The government-taxpayer game

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Abstract. In this paper, we model - quantitatively – a possible realistic interaction between a tax-payer and his Government. We formalize, in a general setting, this strategic interaction. Moreover, we analyze completely a particular realistic sample of the general model. We determine the entire payoff space of the sample game; we find the unique Nash equilibrium of the interaction; we determine the payoff Pareto maximal boundary, *conservative payoff zone* and the *conservative core* of the game (part of Pareto boundary greater than the *conservative payoff vector*). Finally, we suggest possible compromise solutions between the two players. From an economic point of view, the sample chosen gives an example of normative settings, for which **there is no reason (convenience), for the tax-payer, to evade the taxes** or to declare less than his real income. Moreover, the two proposed compromise solutions (which realize the maximum collective gain) could be significantly applied to distinguished tax-payer (big companies and so on).

1. The model

We shall consider a two-player normal form gain game, $G = (f, >)$, representing the rational interaction between the **Government (first player 1)** and a **Taxpayer (second player 2)**, in a country. The payoff function $f : E \times F \to IR^2$ of our game shall be defined upon the bi-strategy space of the game, Cartesian product of the respective strategy spaces of the two players, and with values into the payoff universe IR^2. The two components f_1 and f_2 of the function f are the respective payoff function of the two players.

1.1 Strategy spaces

1.1.1 Government. The unit interval $\mathbf{U} = [0,1]$ is the strategy set of the Government, a probabilistic interval: each element x of the unit interval **U** is the probability that the Government checks the real (true) income of the Taxpayer.

1.1.2 Tax-payer. The compact interval $F = [0,V]$ is the range of possible income declarations of the Taxpayer, each element y in F is a possible income that the Taxpayer may decide to declare to the Government, V is the true income of the taxpayer.

1.2 Payoff function of the Government.

We consider, firstly, the payoff function of the Government, in his interaction with the Taxpayer 2; it is, as usual, a real function f_1: $[0,1] \times [0,V] \to \mathbf{R}$. To construct the payoff function f_1 (that is, to define its "correspondence law") we shall proceed step by step.

1.2.1 First step. Firstly, we consider the case in which the Government checks the Taxpayer declaration, that is, the strategy 1 of the Government. In this case, we have

$$f_1(1, y) = tV + g(t)(V - y) - C,$$

for every strategy y in F, where:

a) the fixed real number t, belonging to IU, is the percentage (the unit interval **U** is now interpreted in a different way), due by the Taxpayer to the Government, upon his real income V. Hence, a first income of the Government, in this case, is the discounted cash flow tV;
b) the real number $g(t) > t$ is the fixed percentage due, by the taxpayer, to the Government upon his own non-declared income $(V – y)$; so that, the Government receives also the discounted cash flow $g(t)(V - y)$;
c) at last, the real number C is the cost afforded by the Government to check the tax evasion.

1.2.2 Second step. Secondly, we consider the case in which the Government does not check (at all) the taxpayer declaration and accept the strategy y of the Tax-payer as his true income, so that, we have:

$$f_1 (0,y) = ty,$$

for every strategy y of the second player.

1.2.3 Third step. To obtain the values of the function f_1, on the remaining part of the bi-strategy space, we shall use the von Neumann mixed-extension method, but only with respect to the first finite strategy space $\{0,1\}$ of the Government. In other terms, we shall consider – for every strategy y of the Taxpayer - the mixed extension of the finite stochastic variable $L(y) : \{0,1\} \to \mathbf{R}$, defined by

$$L(y)(0) = f_1 (0,y) = ty \quad \text{and} \quad L(y)(1) = f_1 (1, y) = tV+ g(t)(V - y) - C ,$$

by using the probabilistic scenarios **only for the actions of the Government** (see later for a robust justification of this probabilistic choice and its applicability). We so have:

$$\begin{aligned} f_1 (x, y) &= \mathbf{E}_{(1-x, x)} (L(y)) = \\ &= \mathbf{E}_{(1-x, x)} (ty , tV + g(t)(V - y) - C) = \\ &= (1- x) ty + x [tV+ g(t) (V- y) - xC] = \\ &= x [tV + g(t)(V- y) - C - ty]+ ty . \end{aligned}$$

1.3 Payoff function of the second player, the Taxpayer.

In general, for the payoff function of the second player, we have:

$$f_2(0,y) = (V- y) + (1 - t)y,$$

for every y in F; indeed, when the government does not check the possible evasion, the Taxpayer net income is:

1) the non-declared income $(V- y)$ (considered as it is, since there are no taxes on it)
2) plus the declared income y minus the tax ty , which player 2 has to pay because of the declaration y.

When the Government decides to check the declaration of the Taxpayer, we obtain:

$$f_2(1, y) =(1-t)V - g(t)(V- y),$$

for every y in F, indeed, when the Government checks the possible evasion, the Taxpayer net income is:

1) the non-declared income $(V- y)$ minus a higher tax $g(t)(V- y)$ - with respect to the usual taxation $t(V- y)$ - because of the evasion;
2) plus the real income V minus the tax tV on the real income V, which player 2 has to pay because the Government, after the check, knows the real income V of the Taxpayer.

1.3.1 Mixed extension. By adopting the von Neumann mixed extension method, as before only on the Government strategies, we obtain:

$$\begin{aligned} f_2 (x,y) &= \mathbf{E}_{(1-x, x)} ((V - y)+(1 - t)y, (1- t)V - g(t)(V- y)) = \\ &= (1- x) ((V - y)+ (1 - t)y)+ x((1- t)V - g(t)(V- y)), \end{aligned}$$

for every pair (x,y) in the bi-stategy space.

1.3 Payoff function of Government-Taxpayer Game.

Resuming the above results, we can finally give the definition of the payoff vector function of our entire game G; it is defined by

$$f(x,y) = (x\,[tV + g(t)(V- y) - C - ty\,] + ty\,,\ \ (1- x)\,[(V - y)+ (1 - t)y] + x[(1- t)V - g(t)(V- y)]),$$

for every (x,y) in the strategic square S.

2. Numerical example

To build up a computable and realistic example, we shall put: t = 25% = ¼ , g(t) = ½ = 50% , C = ¼ and V = 1.

Remark (on the strategy sets). (1) In the above example, we are normalizing the real income V; so that, also the declaration strategy y belongs to the compact unit interval **U** = [0,1], 0 means total Tax Evasion (declaration 0), 1 means No Tax Evasion (the declaration y equals the real income V). **(2)** Any strategy x of the first player belongs to [0,1], but the meaning is completely different, as it is emphasized in the following remark.

Remark (interpretation of strategy spaces). The interpretations of our strategy spaces are obvious and recalled below:

a) the strategy space E is a probabilistic strategy space;
b) the strategy space F (of the second player) is a "money" strategy space;
c) any strategy x of E has a probabilistic meaning: probability 0 means "No check the possible tax evasion of the Taxpayer"; probability 1 means "to check the possible tax evasion of the player 2";
d) from a *frequency* point of view, the probabilistic strategy x is realizable by checking n = x m taxpayer declarations, where m is the total number of taxpayers.

2.1 Payoff functions

Let us see the form of our particular payoff functions.

2.1.1 Payoff function of the Government. In our numerical example, we have:

$$\begin{aligned} f_1(x, y) &= x((1/4)\,V + (1/2)\,(V- y) - (1/4) - (1/4)y)+ (1/4)y = \\ &= x\,((3/4) - (1/2)\,y - (1/4) - (1/4)\,y)) + (1/4)y = \\ &= x\,((1/2) - (3/4)\,y))+ (1/4)\,y = \\ &= x/2 - (3/4)xy + y/4, \end{aligned}$$

for every pair (x,y) in the square $\mathbf{U}^2$.

2.1.2 Payoff function of the taxpayer. In our numerical example, we have:

$$\begin{aligned} f_2(x, y) &= (1- x)\,(1- y +(3/4)y + x((3/4) -(1/2)(1- y))= \\ &= (1- x)\,(1- (1/4)y) + x(1/4)+(y/2)) = \\ &= (1- (y/4))- x + (x/4)\,y + (x/4) + (x/2)\,y = \\ &= 1- (3/4)\,x + (3/4)\,xy - y/4, \end{aligned}$$

for every pair (x,y) in $\mathbf{U}^2$.

2.1.3 Payoff function of the sample game. Concluding, in our numerical example, the payoff function of the entire game is defined by

$$\begin{aligned} f\,(x, y) &= (x/2 + y/4 - (3/4)\,xy\,,\ \ 1 - (3/4)\,x + (3/4)\,xy - y/4) = \\ &= (1/4)\,(2x + y - 3xy, -3x - y + 3xy) + (0,1), \end{aligned}$$

for every strategy profile (x,y) of the game G.

2.1 Tridimensional representation of the game (f,>)

In this subsection, we present a 3D representation of the sample game (f,>). This representation simply consists of the union of the graphs of the two payoff functions. The mostly higher surface is the graph of the Government payoff function, the mostly lower surface is he graph of the Tax-payer payoff function.

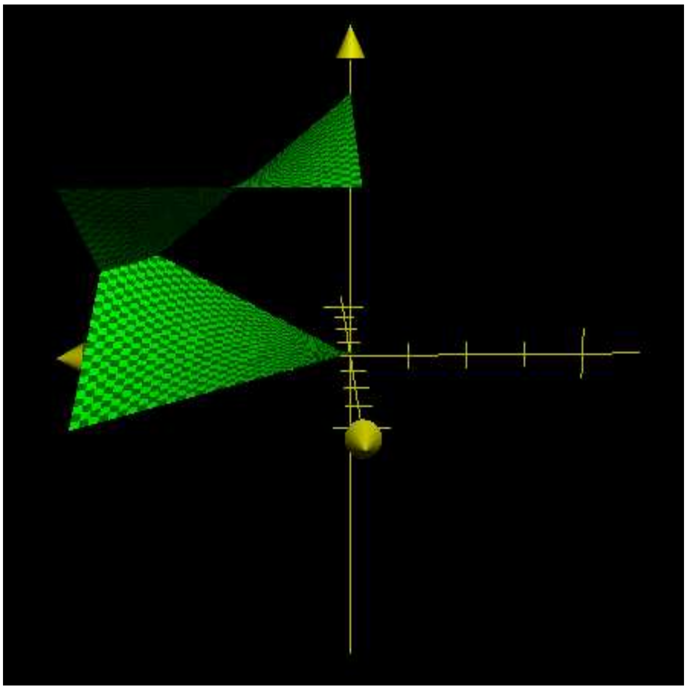

Figure. 01

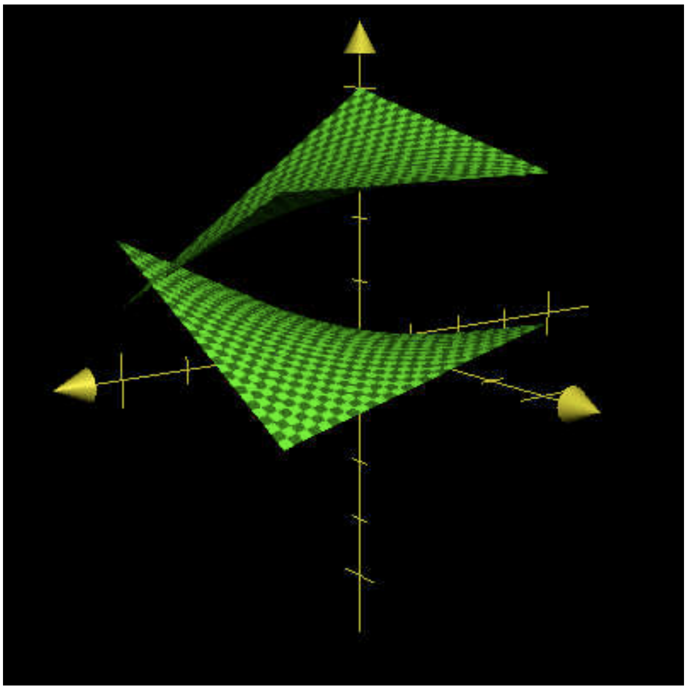

Figure. 02

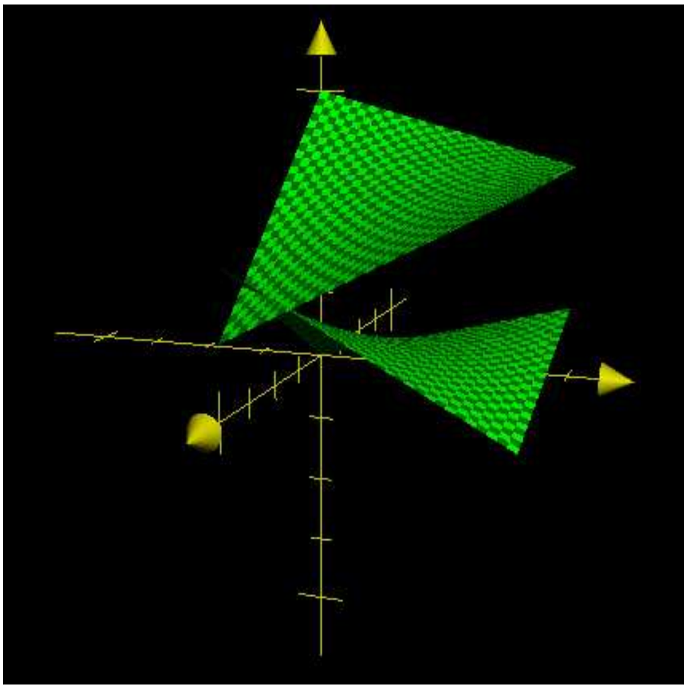

Figure. 03

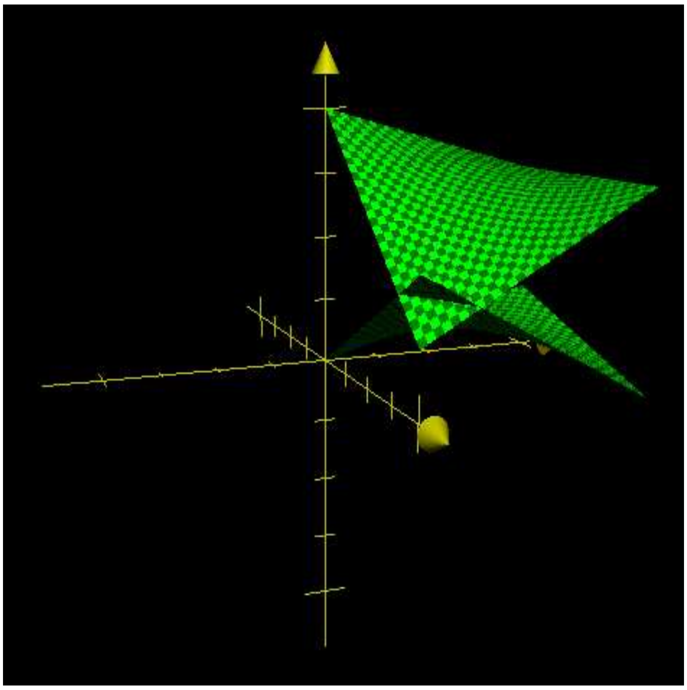

Figure. 04

Note that, there is a connected part of the bi-strategy square on which the Tax-payer function is greater than the Government function. We represent it in the following figure.

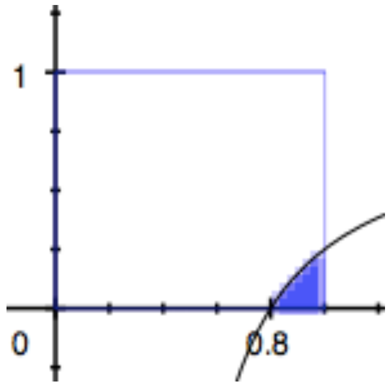

Figure 05.

3. Digression: Why the tax payer should pay the taxes?

Note that our game is a non-zero and non-constant sum game: the aggregate payoff function s of our game G is the real function defined on the bi-strategy space by $s(x, y) = - ¼ x + 1$, for every probability strategy x of the Government. The interpretation is quite clear:

a) 1 is the total income of the tax payer, which is in this context the only effective income of the game
b) and ¼ x is the expense of the Government for checking the evasion, when it decides to employ the strategy x.

Observe, moreover, that the maximum, on the strategy space **U,** of this social sum s is attained at any bi-strategy point (0,y), with y in **U**, and this maximum is 1 (the Taxpayer real income). So that, the maximum collective gain $1 = \max_U (s)$, corresponds to the situation in which the Government does not check the declaration of the Taxpayer (0 expenses for checking it). Of course, in this case, the Taxpayer (if he is aware of the Government strategy) will choose to declare nothing (his strategy 0) and *the total collective payoff remains only in the Taxpayer's hands*. obviously this last situation is a selfish bad scenario for the human society: but this is also the reason at the root of the tax evasion. The **key-solution for the collectivity** is that the tax tV (or ty or g(t)(V-y), and so on…) should be used by the Government "much better" than how much the taxpayer itself can do! In other terms, *to convince the Taxpayer to pay the taxes*, the Government should employ the capitals deriving from taxes at an income rate i_G such that, for any reasonable tax payer individual income rate i_T, one has (for instance in the case of truthful declaration)

$$(1+ i_G)\, tV + (1+ i_T)(1- t)V > (1+ i_T)V,$$

i.e., a rate of income that makes the social sum $(1+ i_G)\, tV + (1+ i_T)(1- t)V$ greater than the potential future income $(1+ i_T)V$, of the Taxpayer.

4. The Complete Analysis of our sample game

In this section we conduct the complete analysis of the game (f, >). At this aim, we observe that the payoff function f, of our numerical example, is viewable as 0.25 g + (0,1), where we have considered the "payoff kernel" g, defined by

$$g(x,y)= (2x + y - 3xy, -3x - y + 3xy),$$

for every bi-strategy (x,y). We shall study only this kernel g, since any information on the game (f, >) is deductible from the game (g, >), by a 0.25 rigid contraction and then by a (0,1) translation.

4.1 The bi-strategy space

As we already saw, the bi-strategy space of our game is the square $\mathbf{U}^2$, it is represented in the following figure.

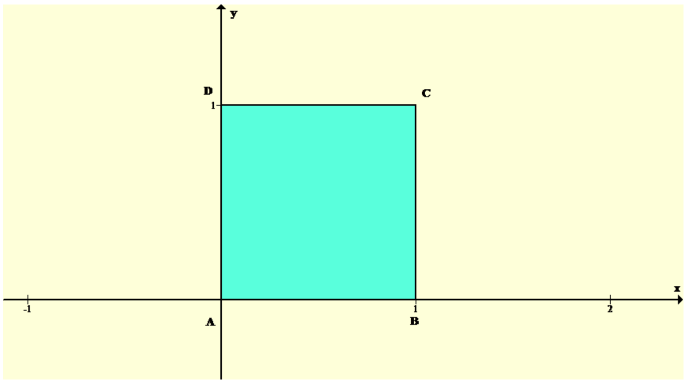

Figure 1. The bi-strategy space (square) S.

4.2 The Payoff space

Our first significant aim is to find the payoff space of the game (g, >), that is the image g(S) of the vector payoff function g (the image of the bi-strategy square S under g). At this aim, we transform the topological boundary fr(S) of the square S, that is the union of the 4 edges of the square; and moreover, we have also to transform the critical zone cr(g) of g (set of bi-strategies at which the Jacobian matrix is not-invertible): the boundary of the image of g, is contained into the union of g(fr(S)) and g(cr(g)):

$$\mathrm{fr}(g(S)) \subseteq g(\mathrm{fr}(S)) \cup g(\mathrm{cr}(g)).$$

So let's go to study the Critical zone of g.

4.2.1 Critical zone. The critical zone of the function g is the set of all bi-strategies (x,y) at wich the Jacobian matrix (family) is not invertible. The Jacobian family of the function g - defined by

$$g(x,y)= (2x + y - 3xy, -3x - y + 3xy),$$

for every bi-strategy (x,y) - is the pair of gradients

$$J(g)(x,y) = (\mathrm{grad}(g_1), \mathrm{grad}(g_2)) =$$

$$= ((2 - 3y , 1 - 3x) , (3y - 3 , 3x - 1)).$$

The Jacobian determinant of the function g at the point (x,y) is the determinant of the vector family J(g)(x,y), that is the real number:

$$\det J(g)(x,y) = (2 - 3y)(3x - 1) + (3y - 3)(3x - 1) =$$

$$= 6x - 2 - 9xy + 3y + 9xy - 3y - 9x + 3 =$$

$$= -3x + 1.$$

So the Jacobian family is not invertible at the point (x,y) if and only the abscissa x is equal to 1/3. Concluding the critical zone of the function g is the segment

$$\mathrm{cr}(g) = \mathbf{U}(0,1) + (1/3,0),$$

or, if you prefer, the segment

$$\mathrm{cr}(g) = [(1/3,0), (1/3,1)].$$

The critical zone is the subset of points (x,y) of the plane with x = 1/3 and y in **U**, it is the segment [P, Q] represented in the following figure 2.

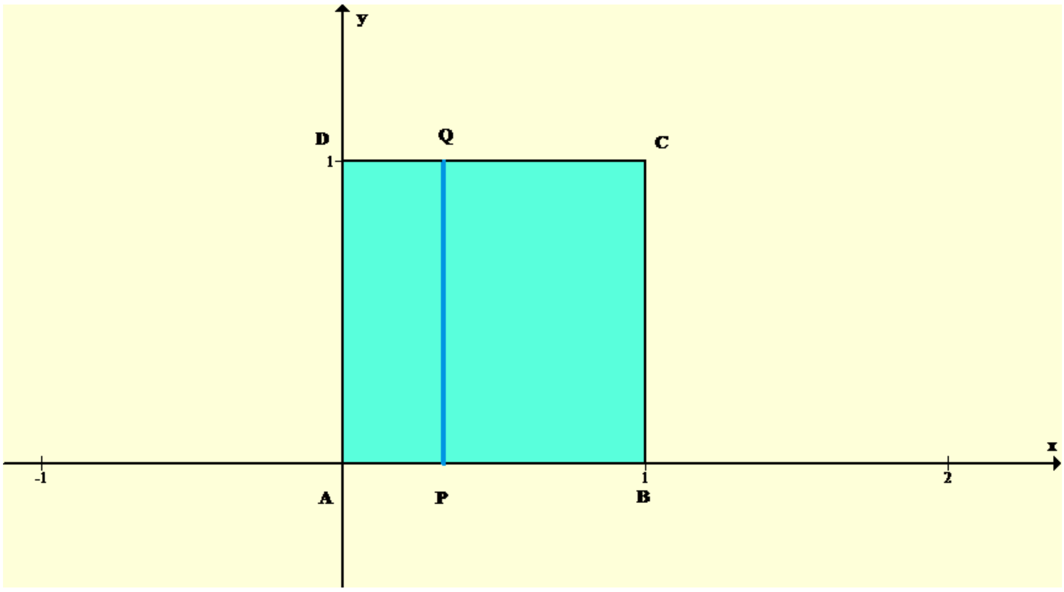

Figure 2. The Critical zone [P,Q] in bi-strategy space.

4.2.2 Payoff space. Since the boundary of the payoff space is contained into the union of the transformation of the boundary of the square S and since the critical zone in the payoff space in reduced to a point, we deduce that the boundary of the payoff space is indeed the transformation of the boundary of the square S, so that we have the following figure 3.

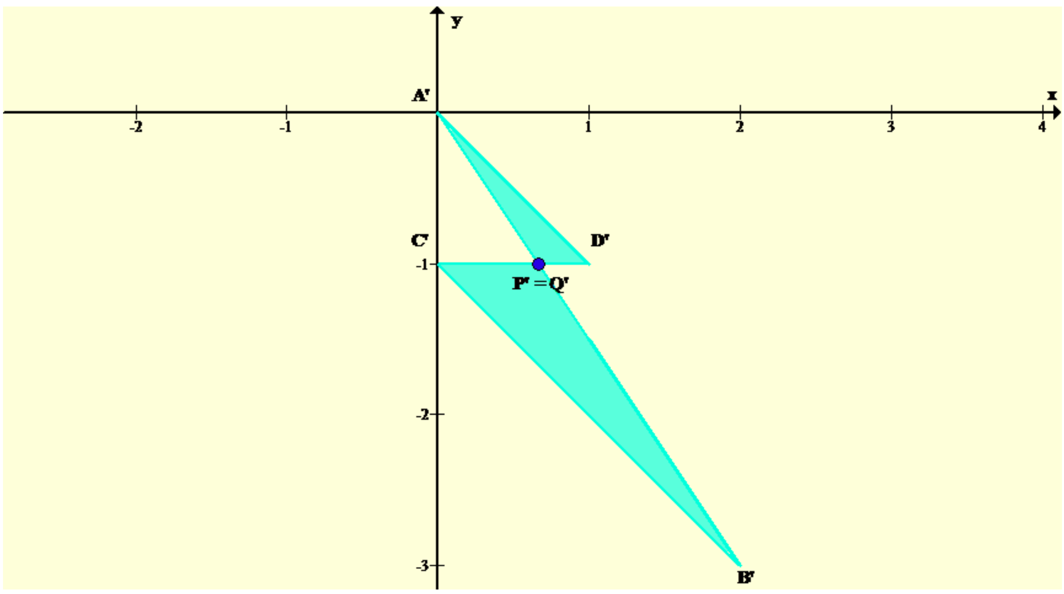

Figure 3. The payoff space of the game.

4.3 Nash equilibrium

The unique Nash equilibrium payoff belongs to the payoff critical zone it is the payoff P'. It's simple to check, by partial derivation. We give all the particulars in Appendix 1.

4.4 Conservative phase

Also the conservative bi-value of the game belongs to the payoff critical zone {P'}, as showed in the following figure. We give all the particulars in Appendix 2.

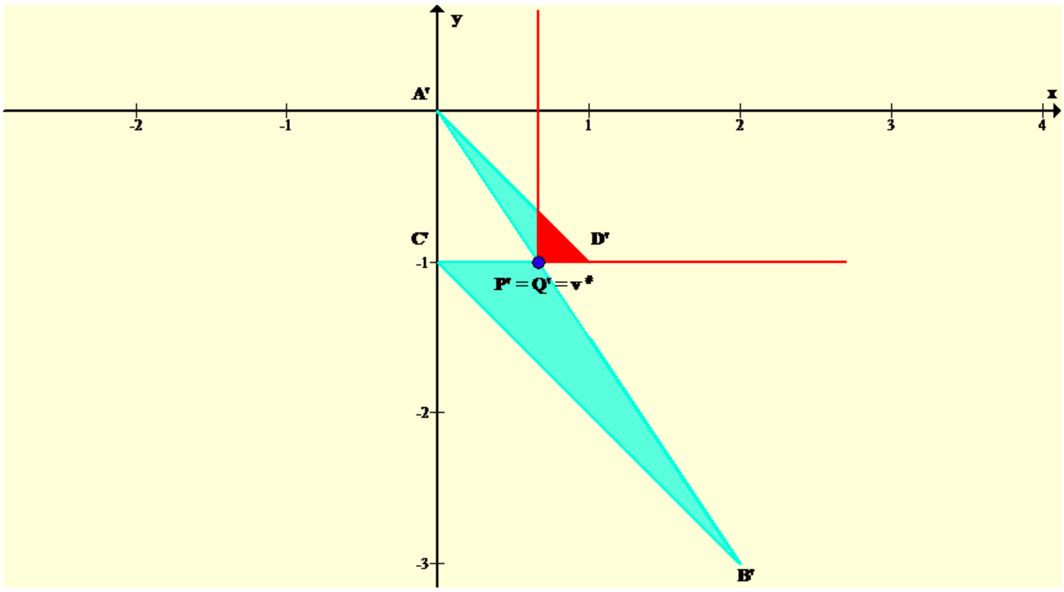

Figure 4. The conservative zone of the game (g, >) in its payoff space.

4.5 Pareto boundary

The Pareto maximal boundary is (straightforwardly) the union of two segments, showed in the below figure. Note that it is bounded but not compact and (obviously) not connected.

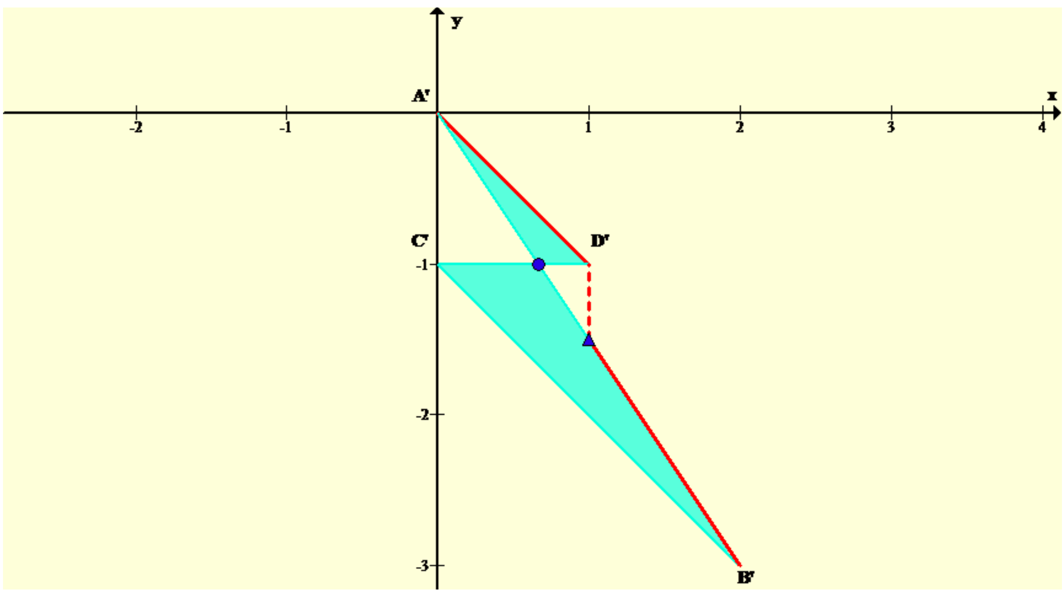

Figure 5. The Pareto boundary in the payoff space (bounded but not closed).

4.6 The conservative core

The core is simply a compact segment, as it is shown in the figure below. It has the important characteristic to be entirely with the maximum collective (aggregate) value of the game.

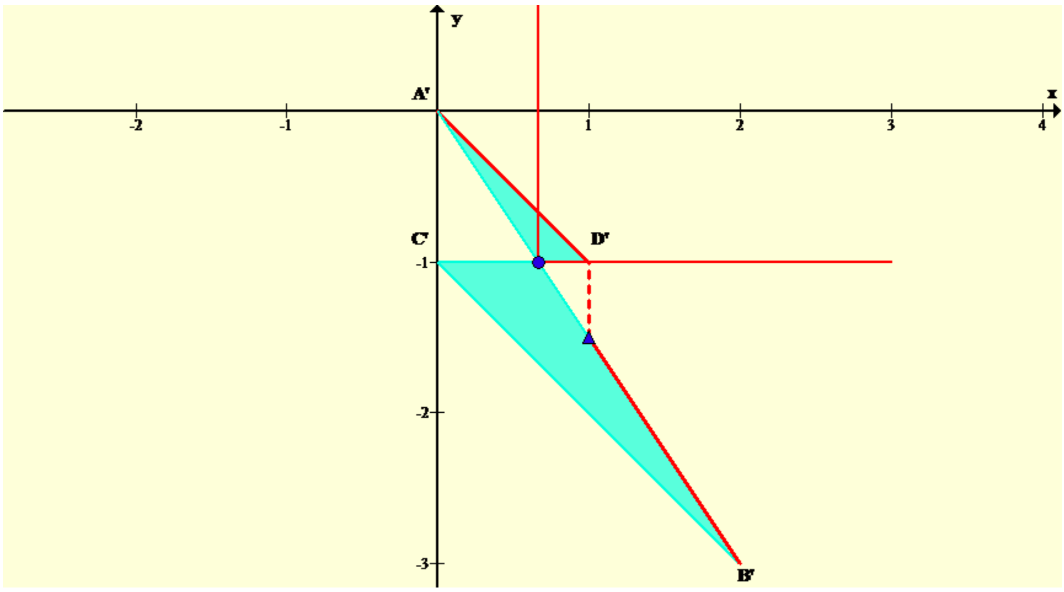

Figure 6. The conservative core (which coincides with the Nash core) in the payoff space.

4.7. Compromise solution

We propose two compromise solutions, maximizing the collective payoff of the game. They are almost coincident. We propose Kalai-Smorodisky solutions with threat point the Nash equilibrium and utopia points, the sup of the game or the sup of the core. They are both win-win solutions with respect to the Nash payoff.

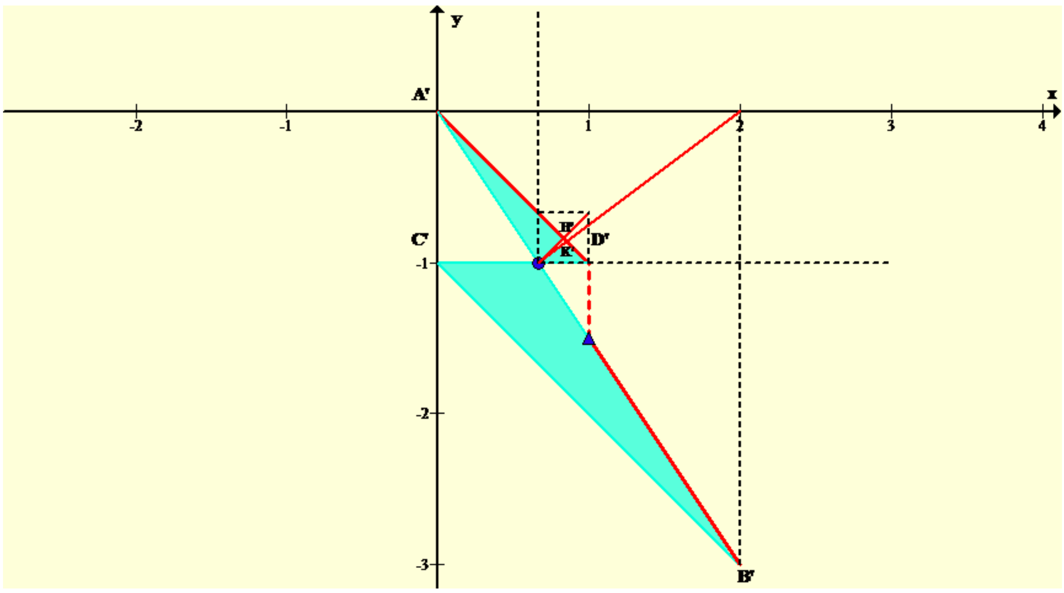

Figure 7. The Kalai-Pareto compromise solution and the core compromise in the payoff space.

5. Conclusions

The novelty of our research are:

1. we model, in a general and applicable framework, the interaction between a Government of a country and any possible relative tax-payer, by using a **realistic probability (frequency) approach to the checking evasion strategy** of the Government;

2. we propose a realistic and realizable (by normative actions and laws) sample, of the general model; in this sample, the **proper solution (Nash equilibrium)** shows a situation in which **the tax payer has no convenience to declare an income inferior than the real one**;
3. we propose, by the way, a measure of the **collective loss in the above Nash equilibrium**, by the **total knowledge of the entire payoff space** of the sample interaction;
4. we show **two (quantitatively close) compromise solutions**, applicable (by binding contracts) in case of **great distinguished tax-payers**, maximizing the collective gain of the society.

Appendix 1. Nash Equilibrium

We have g(x,y)= (2x + y - 3xy, -3x – y + 3xy), for every x,y in U. Then,

$$g_2(x,.)'(y) = \partial_2 g_2(x,y) = -1 + 3x \geq 0,$$

so the section $g_2(x,.)$ is strictly increasing if x > 1/3, it is constant if x = 1/3 and it is strictly decreasing if x < 1/3. Hence, the reaction correspondence B_2 is defined by $B_2(x) = \{0\}$ if x < 1/3, $B_2(x) = F$ if x = 1/3 and $B_2(x) = \{1\}$ if x > 1/3.

$$B_2(x) = \begin{cases} \{0\} & \text{if } x \in [0, 1/3[\\ [0,1] & \text{if } x = 1/3 \\ \{1\} & \text{if } x \in]1/3, 1] \end{cases}$$

Again, we have $g_1(x,y) = 2x + y – xy$, for any x, y in [0,1]. Since

$$g_1(.,y)'(x) = \partial_1 g_1(x,y) = 2 – 3y \geq 0$$

so the section $g_1(.,y)$ is strictly increasing, if y < 2/3, it is constant, if y = 2/3, and it is strictly decreasing, if y > 2/3. Hence, the reaction correspondence B_1 is defined by $B_1(y) = \{1\}$ if y < 2/3, $B_1(y) = E$, if y = 2/3, and $B_1(y) = \{0\}$, if y > 2/3.

$$B_1(y) = \begin{cases} \{1\} & \text{if } y \in [0, 2/3[\\ [0,1] & \text{if } y = 2/3 \\ \{0\} & \text{if } y \in]2/3, 1] \end{cases}$$

Nash Equilibria. We have only one Nash equilibrium, the point N = (1/3, 2/3). The Nash payoff is

$$N' = g(N) = (2(1/3) + (2/3) – 3(1/3)(2/3), – 3(1/3) – (2/3) + 3(1/3)(2/3)) = (2/3, – 1).$$

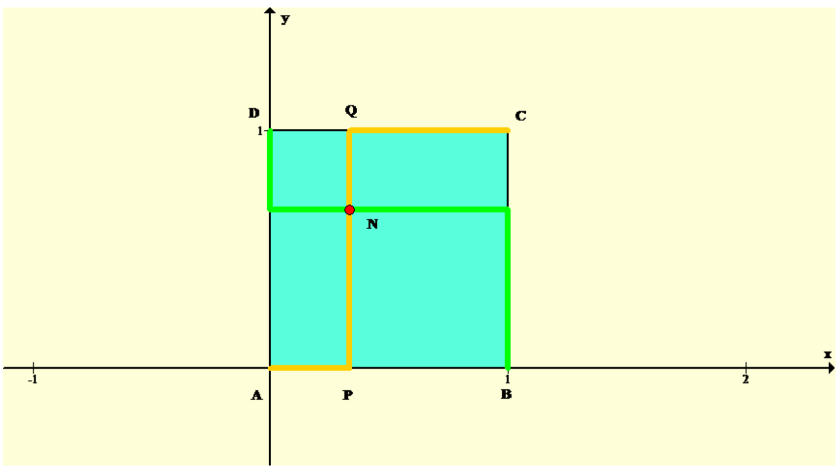

Figure A1. The Best replay graphs with Nash equilibrium.

Appendix 2. Conservative phase.

The conservative value for the first player is

$$v_1^{\#} = \sup_{x \in E} \inf_{y \in F} (2x + y - 3xy) = \sup_{x \in E} g_1^{\#}(x),$$

where $g_1^{\#}(x) = \inf_{y \in F} (2x + y - 3xy)$ is the *worst gain function for the first player*. Now, fixed $x \in [0,1]$, we have

$$g_1(x,.)'(y) = 1 - 3x,$$

so the section $g_1(x,.)$ is strictly increasing if $x < 1/3$, it is constant if $x = 1/3$ and it is strictly decreasing if $x > 1/3$. Hence, *the offensive correspondence* O_2 is defined by $O_2(x) = \{0\}$ if $x < 1/3$, $O_2(x) = F$ if $x = 1/3$ and $O_2(x) = \{1\}$ if $x > 1/3$.

So that **the worst gain function for the Government** is defined by

$$g_1^{\#}(x) = \begin{cases} g_1(x, 0) = 2x & \text{if } x \leq 1/3 \\ g_1(x, 1) = 1 - x & \text{if } x > 1/3 \end{cases}$$

The unique maximum point of the above function is $x^{\#} = 1/3$, this is the **unique conservative strategy of the first player**. **The conservative value of the first player** is $v_1^{\#} = g_1^{\#}(x^{\#}) = 2/3$.

The conservative value for the second player is

$$v_2^{\#} = \sup_{y \in F} \inf_{x \in E} (-3x - y + 3xy) = \sup_{y \in F} g_2^{\#}(y),$$

where

$$g_2^{\#}(y) = \inf_{x \in E} (-3x - y + 3xy)$$

is the *worst gain function for the second player*. Now, fixed $y \in [0,1]$, we have $g_2(.,y)'(x) = -3 + 3y$, so the section $g_2(.,y)$ is strictly decreasing if $y < 1$, it is constant if $y = 1$. Hence, the offensive correspondence O_1 is defined by $O_1(y) = \{1\}$ if $y < 1$, $O_1(y) = E$, if $y = 1$. So that **the worst gain function for the Taxpayer** is defined by $g_2^{\#}(y) = g_1(1, y) = -3 + 2y$, for every y in F. The unique maximum point of the above function (in F) is $y^{\#} = 1$, this is the **unique conservative strategy of the second player**. **The conservative value of the second player** is $v_2^{\#} = g_2^{\#}(y^{\#}) = -1$. **The payoff at the conservative cross** is the Mash Payoff, $g(1/3, 1) = (2/3, -1)$.

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