Not so cheap talk: costly and discrete communication

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Abstract

We model an interaction between an informed sender and an uninformed receiver. As in the classic cheap talk setup, the informed player sends a message to an uninformed receiver who is to take an action which affects the payoffs of both players. However, in our model the sender can communicate only through the use of discrete messages which are ordered by the cost incurred by the sender. We characterize the resulting equilibria without refining out-of-equilibrium beliefs. Subsequently, we apply an adapted version of the *no incentive to seperate* ($NITS$) condition to our model. We show that if the sender and receiver have aligned preferences regarding the action of the receiver then $NITS$ only admits the equilibrium with the largest possible number of induced actions. When the preferences between players are not aligned, we show that $NITS$ does not guarantee uniqueness and we provide an example where an increase in communication costs can improve communication. As we show, this improvement can occur to such an extent that the equilibrium outperforms the Goltsman et al. (2009) upper bound for receiver's payoffs in mediated communication.

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1 Introduction

A person will often use words to convey information about a complex and nuanced reality. However, the set of possible ideas is much larger than the set of single words which can be used to express these ideas. One can use words to express more detailed and nuanced information, but only at a cost to the sender. In this paper, we accommodate these aspects of communication by taking the view that communication is necessarily costly and discrete. We analyze the implications of such communication in a strategic interaction between an informed sender and an uninformed receiver. In our model, the sender learns the state of the world on the unit interval and transmits a discrete and costly message to the receiver. After observing the message, the receiver is to take an action which affects the payoffs of both sender and receiver.

To gain some intuition, consider the communication between an advisor and an advisee. The advisor is better informed than the advisee about the quality of the advisee's current research. Additionally, the advisor has better information regarding many other issues related to the success of the advisee, including the advisee's research interests, presentation style, modes of personal interaction, and so on. Furthermore, the advisor and the advisee have identical preferences in that both want the advisee to have a successful career.

While the advisor will communicate some relevant information to the advisee, the question is, why does the advisor not communicate *everything* which is possibly helpful to the advisee. The answer, we argue, is that communication is costly and discrete. If there was a single word to communicate the full extent of the helpful advice then the advisor would transmit this word. However, there obviously does not exist a single word to communicate the full extent of the advice. Rather, the advisor can construct more detailed messages, however these come at a cost to the advisor. As a result, the advisor will not fully communicate even though the preferences over the advisee's career are identical.

We are motivated to investigate costly and discrete communication for both a priori reasons and for the qualitative insights which the model provides. First, words are scarce and costly to transmit therefore any model which accounts for this would seem to be a productive step towards realism. We are also motivated by qualitative insights which the assumptions yield. Specifically, our model provides a very simple explanation for why full communication does not occur when sender and receiver have perfectly aligned preferences over the action of the receiver. More generally, our model is possibly helpful in any communication setting in which the difference in the preferences of the action of the receiver is not the only factor hindering communication.

In our paper, we first characterize the equilibrium without refining the out-of-equilibrium beliefs. We show that without a refinement, there is a multiplicity of equilibria. We next employ an adapted version of the *no incentive to separate* ($NITS$) condition of Chen et al. (2008). This condition roughly states that if the receiver ever observes an out-of-equilibrium message then the receiver believes that the state is 0. We show that under $NITS$, if there is perfect alignment between the preferences regarding the receiver's action then the equilibria is the one most preferred by the receiver: the state space is partitioned into the largest number of possible elements. This result is analogous to that of Chen et al. (2008) when $NITS$ is applied to the original cheap talk model.

If preferences regarding the receiver's action are not aligned, we show that $NITS$ does not guarantee a unique equilibrium and we show that an increase in communication costs can improve communication. We also show that when preferences are not aligned there exists an equilibrium in which the receiver's payoffs outperform the Goltsman et al. (2009) upper bound for payoffs in efficient, mediated communication.

The balance of the paper is organized as follows. In Section 2, we review the related literature. In Section 3, we introduce the model and in Section 4, we offer some preliminary analysis. In Section 5, we characterize the equilibrium without refinements of the out-of-equilibrium beliefs. In Section 6, we characterize the equilibrium under $NITS$ where there is perfect alignment of preferences and in Section 7, we examine $NITS$ in the case of imperfect alignment. In Section 8, we discuss our modeling choices and in Section 9, we conclude. In the appendix, we offer the proofs which were not presented in the body of the paper. Further, we present numerical examples involving equilibria without a restriction of the out-of-equilibrium beliefs and the equilibria as refined by $NITS$. Finally, we present an example where there does not exist an equilibrium, even in the absence of communication costs, under an alternate, and arguably more reasonable, specification of $NITS$.

2 Related Literature

2.1 Cheap Talk and Related Models

The large strand of cheap talk literature was initiated by Crawford and Sobel (1982) (hereafter referred to as CS). The authors show that for mild differences in the preferences of receiver and sender, meaningful, albeit incomplete, communication can occur. CS shows that in each equilibrium, the state space is partitioned whereby messages induce a unique action within each element of the partition. Our equilibrium is analogous in that a unique action is induced on each partition element.

A number of papers have extended the original CS model. For instance, Morgan and Stocken (2003) extend the CS model to the case where there is uncertainty regarding the difference between the preferences of the sender and receiver. Fischer and Stocken (2001) model a situation where the sender has imperfect information about the state. Blume et al. (2007) modify the CS setup where communication errors (or noise) can occur. In our view, the present paper shares the goal of these papers: to learn the significance of a particular assumption of the CS model. Here we seek to learn the importance of the assumption that messages are plentiful and equally costless.

This literature suggests an investigation into the conditions under which communication can be improved over that in the CS model. Blume et al. (2007) demonstrate that a small amount of noise can improve communication in the CS model. In particular, the authors show that there is an optimal amount of noise which maximizes the receiver's payoffs. Subsequently, Goltsman et al. (2009) study general communication in the CS model. The authors consider mediated communication, whereby a neutral third party (or mediator) will offer a nonbinding recommendation to both of the players. Goltsman et al. (2009) find that the payoffs in the equilibrium with the optimal amount of noise found by Blume et al. (2007) is the optimal outcome in any mediated communication in the CS setting. In other words, the mediator optimally introduces noise to the message of the sender. Within this optimal mediated outcome, Goltsman et al. (2009) identify an upper bound on the payoffs which can be attained by the receiver. Although the settings are different, we identify an equilibrium in which communication costs imply that the receiver can attain a payoff above this upper bound. In our view, this illustrates the significance of communication costs on communication outcomes.

The original CS model exhibits a large number of possible equilibria. As is often the case for multiple equilibria, researchers have sought to reduce the number of equilibria through refinements.[1] Many standard refinements, such as the intuitive criterion and divinity, do not have the ability to refine the number of equilibria of the CS model. Although there are costly messages in our model, the standard refinements only marginally reduce the set of equilibria since, among other features, the model is not monotonic in the sense of Cho and Sobel (1990).

A recent innovation in the refinement of the equilibria in the CS model is the *no incentive to separate* ($NITS$) condition of Chen et al. (2008). $NITS$ restricts attention to equilibria in which it is not the case that the sender at the state $s = 0$ (with a state space of $[0, 1]$) prefers to perfectly reveal the state. In their Proposition 3, the authors show that if the monotonicity condition[2] holds in the CS model (as it does in the commonly used "uniform-quadratic" case) then $NITS$ selects a unique equilibrium which contains the largest possible number of partitions. We present a parallel result: when there are communication costs and the preferences are aligned, our adapted version of $NITS$ admits only the equilibria with the largest number of partitions. Despite that $NITS$ yields uniqueness in both of the above cases, we also show that when preferences are not aligned and there are communication costs, $NITS$ does not guarantee uniqueness.

2.2 Costly and Constrained Communication

We are not the first to introduce costly communication into the CS model. Austen-Smith and Banks (2000) and Kartik (2007) investigate the effect of including both costly and costless messages in the original CS model.[3] These *burning money* papers ask, what happens if we include the option of sending costly messages, in addition to the cheap, plentiful messages of the CS model. By contrast, in our paper the message space is not uncountably infinite, but there are only a finite number of finite-cost messages. The difference between the models can best be seen in the case of perfect alignment of preferences of the action of the receiver. In the burning money setup there would be complete communication, whereas in our setup there

[1] For instance, see Banks and Sobel (1987), Cho and Kreps (1987), Farrell (1993), Kohlberg and Mertens (1987), Matthews et al. (1991).

[2] In the literature, this is commonly refered to as Condition M.

[3] The cost of these messages are unrelated to the unknown state of the world. See Spence (1973) for the classic model of the case where the cost of transmitting a signal varies with the underlying state of the world. Also see Gossner et al. (2006). Kartik et al. (2007) investigate a model of costly lying, credulous receivers, and show that a separating equilibrium can emerge. Mialon and Mialon (2012) analyze a model where costly communication can imply nonliteral speech.

would not be complete communication. Further, the burning money papers find that the inclusion of these additional, costly messages can expand the set of equilibria and that there can be regions of full separation. By contrast, we never find full separation and, in general, the presence of communication costs reduce the informativeness of communication.

In Dewatripont and Tirole (2005) the sender incurs costs in order to effectively communicate information and the receiver incurs costs in absorbing information. In Dewatripont and Tirole, information is either understood or not.[4] By contrast, the states in our model are better characterized by the degree to which they are understood. Additionally, in Dewatripont and Tirole the sender and receiver necessarily have different preferences over the action of the receiver. By contrast, we examine the cases where they are aligned and are unaligned.[5] Lastly, in our model the communication costs are exclusively incurred by the sender. We focus on this case for the following reasons. When communication is discrete, it is not obvious how to best model the cost associated with absorbing and processing messages. Even if a suitable formulation could be found, a higher cost incurred by the receiver would presumably induce a lower correlation between the state and the action. We suspect there exists a profile of communication costs borne exclusively by the sender which would yield an identical distribution of actions as in a model in which both sender and receiver incur communication costs.

There are other models of costly and constrained communication, with shades of understanding.[6] In Cremer et al. (2007) a fixed number of partition elements are optimally arranged in order to minimize communication problems between an informed sender and an uninformed receiver who have identical preferences over the action of the receiver. Like Cremer et al., we find that the equilibrium mapping from the state space to the message space is lumpy in the sense that the precision with which the receiver learns the state is affected by the details of the discrete nature of the communication. Also note that in Cremer et al., the size of the language is exogenously given however in our model the size of the language endogenously emerges due to the costs of communication.

Finally, Sobel (2012) offers a model of costly communication whereby both sender and receiver undertake a costly acquisition of communication capacity. In the paper, Sobel also points out that models of costly communication with aligned preferences can have parallel results to models of costless communication where preferences are not aligned. In particular, Sobel notes that for any communication costs or differences in the preferences of the action taken, full communication is not possible and failure to communicate is always possible. The author also notes that increases in either communication costs or the difference in preferences will decrease the quality of communication. The previously mentioned Proposition 3 in Chen

[4] See Austen-Smith (1994) for another costly communication paper in which information is either understood or not.

[5] Also note that we are not the first to model communication between a sender and receiver who have identical preferences over the receiver's action. See Blume et al. (2007), Blume and Board (2010b), Che and Kartik (2009), Cremer et al. (2007), Jager et al. (2011), Mialon and Mialon (2012), and Morris (2001).

[6] Also see Jager et al. (2011) and Mialon and Mialon (2012). In Calvo-Armengol et al. (2011) the sender transmits a necessarily noisy signal but can affect its precision by a incurring larger communication cost. In our view, this assumption is less appropriate when modeling discrete communication as it is not obvious to us how to model noise when messages are discrete.

et al. (2008) and our Proposition 2 provide another such parallel result between the classes of models.

3 Model

A sender (S) and receiver (R) play a communication game in a single period. Payoffs for both players depend on the receiver's action a, as well as the state of the world s. The state is an element of $[0,1]$. The receiver's action space is $\mathbb{R}$. The receiver's utility is:

$$u^R(a,s) = -(a-s)^2.$$

The receiver has ex-ante beliefs that the state is uniformly distributed on $[0,1]$. The sender observes the state and can communicate some information about the state to R by sending a message m where $m \in \mathcal{M}$. Associated with each message m^i, there is a cost $c(i)$ which the sender incurs when it is transmitted. The cost of communication ($c : \mathbb{N} \Rightarrow \mathbb{R}$) is an increasing function of its index.[7] Further, we require that $c(i+1) - c(i) \geq \psi > 0$. We also assume that $c(0) = 0$. In a slight abuse of notation, we will refer to the case described above as $c > 0$, the case where there is no communication costs as $c = 0$ and the case of both communication costs and the absence of communication cost as $c \geq 0$.[8] The sender's utility is:

$$u^S(a, m^i, s) = -(a - s - b)^2 - c(i)$$

where $b \geq 0$.

The sender's strategy is $\mu : [0,1] \to \Delta\mathcal{M}$ and the receiver's strategy is $\alpha : \mathcal{M} \to \Delta\mathbb{R}$. We seek an equilibrium (μ^*, α^*) such that S chooses the optimal message, R chooses the optimal action and R's beliefs are derived from Bayes' Rule whenever possible. We denote R's beliefs as $\beta(s|m)$.

Definition 1 *For an equilibrium (μ^*, α^*) we require:*

$$\text{for each } s \in [0,1],\ \mu(s) \in \underset{m'}{\text{argmax}}\ u^S(\alpha(m'), m', s)$$

$$\text{for each } m \in \mathcal{M},\ \alpha(m) \in \underset{a'}{\text{argmax}} \int u^R(a', s)\beta(s|m)ds$$

and that R's beliefs are derived from S's strategy using Bayes' Rule whenever possible.

4 Preliminaries

Before we offer a characterization of the equilibria, we introduce some notation and provide a necessary condition for equilibria. Although our equilibria share some of the familiar characteristics of the cheap talk literature, the additional results which emerge will require the flexibility provided by the notation which we now define. Like the CS equilibria, messages are

[7] See Vartiainen (2009) for a related notion of communication costs.

[8] Of course, since there is no outside option, adding a constant amount to the function c would not affect our results. We assume that $c(0) = 0$ in order to render meaningful our notation of $c = 0$ and $c > 0$.

sent on connected, nonoverlapping intervals.[9] Therefore, we may characterize an equilibrium by a set of cutoff states. If there are n messages used in equilibrium, we can list the order of the messages as m_1, ..., m_n. The messages induce a set of cutoff states which we denote:

$$0 = s_1 \leq s_2 \leq ... \leq s_h \leq ... \leq s_n \leq 1 = s_{n+1}.$$

Equilibrium is such that S's messages are sent on intervals of the state space:

$$m_h = \mu^*(s) \text{ for } s \in [s_h, s_{h+1})$$

and R best responds in a straightforward manner:

$$\alpha^*(m_h) = \overline{a}(s_h, s_{h+1}) = \underset{a'}{\operatorname{argmax}} \int_{s^h}^{s^{h+1}} u^R(a', s)\beta(s|m_h)ds. \tag{1}$$

Upon receiving message m_h, as in CS, R has posterior beliefs that the message is uniformly distributed between s_h and s_{h+1}. Therefore we can write the best response of R, $\overline{a}(s_h, s_{h+1})$, as:

$$\overline{a}(s_h, s_{h+1}) = \frac{s_h + s_{h+1}}{2}.$$

Definition 1 implies the arbitrage equation, which is also found in CS. This expression characterizes the equilibrium set of cutoff states:

$$u^S(\overline{a}(s_h, s_{h+1}), m_h, s) = u^S(\overline{a}(s_{h+1}, s_{h+2}), m_{h+1}, s) \text{ for } h \in \{1, ..., n-1\}. \tag{2}$$

Expression (2) describes the state for which S is indifferent between sending message m_h (which is strictly preferred on states (s_h, s_{h+1})) and message m_{h+1} (which is strictly preferred on states (s_{h+1}, s_{h+2})). Now we define λ_h to be the mass of states such that $m_h = \mu(s)$. Since the messages are sent on an interval of the state space and the states are distributed uniformly, $\lambda_h = s_{h+1} - s_h$ when $m_h = \mu(s)$ for $s \in [s_h, s_{h+1})$ and $m_h \neq \mu(s)$ for $s \notin [s_h, s_{h+1})$.

While subscripts refer to the order of the messages, we use superscripts to denote the cost index of the message. Therefore, we denote the least costly message as m^0, the next costly message as m^1 and so on. Correspondingly, we define λ^j to be the mass of states associated with the message which has cost index j. An equilibrium in which there are n actions induced will obviously require that:

$$\lambda_1 + \lambda_2 + ... + \lambda_n = 1 \text{ where } \lambda_h \geq 0 \text{ for every } h \in \{1, ..., n\}. \tag{3}$$

We now provide a necessary condition for an equilibrium. Lemma 1 describes the relative size of two adjacent intervals. As in the CS model, for $b > 0$ the interval size is increasing in its location on the state space. In other words, for $b > 0$ and $c = 0$, the intervals representing larger numbers on the state space are larger than intervals representing smaller numbers. The lemma also shows that the size of the interval is decreasing in the cost of the signal transmitted

[9] See the appendix for the proof of Lemma 6 which shows that the equilibrium strategy for S entails sending a message for states which are conntected intervals and Lemma 7 which shows that the intervals are not partially overlapping.

on that interval.

Lemma 1 *For any equilibrium* (μ^*, α^*) *where* $b \geq 0$ *and* $c \geq 0$ *in which there are* n *actions induced, it must be that:*

$$\left(\lambda_{h+1}^{j}\right)^2 - \left(\lambda_h^i\right)^2 = 4b\left[\lambda_{h+1}^{j} + \lambda_h^i\right] + 4\left[c(i) - c(j)\right] \text{ for } h \in \{1, ..., n-1\}. \tag{4}$$

It is important to note the generality of expression (4). When $b > 0$ and $c = 0$, we are in the CS model because expression (4) easily reduces to expression (21) in CS. Therefore, when $b > 0$ intuition behind the relationship between the interval size and its location on the state space is identical to that provided by CS. Also note that when $c > 0$, costly signals are conserved: the cost of a signal is negatively related to the size of the region of the state space on which it is transmitted. Finally, note that Lemma 1 and the restriction that each message has a unique cost, together imply that there are no equilibrium mixed strategies for the sender. Indeed each sender type has a strict preference to transmit a single message, with the exception of the boundary types.

5 Equilibrium Characterization

In this section we characterize the equilibrium without refining the out-of-equilibrium beliefs. Our first result is that, in equilibrium, the least costly message is always used. Further, for any positive communication costs there will exist an uninformative equilibrium where the least costly message is sent on all states.

Lemma 2 *In any equilibrium* (μ^*, α^*) *where* $c > 0$ *it must be that* $\mu^*(s) = m^0$ *on some states. Also for* $c > 0$*, there will exist an equilibrium* (μ^*, α^*) *such that* $\mu^*(s) = m^0$ *on all states.*

Proof: Consider an equilibrium (μ^*, α^*) in which the least costly message m^0 is not sent on any region of the state space. For any out-of-equilibrium beliefs of the receiver, we will show that there exists a profitable deviation. Suppose that upon observing an out-of-equilibrium message, the receiver believes that the message was transmitted by a sender of type $\widetilde{s}$, which would induce action $\widetilde{s}$. If $\widetilde{s} - b \geq 0$ then the sender type $\widetilde{s} - b$ could deviate and send the message m^0. This deviation message would induce the preferred action of this sender type and could be sent with a smaller cost than the equilibrium message. If $\widetilde{s} - b < 0$ then the sender type 0 could deviate and send message m^0. This deviation would induce the preferred action of the sender type restricted to $[0, 1]$, which is 0. Further m^0 could be transmitted at a smaller cost than the equilibrium message. In either case, there is a profitable deviation for S, and so (μ^*, α^*) cannot be an equilibrium. Now consider strategy pair (μ, α) such that $\mu(s) = m^0$ for all $s \in [0, 1]$. Suppose that an out-of-equilibrium message implies that the receiver believes that it was sent by the sender type 0.5. As a result, an out-of-equilibrium message does not induce an action which is not induced in equilibrium. Therefore, there is no profitable deviation from this completely uninformative strategy pair and so it is an equilibrium.■

We note that there are many other out-of-equilibrium beliefs which, for some parameter values, support the uninformative equilibrium of Lemma 2. However, the beliefs specified in the proof of the lemma support the uninformative equilibrium for all parameter values.

Next, we offer a definition which summarizes the necessary conditions for an equilibrium without refinement of out-of-equilibrium beliefs.

Definition 2 *A strategy pair* (μ, α) *is feasible if it satisfies expressions (1), (3), (4) and additionally the least costly message* m^0 *is sent on some region of the state space.*

We are ready to characterize the equilibria without refining out-of-equilibrium beliefs. As the following proposition shows, each feasible strategy pair (μ, α) will form an equilibrium.

Proposition 1 *If* (μ, α) *is feasible then it is possible to find out-of-equilibrium beliefs such that the strategies form an equilibrium* (μ^*, α^*)*.*

As the above proposition implies, there are many equilibria when the out-of-equilibrium beliefs are not refined. This abundance of equilibria stands in contrast to our results in the following section. There we show that for $b = 0$, the only equilibria which satisfy our version of $NITS$ are the equilibria which have the largest possible number of partitions. Note that Example 3, given in the appendix, illustrates the multiplicity of equilibria when we do not refine out-of-equilibrium beliefs.

6 Alignment of Preferences under $NITS$

Here we focus on the implications of our adaptation of the *no incentive to separate* ($NITS$) condition for case where the preferences regarding the receiver's actions are perfectly aligned ($b = 0$). We begin by noting that when $b = 0$, the arrangement of the signals across the state space does not matter. To see this, we can rewrite expression (4) in Lemma 1 for the case of $b = 0$ as:

$$(\lambda^j)^2 - (\lambda^i)^2 = 4\left[c(i) - c(j)\right]. \tag{5}$$

As expression (5) suggests, the interval size on which a message is sent is exclusively determined by its communication cost and not by its location on the state space. As a result, expression (5) does not contain subscripts.

If the incentives are aligned ($b = 0$) and there are n actions induced in equilibrium then there are $n+1$ sender types which have the largest difference between the equilibrium payoffs and the payoffs which could be achieved if the sender type was identified by the receiver. In other words, we say that these $n + 1$ sender types have the *most incentive* to deviate.[10] Therefore, as a matter of convention, we select one of these $n + 1$ states. Hence, $NITS$ specifies that if an out-of-equilibrium message is observed then R believes that the state is certain to be $s = 0$.

No incentive to separate: Given a strategy pair (μ, α), if R observes $\widetilde{m}$ where $\widetilde{m} \notin \mu([0, 1])$ then R believes that the state is certain to be $s = 0$, $\beta(0|\widetilde{m}) = 1$.

Note that our version of $NITS$ is not identical to the original specification of Chen et al. (2008). Our specification focuses on the out-of-equilibrium beliefs which are implicit in the original specification. The authors motivate their condition by suggesting that, upon

[10] See Lemma 8 in the appendix.

observing an out-of-equilibrium message, a natural place to expect such a deviation is from the *lowest* state.[11] Before the statement of $NITS$, we noted that if $b = 0$ and there are n actions induced in equilibrium then there are $n + 1$ states in which the sender has the most incentive to deviate, should an out-of-equilibrium message lead the receiver to correctly identify the sender type. All of the results involving $b = 0$ would follow if we selected any of the other such n states.[12]

Before we state the main result regarding the equilibria under $NITS$, we provide the following lemmas which are used in the proof of the result. Our next lemma shows that, if a message of a certain cost is used in equilibrium, it must be the case that all lower cost messages are also used in equilibrium. We refer to this as the *no holes* result.

Lemma 3 *Consider an equilibrium (μ^*, α^*) in which m^i is transmitted. Under $NITS$, if $b = 0$ then every m^j where $c(j) < c(i)$ is also used in equilibrium.*

Proof: Suppose that there is an equilibrium (μ^*, α^*) such that $\mu^*(s) = m^i$ with cost $c(i)$ however there does not exist an s' such that $\mu^*(s') = m^j$ and $c(j) < c(i)$. If the signal m^j is observed, R believes that the state is certain to be $s = 0$. On the interval in the state space for which the S sends message m^i, S's payoff cannot be greater than $-c(i)$. By Lemma 8, S has identical payoffs at each of the states for which expression (5) is satisfied, including the states 0 and 1. Therefore, at $s = 0$, the sender has a payoff of less than $-c(i)$ and a profitable deviation is then to send m^j. Therefore, (μ^*, α^*) cannot constitute an equilibrium.■

We compare Lemma 2 with Lemma 3. Unlike the case where there are no refinements of the out-of-equilibrium beliefs, we are guaranteed to not have holes in the equilibrium under $NITS$. For this reason, we amend the definition of feasibility to account for the no holes result.

Definition 3 *A strategy pair (μ, α) is $NITS$-feasible if it satisfies expressions (1), (3), (4) and additionally there does not exist a used message m^i and an unused message m^j such that $c(j) < c(i)$.*

We now show that we are guaranteed a $NITS$-feasible strategy pair with a most costly message m^k such that there does not exist a $NITS$-feasible strategy pair with a most costly message $m^{k'}$ where $k' > k$. If such a k is found then we say that the $NITS$-feasible strategy pair with a most costly message m^k is *maximal*. In the lemma below, we show that if preferences are aligned then there is a $NITS$-feasible strategy pair (μ, α) which is maximal.

Lemma 4 *If $b = 0$ then for any $c > 0$ there always exists a maximal, $NITS$-feasible strategy pair (μ, α).*

Proof: In order to identify the maximal, $NITS$-feasible strategy pair, we start with the strategy pair involving messages m^0 and m^1. We check whether this is $NITS$-feasible. If it

[11] The reader should consult Chen et al. (2008) for further justification of the $NITS$ condition.

[12] However when $b > 0$ the specification of $NITS$ indeed affects the results. As we show in the appendix, if $b > 0$ then we are not guaranteed existence for an alternate, and arguably more reasonable, specification of $NITS$.

is not then the uninformative strategy pair is the maximal, $NITS$-feasible strategy pair. If it is $NITS$-feasible then we check whether the strategy pair involving messages m^0, m^1 and m^2 is $NITS$-feasible. If it is not then the strategy pair involving messages m^0 and m^1 is the maximal, $NITS$-feasible strategy pair. If it is $NITS$-feasible then we check whether the strategy pair involving messages m^0, m^1, m^2 and m^3 is $NITS$-feasible. We continue until we arrive at a k' such that the strategy pair involving messages m^0, ..., $m^{k'}$ is not $NITS$-feasible. To check whether a particular strategy pair (μ, α) is $NITS$-feasible, we rewrite expressions (3) and (4). Consider the case where m^k is the most costly message used. The message which costs $c(k)$ is sent on an interval of size λ^k. The message which costs $c(k-1)$ is sent on an interval of size $\lambda^{k-1} = \sqrt{4\left[c(k) - c(k-1)\right] + (\lambda^k)^2}$. The message which costs $c(k-2)$ is sent on an interval of size $\lambda^{k-2} = \sqrt{4\left[c(k) - c(k-2)\right] + (\lambda^k)^2}$. The message which costs $c(2)$ is sent on an interval of $\lambda^2 = \sqrt{4\left[c(k) - c(2)\right] + (\lambda^k)^2}$. The message which costs $c(1)$ is sent on an interval of size $\lambda^1 = \sqrt{4\left[c(k) - c(1)\right] + (\lambda^k)^2}$. Finally for the costless message, we write $\lambda^0 = \sqrt{4c(k) + (\lambda^k)^2}$. Therefore, when m^k is the most costly message sent, we may write expression (3) as:

$$\sqrt{4c(k) + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(1)\right] + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(2)\right] + (\lambda^k)^2} + \\ ... + \sqrt{4\left[c(k) - c(k-2)\right] + (\lambda^k)^2} + \sqrt{4\left[c(k) - c(k-1)\right] + (\lambda^k)^2} + \lambda^k = 1. \tag{6}$$

If $\lambda^k \geq 0$ then the strategy pair is $NITS$-feasible. However, for $\lambda^k < 0$ the strategy pair is not $NITS$-feasible. To see that we will eventually reach a strategy pair which is not $NITS$-feasible, recall that we require that $c(i+1) - c(i) \geq \psi > 0$ for all $i \in \{0, ..., k\}$. Therefore, we can write the lower bound of each term in the left hand side of expression (6):

$$\sqrt{4k\psi} + \sqrt{4(k-1)\psi} + \sqrt{4(k-2)\psi} + ... + \sqrt{4(2)\psi} + \sqrt{4\psi} > 1. \tag{7}$$

For every ψ, there is a k large enough so that expression (7) is satisfied. Therefore, we are guaranteed a maximal, $NITS$-feasible (μ, α).■

Intuitively, Lemma 4 shows that, for any communication costs when there are aligned preferences, there exists an upper bound on the number of messages used in an equilibrium.[13] It should not come as a surprise that full communication is not $NITS$-feasible when $c > 0$. However, the straightforward characterization of equilibrium under $NITS$ is perhaps surprising, given the complicated nature of characterizing the equilibria without refining out-of-equilibrium beliefs.

We are now ready for the main result of the section. Proposition 2 shows that we are guaranteed an equilibrium under $NITS$. Further, the only equilibria admitted under $NITS$ are the ones which are maximal among the $NITS$-feasible strategy pairs.

Proposition 2 *If $b = 0$ then under $NITS$ an equilibrium (μ^*, α^*) exists and it is a member of the maximal, $NITS$-feasible class.*

[13] Note that a variant of Lemma 4 would hold for the case of $b > 0$. However, it is not necessary for our present purposes and it would require a slightly different proof, therefore we do not provide it.

Proposition 2 shows that under $NITS$, only the strategy pairs with the largest possible number of messages will not have a profitable deviation. The proposition uses the language *class* because when $b = 0$, the ordering of the messages does not matter. Also, one can see the full force of $NITS$ by noting the difference between the multiplicity of equilibria without refining the out-of-equilibrium beliefs and the uniqueness in Proposition 2.

Despite the very different settings in which they occur, our Proposition 2 is parallel to Proposition 3 in Chen et al. (2008). The authors show that in the CS model where monotonicity holds, $NITS$ admits only the equilibrium with the largest possible number of actions induced. In the notation of our model, Chen et al. (2008) show that for $b > 0$ and $c = 0$ in the uniform-quadratic case, NITS uniquely selects the equilibrium with the most partitions. Our Proposition 2 and Proposition 3 of Chen et al. (2008) become more surprising when we provide an example which demonstrates that we are not guaranteed uniqueness when $b > 0$ and $c > 0$.

6.1 Simple Characterization

Here we focus on the case where preferences are perfectly aligned ($b = 0$) and communication costs are linear in the index of the message. In other words, we assume that $c(k) = ck$ where $c > 0$. One benefit of this exercise is that, for general communication costs it is difficult to characterize the threshold level of costs which render a strategy pair (μ, α) $NITS$-feasible. However, in the linear case the characterization is simple. If $c \leq c^*(k)$ then a strategy pair (μ, α) which employs a most costly message m^k is $NITS$-feasible and if $c > c^*(k)$ then such a (μ, α) is not $NITS$-feasible.

Lemma 5 *If $c(k) = ck$ and $c > 0$ then the cutoff cost for a strategy pair involving message m^k is:*

$$c^*(k) = \left(\frac{1}{2 \sum_{j=1}^{k} \sqrt{j}} \right)^2 .$$

Proof: At the largest c such that the strategy pair involving message m^k is $NITS$-feasible, it must be that $\left(\lambda^k\right)^2 = 0$. By expression (5) it must be that, $\left(\lambda^{k-1}\right)^2 = 4c$, $\left(\lambda^{k-2}\right)^2 = 8c$, ..., $\left(\lambda^1\right)^2 = 4(k-1)c$, $\left(\lambda^0\right)^2 = 4kc$. Therefore, we may write expression (6) in the case of linear costs as

$$2\sqrt{c(k)} + 2\sqrt{c(k-1)} + 2\sqrt{c(k-2)} + ... + 2\sqrt{c(2)} + 2\sqrt{c(1)} = 1.$$

and so the lemma is proved.■

See Example 4 in appendix for an application of Lemma 5. There we show how the calculation of c^* reduces the difficulty in identifying the maximal, $NITS$-feasible equilibria. Example 4, when compared with Example 3, also illustrates the utility of $NITS$ in reducing the number of equilibria.

7 Imperfect Alignment of Preferences under $NITS$

Recall Proposition 2 which demonstrated that the only equilibria admitted under $NITS$ are maximal and $NITS$-feasible. In a parallel fashion, Proposition 3 in Chen et al. (2008) shows that in the CS model where monotonicity holds, $NITS$ admits only the equilibrium with the largest possible number of partitions. In the notation of our model, Chen et al. (2008) show that for $b > 0$ and $c = 0$ in the uniform-quadratic case that $NITS$ uniquely selects the equilibrium with the largest number of induced actions. However, as we show below, when $b > 0$ and $c > 0$ we are not guaranteed uniqueness.

Recall that for the case of $b = 0$ and $c > 0$, the order of the messages does not matter as long as their size is governed by expression (5). For the case of $b > 0$ and $c = 0$, the order of the signals themselves does not matter, but it does matter that the size of the intervals are increasing along the state space. However, when $b > 0$ and $c > 0$ there is an interaction between these two effects, which might cause the nonuniqueness which we now describe.

The nonuniqueness can manifest itself in two distinct ways. First, there could exist several equilibria with an identical set of equilibrium messages, however these equilibria differ in their informativeness, as measured by the ex-ante payoffs of the receiver. Second, there can exist equilibria which differ in the set of equilibrium messages. The following example demonstrates this second aspect and the subsequent example demonstrates the first.

Example 1 *Suppose that* $b = 0.245$ *and communication costs are* $c(i) = 0.01i$*. First, there exists an equilibrium* (μ^*, α^*) *where two messages are used. Message* m^0 *is sent on* $s \in [0, 0.03)$ *and the* m^1 *is sent on* $s \in [0.03, 1]$*. The sender's* $s = 0$ *equilibrium payoffs are* $-(0.015 - 0.245)^2 = -0.0529$*, which is greater than the deviation payoffs of* $-(0.245)^2 - 0.02 = -0.080$*. There also exists an equilibrium where* m^0 *is sent for all states. The sender's* $s = 0$ *equilibrium payoffs are* $-(0.5 - 0.245)^2 = -0.065$*, which is greater than the deviation payoffs of* $-(0.245)^2 - 0.01 = -0.070$*.*

The example above shows that when $b > 0$ there can exist equilibria with a different set of messages. Our next example shows that when $b > 0$, there exist equilibria with identical sets of messages yet differ in their informativeness. Also note that the following example shows that when $b > 0$ there exists equilibria where an increase in communication costs will improve communication.

Example 2 *First, consider the costless communication case. When* $b = 0.2$*, and* $c(i) = 0$*, all equilibria are outcome equivalent to the following: a single action is induced on* $s \in [0, 0.1)$ *and a single action is induced on* $s \in [0.1, 1]$*. Message* m_0 *induces* $a = 0.05$ *and message* m_1 *induces* $a = 0.55$*. In this case,* $E[-(a - s)^2] = -0.0608$*. However, when* $b = 0.2$*, and* $c(i) = 0.01i$*, there are two non-outcome equivalent equilibria. In the first equilibrium,* m^0 *is sent on* $s \in [0, 0.12)$ *and* m^1 *on* $s \in [0.12, 1]$*. In the second equilibrium,* m^1 *is sent on* $s \in [0, 0.08)$ *and* m^0 *on* $s \in [0.08, 1]$*. In the first equilibrium,* $E[-(a - s)^2] = -0.0569$ *and in the second,* $E[-(a - s)^2] = -0.0649$*. If the cost of communication is increased to* $c(i) = 0.02i$ *then in the first equilibrium* m^0 *is sent on* $s \in [0, 0.14)$ *and* m^1 *on* $s \in [0.14, 1]$*, implying* $E[-(a - s)^2] = -0.0532$*.*

Example 2 illustrates a setting in which an increase in communication costs can lead to an improvement in communication. Also note that Example 2 contained an instance of two distinct equilibria, which share the set of equilibrium messages yet differ in their informativeness.

Recall that Goltsman et al. (2009) study the optimal mediated communication equilibria. The authors find that the upper bound for the expected payoffs of the receiver in mediated communication is:

$$E[-(a-s)^2] = -\frac{1}{3}b(1-b).$$

In Example 2, this upper bound would be -0.0533. However we note that the last equilibrium described in Example 2 outperforms this upper bound.[14] Keep in mind that these results occur in very different settings. In the Goltsman et al. setting, a mediator is adding an optimal amount of noise to the message of the receiver. In our setting, communication occurs through messages which have a differential cost.

While the settings are very different, it is interesting to note that communication costs can have as positive an influence on communication as that produced by the optimal amount of noise. What is the intuition behind our equilibrium which outperforms the Goltsman et al. (2009) upper bound? Note that there are two effects at work. When $b > 0$, the sender increases the size of the intervals at the upper end of the state space, which reduces the expected payoff to the receiver. However, the communication costs induce the sender to decrease the interval sizes on which the costly signal is sent. In the relevant equilibrium, the costly message is transmitted on the upper end of the state space. Therefore, these effects work in opposite directions, thereby achieving an expected payoff above that of the upper bound for the case where communication is not costly.

Also, note that the effects discussed above are very different from the effects found in the burning money literature. Unlike the burning money literature, it is not the case that there is a larger number of actions induced in equilibrium. Rather, the improvement is due to the fact that the partitions on the state space are more evenly spaced as a result of the communication costs.

8 Discussion of Modeling Choices

Before we proceed to the conclusion, we discuss some of our modeling choices. Our state space is designed to be richer than our message space[15] as the state space is uncountably infinite and there are only a finite number of messages which can be transmitted with a finite cost. We believe that this captures an important aspect of reality: it is impossible to completely communicate the complexity of the real world, one may only increase the precision of communication by expending more costly effort. Also note that the size of the language used in equilibrium arises endogenously. In our view, this captures another important feature of reality: the precision of communication is determined by the costs incurred by the sender.

We assumed that there is only a single message associated with a particular communication cost. This assumption yields several benefits. First, there is no need to restrict attention to pure strategies. In the case where there are several messages of a particular cost, obviously the receiver would not employ mixed strategies in equilibrium, however this is not the case

[14] This possibility was first suggested by Andreas Blume.

[15] This assumption also appears in Jager et al. (2011) and Lipman (2009). Blume and Board (2010a) examine the opposite case where the message space is much larger than the state space.

for the sender. In this case, there exists equilibria in which the sender would use mixed strategies, and $NITS$ would not exclusively admit the equilibrium with the most partitions. Further, even when restricting attention to pure strategies, Proposition 2 would not hold, as we would need an additional restriction to guarantee the selection of the equilibrium with the most partitions.

Perhaps a natural questions is, why not model communication which is necessarily noisy where the sender incurs a communication cost which is decreasing in the variance of the possible messages. Within this possibility, there arise some features which we find unappealing.

First, as Blume et al. (2007) showed, the quality of communication is not monotonic in the amount of noise and also communication is always enhanced by a small amount of noise. Therefore in any model in which the sender can affect the amount of noise, there will be parameters such that the sender would prefer more noise to less noise. Further, this problem is not avoided if the noise is determined by the amount of effort expended by the sender. Therefore, we do not view this possibility as an adequate substitute for our modeling choices.

Second, it is important that the model does not allow back-door communication between sophisticated players. Suppose that the sender would specify the upper and lower bound of the possible states and incur a cost which is decreasing in the size of this interval. In this case, we would have to assume that the receiver is unsophisticated, otherwise the sender would transmit the largest possible interval with the desired state at one endpoint. For instance, if the sender wished to communicate the state, $s = 0.315789215$, the sender could cheaply send the message leading to the possible interval $[0.315789215, 1]$ and the sophisticated receiver would infer that the state is certain to be 0.315789215. To avoid these types of problems, we would either have to model the receiver as unsophisticated or to model communication as we do here.

In both of the above options, the communication does not, in our view, resemble communication which is costly and discrete. Most notably the resulting equilibrium would be a fully separating equilibrium whereby each state would induce a unique action by the sender. By contrast, the equilibrium in our model is a pooling equilibrium in that several states induce identical actions by the sender. This seems to be more consistent with our intuition regarding communication.

We have worked to justify our modeling choices, but at this point it is also natural to wonder about the individual contributions of the assumption of costly communication and that of discrete communication. In other words, we now discuss the individual implications of both of these assumptions. First, consider the case where communication is costly but there are an infinite number of messages which are not discrete. In this setting, there would be a message with a cost between the costs of any two messages. Under these assumptions, there would not exist an equilibrium in which more than one message is transmitted, since for any message greater than the least costly message, there would always exist a less costly message. Therefore, the assumption of costly communication, without a restriction to a discrete set will not lead to satisfactory model of communication.

Second, consider the case where messages are costless but there are a finite number of discrete messages. For the case that $b = 0$, we would see that every message is transmitted so that the number of messages in the message space would be a binding constraint. Therefore, this model does not produce an interesting and nontrivial result. Now consider the case there where we take the limit of the equilibrium where we allow the number of possible messages to go to infinity. For the case that $b = 0$, based on the results of Spector (2000), we suspect that the equilibrium would converge to that of full communication. Hence, in neither the finite nor the infinite message case does the assumption of costless and discrete messages produce a satisfactory model. Therefore, the assumptions of costly communication and discrete communication are both necessary because only together do they imply a model with interesting and nontrivial results.

9 Conclusions

We have modeled an interaction between an informed sender and an uninformed receiver where communication is costly and discrete. We have characterized the equilibria without refining out-of-equilibrium beliefs. When the sender and receiver have aligned preferences over the action of the receiver, we have demonstrated that the no incentive to separate ($NITS$) out-of-equilibrium condition admits only the class of equilibria with the largest possible number of actions induced. This result is parallel to the application of $NITS$ to the uniform-quadratic version of Crawford and Sobel (1982). Finally, for the case that preferences are not aligned, we note that $NITS$ does not identify a unique equilibrium and that an increase in communication costs might improve communication. Further, we show that this improvement can be large enough so that it outperforms the Goltsman et al. (2009) upper bound of the receiver's payoffs in efficient, mediated communication.

There remain interesting questions which are unanswered. For instance, although we suspect that we are guaranteed an equilibrium under $NITS$ when $b > 0$, we have been unable to find a proof. Also, we have modeled the interaction as a single repetition. However, we are interested to learn the equilibrium behavior where the interaction is repeated. There are three possibilities as the relationship is potentially finitely repeated, infinitely repeated or is repeated until the communication attains some threshold. There exists an additional issue, which arises only in the repeated version of the game: presumably there is a relationship between some publicly observable signal and the optimal action for the receiver and also that the sender wishes to teach the receiver this relationship. It would seem interesting to explore this learning. Additionally, we are eager to learn the significance of our assumption of quadratic preferences and a uniform probability distribution. Finally, we are interested to know whether an environment with several heterogenous senders and receivers, would produce a novel matching problem.

Finally, Duffy et al. (2011) tests our model in an experimental setting. Like most communication games, our equilibrium is complicated and this fact makes experimental investigation difficult. However, using the example of other such papers[16] the authors test a simplified version of the theoretical model presented above. Duffy et al. (2011) find that the size of the language arises endogenously, as it does in our paper. This suggests that further study of costly and discrete communication could prove fruitful.

[16] For instance, Cai and Wang (2006) and Kawagoe and Takizawa (2009). Also see Blume et al. (1998) and Blume et al. (2001).

10 Appendix

The appendix is organized as follows. In 10.1, we first prove a few results about the nature of the equilibria. Then we prove the results which appear in the body of the paper. Subsequently in 10.2, we offer two numerical examples which illustrate the equilibria without refinement of out-of-equilibrium beliefs and that under $NITS$. Finally in 10.3, we offer an example where there does not exist an equilibrium for an alternate, and arguably more reasonable, specification of $NITS$.

10.1 Proofs

We now offer Lemma 6, which shows that the intervals must be connected and Lemma 7, which shows that the intervals cannot partially overlap.

Lemma 6 *In any equilibria it cannot be the case that there exists* m *such that* $m \in \mu^*(\underline{s}) = \mu^*(\overline{s})$, $m' \notin \mu^*(\underline{s}) = \mu^*(\overline{s})$, $m' \in \mu^*(s')$ *and* $m \notin \mu^*(s')$ *where* $\underline{s} < s' < \overline{s}$.

Proof: Suppose there exists m such that $m \in \mu^*(\underline{s}) = \mu^*(\overline{s})$, $m' \notin \mu^*(\underline{s}) = \mu^*(\overline{s})$, $m' \in \mu^*(s')$ and $m \notin \mu^*(s')$ where $\underline{s} < s' < \overline{s}$.

If $\alpha(m) = \alpha(m')$ then there exists a profitable deviation for S in choosing the cheaper message. Now suppose that $\alpha(m) \neq \alpha(m')$. If $\alpha(m) < \alpha(m')$ and $m' \in \mu(s')$ as:

$$-(\alpha(m) - s' - b)^2 - c(m) < -(\alpha(m') - s' - b)^2 - c(m')$$

then it must be that:

$$-(\alpha(m) - \overline{s} - b)^2 - c(m) < -(\alpha(m') - \overline{s} - b)^2 - c(m').$$

We have arrived at a contradiction because there is a profitable deviation to m' on $\overline{s}$.

If $\alpha(m) < \alpha(m')$ and $m \in \mu(\overline{s})$ as:

$$-(\alpha(m) - \overline{s} - b)^2 - c(m) > -(\alpha(m') - \overline{s} - b)^2 - c(m')$$

then if must be that:

$$-(\alpha(m) - s' - b)^2 - c(m) > -(\alpha(m') - s' - b)^2 - c(m').$$

We have arrived at a contradiction because there is a profitable deviation to m on s'. The proof for the case of $\alpha(m) > \alpha(m')$ follows in the analogous manner.■

Lemma 7 *In any equilibria it cannot be the case that* $m' \in \mu^*(s')$ *where* $s' \in [s_1, s_3)$ *and* $m'' \in \mu^*(s'')$ *where* $s'' \in [s_2, s_4)$ *where* $s_2 < s_3$.

Proof: Suppose that there was such an equilibrium. The message m' induces action a' and message m'' induced action a''. Therefore the payoff from sending m' is

$$U^S(m') = -(a' - s - b)^2 - c(m')$$

and the payoff from sending message m'' is

$$U^S(m'') = -(a'' - s - b)^2 - c(m'').$$

For $a' \neq a''$ there is only a single state for which

$$U^S(m') = U^S(m'')$$

and therefore it cannot both a' and a'' are sent on $[s_2, s_3)$. For the case of $a' = a''$ it must be that $s_2 < s_1 < s_3 < s_4$. Then there exists a profitable deviation by the sender to select the cheaper message. Therefore there cannot exist such an equilibrium.■

Proof of Lemma 1: If there are $n+1$ distinct actions induced by the sender then it must be that there are n equations in expression (2). If this was not the case then Definition 1 would not hold. A typical such expression would be the cutoff state between intervals such that $m_h^i \in \mu^*(s')$ for $s' \in [s_h, s_{h+1})$, $m_{h+1}^j \in \mu^*(s'')$ for $s'' \in [s_{h+1}, s_{h+2})$:

$$-\left(\frac{s_h + s_{h+1}}{2} - s_{h+1} - b\right)^2 - c(i) = -\left(\frac{s_{h+1} + s_{h+2}}{2} - s_{h+1} - b\right)^2 - c(j).$$

Which we rewrite as:

$$\begin{aligned} -\left(\frac{s_h - s_{h+1}}{2} - b\right)^2 - c(i) &= -\left(\frac{s_{h+2} - s_{h+1}}{2} - b\right)^2 - c(j) \\ -\left(\frac{-\lambda_h^i}{2} - b\right)^2 &= -\left(\frac{\lambda_{h+1}^j}{2} - b\right)^2 + c(i) - c(j) \end{aligned}$$

so that

$$(\lambda_{h+1}^j)^2 - (\lambda_h^i)^2 = 4\left[c(i) - c(j)\right] + 4b\left(\lambda_h^i + \lambda_{h+1}^j\right).■$$

Lemma 6 showed that the intervals must be connected. Lemma 7 showed that the equilibria cannot be partially overlapping. Lemma 1 showed the relative size of the intervals as a function of their position on the state space and the cost of message. Also note that Lemma 1 together with the assumption that a unique cost is associated with each message implies that the sender will not employ a mixed strategy.

Proof of Proposition 1: Suppose that strategy pair (μ, α) is feasible, and therefore message m^0 is transmitted on some portion of the state space. In particular suppose that $\mu(s) = m^0$ for $s \in [\underline{s}, \overline{s})$. Suppose that upon observing an out-of-equilibrium message, the receiver believes that the message was transmitted by a sender of type $\frac{\underline{s}+\overline{s}}{2}$. As a result, an out-of-equilibrium message does not induce an action which is not induced in equilibrium. Given α, there does not exist a profitable deviation from μ regarding the messages used in equilibrium since it satisfies expression (4). There does not exist a profitable deviation for S by sending an out-of-equilibrium message. Given μ which satisfies Lemma 1 there is no profitable deviation for R from α since it satisfies expression (1). Therefore, the strategy pair (μ, α) is an equilibrium.■

Lemma 8 *Consider an equilibrium* (μ^*, α^*). *If* $b = 0$ *and there are* n *actions induced then there are* $n+1$ *solutions to* $min_{s\in[0,1]} U^S(\alpha^*(m), \mu^*(s), s)$.

Proof: Suppose that $U^S(\overline{a}, \widehat{m}, \underline{s}) > U^S(\overline{a}, \widehat{m}, \overline{s})$ where $\mu^*([\underline{s}, \overline{s})) = \widehat{m}$. As the distribution is uniform, $\overline{a}(\underline{s}, \overline{s}) = \frac{\underline{s}+\overline{s}}{2}$. This implies that $\left(\frac{\underline{s}+\overline{s}}{2} - \underline{s}\right)^2 > \left(\frac{\underline{s}+\overline{s}}{2} - \overline{s}\right)^2$, which cannot be the case. Combined with expression (2), we have $n+1$ such solutions.■

Hence, if $b = 0$ and there are n actions induced there are $n+1$ states with the worst ex-post payoff. Naturally these are candidates for reasonable beliefs in the event of an out-of-equilibrium message. Further, any of these $n+1$ states would be sufficient for the results under $NITS$ to hold when $b = 0$.

Proof Proposition 2: First we show that an equilibrium under $NITS$ exists. As Lemma 4 shows, there will always be a maximal, $NITS$-feasible strategy pair (μ, α). Suppose that m^k where $k \in \mathbb{N}$ is the most costly message in this maximal, $NITS$-feasible strategy pair (μ, α). In other words, there is a solution to:

$$\left(\lambda^j\right)^2 - \left(\lambda^i\right)^2 = 4(c(i) - c(j)) \text{ for } i, j \in \{0, .., k\}$$
$$\lambda^i \geq 0 \text{ for } i \in \{0, .., k\}$$
$$\lambda^0 + \lambda^1 + ... + \lambda^k = 1$$

which we can rewrite as:

$$\lambda^0 + \sqrt{\left(\lambda^0\right)^2 - 4c(1)} + ... + \sqrt{\left(\lambda^0\right)^2 - 4c(k-1)} + \sqrt{\left(\lambda^0\right)^2 - 4c(k)} = 1$$
$$\text{where } \left(\lambda^0\right)^2 - 4c(i) \geq 0 \text{ for } i \in \{0, .., k\}$$

However, there does not exist a solution to:

$$\left(\widetilde{\lambda}^j\right)^2 - \left(\widetilde{\lambda}^i\right)^2 = 4(c(i) - c(j)) \text{ for } i, j \in \{0, .., k+1\}$$
$$\widetilde{\lambda}^i \geq 0 \text{ for } i \in \{0, .., k\}$$
$$\widetilde{\lambda}^0 + \widetilde{\lambda}^1 + ... + \widetilde{\lambda}^{k+1} = 1.$$

which we can rewrite as:

$$\widetilde{\lambda}^0 + \sqrt{\left(\widetilde{\lambda}^0\right)^2 - 4c(1)} + ... + \sqrt{\left(\widetilde{\lambda}^0\right)^2 - 4c(k)} + \sqrt{\left(\widetilde{\lambda}^0\right)^2 - 4c(k+1)} = 1$$
$$\text{where } \left(\widetilde{\lambda}^0\right)^2 - 4c(i) \geq 0 \text{ for } i \in \{0, .., k+1\}$$

Therefore, if a strategy pair involving m^k is $NITS$-feasible and maximal, it must be that:

$$4c(k+1) > \left(\lambda^0\right)^2 \geq 4c(k) \tag{8}$$

We need to check that it is not profitable for the sender at $s = 0$, to transmit a message more costly than m^k. By Lemma 8, the equilibrium payoffs for the S who received signal $s = 0$ is:

$$-\left(\frac{\lambda^0}{2} - 0\right)^2 - c(0) = -\left(\frac{\lambda^1}{2} - 0\right)^2 - c(1) = ... = -\left(\frac{\lambda^k}{2} - 0\right)^2 - c(k).$$

All of the messages used in equilibrium will not provide a profitable deviation, therefore we must use an out-of-equilibrium message to find a deviation. Any deviation accomplished by message m^{k+x} where $x > 1$ can be accomplished by sending message m^{k+1}. Therefore, the least costly (and therefore best candidate) out-of-equilibruim message is the message m^{k+1}. If such a message is sent, R would have beliefs that the message was sent by state $s = 0$. Sending this message yields a payoff of $-c(k+1)$. Therefore, the signal will be profitable when:

$$-c(k+1) \geq -\left(\frac{\lambda^0}{2} - 0\right)^2$$

which we rewrite as:

$$\left(\lambda^0\right)^2 \geq 4c(k+1). \tag{9}$$

However, we have arrived at a contradiction as it cannot be the case that both (8) and (9) can hold. Therefore, there does not exist a deviation from the maximal, $NITS$-feasible strategy pair where m^k is the most costly message.

Now we will show that if $b = 0$ and there is an equilibrium with a most costly message m^k then $NITS$ does not admit an equilibrium with a cheaper most costly message. For the case of $k > 0$, suppose that a strategy pair with a most costly message m^k is $NITS$-feasible and a strategy pair with a most costly message m^{k+1} is not[17], and so expression (8) holds. Consider a candidate strategy pair involving a most costly message of $m^{k'}$ where $k' < k$. This candidate equilibrium is characterized by:

$$\left(\widehat{\lambda}^j\right)^2 - \left(\widehat{\lambda}^i\right)^2 = 4(c(i) - c(j)) \text{ for } i, j \in \{0, .., k'\}$$

$$\widehat{\lambda}^{k'} > 0$$

$$\widehat{\lambda}^0 + \widehat{\lambda}^1 + ... + \widehat{\lambda}^{k'} = 1.$$

Each of the intervals in the candidate equilibrium are larger than their corresponding intervals in the original equilibrium. Namely that $\widehat{\lambda}^i > \lambda^i$ for $i \in \{0, ..., k'\}$. To see this, note that the difference between the size of the intervals on which messages m^i for $i \in \{0, ..., k'\}$ are sent are identical in the original and candidate equilibria. However, for the original equilibria there are additional intervals to accommodate and so each of the intervals in the original equilibria must be smaller than their counterpart in the candidate equilibrium. Therefore

$$\widehat{\lambda}^i > \lambda^i \geq 4c(k).$$

[17] For the case that $k = 0$, then there is no possible equilibria in which a less costly signal is used.

So we can write the equilibrium payoffs as:

$$U^S = -\left(\frac{\left(\widehat{\lambda}^0\right)^2}{2} - 0\right)^2 < -c(k).$$

Deviation payoffs are $-c(k)$, therefore equilibrium payoffs are less than deviation payoffs and so an equilibrium involving a most costly message of $m^{k'}$ cannot exist where there exists an equilibrium with a most costly message of m^k.

To see that each $NITS$-feasible strategy pair involving a most costly message m^k uniquely determines the values of λ, we can rewrite expression (3) as:

$$\begin{aligned} &\sqrt{4c(k) + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(1)] + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(2)] + \left(\lambda^k\right)^2} + ... \\ &+\sqrt{4[c(k) - c(k-2)] + \left(\lambda^k\right)^2} + \sqrt{4[c(k) - c(k-1)] + \left(\lambda^k\right)^2} + \lambda^k = 1. \end{aligned} \tag{10}$$

The left hand side of expression (10) is strictly increasing in λ^k and therefore must only hold for a single value of λ^k. And so the proposition is proved.■

10.2 Numerical Examples

In the interest in keeping the examples as simple as possible, we restrict attention to the case of perfect alignment of preferences ($b = 0$) and a linear cost function. We now provide an example of the set of equilibria without a refinement of the out-of-equilibrium beliefs.

Example 3 *Consider the case where $c(i) = 0.01i$ and $b = 0$. First, by Lemma 2 there is an equilibrium where m^0 is sent on all states. Next, we consider the set of equilibria in which two messages are sent. By Lemma 2, one of the messages must be the least costly message, m^0. For instance, there is an equilibrium where where m^0 is sent on states $[0, 0.52)$ and m^1 on states $[0.52, 1]$. There is another such equilibrium where m^1 is sent on states $[0, 0.48)$ and m^0 on states $[0.48, 1]$. Note that in both of these equilibria $\lambda^0 = 0.52$ and $\lambda^1 = 0.48$. There are also 2 equilibria involving m^0 and m^2 where $\lambda^0 = 0.54$ and $\lambda^2 = 0.46$. Additionally, there are 2 equilibria involving m^0 and m^3 where $\lambda^0 = 0.56$ and $\lambda^3 = 0.44$. In fact, this continues in this fashion for m^1 through m^{24}. In other words, there are equilibria involving two messages m^0 and m^{k_1} where $k_1 \in \{1, .., 24\}$. Next, we consider the case where there are three messages sent in equilibrium. There are 6 equilibria involving m^0, m^1 and m^2 where $\lambda^0 = 0.392$, $\lambda^1 = 0.337$, and $\lambda^2 = 0.271$. There also exist 6 equilibria involving m^0, m^1 and m^3 where $\lambda^0 = 0.413$, $\lambda^1 = 0.362$, and $\lambda^3 = 0.225$. Likewise, there also exists 6 equilibria involving m^0, m^2 and m^3 where $\lambda^0 = 0.427$, $\lambda^2 = 0.321$, and $\lambda^3 = 0.251$. There also exists many more equilibria in which 3 messages are sent. We now consider the case where 4 messages are sent in equilibrium. For instance, there are 24 equilibria involving m^0, m^1, m^2 and m^3 where $\lambda^0 = 0.363$, $\lambda^1 = 0.303$, $\lambda^2 = 0.227$ and $\lambda^3 = 0.107$. There also exists 24 equilibria involving m^0, m^2 , m^3 and m^4 where $\lambda^0 = 0.408$, $\lambda^2 = 0.294$, $\lambda^3 = 0.216$ and $\lambda^4 = 0.082$. Correspondingly, there also exists more equilibria in which 4 messages are sent. There does not exist an equilibrium in which 5 or more messages are sent.*

In contrast to the multiplicity of equilibria in Example 3, in the Example 4 we show that $NITS$ admits only the class of equilibria with the largest possible number of actions induced. The example also illustrates the utility of Lemma 5 whereby we are able to easily identify that the maximal, $NITS$-feasible strategy will involve 4 messages.

Example 4 *Consider the case where* $c(i) = 0.01i$ *and* $b = 0$. *Note that:*

$$c^*(4) = 0.00662 < 0.01 < c^*(3) = 0.0145.$$

Therefore we can restrict attention to equilibria in which messages m^0, m^1, m^2 *and* m^3 *are used. There are no* $NITS$*-feasible strategy pairs* (μ, α) *for the case of more than four messages. There is a monotonic equilibrium where* m^0 *is sent on states* $[0, 0.363)$, m^1 *on states* $[0.363, 0.665)$, m^2 *on* $[0.665, 0.892)$ *and* m^3 *on* $[0.892, 1]$. *The remaining* 23 *equilibria require that* $\lambda^0 = 0.363$, $\lambda^1 = 0.302$, $\lambda^2 = 0.227$ *and* $\lambda^3 = 0.108$. *Only the* 24 *equilibria which involve* m^0, m^1, m^2 *and* m^3, *are admitted under* $NITS$.

10.3 Example of non-existence of equilibrium under an alternate specification of $NITS$

Here we provide an example where there does not exist an equilibrium for an alternate, and arguably more reasonable, specification of $NITS$. Recall that upon observing an out-of-equilibrium message, the receiver believes that the state is $s = 0$. A common justification for these beliefs is that $s = 0$ is the *lowest* state. However, in general that state does not yield the lowest ex-ante payoffs for the sender. Specifically, the state $s = 0$ shares with other the states the distinction of the smallest ex-ante payoffs for the case of $b = 0$. However, this is not true for the case of $b > 0$. When $b > 0$, the sender at state $s = 1$ has the lowest ex-ante payoffs, and this state would therefore seem to be the best candidate for beliefs upon observing an out-of-equilibrium message. Although these beliefs appear to be reasonable, as the following example shows, under these beliefs we are not guaranteed an equilibrium.

Example 5 *Suppose that* $c = 0$ *and* $b = 0.2$. *Upon observing an out-of-equilibrium message, we assume that the receiver believes that the state is* $s = 1$. *Consider the strategy pair* (μ, α) *in which one message is sent on all states. This induces an optimal action of* R *of* $a = 0.5$. *The payoff of the sender at state* $s = 1$ *is* -0.49, *whereas the payoff to the sender at state* $s = 0$ *is* -0.09. *If the sender at state* $s = 1$ *transmits an out-of-equilibrium message then a payoff of* -0.04 *can be attained and so a profitable deviation exists. Therefore, the uninformative equilibrium cannot exist. Consider the strategy pair* (μ, α) *in which two messages are sent. According to expression (4) one message is sent on states* $[0, 0, 1)$ *and the other is sent on states* $[0.1, 1]$. *The payoff of the sender at state* $s = 1$, *is* -0.4225 *and the payoff of the sender at state* $s = 0$ *is* -0.0225. *If the sender at state* $s = 1$ *transmits an out-of-equilibrium message then a payoff of* -0.04 *can be attained and so a profitable deviation exists. Note that in both the strategy pair in which one message is transmitted and the strategy pair in which two messages are transmitted, the* S *at the state* $s = 1$ *obtains the smallest payoffs, and in this sense is an appropriate candidate for the origin of an out-of-equilibrium message. On the other hand, there does not exist an equilibrium under this alternate specification of* $NITS$ *when either one or two messages are transmitted. Additionally, a strategy pair* (μ, α) *involving three messages cannot satisfy expressions (1), (3) and (4). Hence, there cannot* *exist an equilibrium with in which three or more messages are used. Therefore, there does not exist an equilibrium under this alternate specification of $NITS$.*

Although for $b > 0$ we are not guaranteed an equilibrium under this alternate specification of $NITS$, we are guaranteed an equilibrium for the case of $b = 0$.

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