot u(t, \hat{y}_t) \geq p \cdot u(t, \hat{y}_t)$  for every  $p \in P$ .

Similarly to the finite case, we denote by  $CCE_u(C)$  the set of all CCE in the game with a continuum of players  $u$  w.r.t. the categorization profile  $C$ .  $NE_u$  is the set of Nash equilibria of  $u$ . The following is the analogue of Lemma 1 for the case of a continuum of players.

**Lemma 2.** For every game with a continuum of players  $u$ ,

- (i) If  $C$  refines  $C'$  then  $CCE_u(C) \subseteq CCE_u(C')$ .
- (ii)  $NE_u \subseteq CCE_u(C)$  for every categorization profile  $C$  in  $u$ .

<sup>8</sup>Since we are interested in games which are not necessarily anonymous, the models of Mas-Colell (1984) and of Rath (1992) are not suitable here.

<sup>9</sup> $\text{conv}(A)$  denotes the convex hull of the set  $A$ .

<sup>10</sup>In this non-atomic setting it is not important for our purposes whether the partition is of  $T$  or of  $T \setminus \{t\}$ . We also assume that the measure of each set in the partition is strictly positive.



#### 4. SUFFICIENT CATEGORIZATION PROFILES

The current section contains the main results of the paper. It deals with a property of categorization profiles which we call *sufficiency*. A categorization profile is sufficient if a best response to every consistent conjecture of every player is also a best response to the actual profile of actions. When an agent categorizes her opponents according to a sufficient categorization she maintains her utility level with significantly less mental effort. Exact and approximated sufficiency are formally defined as follows.

**Definition 3.** *Fix a game (either finite or non-atomic) and let  $\varepsilon \geq 0$ . A categorization profile  $C$  is  $\varepsilon$ -sufficient if every CCE w.r.t.  $C$  is an  $\varepsilon$ -Nash equilibrium<sup>11</sup> of the game. A categorization profile is sufficient if it is 0-sufficient.*

The rest of this section discusses sufficient conditions for the existence of sufficient categorization profiles. Of course, the finest categorization (in the finite case) in which every category consists of only one agent is always sufficient. What we show, however, is that for a wide family of games there are also non-trivial sufficient categorization profiles. We start with finite games and then move on to the non-atomic case.

**4.1. Sufficiency in finite games.** We start with some notation. Fix a game  $G$ . For a profile of actions  $\underline{s} = (s_1, \dots, s_n) \in S$  and two players  $j, k \in N$  with  $S_j = S_k$ , let  $\underline{s}^{jk}$  be the profile of actions in which every player other than  $j$  and  $k$  plays the same as in  $\underline{s}$  and players  $j$  and  $k$  exchange their choices. That is, player  $j$  plays  $s_k$ , player  $k$  plays  $s_j$  and every player  $l \in N \setminus \{j, k\}$  plays  $s_l$ . For a player  $i \in N$ , we say that the players  $j, k \in N \setminus \{i\}$  are *exchangeable for  $i$*  (denoted  $j \sim_i k$ ) if  $S_j = S_k$  and  $u_i(\underline{s}) = u_i(\underline{s}^{jk})$  for every  $\underline{s} \in S$ .

If  $j \sim_i k$  then player  $i$  only cares about the pair of actions taken by players  $j$  and  $k$ . She is not concerned with who plays what. Thus, assuming that  $i$  observes the distribution of actions in each cell of her categorization, it is natural for her to put  $j$  and  $k$  in the same cell.

It is easy to verify that  $\sim_i$  is transitive and symmetric over  $N \setminus \{i\}$ . Let  $\hat{C}_i$  be the partition of  $N \setminus \{i\}$  to the equivalence classes of  $\sim_i$  and let  $\hat{C} = (\hat{C}_1, \dots, \hat{C}_n)$ . Notice that our notation neglects the dependence of the categorization profile  $\hat{C}$  on the game  $G$ . This is so since it will always be clear what is the relevant game. Notice also that  $\hat{C}$  is endogenous: Nothing besides the description of the game is required in order to determine it.

If players were only allowed to play pure strategies and, in addition, players would always conjecture that their opponents play pure strategies then  $\hat{C}$  would have been

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<sup>11</sup>A strategy profile constitute an  $\varepsilon$ -Nash equilibrium in a finite game if no player can gain more than  $\varepsilon$  by deviating. In the non-atomic case the same should hold almost everywhere.

sufficient (see Lemma 6 in subsection 8.1). However, since players may randomize some conditions on the game must be added in order to maintain the sufficiency of  $\hat{C}$ . Although restricting the generality of our discussion, these conditions are valid for a wide family of games. The following definitions will be useful.

**Definition 4.** *The degree of anonymity of a finite game  $G$  is the number  $d(G) = \min_{i \in N} \min_{B \in \hat{C}_i} |B|$ .*

**Definition 5.** *Fix a finite set of actions  $\mathbb{S}$ . Let  $\Gamma(\mathbb{S})$  denote a family of normal form games such that for every game  $G \in \Gamma(\mathbb{S})$  and for every  $i \in N$ ,  $S_i \subseteq \mathbb{S}$ .*

*(a)  $\Gamma(\mathbb{S})$  is uniformly bounded if there is a number  $M > 0$  such that for every  $G \in \Gamma(\mathbb{S})$  and for every utility function  $u_i \in G$ ,  $|u_i| \leq M$ .*

*(b)  $\Gamma(\mathbb{S})$  exhibits a diminishing effect of a single player if there is a positive constant  $M$  such that  $|u_i(\underline{s}) - u_i(s'_j ; s_{-j})| \leq \frac{M}{|N|}$  for every  $G \in \Gamma(\mathbb{S})$ , every two players  $i, j \in N$ , every  $\underline{s} \in S$  and every  $s'_j \in S_j$ .*

*(c)  $\Gamma(\mathbb{S})$  has a logarithmic degree of anonymity if for every  $r, \varepsilon > 0$  there is  $n_0$  such that  $\frac{|N|}{d(G)} e^{-rd(G)} < \varepsilon$  for every game  $G \in \Gamma(\mathbb{S})$  with  $|N| > n_0$ .*

**Theorem 1.** *Consider a family  $\Gamma(\mathbb{S})$  of normal form games which is uniformly bounded, exhibits a diminishing effect of a single player and has a logarithmic degree of anonymity. For every  $\varepsilon > 0$  there exists  $n_0$  such that if  $G \in \Gamma(\mathbb{S})$  satisfies  $|N| > n_0$  then the categorization profile  $\hat{C}$  in  $G$  is  $\varepsilon$ -sufficient.*

Roughly speaking, Theorem 1 states that, under the conditions of uniform boundedness and diminishing effect of a single player, if each of the relations  $\{\sim_i\}_{i \in N}$  partition the set of players into a small number of large sets then every CCE w.r.t. this categorization profile is almost a Nash equilibrium. Note that, by Lemma 1, the same holds for every CCE w.r.t. any categorization profile finer than  $\hat{C}$ .

We illustrate the result of Theorem 1 with the following two examples.

**Example 1. (Village versus beach)** *This example is taken from Kalai (2004, Example 1). The universal set of actions is  $\mathbb{S} = \{v \text{ (village), } b \text{ (beach)}\}$ . The family  $\Gamma(\mathbb{S})$  contains games with  $|N| = 2n$  ( $n \in \mathbb{N}$ ) players of which  $n$  are ‘males’ and  $n$  are ‘females’. The payoff of a male is equal to the proportion of females his choice matches and the payoff of a female is equal to the proportion of males her choice mismatches.*

The categorization  $\hat{C}_i$  of every player lumps together players of the same gender. Indeed, the payoff of every player is not changed if two males (or females) exchange their choices. Notice that the family  $\Gamma(\mathbb{S})$  is uniformly bounded (by  $M = 1$ ), exhibits a diminishing effect of a single player (again, with  $M = 1$ ) and has a logarithmic degree of anonymity (since  $d(G) = \frac{|N|-2}{2}$ ). Thus, by Theorem 1, when the number of players becomes large every CCE w.r.t. the profile  $\hat{C}$  is almost Nash.

As a matter of fact, in this particular example Theorem 1 is redundant and a stronger result can be achieved by a much simpler argument. The reason is that the signal that every player observes is the expected proportions of males and females in each of the locations  $v$  and  $b$ . But from this signal a player can deduce his/her payoff for every possible choice. Thus, if a player's choice is optimal w.r.t. some consistent conjecture then it is also optimal w.r.t. the true strategy profile of his/her opponents. It follows that in the village versus beach game, *no matter what is the number of players*,  $\hat{C}$  is sufficient (and not just  $\varepsilon$ -sufficient).

**Example 2. (A generalized village versus beach)** Let  $\mathbb{S}$  be as in the previous example and fix two Lipschitz and non-decreasing functions  $f, g : [0, 1] \rightarrow \mathbb{R}$ . We consider games of the following form. For each player  $i \in N$  there is a set  $F^i$  of  $i$ 's friends and a set  $E^i$  of  $i$ 's enemies ( $F^i \cap E^i = \emptyset$ )<sup>12</sup>. The payoff to player  $i$  is  $f(p) + g(q)$  where  $p$  is the proportion of  $i$ 's friends that her choice matches and  $q$  is the proportion of  $i$ 's enemies that her choice mismatches.

It is clear that, without any further restrictions on the sets of friends and enemies, a family  $\Gamma(\mathbb{S})$  of games in the above form will be uniformly bounded and will exhibit a diminishing effect of a single player. In order to make sure that the family of games has a logarithmic degree of anonymity we need that, for every  $i \in N$ , the sets  $F^i$  and  $E^i$  are not too small in comparison to  $N$ . For example, it is sufficient for this purpose that there is a positive constant  $\rho$  such that  $|E^i|, |F^i| \geq |N|^\rho$  for every  $i \in N$  and for every game in the family<sup>13</sup>.

We emphasize that the simple argument of the previous example is not valid in this new scenario since consistent conjectures may lead to wrong actions. Indeed, assume that  $F^1 = \{2, 3, 4\}$ ,  $E^1 = \emptyset$ , and  $F^i = E^i = \emptyset$  for every player  $i > 1$ . Moreover, assume that  $f(p) = p^3$ . The following strategy profile is a CCE (w.r.t.  $\hat{C}$ ) which doesn't become close to Nash as the number of players increases. Player 1 plays  $v$ , players 2 and 3 play  $v$  with probability  $3/4$  and  $b$  with probability  $1/4$ , and player 4 plays  $b$  (the strategies of the other players are arbitrary). The true payoff to player 1 in this case is  $0 \cdot f(1) + 9/16 \cdot f(2/3) + 6/16 \cdot f(1/3) + 1/16 \cdot f(0) = 13/72$ , whereas if she would switch to  $b$  she will get  $0 \cdot f(0) + 9/16 \cdot f(1/3) + 6/16 \cdot f(2/3) + 1/16 \cdot f(1) = 14/72$ . Thus, the action of player 1 is suboptimal, independently of the number of players in the game.

To see that the above profile is a CCE notice that one of the consistent conjectures of player 1 is that players 2,3 play  $v$  with probability  $1/4$  and  $b$  with probability  $3/4$ , and player 4 plays  $v$ . If this is the belief that player 1 has then it is optimal for her

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<sup>12</sup>We do not assume that the relations 'to be a friend of' and 'to be an enemy of' are symmetric nor transitive.

<sup>13</sup>In fact, all we need in this particular example is that for every  $t \in \mathbb{R}$  there is  $k \in \mathbb{N}$  such that  $|E^i|, |F^i| > t$  for every  $i \in N$  and for every game in the family with  $|N| > k$ . See subsection 6.4.

to play  $v$  since, by symmetry, this would give her a payoff of  $14/72$ . Switching to  $b$ , however, would reduce the payoff to  $13/72$ .

**4.2. Sufficiency in non-atomic games.** When there is a continuum of players it will be meaningless to define a relation  $\sim_t$  analogous to the relation  $\sim_i$  in the finite case. What we need in order to insure that a categorization profile will be sufficient is that the utility of every player only depends on the distribution of actions in each set of her partition. No other assumptions should be made and the sufficiency obtained is not approximated as in the finite case. Thus, we have the following result.

**Theorem 2.** *Let  $u$  be a game with a continuum of players. If  $C = \{C_t\}_{t \in T}$  is a categorization profile such that, for every  $t \in T$ ,  $u(t, \hat{x})$  depends only on  $\{\int_B \hat{x} d\lambda\}_{B \in C_t}$  then  $C$  is sufficient.*

**Example 3. (A non-atomic generalized village versus beach)** *The following example is taken (with cosmetic changes) from Schmeidler (1973)<sup>14</sup>. The number of possible actions for every player is  $n = 2$ . For  $i = 1, 2$  the utility of player  $t \in T$  when she chooses  $e_i$  and when the  $T$ -strategy is  $\hat{x}$  is  $u^i(t, \hat{x}) = -\int_0^t \hat{x}_i d\lambda$ , where  $\hat{x}_i$  is the  $i$ 'th component of the vector function  $\hat{x}$ . Thus, every player  $t$  prefers the action which was less frequently used by her predecessors - the players  $[0, t)$ . This situation can be seen as a continuous analogue of the generalized village versus beach game, where for every player  $t \in T$  the set of enemies is  $[0, t)$  and the set of friends is empty.*

*Notice that the utility of every player  $t$  depends only on the integral  $\int_0^t \hat{x} d\lambda$ . Thus, by Theorem 2 the profile of categorizations  $C = \{C_t\}_{t \in T}$  defined by  $C_t = \{[0, t], (t, 1]\}$  is sufficient.*

The question naturally arises is how 'common' are games with the property that the utility of every player depends only on the average behavior of a finite number of groups of the participating players. Our next aim is to show that the set of games with this property is dense within the set of all non-atomic games. This implies that every non-atomic game can be approximated by a game in which a sufficient categorization profile exists.

We denote by  $Y$  the set of all possible (continuous) utility functions of a player. That is  $Y = \{v : \hat{P} \rightarrow \mathbb{R}^n \mid v \text{ is continuous}\}$ . Since  $\hat{P}$  is a compact we can define a norm in  $Y$  by  $\|v\| = \sup_{\hat{x} \in \hat{P}} \|v(\hat{x})\|$ , where  $\|v(\hat{x})\|$  is the Euclidean norm of  $\mathbb{R}^n$ . A non-atomic game  $u$  specifies the utility function of every player and is therefore an element of the product space  $Y^T$ . The set of all non-atomic games is denoted by  $U \subseteq Y^T$  (since  $u$  should be a measurable function of  $t$  not every element of  $Y^T$  is a game). Let  $\tilde{U} \subseteq U$  be the set of all games  $u$  with the property that, for each

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<sup>14</sup>The original purpose of this example was to show that not every non-atomic game has a pure strategy equilibrium.

player  $t$ , there is a finite and measurable partition  $C_t$  of  $T$  such that  $u(t, \hat{x}) = u(t, \hat{y})$  whenever  $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$  for every  $B \in C_t$ .

**Theorem 3.**  $\tilde{U}$  is dense in  $U$ .

## 5. CCE AND SOCIAL EFFICIENCY

The previous section considered the case in which every player categorizes her opponents “correctly” in the sense that players within each category are anonymous in the eyes of the categorizer. The aim of the current section is to study some of the effects that “wrong” categorizations may have. In particular, we are interested in the social efficiency of profiles of strategies which constitute a CCE in comparison to the efficiency of profiles which are Nash equilibria.

There may be various reasons why agents categorize others according to payoff irrelevant criteria (that is, not according to the partitions  $\hat{C}$ ). First, it may be that the partition  $\hat{C}_i$  contains too many elements for player  $i$  to handle. If player  $i$  has a limited computational ability then the number of different categories that she can create in her mind is bounded. Thus, she cannot sort her opponents optimally if the number of categories she needs to do so is greater than her ability<sup>15</sup>.

Another reason for sub-optimal categorization may be lack of information. Namely, player  $i$  may not know the effect that the actions taken by player  $j$  have on her payoff. This naturally brings up the question of how players should categorize in a game with incomplete information, which we will not discuss here.

As opposed to the previous section we do not pursue here general results. Rather, we restrict attention to two examples which reflect the implications that categorization can have on social efficiency. The first example shows how CCE may cause *all* the players to lose utility in comparison to their equilibrium payoffs (thus decreasing the social efficiency of the strategy profile). In the second example it is shown that a CCE may be more socially efficient than any Nash equilibrium. Both examples are of non-atomic *congestion games*<sup>16</sup> and can also be seen as special cases of the (generalized non-atomic) village versus beach game.

**Definition 6.** Let  $u$  be a non-atomic game. The social efficiency of a strategy profile  $\hat{x} \in \hat{P}$  is  $eff(\hat{x}) = \int_T \hat{x}(t)u(t, \hat{x})d\lambda(t)$ .

**Example 4.** Consider the following non-atomic game with 2 possible actions ( $n = 2$ ). If  $t \in [0, \frac{1}{2})$  then  $u^1(t, \hat{x}) = \int_0^{\frac{1}{2}} \hat{x}_2 d\lambda$  and  $u^2(t, \hat{x}) = \frac{1}{2} - u^1(t, \hat{x})$ . For  $t \in [\frac{1}{2}, 1]$  the utility function is  $u^1(t, \hat{x}) = \int_{\frac{1}{2}}^1 \hat{x}_2 d\lambda$  and  $u^2(t, \hat{x}) = \frac{1}{2} - u^1(t, \hat{x})$ . We call the players in the interval  $[0, \frac{1}{2})$  type 1 players and those in  $[\frac{1}{2}, 1]$  are called type 2 players.

<sup>15</sup>Recall that one of the reasons for the need to categorize in the first place is to save mental resources. The issue of “optimal” categorization when there is a bound on the number of categories seems to be of self interest.

<sup>16</sup>For a general study of social optimality in non-atomic congestion games see Milchtaich (2004).

Thus, each player is trying to avoid players with his own type and is careless about the choices of players from the other type.

Since payoffs depend only on the distribution of actions within each type of players we may w.l.o.g. restrict attention to pure strategies. Denote by  $p_1(\hat{x}) = \lambda(\{t \in [0, \frac{1}{2}) : \hat{x}(t) = e_1\})$  and  $p_2(\hat{x}) = \lambda(\{t \in [\frac{1}{2}, 1] : \hat{x}(t) = e_1\})$  the proportions of players of types 1 and 2 respectively who choose the first action according to  $\hat{x}$ . Then the social efficiency of a  $T$ -strategy  $\hat{x}$  is  $eff(\hat{x}) = 2p_1(\hat{x})(\frac{1}{2} - p_1(\hat{x})) + 2p_2(\hat{x})(\frac{1}{2} - p_2(\hat{x}))$ . Notice that, in any equilibrium  $\hat{x}$ , it must be that  $p_1(\hat{x}) = \frac{1}{4}$  and  $p_2(\hat{x}) = \frac{1}{4}$ . Therefore, the social efficiency in every equilibrium is  $\frac{1}{4}$ . Moreover, every equilibrium is *socially optimal* in the sense that there is no profile of strategies  $\hat{x}$  with  $eff(\hat{x}) > \frac{1}{4}$ .

By Theorem 2 the categorization profile defined by  $C_t = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$  for every  $t \in T$ , as well as any finer categorization profile, is sufficient. However, assume that players categorize their opponents differently and that the categorization of all the players is the same. For instance, this corresponds to the case where players are categorizing according to some publicly observed property (such as gender or skin color). For simplicity we restrict attention to the case in which the (common) categorization has only two elements (say,  $M$ =Males and  $F$ =Females) each of which have a measure of  $\frac{1}{2}$ .

Let  $\alpha = \lambda(M \cap [0, \frac{1}{2}))$  be the measure of the set of type 1 males, and let  $g(\alpha) = \min \{eff(\hat{x}) ; \hat{x} \in CE_u(\{M, F\})\}$  be the *lowest* social efficiency of a CCE<sup>17</sup>. We have

**Proposition 1.**

$$g(\alpha) = \begin{cases} \frac{1}{4} - 16\alpha^2 & 0 \leq \alpha \leq \frac{1}{8} \\ 0 & \frac{1}{8} \leq \alpha \leq \frac{3}{8} \\ \frac{1}{4} - 16(\frac{1}{2} - \alpha)^2 & \frac{3}{8} \leq \alpha \leq \frac{1}{2} \end{cases}$$

Specifically, if the payoff relevant partition (type 1 versus type 2) and the actual categorization (males versus females) are independent ( $\alpha = \frac{1}{4}$ ) or not “too dependent” ( $\frac{1}{8} \leq \alpha \leq \frac{3}{8}$ ) then there exists a CCE in which all the players get the worst possible payoff. As  $\alpha$  approaches 0 or  $\frac{1}{2}$  the social efficiency of any CCE increases to the optimal level.

**Example 5.** Consider the following non-atomic game with 2 possible actions ( $n = 2$ ). If  $t \in [0, \frac{3}{4})$  then  $u^1(t, \hat{x}) = \frac{1}{4} + \int_0^{\frac{3}{4}} \hat{x}_2 d\lambda + 2 \int_{\frac{3}{4}}^1 \hat{x}_2 d\lambda$  and  $u^2(t, \hat{x}) = \int_0^{\frac{3}{4}} \hat{x}_1 d\lambda + 2 \int_{\frac{3}{4}}^1 \hat{x}_1 d\lambda$ . For  $t \in [\frac{3}{4}, 1]$  the utility function is  $u^1(t, \hat{x}) = 2 \int_0^{\frac{3}{4}} \hat{x}_2 d\lambda + \int_{\frac{3}{4}}^1 \hat{x}_2 d\lambda$  and  $u^2(t, \hat{x}) = 1 + 2 \int_0^{\frac{3}{4}} \hat{x}_1 d\lambda + \int_{\frac{3}{4}}^1 \hat{x}_1 d\lambda$ . Players in the interval  $[0, \frac{3}{4})$  are called type 1 players and those in  $[\frac{3}{4}, 1)$  type 2 players. The payoff to players of each type equals the proportion of players of their own type that their choice mismatches plus twice

<sup>17</sup>One can convince herself that  $g$  indeed depends only on  $\alpha$  and not on the choice of the sets  $M$  and  $F$ .

the proportion of players of the other type that their choice mismatches. In addition, players of type 1 get  $\frac{1}{4}$  if they choose the first action and players of type 2 get 1 if they choose the second action.

As in the previous example, we restrict attention to pure strategy profiles and denote  $p_1(\hat{x}) = \lambda(\{t \in [0, \frac{3}{4}) : \hat{x}(t) = e_1\})$  and  $p_2(\hat{x}) = \lambda(\{t \in [\frac{3}{4}, 1] : \hat{x}(t) = e_1\})$  the proportions of players of types 1 and 2 respectively who choose the first action according to  $\hat{x}$ . the efficiency of a  $T$ -strategy  $\hat{x}$  is then given by  $eff(\hat{x}) = -2p_1(\hat{x})^2 - 2p_2(\hat{x})^2 - 8p_1(\hat{x})p_2(\hat{x}) + \frac{11}{4}p_1(\hat{x}) + \frac{10}{4}p_2(\hat{x}) + \frac{1}{4}$ . It is not hard to verify that there is a unique equilibrium in this game. Namely, all type 1 players choose the first action ( $p_1(\hat{x}) = 3/4$ ) while all type 2 players choose the other option ( $p_2(\hat{x}) = 0$ ). The social efficiency of the equilibrium strategy is equal to  $\frac{19}{16} = 1.1875$ .

However, assume that the categorization of all the players is trivial. That is,  $C_t = \{T\}$  for all  $t \in T$ . In this case there is a CCE which is more efficient than the Nash equilibrium. Indeed, it is a simple exercise to check that the set of CCE in this case is  $\{\hat{x} : \frac{1}{2} \leq p_1(\hat{x}) \leq \frac{3}{4}, p_2(\hat{x}) = 0\}$ . Taking  $\hat{x}$  to be a profile with  $p_1(\hat{x}) = \frac{11}{16}$  and  $p_2(\hat{x}) = 0$  gives a CCE with social efficiency of  $\frac{153}{128} \cong 1.1953$ . Moreover, such a profile is socially optimal.

## 6. DISCUSSION

**6.1. Simple conjectures: A refinement.** In a CCE the conjecture of an agent is limited only by the signal she observed (and by the restriction that agents play independently of each other). One may want to restrict the belief that an agent can have even more by requiring that it will be simple in some sense<sup>18</sup>. By so doing, a refinement of CCE can be obtained.

Among all the possible conjectures of a player there is one which can quite naturally be considered as the simplest. Namely, the conjecture in which all players in each cell of her partition are playing the same strategy. A player holding this belief can be seen as having a prototypical agent for each set in her partition. All the players in each set are playing the same as their representing prototype. The common strategy in each cell is then uniquely determined by the signal the player observed.

We would like to make this last idea formal both in finite games and in games with a continuum of players. First, in the finite model, let  $f_{C_i}(\sigma_{-i}) \in F_{C_i}(\sigma_{-i})$  be the (unique) strategies vector of players other than  $i$  in which  $\sigma_j = \sigma_k$  whenever  $j$  and  $k$  are in the same cell of the partition  $C_i$ . The refinement of CCE, which we call *prototypical equilibrium* requires that every player  $i$  plays optimally against the strategy profile  $f_{C_i}(\sigma_{-i})$ . Formally,

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<sup>18</sup>This idea is certainly not new. Eliaz (2003) and Spiegler (2002 and 2004) are examples of papers in which the solution concept takes into account the complexity of the belief of an agent about what others will do.

**Definition 7.**  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a prototypical equilibrium w.r.t. the categorization profile  $C = (C_1, \dots, C_n)$  if, for every  $i \in N$ ,  $\sigma_i$  is a best response to  $f_{C_i}(\sigma_{-i})$ .

The definition in the non-atomic setting is similar. First, for a partition  $R$  of  $T$  and a player  $t \in T$ ,  $R(t)$  denotes the cell of  $R$  which contains  $t$ . If  $\hat{x} \in \hat{P}$  is a  $T$ -strategy, let  $f_R(\hat{x})$  denote the  $T$ -strategy defined by  $f_R(\hat{x})(t) = \frac{1}{\lambda(R(t))} \int_{R(t)} \hat{x} d\lambda$ . Notice that always  $f_R(\hat{x}) \in F_R(\hat{x})$ .

**Definition 8.**  $\hat{x} \in \hat{P}$  is a prototypical equilibrium w.r.t. the categorization profile  $C = \{C_t\}_{t \in T}$  if  $\hat{x}(t)$  is a best response to  $f_{C_t}(\hat{x})$  for almost every  $t \in T$ .

We emphasize that a prototypical equilibrium is always a CCE (w.r.t. the same categorization profile). However, unlike CCE, it may well be that a Nash equilibrium is not a prototypical equilibrium. Existence of a prototypical equilibrium in every game (finite or infinite) is guaranteed by a standard fixed point argument.

**6.2. Correlated conjectures.** The sufficiency result of subsection 4.1 relies heavily on the assumption that a player takes into account only *independent* profiles of strategies of her opponents. That is, we rule out the possibility that some player thinks that other players correlate their strategies, even though this correlated strategy might be consistent with the signal that this player observes. The fact that correlated conjectures are not allowed enables us to use the power of the laws of large numbers, which otherwise fail.

To emphasize this point we return to the generalized village versus beach game (Example 2). Assume that, for some player  $i \in N$ ,  $F^i = N \setminus \{i\}$  and  $E^i = \emptyset$ , and that  $F^j = E^j = \emptyset$  for all other players  $j$ . As opposed to Example 2, we do not assume that the function  $f$  is non-decreasing<sup>19</sup>. Specifically, consider the function  $f$  defined by  $f(x) = 3x$  for  $0 \leq x \leq \frac{1}{3}$  and  $f(x) = \frac{4}{3} - x$  for  $\frac{1}{3} \leq x \leq 1$ .

If correlated conjectures were allowed then the following profile of strategies would constitute a CCE w.r.t. the categorization profile  $\hat{C}$ . Player  $i$  plays  $v$  with probability 1 and every other player plays  $v$  with probability  $\frac{2}{3}$  and  $b$  with probability  $\frac{1}{3}$ . Indeed, a consistent (correlated) conjecture of player  $i$  is that either all the players play  $v$  (with probability  $\frac{2}{3}$ ) or all the players play  $b$  (with probability  $\frac{1}{3}$ ). For this conjecture the best response of  $i$  is  $v$  since  $\frac{2}{3}f(1) + \frac{1}{3}f(0) > \frac{2}{3}f(0) + \frac{1}{3}f(1)$ . However, this profile of strategies does not become approximately Nash as the number of players increases. This is because the payoff to player  $i$  will be close to  $f(\frac{2}{3}) = \frac{2}{3}$  while deviating to  $b$  would result in a payoff close to  $f(\frac{1}{3}) = 1$ .

**6.3. Pure equilibrium.** The fact that Nash equilibrium in pure strategies may fail to exist is seen by many as a drawback of this solution concept. Even in games with a continuum of players, a certain degree of anonymity is required in order

<sup>19</sup>A possible interpretation is that a player wants to be with her friends, but not with too many of them.



to insure the existence of a pure equilibrium (see Remarks 2 and 3 in Schmeidler, 1973). The reason that players need to randomize in equilibrium is to hide their action from their opponents. The same goal can be achieved by using the CCE solution concept. The fact that an agent cannot predict accurately her opponents behavior eliminates in some cases the need for randomization in equilibrium. This phenomenon is demonstrated in the following theorem.

**Theorem 4.** *Let  $u$  be a game with a continuum of players, and let  $C^*$  be a partition of  $T$ . Assume that  $C = \{C_t\}_{t \in T}$  is a profile of categorizations such that  $C^*$  is finer than  $C_t$  for every  $t \in T$ . Then there is a pure CCE w.r.t.  $C$ <sup>20</sup>.*

To illustrate the idea consider Example 3. Assume that  $C_t = \{T\}$  for every  $t \in T$ . Thus, the signal to every player is just the average behavior of the entire set of players. The pure  $T$ -strategy  $\hat{x}$  defined by  $\hat{x}(t) = e_1$  for  $0 \leq t \leq \frac{1}{2}$  and  $\hat{x}(t) = e_2$  for  $\frac{1}{2} < t \leq 1$  is a CCE w.r.t.  $C = \{C_t\}_{t \in T}$ . Indeed, for every  $t \in T$ ,  $F_{C_t}(\hat{x})$  contains a  $T$ -strategy for which  $\hat{x}(t)$  is a best response.

Finally, we note that the condition that  $C_t$  is coarser than some  $C^*$  for all  $t \in T$  is necessary for the theorem to hold. This can be seen by considering the categorization profile  $C_t = \{[0, t], (t, 1]\}$ ,  $t \in T$  in the above example.

**6.4. On the degree of anonymity.** The number  $d(G)$  describes the size of the smallest set in the partitions  $\hat{C} = \{\hat{C}_i\}_{i \in N}$ . The payoff to every player  $i \in N$  depends only on the proportion of players playing each action in each of the sets in  $\hat{C}_i$ . Thus, the ratio between  $d(G)$  and  $|N|$  is, in some sense, a measure of the degree of anonymity in the game  $G$ . If this ratio is large then the payoff function of every player is relatively robust to opponents exchanging their actions with one another. In other words, it is less important who plays what.

In order for Theorem 1 to hold, the games in the family  $\Gamma(\mathbb{S})$  should have a sufficient degree of anonymity. The formulation of this is provided by Definition 5 (c). The reason that we call the required degree of anonymity *logarithmic* is that it only requires that  $\ln |N| = o(d(G))$  uniformly in the family  $\Gamma(\mathbb{S})$ .

Reading the proof of Theorem 1, one can see that the logarithmic degree of anonymity can be replaced by the following two conditions combined. The first is that<sup>21</sup>  $d(G) \rightarrow \infty$  as  $|N| \rightarrow \infty$  (with no specification of the rate of convergence). The second is that  $|\hat{C}_i| \leq M$  for some constant  $M > 0$  for every  $i \in N$  and for every  $G \in \Gamma(\mathbb{S})$ . It is important to note that these conditions do not imply and are not implied by the logarithmic degree of anonymity condition.

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<sup>20</sup>Reading the proof, one can see that existence of a pure prototypical equilibrium w.r.t.  $C$  is also guaranteed under the conditions of the theorem.

<sup>21</sup>Formally, for every  $k$  there is  $n_0 = n_0(k)$  such that  $d(G) > k$  for every game  $G \in \Gamma(\mathbb{S})$  with  $|N| > n_0$ .

**6.5. Self categorization.** Throughout the discussion of finite games the partition of player  $i$  is of the set  $N \setminus \{i\}$ . Thus,  $i$  doesn't include herself in any of the groups of her categorization. The reason for this modeling choice is the common assumption that every agent knows the action he plays. Inserting  $i$  into one of the cells of her partition  $C_i$  (say  $B$ ) can create a situation in which  $i$ 's conjecture (about what the players in  $B$  are playing) is consistent with her signal (the expected behavior within  $B$ ) but not with the action which she actually plays. Leaving  $i$  out of her own partition prevents such an awkward situation.

It should be noted, however, that individuals do not exclude themselves from their categorical perception of the society. In fact, self categorization and identity are among the most studied subjects in social psychology (for references see Ellemers et al., 2002). The social categories to which one belongs and the way these categories are seen by the society can have significant implications on one's choices. This important issue is not addressed by the current paper.

## 7. RELATED LITERATURE

Our main result (Theorem 1) is inspired by the works of Kalai (2004, 2005) on the robustness of equilibria in large games. There, it is shown that when the number of players is large Nash equilibria of a wide family of games is immune to many modifications of the game. These modifications include various extensive form versions of the game such as sequential play (instead of simultaneous play) and versions in which players can revise their initial choices. The importance of such results is that they help to overcome difficulties in the modeling of real-life situations when the modeler has missing details. Since in general it is impossible to construct an extensive version of a game in which Nash equilibria corresponds to CCE of the original game, the results of Kalai do not imply ours.

It is interesting to compare our model and assumptions to those of Kalai. A first difference is that we study only complete information games while Kalai allows players to be of several *types* (though a key assumption in his paper is that types are drawn independently from some universal finite set). Another difference is in the anonymity and continuity assumptions used. In Kalai's paper the payoff to a player depends on his own type and action and on the empirical distribution of type-action characters of the other players. This implies that, with probability 1, *the categorization  $\hat{C}_i$  will be the same for all the players*<sup>22</sup>. In the current paper 'types' of players are *subjective* in the sense that the categorization  $\hat{C}_i$  is likely to depend on  $i$ . Moreover, the number of different 'types' (elements of the categorization  $\hat{C}_i$ ) is

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<sup>22</sup>More accurately, for every two players  $i, j$  the partitions  $\hat{C}_i$  excluding player  $j$  and  $\hat{C}_j$  excluding player  $i$  will be identical. Notice, however, that when different profiles of types are realized this common categorization may very well change.

not bounded and can grow to infinity as  $|N|$  grows to infinity (but not too fast since logarithmic degree of anonymity must be maintained)<sup>23</sup>.

The condition of diminishing effect of a single player (Definition 5 (b)) is slightly different than the uniform equicontinuity condition of Kalai (2004, Definition 3). None of them implies the other. Uniform equicontinuity implies semi-anonymity in the sense of Kalai (2004, Definition 2) which will make our analysis trivial. But the diminishing effect condition requires that the change in payoff when one player change his action will be *inversely proportional* to the number of players in the game. This property is not implied by uniform equicontinuity.

The concept of Nash equilibrium in normal form games with a continuum of players is originated in the work of Schmeidler (1973). The main result there is the existence of a pure equilibrium when the game is anonymous. Similar results in different models were obtained by Mas-Colell (1984) and by Rath (1992). Theorem 4 shows that CCE can in some cases eliminate the need for randomization even if the game is not anonymous. A comprehensive survey of the literature about games with a continuum of players can be found in Ali Khan and Sun (2002).

CCE is a special case of Battigalli and Guaitoli's (1988) *Conjectural Equilibrium* (CE). Therefore, this paper can be seen as contributing to the literature originates in the works of Bernheim (1984) and Pearce (1984)<sup>24</sup>, who defined the concept of *rationalizability*. Rubinstein and Wolinsky (1994) generalized rationalizability to the case where each agent observes some exogenously defined signal, which is a function of the actions taken by the agents. A profile of actions is a *Rationalizable Conjectural Equilibrium* (RCE) if every player's action is optimal w.r.t. some consistent conjecture about the actions of her opponents. A consistent conjecture is one that can't be contradicted by the realized signal that the agent observed when combined with common knowledge of rationality. When the signal to every player contains only her own action the RCE concept boils down to rationalizability, while when the signal to every player is the entire profile of actions the RCE is the same as Nash equilibrium. CE of Battigalli and Guaitoli (1988) is a weaker solution concept than RCE, since the common knowledge of rationality assumption is removed. Thus, with the same signal functions, a profile of strategies may be a CE but not an RCE.

One may ask why we use CE and not RCE as our solution concept. Two reasons justify this choice. The first is that the games we analyze have many players. It is natural to assume that in this case players do not "get into the head" of their opponents and draw conclusions which change their beliefs, simply because it is too

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<sup>23</sup>As noted in subsection 6.4, if the size of the partitions  $\hat{C}_i$  is uniformly bounded in  $\Gamma(\mathbb{S})$  then logarithmic degree of anonymity can be replaced by a weaker condition.

<sup>24</sup>Bernheim's paper is more relevant to the current work since it highlights normal form games. Pearce, on the other hand, emphasizes extensive form games.

complicated to do so. Second, every RCE is also a CE. Thus, our main results wouldn't change had we define CCE using RCE and not CE.

Each one of these solution concepts is weaker than Nash equilibrium. In the final section of their paper Rubinstein and Wolinsky (1994) suggest that the plausibility of Nash equilibrium increases when every RCE is also Nash. They write

“... In games with this property the Nash equilibrium concept is more compelling, because in a sense the equilibrium requires less information on the part of the players. It may therefore be of interest to identify conditions under which, for some natural signal function such as one's own payoff, RCE and Nash equilibria are equivalent.”

Theorems 1 and 2 provide precisely the kind of conditions that Rubinstein and Wolinsky are talking about. The signal function of a player, however, is not her own payoff but the expected behavior of groups of her opponents.

Attempts to weaken the assumption that agents predict accurately the actions of their opponents have been made in settings other than normal form games. For extensive form games Fudenberg and Levine's (1993) *self-confirming equilibrium* is based on the fact that agents' beliefs are correct only along the equilibrium path of play<sup>25</sup>. In Jehiel's (2005) *analogy-based expectation equilibrium* agents bundle nodes in which other players should move and can only tell what is the average behavior in every class of nodes. In fact, our prototypical equilibrium is a special case of Jehiel's concept. For repeated games, Kalai and Lehrer (1993a, 1993b) introduced the notion of *subjective equilibrium* where player's beliefs are not contradicted by the observed choices of their opponents.

Some recent papers discuss the implications of categorical thinking for decision making. Fryer and Jackson (2004) develop a model of how past experiences are sorted into categories and show that certain biases in decision making emerge from this process. Peşki (2006) shows that in symmetric environments categorization is an optimal way for predicting properties of future instances based on past instances. Finally, for surveys about categorization in social psychology see Fiske (1998) and Macrae and Bodenhausen (2000).

## 8. PROOFS

**8.1. Proof of Theorem 1.** Fix a family of games  $\Gamma(\mathbb{S})$  which is uniformly bounded (by a constant  $M > 0$ ), exhibits a diminishing effect of a single player (w.l.o.g. with the same constant  $M$ ) and has a logarithmic degree of anonymity.

The proof of the theorem is divided into several lemmas. For the following Lemmas 3-7 and 9 fix a game  $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}) \in \Gamma(\mathbb{S})$ , a player  $i \in N$ , a profile of strategies  $\sigma$  and a positive number  $\delta$ . Assume that  $\hat{C}_i = \{B_1, \dots, B_m\}$ . For every

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<sup>25</sup>See also Dekel et al. (1999).

$j = 1, \dots, m$  and for every  $s \in \mathbb{S}$  let

$$D_\delta(B_j, s) = \left\{ s_{-i} \in S_{-i} : \left| \frac{\#\{l \in B_j : s_l = s\}}{|B_j|} - \sigma^{B_j}(s) \right| \geq \delta \right\}$$

and

$$D_\delta = \bigcup_{j=1}^m \bigcup_{s \in \mathbb{S}} D_\delta(B_j, s).$$

Let  $\mathbb{P}_{\sigma_{-i}}$  denote the probability measure on  $S_{-i}$  induced by the profile of strategies  $\sigma_{-i}$ .

**Lemma 3.**  $\mathbb{P}_{\sigma_{-i}}(D_\delta(B_j, s)) \leq 2e^{-2|B_j|\delta^2}$  For every  $j = 1, \dots, m$  and for every  $s \in \mathbb{S}$ .

*Proof.* For fixed  $1 \leq j \leq m$  and  $s \in \mathbb{S}$  consider the sequence of independent random variables  $(X_l)_{l \in B_j}$  defined by  $X_l = 1$  if player  $l$  realized strategy is  $s$  and  $X_l = 0$  otherwise. Let  $X = \sum_{l \in B_j} X_l$ . We have  $D_\delta(B_j, s) = \left\{ \left| \frac{X}{|B_j|} - \sigma^{B_j}(s) \right| \geq \delta \right\}$  where  $\sigma^{B_j}(s)$  is the expected value of the variable  $\frac{X}{|B_j|}$ . By a classical bound of Hoeffding (see for instance Petrov, 1975 chapter III) the probability of this event is not greater than  $2e^{-2|B_j|\delta^2}$ . ■

**Lemma 4.**  $\mathbb{P}_{\sigma_{-i}}(D_\delta) \leq 2|\mathbb{S}|me^{-2d(G)\delta^2}$ .

*Proof.* Using the previous lemma,

$$\mathbb{P}_{\sigma_{-i}}(D_\delta) \leq \sum_{j=1}^m \sum_{s \in \mathbb{S}} 2e^{-2|B_j|\delta^2} = 2|\mathbb{S}| \sum_{j=1}^m e^{-2|B_j|\delta^2} \leq 2|\mathbb{S}|me^{-2d(G)\delta^2}.$$

■

**Lemma 5.** If  $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$  then  $|\mathbb{P}_{\sigma_{-i}}(D_\delta) - \mathbb{P}_{\tau_{-i}}(D_\delta)| \leq 2|\mathbb{S}|me^{-2d(G)\delta^2}$ .

*Proof.*  $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$  means that  $\sigma^{B_j}(s) = \tau^{B_j}(s)$  for every  $j = 1, \dots, m$  and for every  $s \in \mathbb{S}$ . Thus, using Lemma 4 with  $\tau_{-i}$  instead of  $\sigma_{-i}$ , we have  $\mathbb{P}_{\tau_{-i}}(D_\delta) \leq 2|\mathbb{S}|me^{-2d(G)\delta^2}$ . It follows that both  $\mathbb{P}_{\tau_{-i}}(D_\delta)$  and  $\mathbb{P}_{\sigma_{-i}}(D_\delta)$  are in the interval  $[0, 2|\mathbb{S}|me^{-2d(G)\delta^2}]$ . ■

**Lemma 6.** Fix two profiles of actions  $\underline{s}, \underline{s}' \in S$ . If  $s_{-i} \in F_{\hat{C}_i}(s'_{-i})$  then  $u_i(\underline{s}) = u_i(\underline{s}')$ .

*Proof.* Assume that  $\hat{C}_i = \{B_1, \dots, B_m\}$ .  $s_{-i} \in F_{\hat{C}_i}(s'_{-i})$  means that  $\frac{\#\{l \in B_j : s_l = s\}}{|B_j|} = \frac{\#\{l \in B_j : s'_l = s\}}{|B_j|}$  for every  $s \in \mathbb{S}$  and for every  $j = 1, \dots, m$ . Thus, for every  $j = 1, \dots, m$ , there is a permutation of players' names in the set  $B_j$  which transforms the restriction of  $\underline{s}$  to  $B_j$  to the restriction of  $\underline{s}'$  to  $B_j$ . However, every such permutation of players in the set  $B_j$  can be achieved by a sequence of exchanges of pairs of players. By the definition of the partition  $\hat{C}_i$ , every such exchange doesn't affect the payoff of player  $i$ . The assertion follows. ■

**Lemma 7.** Let  $s_{-i}, s'_{-i}$  be two action vectors in  $D_\delta^c$  (the complement of  $D_\delta$  w.r.t.  $S_{-i}$ ). Then  $|u_i(s_i ; s_{-i}) - u_i(s_i ; s'_{-i})| \leq \delta M$  for every  $s_i \in S_i$ .

*Proof.* Since both  $s_{-i}$  and  $s'_{-i}$  are in  $D_\delta^c$  it follows that  $|\#\{l \in B_j : s_l = s\} - \#\{l \in B_j : s'_l = s\}| \leq 2\delta|B_j|$  for every  $j = 1, \dots, m$  and for every  $s \in \mathbb{S}$ . Thus, there is a vector of actions  $s''_{-i} \in F_{\hat{C}_i}(s'_{-i})$  such that  $s''_{-i}$  is obtained from  $s_{-i}$  by no more than  $\sum_{j=1}^m \delta|B_j| = \delta(|N| - 1)$  changes in players' actions. By the previous lemma, we have that  $|u_i(s_i ; s_{-i}) - u_i(s_i ; s'_{-i})| = |u_i(s_i ; s_{-i}) - u_i(s_i ; s''_{-i})|$ . By assumption, the influence of some player  $l \neq i$  changing her action on the utility function  $u_i$  is no greater than  $\frac{M}{|N|}$ . It follows that  $|u_i(s_i ; s_{-i}) - u_i(s_i ; s''_{-i})| \leq \delta(|N| - 1)\frac{M}{|N|} \leq \delta M$ . ■

**Lemma 8.** *Let  $\Omega$  be a finite set,  $P, Q$  two probability measures on  $\Omega$ ,  $X : \Omega \rightarrow \mathbb{R}$  a random variable and  $\varepsilon > 0$ . Let  $A \subseteq \Omega$  be an event such that  $|P(A) - Q(A)| \leq \varepsilon$ , and assume that  $r \leq X(\omega) \leq R$  for every  $\omega \in A$  ( $r \leq R$  are two constants). Then  $|\sum_{\omega \in A} X(\omega)(P(\omega) - Q(\omega))| \leq R - r + \varepsilon \max(|r|, |R|)$ .*

*Proof.* Denoting  $\mathbb{E}_P$  ( $\mathbb{E}_Q$ ) the expectation operator w.r.t. to the measure  $P$  ( $Q$ ), one has

$$\begin{aligned} & \left| \sum_{\omega \in A} X(\omega)(P(\omega) - Q(\omega)) \right| = |P(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_Q(X|A)| \\ & \leq |P(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_P(X|A)| + |Q(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_Q(X|A)| \\ & = |P(A) - Q(A)||\mathbb{E}_P(X|A)| + Q(A)|\mathbb{E}_P(X|A) - \mathbb{E}_Q(X|A)| \\ & \leq \varepsilon \max(|R|, |r|) + R - r. \end{aligned}$$

■

**Lemma 9.** *If  $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$  then  $|u_i(s_i ; \sigma_{-i}) - u_i(s_i ; \tau_{-i})| \leq \delta M + 6M|\mathbb{S}|me^{-2d(G)\delta^2}$  for every  $s_i \in S_i$ .*

*Proof.*

$$\begin{aligned} |u_i(s_i ; \sigma_{-i}) - u_i(s_i ; \tau_{-i})| & \leq \left| \sum_{s_{-i} \in D_\delta} u_i(s_i ; s_{-i}) (\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})) \right| \\ & \quad + \left| \sum_{s_{-i} \in D_\delta^c} u_i(s_i ; s_{-i}) (\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})) \right|. \end{aligned}$$

The first sum can be estimated by

$$\begin{aligned} & \left| \sum_{s_{-i} \in D_\delta} u_i(s_i ; s_{-i}) (\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})) \right| \\ & \leq M \cdot \sum_{s_{-i} \in D_\delta} |\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})| \leq M \cdot (\mathbb{P}_{\sigma_{-i}}(D_\delta) + \mathbb{P}_{\tau_{-i}}(D_\delta)) \\ & \leq M \cdot 4|\mathbb{S}||\hat{C}_i|e^{-2d(G)\delta^2}, \end{aligned}$$

where the first inequality is due to the fact that  $\Gamma(\mathbb{S})$  is uniformly bounded by  $M$ , and the third inequality is by Lemmas 4 and 5. In order to estimate the second

sum we use Lemma 8 with  $\Omega = S_{-i}$ ,  $P = \mathbb{P}_{\sigma_{-i}}$ ,  $Q = \mathbb{P}_{\tau_{-i}}$ ,  $X(\omega) = u_i(s_i ; \omega)$  and  $\varepsilon = 2|\mathbb{S}|me^{-2d(G)\delta^2}$ . Notice that, by Lemma 7, the utility  $u_i(s_i ; \cdot)$  is bounded in an interval of length not larger than  $\delta M$ . Thus,

$$\left| \sum_{s_{-i} \in D_\delta^c} u_i(s_i ; s_{-i}) (\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})) \right| \leq \delta M + M \cdot 2|\mathbb{S}|me^{-2d(G)\delta^2}.$$

Summing up the two inequalities gives the desired bound. ■

We are now in a position to prove Theorem 1. It should be shown that, for a given  $\varepsilon > 0$ , if  $G \in \Gamma(\mathbb{S})$  has a large enough number of players then every CCE in  $G$  w.r.t. the categorization profile  $\hat{C}$  is an  $\varepsilon$ -Nash equilibrium. Assume that  $\sigma$  is a CCE w.r.t.  $\hat{C}$ . Then, for every  $i \in N$ , there is a strategy profile  $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$  such that  $\sigma_i$  is a best response to  $\tau_{-i}$ .

Notice that  $|\hat{C}_i| \leq \frac{|N|}{d(G)}$  for every  $i \in N$ . Fix  $0 < \delta < \frac{\varepsilon}{4M}$  and then take  $n_0$  large enough so that  $6M|\mathbb{S}|\frac{|N|}{d(G)}e^{-2d(G)\delta^2} < \frac{\varepsilon}{4}$  for any game  $G \in \Gamma(\mathbb{S})$  with  $|N| > n_0$ . This is possible since  $\Gamma(\mathbb{S})$  has a logarithmic degree of anonymity. By Lemma 9,  $|u_i(s_i ; \sigma_{-i}) - u_i(s_i ; \tau_{-i})| < \frac{\varepsilon}{2}$  for any  $G \in \Gamma(\mathbb{S})$  with  $|N| > n_0$ . Finally, since  $\sigma_i$  is a best response to  $\tau_{-i}$ , we have for every  $i \in N$  and  $s_i \in S_i$

$$u_i(s_i ; \sigma_{-i}) \leq u_i(s_i ; \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i ; \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i ; \sigma_{-i}) + \varepsilon.$$

**8.2. Proof of Theorem 2.** Assume that  $\hat{x} \in \hat{P}$  is a CCE w.r.t. a categorization profile  $C = \{C_t\}_{t \in T}$  which satisfy the conditions of the theorem. For every player  $t \in T$ , the payoff function  $u(t, \cdot)$  is constant on the set  $F_{C_t}(\hat{x})$ . Also, it is clear that  $\hat{x} \in F_{C_t}(\hat{x})$ . Thus, since for almost every  $t \in T$   $\hat{x}(t)$  is a best response to some  $\hat{y} \in F_{C_t}(\hat{x})$ , it follows that, for almost every  $t \in T$ ,  $\hat{x}(t)$  is also a best response to  $\hat{x}$ . This means that  $\hat{x}$  is also a Nash equilibrium.

**8.3. Proof of Theorem 3.** Let  $\Pi_t : Y^T \rightarrow Y$  be the projection function on the coordinate  $t$ . Since we work with the product topology in  $Y^T$  it will be sufficient to show that  $\Pi_t(\tilde{U})$  is dense in  $Y$  for every  $t \in T$ . Since  $\Pi_t(\tilde{U})$  is independent of  $t$  we denote  $\mathcal{A} = \Pi_t(\tilde{U})$ .

Recall that  $\mathcal{A}$  is the set of all continuous functions  $v : \hat{P} \rightarrow \mathbb{R}^n$  with the property that there exists a finite measurable partition  $R$  of  $T$  such that  $v(\hat{x}) = v(\hat{y})$  whenever  $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$  for every  $B \in R$ . In order to prove that  $\mathcal{A}$  is dense in  $Y$  it is sufficient to do so for each one of the  $n$  coordinates of  $v$  separately. Thus, with abuse of notation, we let  $Y = \{v : \hat{P} \rightarrow \mathbb{R} \mid v \text{ is continuous}\}$  and  $\mathcal{A} \subseteq Y$  is the class of (real valued) functions which depend only on the integral of the  $T$ -strategy over a finite number of sets in  $T$ .

**Claim 1.**  $\mathcal{A}$  is a vector subspace of  $Y$ .

*Proof.* Let  $v_1, v_2 \in \mathcal{A}$ . Then there are finite partitions  $R_1, R_2$  of  $T$  such that  $v_1$  depends only on  $\{\int_B \hat{x} d\lambda\}_{B \in R_1}$  and  $v_2$  depends only on  $\{\int_B \hat{x} d\lambda\}_{B \in R_2}$ . Let  $R$  be a finite partition which is finer than both  $R_1$  and  $R_2$ . If  $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$  for every  $B \in R$  then  $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$  for every  $B \in R_1 \cup R_2$ . Thus, for such  $\hat{x}, \hat{y} \in \hat{P}$ ,  $v_1(\hat{x}) + v_2(\hat{x}) = v_1(\hat{y}) + v_2(\hat{y})$ . This implies that  $v_1 + v_2 \in \mathcal{A}$ . Finally, if  $\alpha \in \mathbb{R}$  then obviously  $\alpha v_1 \in \mathcal{A}$  ■

**Claim 2.**  $\mathcal{A}$  is a subalgebra of (the algebra)  $Y$ . Moreover,  $\mathcal{A}$  contains the constant functions.

*Proof.* To prove that if  $v_1, v_2 \in \mathcal{A}$  then  $v_1 \cdot v_2 \in \mathcal{A}$  repeat the argument of Claim 1. Also, it is clear that every constant function is in  $\mathcal{A}$ . ■

**Claim 3.**  $\mathcal{A}$  separates points of  $\hat{P}$ . That is, for every  $\hat{x} \neq \hat{y} \in \hat{P}$  there is  $v \in \mathcal{A}$  such that  $v(\hat{x}) \neq v(\hat{y})$ .

*Proof.* Fix  $\hat{x} \neq \hat{y} \in \hat{P}$ . For  $i = 1, \dots, n$  define the sets  $B_i^1 = \{t \in T : \hat{x}_i(t) > \hat{y}_i(t)\}$  and  $B_i^2 = \{t \in T : \hat{x}_i(t) < \hat{y}_i(t)\}$ .  $\hat{x} \neq \hat{y}$  implies that there is  $1 \leq i \leq n$  such that  $\lambda(B_i^1) > 0$  or  $\lambda(B_i^2) > 0$ . Assume w.l.o.g. that  $\lambda(B_1^1) > 0$ . Consider the function  $v \in Y$  defined by  $v(\hat{z}) = \int_{B_1^1} \hat{z}_1 d\lambda$  for every  $\hat{z} \in \hat{P}$ .

First, notice that  $v \in \mathcal{A}$  since it only depends on the integral of the  $T$ -strategy over the set  $B_1^1$ . Also, by construction,  $\hat{x}_1(t) > \hat{y}_1(t)$  for every  $t \in B_1^1$ . Since  $B_1^1$  has a positive measure it must be that  $\int_{B_1^1} \hat{x}_1 d\lambda > \int_{B_1^1} \hat{y}_1 d\lambda$ . Thus,  $v(\hat{x}) > v(\hat{y})$  and the claim is proved. ■

Finally, recall that  $\hat{P}$  is a compact set. By the previous claims the conditions of the Stone-Weierstrass theorem (see e.g. Ha, 2006, Theorem 5.4.1 in page 398) hold. Thus, we can conclude that  $\mathcal{A}$  is dense in  $Y$ . This proves the theorem.

**8.4. Proof of Proposition 1.** First, it is clear that  $g$  is symmetric around  $\frac{1}{4}$ , so  $g(\alpha) = g(\frac{1}{2} - \alpha)$  for every  $\alpha \in [0, \frac{1}{4}]$ . It is therefore sufficient to compute  $g$  on the interval  $[0, \frac{1}{4}]$ .

Fix  $\frac{1}{8} \leq \alpha \leq \frac{1}{4}$  and consider the  $T$ -strategy where all type 1 players choose  $e_1$  and all type 2 players  $e_2$ . Obviously, the efficiency of such profile is 0. We claim that this is a CCE. To see this, let  $F'$  be a set of type 1 females with measure  $\alpha$ , and let  $M'$  be a set of type 2 males with measure  $\alpha$ . Then the  $T$ -strategy  $\hat{Y}$  defined by  $\hat{Y}(t) = e_1$  for  $t \in ([0, \frac{1}{2}) \setminus (M \cup F')) \cup M' \cup (F \cap [\frac{1}{2}, 1])$  and  $\hat{Y}(t) = e_2$  otherwise is a consistent conjecture for every player. The best response for such a belief is  $e_1$  for type 1 players and  $e_2$  for type 2 players.

Next, consider the case where  $0 \leq \alpha \leq \frac{1}{8}$ . Let  $F'$  be a set of type 1 females with measure  $\frac{1}{4} + \alpha$ , and let  $M'$  be a set of type 2 males with measure  $\frac{1}{4} + \alpha$ . Define  $\hat{x}$  by  $\hat{x}(t) = e_1$  for  $t \in F' \cup (M \cap [0, \frac{1}{2})) \cup ([\frac{1}{2}, 1] \setminus (F \cup M'))$  and  $\hat{x}(t) = e_2$  otherwise. We have  $p_1(\hat{x}) = \frac{1}{4} + 2\alpha$  and  $p_2(\hat{x}) = \frac{1}{4} - 2\alpha$  so  $eff(\hat{x}) = 4(\frac{1}{4} + 2\alpha)(\frac{1}{4} - 2\alpha) = \frac{1}{4} - 16\alpha^2$ .



To see that  $\hat{x}$  is a CCE, consider the following  $T$ -strategy  $\hat{y}$ . The set  $M \cap [0, \frac{1}{2})$  of type 1 males play  $e_2$  and not  $e_1$  as in  $\hat{x}$ . Instead, a set of measure  $\alpha$  of type 2 males who played  $e_2$  according to  $\hat{x}$  is switching to  $e_1$ . Similarly, the set of type 2 females is switching from  $e_2$  in  $\hat{x}$  to  $e_1$  in  $\hat{y}$  and a set of measure  $\alpha$  of type 1 females is switching from  $e_1$  to  $e_2$ . It is clear that  $\hat{y} \in F_{C_t}(\hat{x})$  for every player  $t$ . Moreover,  $p_1(\hat{y}) = \frac{1}{4}$  and  $p_2(\hat{y}) = \frac{1}{4}$  so for every player both actions give the same payoff (thus both are best responses to  $\hat{y}$ ).

Finally, notice that the efficiency of a  $T$ -strategy  $\hat{x}$  is decreasing in the distance of  $p_1(\hat{x})$  and  $p_2(\hat{x})$  from  $\frac{1}{4}$ . It can therefore be verified that there is no categorial equilibrium which yield a lower social efficiency.

**8.5. Proof of Theorem 4.** Before proving the theorem we need the following lemma.

**Lemma 10.** *For every finite partition  $R$  of  $T$  the map  $f_R : \hat{P} \rightarrow \hat{P}$  as defined in subsection 6.1 is continuous in the  $L_1$  weak topology on  $\hat{P}$ .*

*Proof.* Since  $f_R$  is linear, continuity in the weak topology is equivalent to continuity in the metric topology (see Dunford and Schwartz, 1988, Theorem 15, page 422). The latter is guaranteed by the inequality  $\int_T |f_R(\hat{x}) - f_R(\hat{y})| d\lambda \leq \int_T |\hat{x} - \hat{y}| d\lambda$ , which follows from the fact that  $f_R$  is the projection to the subspace of  $T$ -strategies which are measurable w.r.t. the partition  $R$ . ■

Fix a game  $u$  and a categorization profile  $C = \{C_t\}_{t \in T}$ , with  $C_t$  coarser than some  $C^*$  for every  $t \in T$ . By Lemma 2, it is sufficient to show that there is a pure CCE w.r.t. the categorization profile  $\{C_t = C^*\}_{t \in T}$ .

Consider the map  $f_{C^*} : \hat{P} \rightarrow \hat{P}$  as defined in subsection 6.1. By Lemma 10 it is continuous. It follows that the game  $\bar{u}$  defined by  $\bar{u}(t, \hat{x}) = u(t, f_{C^*}(\hat{x}))$  satisfies assumptions (1) and (2) of subsection 3.2. Moreover,  $\bar{u}$  is an anonymous game, so by Theorem 2 in Schmeidler (1973, see also Remark 2 there) there is a pure Nash equilibrium in  $\bar{u}$ .

Denote by  $\hat{x}$  one such pure equilibrium of  $\bar{u}$ . We claim that  $\hat{x}$  is a prototypical equilibrium (and therefore a CCE) of the game  $u$ . Indeed, for almost every  $t \in T$ ,  $\hat{x}(t) \cdot \bar{u}(t, \hat{x}) \geq p \cdot \bar{u}(t, \hat{x})$  for all  $p \in P$ . For every such  $t$ , by definition of  $\bar{u}$ ,  $\hat{x}(t) \cdot u(t, f_{C^*}(\hat{x})) \geq p \cdot u(t, f_{C^*}(\hat{x}))$  for all  $p \in P$ . This is exactly the definition of a prototypical equilibrium w.r.t. the categorization profile  $\{C_t = C^*\}_{t \in T}$ .

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