Strategic interactions, incomplete information and learning

Michele Berardi

6. May 2012

Strategic interactions, incomplete information and learning*

Michele Berardi
The University of Manchester

May 6, 2012

Abstract

In a model of incomplete, heterogeneous information, with externalities and strategic interactions, we analyze the possibility for learning to act as coordination device. We build on the framework proposed by Angeletos and Pavan (2007) and extend it to a dynamic multi-period setting where agents need to learn to coordinate. We analyze conditions under which adaptive and eductive learning obtain, and show that adaptive learning conditions are less demanding than the eductive ones: in particular, when actions are strategic substitutes, the equilibrium is always adaptively learnable, while it might not be eductively so. In case of heterogeneous preferences, moreover, convergence only depends on the average characteristic of agents in the economy. We also show that adaptive learning dynamics converge to the game theoretical strategic equilibrium, which means that agents can learn to act strategically in a simple and straightforward way.

Key words: Learning, heterogeneity, interaction, coordination.
JEL classification: C62, C73, D83.

*I would like to thank participants at the 2011 CDMA Conference "Expectations in Dynamic Macroeconomic Models", and in particular George Evans, Roger Guesnerie, Albert Marcet and Yang Lu, for useful comments and discussions. All remaining errors are my own.

Strategic interactions, incomplete information and learning

1 Introduction

In recent years a growing literature has analyzed macroeconomic models under learning dynamics (for an authoritative treatise, see Evans and Honkapohja, 2001) by following the common practice of replacing the expectational operators, which arise at the semi-reduced form of the model after aggregation and linearization of microfounded equations describing agents' optimal behavior, with an explicit expectations formation mechanism that is meant to represent the evolution of agents' beliefs under adaptive learning.

While this practice is valid to a first approximation and has indeed delivered useful insights into the properties of economic models under learning, it neglects the fact that behind a macro model there is often hidden, at the micro level, a component of coordination. This tension between micro coordination and macro outcomes is easily resolved under the assumption of rational expectations, which deliver a fixed point in the coordination problem. But once agents are deprived of full rationality, as it happens in the learning literature, the issue of coordination becomes critical and its solution is connected at a deep level with the learning activity of agents and the dynamics of their beliefs, which ultimately affect aggregate outcomes.

A typical example is Muth's price model, where firms need to coordinate their production decisions based on information conveyed by prices. Carton and Guse (2010) consider a game theoretic version of this model, and show how adaptive learning and replicator dynamics learning can give rise to rather different outcomes when firms have a discrete set of possible production levels. The learning mechanism used by agents therefore affects the solution to the coordination problem implicit in their production decisions. More in general, there are a number of macroeconomic models that lay hidden underneath a coordination problem among agents and that rely on the assumption that such a problem has been somehow solved.

The aim of the present work is to consider such coordination problems explicitly and investigate conditions under which agents can learn to coordinate. To this end, we use a setting proposed by Angeletos and Pavan (2007), which neatly captures the need for agents to forecast other agents' actions in order to maximize their own utility.

In a model where individual utility depends not only on a fundamental of the economy but also on the aggregate action in the population, agents need to anticipate other people's behavior in order to decide their own action. In such setting, we investigate whether agents can learn to coordinate on the best strategy using either adaptive or eductive learning. With the first, agents rely on the information observable at the aggregate level and use statistical techniques in order to process such information and form expectations about future aggregate actions; with the second, instead, agents engage in a mental process of reasoning to try to determine their best response to other agents' actions: coordination on an equilibrium is achieved if repeated deletion of dominated strategies leads to a unique action for all agents.

Of course, the possibility for agents to learn and coordinate on an equilibrium depends on the amount and quality of information available to them. In our analysis, we will first assume that the fundamental process determining the state of the economy is observable to agents and we will focus solely on the problem of coordination: besides knowledge of the fundamental, in fact, agents will need to have some expectation about the average action in the economy in order to decide their best strategy. The way they form such expectations about other agents' actions will determine their own actions and therefore the possibility of convergence to an equilibrium.

We will then build on the global games literature and assume that the fundamental itself is not observable to agents but that they have access to noisy private and public signals about such fundamental. Given this information, agents need to chose their optimal action, taking into account the fact that everybody else in the economy is also doing the same.

In both settings, we will consider in turn adaptive and eductive learning. Under adaptive learning, agents rely on observables and econometric techniques in order to infer missing information and inform their decisions about actions, while under eductive learning agents rely on a mental process of reasoning that, by iteratively deleting dominated strategies, tries to single out a Nash equilibrium for the economy.

The framework we propose will allow us to investigate the interaction between the problem of learning, as usually addressed in the macro literature, and that of coordination. We will show how adaptive and eductive learning can in fact act as coordination devices in a model with heterogeneous information and strategic interactions. The key parameter that governs learnability will turn out the be the private value of coordination: only if agents don't overreact to the expected actions of others, they will be able to coordinate on an equilibrium. Interestingly, adaptive learning can guide agents towards the game theoretical strategic equilibrium of the model, without them having to engage in higher order thinking but solely relying on information observable in the economy. This key result shows how powerful this mechanism is in guiding agents' actions towards equilibrium.

Lastly, we will consider the issue of coordination based on a sunspot variable, one that, though unrelated to fundamentals, could affect the economy simply because agents deem it relevant and use it in their forecasts. We will show, though, that in the present framework agents can not learn to coordinate based on a sunspot component.

1.1 Related literature

Our contribution is related and builds on a number of works, and it merges concepts from different strains of literature. The most directly related work, in terms of the basic framework used, is Angeletos and Pavan (2007), who introduce a general setting in which agents' best actions depend on the aggregate action in the economy, and agents must solve a coordination problem in order to maximize their utility. They find that the value agents attach to coordination is crucial in determining the equilibrium and welfare properties of the economy.

The information structure for our economy is borrowed from the literature on global games, i.e., coordination games of incomplete information. Morris and Shin (1998, 2001) famously showed that some degree of uncertainty about the fundamentals can be beneficial as it solves the problem of multiple equilibria in the economy. Angeletos, Hellwig, Pavan (2007) then extended the static framework of global games to allow agents to take (binary) actions repeatedly over many periods and to learn about the underlying fundamentals: they show that in this dynamic setting multiplicity of equilibria can emerge under the same conditions that would guarantee uniqueness in the static benchmark. We will not touch upon this aspect though in the present work and only focus on a setting where there is a unique fundamental symmetric equilibrium for the economy.

The spirit of the paper is close to several works in the game theoretical literature, though it takes a more macro oriented approach. Marimon and McGrattan (1992), in a critical review of adaptive learning in repeatedly played strategic form games, show that if agents use adaptive learning rules with inertia and experimentation, the strategy played converges to a subset of rationalizable strategies. Beggs (2009) considers adaptive learning in Bayesian games with binary actions, a framework that includes many of the applications of the theory of global games, and presents conditions under which convergence obtains. Crawford (1995) shows how results from experiments in coordination games can be explained by assuming that agents learn to coordinate using simple linear adjustment rules.

We also refer to concepts from the literature on rationalizable equilibria. Guesnerie (1992) first considered the problem of how a rational expectations equilibrium can emerge as the outcome of the mental process of iterated deletion of dominated strategies by rational agents concerned with maximizing their own utility while recognizing that all other agents in the economy are doing the same. Evans and Guesnerie (1993) then examined the connection between expectational stability (adaptive learning) and strong rationality (eductive learning) by embedding a linear rational expectations model into a game-theoretic framework.

Also relevant to our work is the literature on coordination and higher order beliefs, though we leave the explicit consideration of such a problem in the contest of adaptive learning to future research. Important and related works in this area are Townsend (1983) and Marcet and Sargent (1989): in the former, firms face the problem of forecasting the forecasts of others, and this gives rise to an infinite regress problem which is then solved by Marcet and Sargent (1989) by using adaptive learning to compute the relevant equilibrium for the model.

Lastly, we build on the literature on sunspot equilibria. The possibility of an economy being driven by sunspot variables, i.e., variables unrelated to fundamentals, has received a lot of attention in the literature, at least since the works of Azariadis (1981), Cass and Shell (1983) and Guesnerie (1986). In relation to learning, the possibility of sunspot equilibria to be stable under learning dynamics has been considered in Woodford (1990), Evans and Honkapohja (1994), Evans and Honkapohja (2003) and Evans and McGough (2005). The general message that can be taken from this literature is that, though sunspot equilibria can be learnable, this usually requires rather strict conditions, and the outcome depends on the representation used by agents.

1.2 Plan of the paper

The plan of the paper is as follows: Section 2 introduces the basic model and shows the symmetric equilibrium under full information and rationality; Section 3 introduces learning when there is full information about the fundamental but uncertainty about other agents' actions; Section 4 analyzes learning when there is incomplete and private information about the fundamental; Section 5 considers the possibility of agents using a sunspot variable to coordinate; Section 6 discusses the main results of the paper; and Section 7 concludes.

2 The model

The basic framework is borrowed from Angeletos and Pavan (2007), though we introduce time and make it a multi-period dynamic setting. Moreover, we will allow for heterogeneity in preferences among agents.

There is a continuum of agents on the unit interval, indexed by i, and each agent i needs to choose his own action k_t^i in order to maximize his utility, which depends on an exogenous fundamental θ_t and on the actions of other agents.

The utility of each agent i is given by

$$U_t = U(k_t^i, K_t, \sigma_{k,t}, \theta_t) \tag{1}$$

where

$$\begin{aligned} K_t &= \int_0^1 k_t^i di \qquad (2)\\ \sigma_{k,t} &= \left[\int_0^1 \left(k_t^i - K_t\right)^2 di\right]^{\frac{1}{2}} \qquad (3) \end{aligned}$$

and U is quadratic with partial derivatives $U_{k\sigma} = U_{K\sigma} = U_{\theta\sigma} = 0$ and $U_\sigma(k, K, 0, \theta) = 0$ for all (k, K, θ). This means that the dispersion of actions in the population has only a second order, non strategic effect on individual utility. Technically, it means that utility is separable in σ.

The exogenous fundamental is defined by

$$\theta_t = \theta + \varepsilon_t, \tag{4}$$

where ε_t is an i.i.d. shock, normally distributed with mean zero and variance σ_ε^2.

Since each agent i chooses k_t^i in order to maximize his own utility, given his expectations of other agents' actions and of the fundamental, we have

$$k_t^i = \arg\max_k E_t^i\left[U(k_t^i, K_t, \sigma_{k,t}, \theta_t)\right]. \tag{5}$$

Following the argument in Angeletos and Pavan (2007), it is possible to show that the solution to this problem for the generic agent i must solve

$$k_t^i = \alpha E_t^i K_t + (1-\alpha) E_t^i \kappa(\theta_t) \tag{6}$$

where

$$\alpha \equiv -\frac{U_{kK}}{U_{kk}} \tag{7}$$

and $\kappa(\theta_t)$ is the full information solution given in Section 2.1 below. Parameter α represents the private value of coordination: individual actions are strategic complements if $\alpha > 0$, and strategic substitutes if $\alpha < 0$. In the course of this work we will consider also the case where agents have individual utility functions that differ from each other in the value they assign to coordination and agents are therefore characterized by individual α^i.

In order to decide their best action, agents need to form expectations to be put into (6): the aim of this work is to analyze the coordination problem for agents under different assumptions about information sets and expectations formation processes for agents.

2.1 Equilibrium under full information and rationality

If agents are homogeneous, rational and they all observe θ_t, the problem reduces to

$$k_t^* = \arg\max_k E_t \left[U(k_t, k_t, 0, \theta_t) \right] \tag{8}$$

and with a quadratic utility function, the solution $k_t^* = \kappa(\theta_t)$ is linear (see Angeletos and Pavan, 2007):

$$\kappa(\theta_t) = \kappa_0 + \kappa_1 \theta_t, \tag{9}$$

with

$$\kappa_0 = -\frac{U_k\,(0,0,0,0)}{U_{kk} + U_{kK}} \tag{10}$$

$$\kappa_1 = -\frac{U_{k\theta}}{U_{kk} + U_{kK}}. \tag{11}$$

In this case agents have no uncertainty and their optimal action depends on their own preferences (through κ_0 and κ_1) and on an observable exogenous component (θ_t). They can therefore implement their optimal policy (9). Following Angeletos and Pavan (2007, Supplement), assuming $U_{kk} < 0$ and $-U_{kK}/U_{kk} < 1$ ensures uniqueness and boundedness of equilibrium under complete information.[1]

In the course of this work we will consider in particular the case where[2]

$$\kappa_0 = 0 \tag{12}$$

$$\kappa_1 = 1. \tag{13}$$

In this case the full information solution (9) reduces to

$$k_t^* = \theta_t. \tag{14}$$

An instance of this setting is the Morris and Shin (2002)'s beauty contest model outlined in the Appendix.

Note that (14) is the only equilibrium under complete information and rationality, for any value of $\alpha < 1$. In the course of this paper we will consider the possibility of agents being heterogeneous in their preferences, i.e., having heterogeneous α^i. Under complete information and rationality, given the restrictions assumed on κ_0 and κ_1, this would not affect agents' optimal action, which

[1] To be precise, the model admits a unique solution for any value $-U_{kK}/U_{kk} \neq 1$: for $-U_{kK}/U_{kk} > 1$, though, uniqueness derives from assuming that the action space is unbounded.

[2] Note that there is no loss of generality in this assumption, as it is always possible to redefine a new $\tilde{\theta}_t = \kappa(\theta_t)$ and work with this new process. See Angeletos and Pavan (2007, Supplement).

would still be given, for all agents, by (14). Things could be different with a more generic utility function that would make κ_0 and κ_1 in (9) dependent on α: in this case, under heterogeneous preferences, actions would differ across agents even with complete information. We will neglect this complication in this work and simply focus on results under restrictions (12)-(13). It follows that the optimal action for each agent i will have to satisfy the equation

$$k_t^i = \alpha E_t^i K_t + (1-\alpha) E_t^i \theta_t. \tag{15}$$

3 Complete information about fundamentals and learning

We have seen above the equilibrium fixed point of the model if agents are fully rational. In particular, this requires agents i) to have knowledge about the fundamental process θ_t and to be aware of the fact that everybody else in the economy does as well; and ii) to know that everybody has the same utility function and therefore will behave alike.

In this section we maintain the hypothesis about knowledge of the fundamental, but relax the assumption about full knowledge of others' preferences. Agents therefore need to learn about each other's actions.

Agents, while still observing θ_t, face uncertainty about aggregate action K_t. It follows from (15) that the action of each agent i must satisfy the condition

$$k_t^i = \alpha E_t^i K_t + (1-\alpha) \theta_t. \tag{16}$$

This requires agents to have expectations about K_t at each time t. Given (16), the aggregate model for the economy is

$$K_t = \int_0^1 k_t^i di = \int_0^1 \alpha E_t^i K_t di + \int_0^1 (1-\alpha) \theta_t di = \int_0^1 \alpha E_t^i K_t di + (1-\alpha) \theta_t. \tag{17}$$

3.1 Adaptive learning

We assume first that agents form their expectations as adaptive learners and use information about observables to try to predict what current aggregate action will be. Besides the fundamental, we assume here that also past aggregate actions are observable to agents with one period delay: after each agent has played his own action and the economy has aggregated them all together, aggregate outcomes become observable to everybody. It seems a natural choice for agents to try and use such information about past aggregate actions in order to predict what current actions will be, and therefore we allow agents to do so, even though ex post it will turn out that such information about past aggregate actions is not actually useful. In the terminology of adaptive learning, the forecasting model or perceived law of motion (PLM) we endow agents with will turn out to be overparameterized, as it contains more variables than actually necessary.

The PLM for agents is therefore represented by

$$E_t^i K_t = a_t^i + b_t^i K_{t-1} + c_t^i \theta_t. \tag{18}$$

where parameters a, b, c are updated using econometric techniques such as recursive least squares (RLS) and agents use their most recent estimates of such parameters to compute $E_t^i K_t$. Based on this value, they then choose k_t^i according to (16). Note that k_t^i is computed at each time t according to the anticipated utility model of Kreps (1998), i.e., taking the most recent parameter estimates as given and fixed. In principle, knowing that they are involved in a repeated game, agents might find it convenient to act suboptimally today in order to speed up the learning process and converge faster to equilibrium. We will not consider this possibility here.

Once k_t^i has been chosen, $\forall i$, the economy aggregates actions and K_t is determined. Parameters a, b, c can then be updated using standard statistical methods based on this new value for aggregate data. The question is: does $k_t^i \to k_t^*$ over time, i.e., can agents learn to coordinate on k_t^*?

Since agents use model (18) to form expectations about K_t and then, on the basis of those expectations and the observed θ_t, decide their optimal action, k_t^i must have a (linear) representation of the form (obtained by plugging (18) into (16))

$$k_t^i = \phi_0^i + \phi_1^i \theta_t + \phi_2^i K_{t-1} \tag{19}$$

with

$$\begin{aligned} \phi_0^i &= \alpha a^i \\ \phi_1^i &= (1-\alpha) + \alpha c^i \\ \phi_2^i &= \alpha b^i. \end{aligned}$$

Aggregating actions across agents in the economy we obtain the actual law of motion (ALM):

$$K_t = \int_0^1 k_t^i di = \alpha \int_0^1 a^i di + \left((1-\alpha) + \alpha \int_0^1 c^i di \right) \theta_t + \alpha \left(\int_0^1 b^i di \right) K_{t-1}. \tag{20}$$

Agents update parameters in their PLM (18) using forecast errors, according to the RLS algorithm

$$\begin{aligned} \begin{bmatrix} a_{t+1}^i \\ b_{t+1}^i \\ c_{t+1}^i \end{bmatrix} &= \begin{bmatrix} a_t^i \\ b_t^i \\ c_t^i \end{bmatrix} + t^{-1} R_t^{-1} w_t \left(K_t - E_t^i K_t \right) & (21) \\ R_t &= R_{t-1} + t^{-1} \left(w_t w_t' - R_{t-1} \right) & (22) \end{aligned}$$

with

$$w_t = \begin{bmatrix} 1 \\ K_{t-1} \\ \theta_t \end{bmatrix}$$

representing the vector of regressors and

$$K_t - E_t^i K_t = \left(\alpha \int_0^1 a^i di - a^i \right) + \left(\alpha \int_0^1 b^i di - b^i \right) K_{t-1} + \left((1-\alpha) + \alpha \int_0^1 c^i di - c^i \right) \theta_t$$

the forecast error.

Results from stochastic approximation theory show that the limiting behavior of stochastic recursive algorithms of the form (21)-(22) is well approximated by the behavior of a set of ordinary differential equations (ODEs) that can be obtained by mapping parameters in the PLM of agents to those in the ALM (for a detailed discussion of the techniques involved, see Evans and Honkapohja, 2001). If an equilibrium (fixed point) is stable under such set of differential equations, it is said to be E-stable. This is the concept of stability under adaptive learning that we will use throughout this paper.[3]

Mapping parameters in (18) to those in (20) gives rise to the following system of differential equations for each agent i, that represent the evolution of parameters in agents' forecasting models:

$$\dot{a}^i = \int_0^1 \alpha a^i di - a^i \tag{23}$$

$$\dot{b}^i = \int_0^1 \alpha b^i di - b^i \tag{24}$$

$$\dot{c}^i = 1 - \alpha + \int_0^1 \alpha c^i di - c^i. \tag{25}$$

Note that there is a continuum of systems of differential equations, with three equations for each agent i. We can find stability conditions for the learning process of each agent by computing the derivatives $\frac{d\dot{a}^i}{da^i}$, $\frac{d\dot{b}^i}{db^i}$, $\frac{d\dot{c}^i}{dc^i}$. Using Leibniz's rule, we can see that stability of equations (23)-(25) requires $\alpha < 1$. Remember that α is the private value of coordination: this condition says that such value must not be too high. It also implies that when agents give negative value to coordination (i.e., $\alpha < 0$), the system is stable: agents, trying to move away from each other, induce stability under adaptive learning dynamics.

Proposition 1 *Under adaptive learning, the fundamental symmetric equilibrium is learnable if* $\alpha < 1$.

Proposition 1 says that the private value of coordination must not be too large for convergence to obtain: if agents value coordination too much, they overreact to their expectations of other agents' actions and the economy does not converge to the fundamental symmetric equilibrium.

Solution values for parameters are, after learning has converged and agents all have the same expectations:

$$a^{eq} = 0$$

$$b^{eq} = 0$$

$$c^{eq} = 1,$$

which imply that the economy converges to the fundamental symmetric equilibrium

$$K_t = \theta_t \tag{26}$$

since all agents implement the action $k_t^i = k_t^* = \theta_t$.

[3] For a detailed explanation of the techniques involved, see Evans and Honkapohja (2001).

Looking at PLM (18), we can see now that it is overparameterized with respect to the ALM in equilibrium, as given by equation (26). In the terminology of the adaptive learning literature, this means that such equilibrium is strongly E-stable with respect to this overparameterization, as agents learn to discard from their forecasting model additional variables that do not enter into the fundamental solution.

3.1.1 Heterogeneous preferences

Assume now that agents are heterogeneous in their preferences, so that each agent has his own α^i and his optimal action is therefore given by

$$k_t^i = \alpha^i E_t^i K_t + \left(1 - \alpha^i\right) E_t^i \theta_t. \tag{27}$$

Then the system (23)-(25) becomes

$$\dot{a}^i = \int_0^1 \alpha^i a^i di - a^i \tag{28}$$

$$\dot{b}^i = \int_0^1 \alpha^i b^i di - b^i \tag{29}$$

$$\dot{c}^i = 1 - \alpha^i + \int_0^1 \alpha^i c^i di - c^i, \tag{30}$$

and stability of the learning process for each agent i therefore requires $\int_0^1 \alpha^i di < 1$. This means that we do not need all agents to value coordination in the same way, but only that on average the value they attach to coordination is small enough.

Proposition 2 *With heterogeneous α^i, adaptive learning converges if $\int_0^1 \alpha^i di < 1$, i.e., if the average value of coordination in the population is less than one.*

Proposition 2 says that when preferences are heterogeneous, as long as the average value of coordination is less than one, the learning process of *all* agents converges, even for those agents that have $\alpha^i \geq 1$, since the evolution of other agents' expectations (and therefore actions) acts as stabilizer. This result is very important and must be stressed: learning conditions for each agent depend not on individual preferences but on the average in the population, since it is this average value that governs the dynamics of the underlying variables agents are trying to learn about.

3.2 Eductive learning

Eductive learning was first introduced by Guesnerie (1992) as a way to investigate whether rational and fully informed agents could coordinate on the rational expectations equilibrium with a process of mental reasoning, that would lead them to exclude alternative outcomes thanks to the notion of rationalizable strategies. Evans and Guesnerie (1993) showed the connection between eductive learning and adaptive learning in a cobweb model: while adaptive learning requires $\beta < 1$, where β measures the feedback from expectations to prices, for eductive learning to obtain it is necessary instead that $\mid \beta \mid < 1$. Eductive learning conditions are therefore more stringent in this framework.

In our setting, eductive learning requires agents to be able to coordinate on a strategy by reasoning about what would be best for other agents to do and then implement their best response to such behavior. Suppose agent i thinks that everybody else is implementing the aggregate action K_0; then his best reply, according to (16), would be

$$k_1^i = \alpha K_0 + (1 - \alpha)\,\theta_t.$$

Now, since this holds for any agent i, the aggregate action that follows, K_1, would be

$$K_1 = \alpha K_0 + (1 - \alpha)\,\theta_t$$

which in turn would imply a best response from each agent that would give rise to aggregate action K_2

$$K_2 = \alpha K_1 + (1 - \alpha)\,\theta_t.$$

This mental process defines a difference equation for the aggregate action K (and for a *given* θ_t)

$$K_n = \alpha K_{n-1} + (1 - \alpha)\,\theta_t \tag{31}$$

which is stable for $|\alpha| < 1$, and in this case it converges to the symmetric full information equilibrium $K_t = \theta_t$.

Proposition 3 *Under eductive learning, the economy converges to the symmetric full information equilibrium if* $|\alpha| < 1$.

In the model under consideration, therefore, eductive and adaptive learning conditions differ from each other, similarly to what happens for the cobweb model. This is in fact not a surprise, since our model, once θ_t is assumed to be observable, is isomorphic to a cobweb model.

3.2.1 Heterogeneous preferences

Suppose now that agents are heterogeneous in their α^i. It is easy to verify that in this case eductive learning would require $\left|\int_0^1 \alpha^i di\right| < 1$, i.e., the average private value of coordination must be less than one in absolute value.

Proposition 4 *Under eductive learning with heterogeneity, the economy converges to the symmetric full information equilibrium if* $\left|\int_0^1 \alpha^i di\right| < 1$.

This result states that also in the case of eductive learning, it is sufficient that the condition for stability holds on average in the population.

4 Learning with incomplete and private information

We are now interested in understanding the problem of coordination when agents do not directly observe the fundamental process driving the economy but have to learn about it from imperfect signals. In order to decide their best strategy, agents therefore need now to form expectations about a fundamental exogenous component and about other agents' actions.

Following the literature on global games (see, e.g., Morris and Shin (2001)), we assume that agents do not observe the fundamental process θ_t but receive instead noisy private (x_t^i) and public (y_t) signals. The stochastic processes involved are therefore:

$$\theta_t = \theta + \varepsilon_t \tag{32}$$

$$y_t = \theta_t + u_t \tag{33}$$

$$x_t^i = \theta_t + v_t^i \tag{34}$$

where ε, u, v^i are i.i.d. shocks, normally distributed with mean zero and variances σ_ε^2, σ_u^2 and σ_v^2 respectively. The first is a noise in the drawn made by nature at the beginning of each period to determine the fundamental, while u and v^i are observational noise in the public and private signals.

Starting from the optimality condition

$$k_t^i = \alpha E_t^i K_t + (1-\alpha) E_t^i \theta_t, \tag{35}$$

agents will now need to form expectations both on the fundamental θ_t and on aggregate action K_t in order to implement their individual best action.

Angeletos and Pavan (2007) show in their static setting that in the case of agents not observing θ, but instead receiving a private signal x and a public signal y, agents' optimal action has a linear representation of the form

$$k(x,y) = \kappa_0 + \kappa_1 \left[(1-\gamma) x + \gamma z\right] \tag{36}$$

with

$$z = E\left[\theta \mid y\right]$$

and

$$\begin{aligned} \gamma &= \delta + \frac{\alpha\delta(1-\delta)}{1-\alpha(1-\delta)} \\ \alpha &= -\frac{U_{kK}}{U_{kk}} \\ \delta &= \frac{\sigma_y^{-2} + \sigma_\theta^{-2}}{\sigma_x^{-2} + \sigma_y^{-2} + \sigma_\theta^{-2}}. \end{aligned}$$

Would this strategy be learnable by agents in a repeated game? Note that while α is a behavioral parameter, that depends on the preferences of agents, δ represents characteristics of the economy (the variances of the various shocks), and it is rather farfetched to assume that agents know exactly these values.

We will now investigate whether agents, through adaptive and eductive learning, can learn to implement their best strategy.

4.1 Adaptive learning

As the only information available to agents are the private and public signals, it is natural to assume that they use such information to help solve their coordination problem.[4] Under adaptive learning, therefore, agents use their private (x_t^i) and the public (y_t) signal to learn about the fundamental θ_t and aggregate action K_t, according to the PLMs:

$$\begin{aligned} E_t^i K_t &= E^i(K_t \mid x_t^i, y_t) = a_K^i + b_K^i x_t^i + c_K^i y_t && (37) \\ E_t^i \theta_t &= E^i(\theta_t \mid x_t^i, y_t) = a_\theta^i + b_\theta^i x_t^i + c_\theta^i y_t && (38) \end{aligned}$$

which imply, from (35),

$$k_t^i = \alpha E_t^i K_t + (1-\alpha) E_t^i \theta_t = \phi_c^i + \phi_x^i x_t^i + \phi_y^i y_t. \tag{39}$$

where

$$\begin{aligned} \phi_c^i &= \alpha a_K^i + (1-\alpha) a_\theta^i \\ \phi_x^i &= \alpha b_K^i + (1-\alpha) b_\theta^i \\ \phi_y^i &= \alpha c_K^i + (1-\alpha) c_\theta^i. \end{aligned}$$

Aggregating over agents, we then obtain

$$\begin{aligned} K_t &= \int_0^1 k_t^i di = \alpha \left(\int_0^1 a_K^i di + \int_0^1 b_K^i x_t^i di + \int_0^1 c_K^i di y_t \right) + \\ &\quad + (1-\alpha) \left(\int_0^1 a_\theta^i di + \int_0^1 b_\theta^i x_t^i di + \int_0^1 c_\theta^i di y_t \right) \\ &= [\alpha \bar{a}_K + (1-\alpha) \bar{a}_\theta] + [\alpha \bar{c}_K + (1-\alpha) \bar{c}_\theta] y_t + \alpha \int_0^1 b_K^i x_t^i di + (1-\alpha) \int_0^1 b_\theta^i x_t^i di, && (40) \end{aligned}$$

where $\bar{a}_K = \int_0^1 a_K^i di$, $\bar{a}_\theta = \int_0^1 a_\theta^i di$, $\bar{c}_K = \int_0^1 c_K^i di$, $\bar{c}_\theta = \int_0^1 c_\theta^i di$. Since agents have private information, learning is heterogeneous and the last two terms in (40) can not be reduced down to averages. We therefore have

$$K_t = [\alpha \bar{a}_K + (1-\alpha) \bar{a}_\theta] + [\alpha \bar{c}_K + (1-\alpha) \bar{c}_\theta] y_t + \int_0^1 \left[\alpha b_K^i + (1-\alpha) b_\theta^i \right] x_t^i di. \tag{41}$$

Since θ_t is exogenous, parameters in equation (38) will converge over time to their ordinary least squares estimates (i.e., conditions $E\left(\theta_t - E_t^i \theta_t\right) = 0$, $E\left[x_t^i \left(\theta_t - E_t^i \theta_t\right)\right] = 0$ and

[4] We could also allow agents to use past aggregate actions in their PLMs, but we have seen previously that such variable is not actually useful for coordination and agents learn to discard it from their forecasting model.

$E\left[y_t\left(\theta_t - E_t^i\theta_t\right)\right] = 0$ will hold in equilibrium):

$$\begin{aligned} a_\theta^i &\rightarrow \frac{\sigma_\varepsilon^{-2}\theta}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} := a_\theta^{eq} && (42)\\ b_\theta^i &\rightarrow \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} := b_\theta^{eq} && (43)\\ c_\theta^i &\rightarrow \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} := c_\theta^{eq}. && (44)\end{aligned}$$

As for parameters in the PLM for K_t, if agents update their estimates using RLS, the evolution of parameters over time is represented by the stochastic recursive algorithm:

$$\begin{aligned} \varphi_{t+1}^i &= \varphi_t^i + t^{-1}\left(R_t^i\right)^{-1} w_t^i\left(K_t - E_t^iK_t\right) && (45)\\ R_t^i &= R_{t-1}^i + t^{-1}\left(w_t^i w_t^{i\prime} - R_{t-1}^i\right), && (46)\end{aligned}$$

where

$$\varphi^i = \begin{bmatrix} a_k^i \\ b_k^i \\ c_k^i \end{bmatrix},\ w_t^i = \begin{bmatrix} 1 \\ x_t^i \\ y_t \end{bmatrix}.$$

Since the PLM for each agent turns out to be misspecified with respect to the ALM, as the former depends on individual x_t^i and the latter on their population weighted average, we can not map one to one parameters from the PLM to the ALM as we did previously but we need instead to project the PLM onto the ALM to find the ODEs that govern the dynamics for agents' beliefs. Using stochastic approximation theory we have

$$\begin{aligned} \dot{\varphi}^i &= \frac{d\varphi^i}{d\tau} = \lim_{t\to\infty} EQ(t,\varphi^i,z_t^i)\\ Q(t,\varphi^i,z_t^i) &= \left(R_t^i\right)^{-1} w_t^i\left(K_t - E_t^iK_t\right),\end{aligned}$$

where $z_t^i = \left[w_t^{i\prime}\ \theta_t\right]'$ and expectations are taken over the invariant joint distribution of z_t^i for fixed φ^i.

Since

$$\begin{aligned} K_t - E_t^iK_t &= \left[\alpha\bar{a}_K + (1-\alpha)\,\bar{a}_\theta\right] + \left[\alpha\bar{c}_K + (1-\alpha)\,\bar{c}_\theta\right] y_t + \\ &\quad + \int_0^1 \left[\alpha b_K^i + (1-\alpha)\, b_\theta^i\right] x_t^i di - a_K^i - b_K^i x_t^i - c_K^i y_t,\end{aligned}$$

we have

$$\lim_{t\to\infty} EQ(.) = \lim E\left[\left(R_t^i\right)^{-1} w_t^i\left(\begin{bmatrix}1 & x_t^i & y_t\end{bmatrix}\begin{bmatrix}\alpha\bar{a}_K + (1-\alpha)\,\bar{a}_\theta - a_K^i \\ -b_K^i \\ \alpha\bar{c}_K + (1-\alpha)\,\bar{c}_\theta - c_K^i\end{bmatrix}\right) + \left(R_t^i\right)^{-1} w_t^i \int_0^1 \left[\alpha b_K^i + (1-\alpha)\, b_\theta^i\right] x_t^i di\right].$$

By denoting

$$R^{-1} := \lim_{t\to\infty} E\left(R_t^i\right)^{-1} = \begin{bmatrix} 1 & \theta & \theta \\ \theta & \theta^2+\sigma_\varepsilon^2+\sigma_v^2 & \theta^2+\sigma_\varepsilon^2 \\ \theta & \theta^2+\sigma_\varepsilon^2 & \theta^2+\sigma_\varepsilon^2+\sigma_u^2 \end{bmatrix}^{-1}$$

and noting that $a_\theta^i = a_\theta^{eq}$, $b_\theta^i = b_\theta^{eq}$ and $c_\theta^i = c_\theta^{eq}$ in the limit, we then obtain

$$\frac{d\varphi^i}{d\tau} = \begin{bmatrix} \alpha\bar{a}_K + (1-\alpha)\,a_\theta^{eq} - a_K^i \\ -b_K^i \\ \alpha\bar{c}_K + (1-\alpha)\,c_\theta^{eq} - c_K^i \end{bmatrix} + R^{-1}\left[(1-\alpha)\,b_\theta^{eq}\right] Ew_t^i \int_0^1 x_t^i di + R^{-1}\alpha E w_t^i \int_0^1 b_K^i x_t^i di,$$

which, denoting $B := \left[(1-\alpha)\,b_\theta^{eq}\right]$, leads to

$$\begin{aligned}
\dot{a}_K^i &= \alpha\bar{a}_K + (1-\alpha)\,a_\theta^{eq} - a_K^i + BR_{11}^{-1}E\int_0^1 x_t^i di + BR_{12}^{-1}Ex_t^i\int_0^1 x_t^i di + BR_{13}^{-1}Ey_t\int_0^1 x_t^i di + \\
&\quad + R_{11}^{-1}\alpha E\int_0^1 b_K^i x_t^i di + R_{12}^{-1}\alpha Ex_t^i\int_0^1 b_K^i x_t^i di + R_{13}^{-1}\alpha Ey_t\int_0^1 b_K^i x_t^i di \\
\dot{b}_K^i &= -b_K^i + BR_{21}^{-1}E\int_0^1 x_t^i di + BR_{22}^{-1}Ex_t^i\int_0^1 x_t^i di + BR_{23}^{-1}Ey_t\int_0^1 x_t^i di + \\
&\quad + R_{21}^{-1}\alpha E\int_0^1 b_K^i x_t^i di + R_{22}^{-1}\alpha Ex_t^i\int_0^1 b_K^i x_t^i di + R_{23}^{-1}\alpha Ey_t\int_0^1 b_K^i x_t^i di \\
\dot{c}_K^i &= \alpha\bar{c}_K + (1-\alpha)\,c_\theta^{eq} - c_K^i + BR_{31}^{-1}E\int_0^1 x_t^i di + BR_{32}^{-1}Ex_t^i\int_0^1 x_t^i di + BR_{33}^{-1}Ey_t\int_0^1 x_t^i di + \\
&\quad R_{31}^{-1}\alpha E\int_0^1 b_K^i x_t^i di + R_{32}^{-1}\alpha Ex_t^i\int_0^1 b_K^i x_t^i di + R_{33}^{-1}\alpha Ey_t\int_0^1 b_K^i x_t^i di.
\end{aligned}$$

Because expectations are taken over the distribution of z_t^i for fixed belief parameters φ^i, it follows that

$$\begin{aligned}
\dot{a}_K^i &= \alpha\bar{a}_K + (1-\alpha)\,a_\theta^{eq} - a_K^i + BR_{11}^{-1}\theta + BR_{12}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + BR_{13}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + \\
&\quad + R_{11}^{-1}\alpha\bar{b}_K\theta + R_{12}^{-1}\alpha\left[\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right)\right] + R_{13}^{-1}\alpha\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right) \\
\dot{b}_K^i &= -b_K^i + BR_{21}^{-1}\theta + BR_{22}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + BR_{23}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + \\
&\quad + R_{21}^{-1}\alpha\bar{b}_K\theta + R_{22}^{-1}\alpha\left[\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right)\right] + R_{23}^{-1}\alpha\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right) \\
\dot{c}_K^i &= \alpha\bar{c}_K + (1-\alpha)\,c_\theta^{eq} - c_K^i + BR_{31}^{-1}\theta + BR_{32}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + BR_{33}^{-1}\left(\theta^2+\sigma_\varepsilon^2\right) + \\
&\quad + R_{31}^{-1}\alpha\bar{b}_K\theta + R_{32}^{-1}\alpha\left[\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right)\right] + R_{33}^{-1}\alpha\bar{b}_K\left(\theta^2+\sigma_\varepsilon^2\right),
\end{aligned}$$

where $\bar{b}_K = \int_0^1 b_K^i di$. We then have

$$\begin{aligned}
\dot{a}_K^i &= \alpha\bar{a}_K + (1-\alpha)\,a_\theta^{eq} + \Delta_a - a_K^i && (47) \\
\dot{b}_K^i &= \Delta_b - b_K^i && (48) \\
\dot{c}_K^i &= \alpha\bar{c}_K + (1-\alpha)\,c_\theta^{eq} + \Delta_c - c_K^i, && (49)
\end{aligned}$$

where

$$\begin{aligned}\Delta_a &= \tilde{B}R_{11}^{-1}\theta + \tilde{B}\left(R_{12}^{-1}+R_{13}^{-1}\right)\left(\theta^2+\sigma_\varepsilon^2\right)\\ \Delta_b &= \tilde{B}R_{21}^{-1}\theta + \tilde{B}\left(R_{22}^{-1}+R_{23}^{-1}\right)\left(\theta^2+\sigma_\varepsilon^2\right)\\ \Delta_c &= \tilde{B}R_{31}^{-1}\theta + \tilde{B}\left(R_{32}^{-1}+R_{33}^{-1}\right)\left(\theta^2+\sigma_\varepsilon^2\right)\end{aligned}$$

with

$$\tilde{B} := \left[\alpha\bar{b}_K + (1-\alpha)\, b_\theta^{eq}\right].$$

It can be shown that

$$\Delta_a = \tilde{B}\theta\frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} \tag{50}$$

$$\Delta_b = \tilde{B}\frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} \tag{51}$$

$$\Delta_c = \tilde{B}\frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}}. \tag{52}$$

Stability of each system of three ODEs (one for each agent i) is governed by the Jacobian

$$J = \begin{bmatrix} \frac{\delta\dot{a}^i}{\delta a^i} & \frac{\delta\dot{a}^i}{\delta b^i} & 0 \\ 0 & \frac{\delta\dot{b}^i}{\delta b^i} & 0 \\ 0 & \frac{\delta\dot{c}^i}{\delta b^i} & \frac{\delta\dot{c}^i}{\delta c^i} \end{bmatrix}, \tag{53}$$

whose eigenvalues are the diagonal elements

$$\begin{aligned}\frac{\delta\dot{a}^i}{\delta a^i} &= \alpha - 1\\ \frac{\delta\dot{b}^i}{\delta b^i} &= \alpha\frac{\delta\Delta_b}{\delta\tilde{B}} - 1\\ \frac{\delta\dot{c}^i}{\delta c^i} &= \alpha - 1.\end{aligned}$$

It can be seen from (51) that

$$\frac{\delta\Delta_b}{\delta\tilde{B}} = \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}},$$

and therefore conditions for learnability are

$$\alpha < 1 \tag{54}$$

$$\alpha\frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} < 1. \tag{55}$$

Since the first implies the second (because $0 < \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}} \leq 1$), the system is stable when $\alpha < 1$.

Proposition 5 *Under incomplete private information and adaptive learning, learning dynamics converge if $\alpha < 1$.*

We can therefore see that under incomplete information the condition for adaptive learning to converge is the same as the one we derived under full information about the fundamental. Even though convergence depends only on α, we can see that now the relative precision of signals affects the size of one eigenvalue of the system and it will therefore affect the dynamics of the system over the convergence path towards equilibrium.

4.2 Heterogeneous preferences

Suppose now that agents are heterogeneous in their α^i. Going through the previous reasoning, only now with heterogeneous α^i, it is possible to show that stability under learning depends on $\int_0^1 \alpha^i di$: again, the average value of coordination has to be less than one.

Proposition 6 *Under incomplete private information and adaptive learning with heterogeneous preferences, learnability obtains if* $\int_0^1 \alpha^i di < 1$, *i.e., if the average value of coordination is less than one.*

4.3 Equilibrium under adaptive learning

Because of linearity in (47)-(49), in Section 4.1 we were able to derive conditions for learning dynamics to converge without knowing the fixed point for the system of ODEs. We will compute now equilibrium values for parameters in agents' PLMs, and therefore determine optimal actions for agents: this will allow us to show that the equilibrium we obtain under adaptive learning and incomplete information is the same as the one derived by Angeletos and Pavan (2007). This result means that, by learning statistically, agents are able to take into account the strategic component of their interactions and coordinate on the game theoretical equilibrium, without the need of any knowledge or information about other agents' beliefs.

Equilibrium points for the learning algorithm of agents are resting points of the system (47)-(49). The symmetric solution for each agent i is:

$$a_K^{eq} = a_\theta^{eq}\left(1+\frac{\tilde{B}^{eq}}{1-\alpha}\right) \tag{56}$$

$$b_K^{eq} = b_\theta^{eq}\frac{(1-\alpha)\,\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+(1-\alpha)\,\sigma_v^{-2}} \tag{57}$$

$$c_K^{eq} = c_\theta^{eq}\left(1+\frac{\tilde{B}^{eq}}{1-\alpha}\right), \tag{58}$$

where

$$\tilde{B}^{eq} = \frac{(1-\alpha)\,\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+(1-\alpha)\sigma_v^{-2}}.$$

These equilibrium belief parameters imply the following coefficients in equation (39) representing the best action for generic agent i:

$$\phi_c^{eq} = \alpha a_K^{eq} + (1-\alpha)\, a_\theta^{eq} = \frac{\sigma_\varepsilon^{-2}\theta}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + (1-\alpha)\sigma_v^{-2}} \tag{59}$$

$$\phi_x^{eq} = \alpha b_K^{eq} + (1-\alpha)\, b_\theta^{eq} = \frac{(1-\alpha)\,\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + (1-\alpha)\sigma_v^{-2}} \tag{60}$$

$$\phi_y^{eq} = \alpha c_K^{eq} + (1-\alpha)\, c_\theta^{eq} = \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + (1-\alpha)\sigma_v^{-2}}. \tag{61}$$

Comparing equilibrium values (36) from Angeletos and Pavan (2007) with the ones found here under learning and given by (59-61), it is straightforward to show (once allowed from the change of variable from z to y) that the two solutions are exactly the same.

By learning adaptively from data agents converge to the same strategic equilibrium derived through game theoretical reasoning. Under adaptive learning and incomplete information, therefore, agents are able to take into account the strategic component of their interactions and coordinate on their best strategy.

Proposition 7 *Under incomplete private information and adaptive learning, if learning dynamics converge, the economy converges to the strategic equilibrium as defined in Angeletos and Pavan (2007).*

Moreover, by looking at equations (59)-(61) we can immediately see that the strategic component implicit in agents' utility affects the solution: in particular, if $\alpha > 0$, i.e., actions are strategic complements, agents put more weight on public information, while if $\alpha < 0$, i.e., actions are strategic substitutes, agents put more weight on private information.

4.4 Eductive learning

We consider now whether agents could learn the game theoretical equilibrium through a mental process of reasoning about best reply strategies. Suppose agent i believes that a generic agent j will follow the strategy

$$k_t^j = \phi_c + \phi_x x_t^j + \phi_y y_t. \tag{62}$$

Then agent's i expected average action in the economy is

$$\begin{aligned} E_t^i K_t &= \phi_c + \phi_x E_t^i \theta_t + \phi_y y_t \\ &= \phi_c + \phi_x \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}}\theta + \phi_x \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} x_t^i + \left[\phi_x \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} + \phi_y\right] y_t \end{aligned}$$

and his best reply to it will be

$$\begin{aligned} k_t^i &= \alpha E_t^i K_t + (1-\alpha)\, E_t^i \theta_t \\ &= \alpha\phi_c + (\alpha\phi_x + 1 - \alpha)\, \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}}\theta + (\alpha\phi_x + 1 - \alpha)\, \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} x_t^i + \\ &\quad + \left((\alpha\phi_x + 1 - \alpha)\, \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} + \alpha\phi_y\right) y_t. \end{aligned}$$

But then, since agent i realizes that everybody else is doing the same reasoning, he will take this new action as the action implemented by a generic agent j, and again compute his own best reply to the ensuing aggregate action. Iteration on this reasoning defines three difference equations in notional time in the parameter space

$$\phi_{c,n+1} = \alpha\phi_{c,n} + \left(\alpha\phi_{x,n} + 1 - \alpha\right)\frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}}\theta \tag{63}$$

$$\phi_{x,n+1} = \left(\alpha\phi_{x,n} + 1 - \alpha\right)\frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} \tag{64}$$

$$\phi_{y,n+1} = \alpha\phi_{y,n} + \left(\alpha\phi_{x,n} + 1 - \alpha\right)\frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}}. \tag{65}$$

It is immediate to show that the equilibrium values for these equations are those given in (59-61). Moreover, conditions for eductive learning to converge are

$$|\alpha| < 1 \tag{66}$$

$$\left|\alpha\frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}}\right| < 1, \tag{67}$$

and since the first implies the second, they reduce to $|\alpha| < 1$.

Proposition 8 *Under incomplete private information and homogeneous preferences, eductive learning stability obtains if* $|\alpha| < 1$.

We can see that both under adaptive and eductive learning, the relative precision of signals, summarized by $\frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2}+\sigma_u^{-2}+\sigma_v^{-2}}$, enters into conditions for stability, but it does not affect whether asymptotic convergence obtains. Things would be different, though, if agents were to have additional information about other agents' actions: in this case, in fact, asymptotic convergence under eductive learning would be crucially affected by the relative precision of public and private signals.

To see this point, suppose for simplicity that $\varepsilon_t = 0$ and that agent i believed the generic agent j was acting according to[5]

$$k_t^j = \phi x_t^j + (1-\phi)\, y_t. \tag{68}$$

This equation imposes a restriction across weights on the two signals, and therefore assumes agents have some knowledge about other agents' behavior: in particular, it implies that agents know that the optimal action for a generic agent j is determined by a weighted average of the public and private signals. In this case, the condition for eductive stability would reduce to $\left|\alpha\frac{\sigma_v^{-2}}{\sigma_u^{-2}+\sigma_v^{-2}}\right| < 1$. If the noise in the public signal increases and ultimately makes the signal useless ($\sigma_u^{-2} = 0$), this condition reduces to $|\alpha| < 1$: the public signal, therefore, makes it easier for agents to coordinate, as it makes the eductive learning condition less stringent on the private value of coordination. On the other hand, if the noise in the private signal increases and ultimately makes such a signal useless ($\sigma_v^{-2} = 0$), the equilibrium becomes eductively stable for any value of the private value of

[5] This is the "guess" used by Morris and Shin (2002) in order to find the optimal strategy for agents in their model. The argument they use to find the solution, by looking for the fixed point of a map from perceptions to actions, is similar to the one used here, even though they don't give it an eductive learning interpretation.

coordination α, as all agents use only the public signal in deciding their actions, which makes the coordination problem trivial in this case.

4.5 Heterogeneous preferences

We assume now that agents are heterogeneous in their α^i. Going through the same reasoning as in the previous section, only now with heterogeneous α^i, it is easy to show that stability under eductive learning obtains now if $\left|\int_0^1 \alpha^i di\right| < 1$.

Proposition 9 *Under incomplete private information and heterogeneous preferences, eductive learning stability obtains if* $\left|\int_0^1 \alpha^i di\right| < 1$.

Again, only the average value of coordination in the economy matters for convergence.

5 Sunspot coordination

We now investigate whether in the incomplete information framework under consideration it could be possible for agents to use a sunspot variable, one that is uncorrelated with fundamentals, to gain information on other agents' actions and facilitate coordination.

In the previous sections we allowed agents to use two signals, one private and one public: both signals turned out to be useful for agents in implementing their optimal strategy, but both signals had the property of being correlated with the fundamental process θ_t. We want instead to see now if a signal that is uncorrelated with the fundamental but has the property of being observed by all agents and it is therefore a common signal, could be exploited by agents for coordination.

5.1 Adaptive learning

We first consider the problem of coordination with sunspot under adaptive learning. We continue to assume that agents know their own preferences and are therefore able to realize that their optimal action is given by (35). In addition to the public and private signals considered before, though, now an additional variable ξ_t is observed by everybody in the economy and is allowed to enter into the forecasting model for agents.

Once agents condition their forecasts on the sunspot component ξ_t, which is i.i.d. and independent from x_t^i, y_t and θ_t, PLMs (37)-(38) are modified as follows:

$$E_t^i K_t = E^i(K_t \mid x_t^i, y_t, \xi_t) = a_K^i + b_K^i x_t^i + c_K^i y_t + d_K^i \xi_t \tag{69}$$

$$E_t^i \theta_t = E^i(\theta_t \mid x_t^i, y_t, \xi_t) = a_\theta^i + b_\theta^i x_t^i + c_\theta^i y_t + d_\theta^i \xi_t. \tag{70}$$

With these expectations formation models, the temporary equilibrium for the economy would then be

$$\begin{aligned} K_t = {} & [\alpha \bar{a}_K + (1-\alpha)\bar{a}_\theta] + [\alpha \bar{c}_K + (1-\alpha)\bar{c}_\theta] y_t + \int_0^1 \left[\alpha b_K^i + (1-\alpha) b_\theta^i\right] x_t^i di + \\ & + \left[\alpha \bar{d}_K + (1-\alpha)\bar{d}_\theta\right] \xi_t. \end{aligned} \tag{71}$$

Since θ_t is exogenous and independent of ξ_t, and the sunspot component is independent from the other regressors, it is immediate to show that over time estimates for d_θ^i in (70) would converge to zero. As for the sunspot parameter in PLM (69) for aggregate action K_t, the map from PLM (69) to ALM (71) for this parameter gives rise to the ODE

$$\dot{d}_K^i = \alpha \bar{d}_K + (1-\alpha)\,\bar{d}_\theta - d_K^i, \tag{72}$$

where $\bar{d}$ represents population averages. Since in equilibrium $\bar{d}_\theta = 0$, it follows that the only symmetric solution, for generic α, is $d_K^i = 0$, $\forall i$, and its stability requires $\alpha < 1$. This means that even if agents allow for aggregate actions to depend on an extraneous component and use such component in deciding their optimal action, they will learn over time to discard it under the same condition that ensures stability of the fundamental equilibrium.

Note that this result would carry over to a setting with heterogeneous preferences: even if agents were to hold different α^i, equilibrium under learning would imply $d_K^i = 0$, $\forall i$, and the condition for stability under learning would be $\int_0^1 \alpha^i di < 1$.

The literature on sunspot and adaptive learning has found that learnability of a sunspot equilibrium often depends on its representation (see,e.g., Evans and McGough 2005). In the present setting, though, the representation of the sunspot does not matter, as agents do not need to project it ahead in order to compute their optimal action.

Proposition 10 *Under incomplete information and adaptive learning, agents can not coordinate on an equilibrium with sunspots. Agents learn to discard the sunspot component from their model, and the economy converges to the fundamental equilibrium, if $\alpha < 1$ or, under heterogeneous preferences, if $\int_0^1 \alpha^i di < 1$.*

5.2 Eductive learning

We consider now the issue of sunspot equilibria from an eductive learning perspective. Suppose agent i believes that a generic agent j will follow the strategy

$$k_t^j = \phi_c + \phi_x x_t^j + \phi_y y_t + \phi_\xi \xi_t. \tag{73}$$

Then agent i's expected average action in the economy is

$$\begin{aligned} E_t^i K_t &= \phi_c + \phi_x E_t^i \theta_t + \phi_y y_t + \phi_\xi \xi_t \\ &\phi_c + \phi_x \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} \theta + \phi_x \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} x_t^i + \left[\phi_x \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} + \phi_y\right] y_t + \phi_\xi \xi_t \end{aligned}$$

and his best reply action will be

$$\begin{aligned} k_t^i &= \alpha E_t^i K_t + (1-\alpha)\, E_t^i \theta_t \\ &\alpha \phi_c + (\alpha \phi_x + 1 - \alpha) \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} \theta + (\alpha \phi_x + 1 - \alpha) \frac{\sigma_v^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} x_t^i + \\ &+ \left((\alpha \phi_x + 1 - \alpha) \frac{\sigma_u^{-2}}{\sigma_\varepsilon^{-2} + \sigma_u^{-2} + \sigma_v^{-2}} + \alpha \phi_y \right) y_t + \alpha \phi_\xi \xi_t. \end{aligned}$$

But then, since agent i realizes that everybody else is doing the same reasoning, he will take this new action as the action implemented by a generic agent j, and again compute his own best reply to the ensuing aggregate action. Iteration on this reasoning defines four difference equations in notional time in the parameter space, the first three given as before by (63)-(65), plus the additional equation for the evolution of the parameter attached to the sunspot variable:

$$\phi_{\xi,n+1} = \alpha\phi_{\xi,n}. \tag{74}$$

Condition for stability of this difference equation, which is independent from the other three, is again

$$|\alpha| < 1,$$

or, in case of heterogeneous preferences,

$$\left|\int_0^1 \alpha^i di\right| < 1$$

and agents learn to discard the sunspot component, which does not affect actions in equilibrium.

Proposition 11 *Under incomplete information and eductive learning, agents can not coordinate on an equilibrium with sunspots. The economy converges to the fundamental equilibrium if* $|\alpha| < 1$ *or, under heterogeneous preferences, if* $\left|\int_0^1 \alpha^i di\right| < 1$.

6 Discussion

The basic framework used here to analyze the issues of learning and coordination can be interpreted as representing a number of specific economic models. For example, it could be interpreted as a model of investment and production complementarities, where the return on investment for each firm depends not only on their own productivity but also on how much investment is done by other firms in the same sector; or again, it could represent a beauty contest economy where financial investors try to outbid each other on an asset whose value depends not only on its fundamental, but also on what agents are willing to pay for it.

Our results show that in all these cases agents are able to learn to coordinate on the fundamental equilibrium, provided a certain condition on their preferences holds. The specific condition required, though, depends on whether, in order to predict other agents' actions, they engage in a mental process of higher order thinking (eductive learning) or if instead they rely on the gathering and processing of external information (adaptive learning).

In particular, we have shown that, both under perfect and imperfect information about the fundamental process driving the economy, conditions for adaptive learning are less stringent than those for eductive learning. It is interesting to note that under adaptive learning it makes a difference whether actions are strategic substitutes or complements, while for eductive learning this distinction does not matter. Adaptive learning, in fact, requires $\alpha < 1$: when actions are strategic substitutes ($\alpha < 0$), the equilibrium is therefore learnable, while if actions are complements ($\alpha > 0$), the equilibrium might not be learnable. This distinction does not emerge instead for eductive learning, which requires $|\alpha| < 1$: even if actions are substitutes, the equilibrium might not be learnable.

Results on adaptive learning show that by solely rely on past observables, and without the need to engage in a mental process of guessing and outguessing each other, agents can learn to implement their optimal, game theoretical strategy. Marcet and Sargent (1989) showed that the problem of forecasting the forecasts of others in environments where there is private information could be solved by agents using adaptive learning on a reduced form of the model. Our result goes in the same direction in showing that when agents need to forecast other agents' actions, they are able to coordinate on the rational expectations equilibrium by relying solely on adaptive learning based on the observables of the economy.

An implication of our result is that, while agents coordinate on their best action from the individual perspective, in all cases where a private value for coordination (α) different from zero is socially inefficient, adaptive learning dynamics drive the economy towards the socially inefficient equilibrium. For example, Angeletos and Pavan (2007) show that in beauty contest economies private motives for coordination are not warranted from a social perspective, and the equilibrium that emerges under incomplete private information is inefficient.

In Section 5 we have then considered the possibility of agents' coordination through a sunspot variable, and we have shown that learning dynamics (both eductive and adaptive) rule out such possibility in the contest of the present model: even if agents use an extraneous variable to try improve their performance, over time they learn to discard such component as irrelevant for the economy, provided the conditions for learnability of the fundamental equilibrium hold.

7 Conclusions

In this paper we have considered the problem of learning and coordination for agents when their actions are strategic complements or substitutes. Under complete information about the exogenous fundamental, but uncertainty about other players' actions, both under adaptive learning and eductive learning, agents can learn the fundamental, symmetric equilibrium, but specific conditions for learnability differ. In case of eductive learning, the required condition is that agents do not value coordination too much or too little, because in both cases they would generate instability. In case of adaptive learning, instead, the requirement is only that agents do not value coordination too much. Adaptive learning therefore converges for a larger set of economies. In a setting with heterogeneous agents, moreover, we find that what matters for convergence is only the average characteristic of the population, in all cases.

Under incomplete and private information about the fundamental, we find that both under adaptive and eductive learning, conditions for learnability are the same as the ones we found under complete information: incomplete information therefore does not impact on the conditions for learnability. Interestingly, even under adaptive learning, agents' beliefs converge towards the optimal values implied by the game theoretical, strategic equilibrium: adaptive learning, therefore, leads agents to incorporate strategic considerations into their actions, without them having to engage in a process of higher order thinking. Our work therefore confirms and strengthens the result of Marcet and Sargent (1989) that adaptive learning is a powerful tool in solving the problem of beliefs coordination.

Finally, we have shown that sunspot components are not learnable by agents in this setting, and can not therefore enter in the solution under learning dynamics.

8 Appendix

An instance of the setting laid out in Section 2 is the beauty contest framework used by Morris and Shin (2002):

$$U_t = -L_t = -E_t^i \left[\eta \left(k_t^i - K_t \right)^2 + \beta \left(k_t^i - \theta_t \right)^2 + \sigma_k^2 \right]. \tag{75}$$

By solving agent's maximization problem, we obtain the optimal action

$$k_t^i = \frac{\eta}{\eta + \beta} E_t^i K_t + \frac{\beta}{\eta + \beta} E_t^i \theta_t$$

or, defining $\alpha \equiv \frac{\eta}{\eta+\beta}$,

$$k_t^i = \alpha E_t^i K_t + (1 - \alpha) E_t^i \theta_t. \tag{76}$$

Using loss function (75), the restrictions necessary for uniqueness and boundedness of equilibrium correspond to $(\eta + \beta) > 0$ and $\alpha < 1$.

References

[1] Angeletos, G.-M., Pavan, A., 2007. Efficient use of information and social value of information. Econometrica 75, 1103-1142.

[2] Angeletos, G.-M., Pavan, A., 2007. Supplement to "Efficient use of information and social value of information" (Econometrica 75, 1103-1142). Econometrica Supplementary Material.

[3] Angeletos, G.-M., Hellwig, C., Pavan, A., 2007. Dynamic global games of regime change: learning, multiplicity, and the time of attack. Econometrica 75, 711-756.

[4] Azariadis, C., 1981. Self-fulfilling prophecies. Journal of Economic Theory 25, 380-396.

[5] Beggs, A., 2009. Learning in Bayesian games with binary actions. The BE Journal of Theoretical Economics - Advances, 33.

[6] Branch, W. A., McGough, B., 2009. A New Keynesian model with heterogeneous expectations. Journal of Economic Dynamics and Control 33, 1036-1051.

[7] Carton, J., Guse, E., 2010. Replicator Dynamic Learning in Muth's Model of Price Movements. Working Papers 10-18, Department of Economics, West Virginia University.

[8] Cass, D., Shell, K., 1983. Do sunspot matter? Journal of Political Economy 91, 193-227.

[9] Crawford, V. P., 1995. Adaptive dynamics in coordination games. Econometrica 63, 103-143.

[10] Evans, G. W., Guesnerie, R., 1993. Rationalizability, strong rationality, and expectational stability. Games and Economics Behavior 5, 632-646.

[11] Evans, G.W., Honkapohja, S., 1994. On the local stability of sunspot equilibria under adaptive learning rules. Journal of Economic Theory 64, 142-161.

[12] Evans, G.W., Honkapohja, S., 2001. *Learning and Expectations in Macroeconomics.* Princeton University Press.

[13] Evans, G.W., Honkapohja, S., 2003. Expectational stability of stationary sunspot equilibria in a forward-looking model. Journal of Economic Dynamics and Control 28, 171-181.

[14] Evans, G.W., McGough, B., 2005. Stable sunspot solutions in models with predetermined variables. Journal of Economic Dynamics and Control 29, 601-625.

[15] Guesnerie, R., 1992. An exploration of eductive justifications of the rational expectations hypothesis. American Economic Review 82, 1254-78.

[16] Guesnerie, R., 1986. Stationary sunspot equilibria in an N-commodity world, Journal of Economic Theory 40, 103-128.

[17] Kreps, D., 1998. Anticipated Utility and Dynamic Choice. In D.P. Jacobs, E. Kalai, and M. Kamien, eds., Frontiers of Research in Economic Theory, Cambridge: Cambridge University Press, 242–74.

[18] Marcet, A., Sargent, T.J., 1989. Convergence of least-squares learning in environments with hidden state variables and private information. Journal of Political Economy, 1306-1322.

[19] Marimon, R., McGrattan, E., 1992. On adaptive learning in strategic games. Economics Working Paper 24, Universitat Pompeu Fabra.

[20] Morris, S., Shin, H. S., 1998. Unique equilibrium in a model of self-fulfilling currency attacks. American Economic Review 88, 587-597.

[21] Morris, S., Shin, H. S., 2001. Global games: theory and applications. Cowles Foundation Discussion Papers 1275R.

[22] Morris, S., Shin, H. S., 2002. The social value of public information. American Economic Review 92, 1521-1534.

[23] Townsend, R.M., 1983. Forecasting the forecasts of others. Journal of Political Economy 91, 546-588.

[24] Woodford, M., 1990. Learning to believe in sunspots. Econometrica 58, 277-307.