Savings for retirement under liquidity constraints: a note

Corsini, Lorenzo and Spataro, Luca

University of Pisa, Department of Economics

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Lorenzo Corsini* and Luca Spataro†

Abstract

Pension systems often entail some compulsory saving over which individuals have some degree of choice in terms of the pension plan in which to invest. Our contribution analyses whether the choice between alternative plans is affected by the presence of liquidity constraints during working life. We show that liquidity constraints obviously affect the amount saved and consumed during working life but they do not affect the decision on which pension plan to choose. In fact we prove that the analytical conditions that determine the choice between different plans are the same in the constrained and unconstrained case.

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1. Introduction

Modern pension systems are typically designed according to a multi-pillar structure. Usually, one or more of these pillars entail compulsory saving over which workers has some degree of choice in terms of the pension plan in which to invest. Given the existence of compulsory savings, agents could find it optimal to indebt themselves in order to off-set too large compulsory rates of contribution. Their saving decisions might thus be affected by the presence of incomplete financial markets that prevent them from borrowing the desired amount. In turn this implies that, when choosing their pension plan, liquidity constraints could become relevant and affect the investment choice. The aim of our work is to analyse what happens to agents' decision on pension plans when liquidity constraints are binding.

A vast literature has stressed how liquidity constraints have a direct effect on the amounts consumed and saved for retirement (for a review of this literature see Magnussen 1994) but not much has been said on how they affect the destination of those savings. Contributions from Dutta et al. (2000), Wagener (2003), Matsen and Thogersen (2004) De Menil et al. (2006) and Corsini and Spataro (2011) cover the topic of decisions on retirement for saving and on pension plans but none of them focus on the role of liquidity constraints nor examine in details the emergence of corner solutions.

* Corresponding author. Dipartimento di Scienze Economiche, Università di Pisa. Via Ridolfi 10, 56124 Pisa, Italy. Tel: +39 0502216220 Fax: +39 0502216384. Email: lcorsini@ec.unipi.it.
† Dipartimento di Scienze Economiche, Università di Pisa and CHILD. Email: l.spataro@ec.unipi.it.

Our contribution extends a model of optimal choice on pension plans developed by Corsini and Spataro (2011), and differently from it, focuses on (and allows for) the corner solutions that emerge in the presence of liquidity constraints and assess whether these constraints affect individual decisions.

Our results show that while liquidity constraints obviously condition the decisions of individuals in terms of the amounts saved and consumed, they do not affect the decisions in terms of the pension plan chosen. The work is organized as follows: section 2 develops the basic model with complete financial market; section 3 tackles the central issue of our contribution, introducing incomplete financial markets and exploring in details the role of liquidity constraints and section 4 concludes.

2. Saving decisions under complete financial markets

We imagine an economy where agents live two periods: in the first period they work receiving a wage *w* and consuming part of their income; in the second period they retire, consuming what they have saved. Saving is partly voluntary, cumulated at the risk-free rate r_S, and partly compulsory in that the pension system forces individuals to save a fixed contribution rate γ in a pension plan of their choice. For sake of simplicity we imagine that only two plans exist: (*i*) a safe plan *S* with the risk-free rate of return r_S and (*ii*) a risky plan *R* whose returns are normally distributed with mean r_R and variance σ_R^2; the analysis can be extended to more options on retirement savings, without qualitatively changing the results. Agents choose how much to consume and to save and which pension plan to adopt. Basically, individuals compute their expected indirect utility under the two plans and then choose the one bestowing the highest indirect utility. Thus we first compute the expected indirect utility for each plan: this is done solving a maximization problem with respect to first period consumption (and saving). In the presence of complete financial markets an individual under the generic plan *i* faces the following problem:

$$1) \quad \begin{cases} \max\limits_{c_1} E[U(c_1,c_2)] \\ s.t.\ c_2 \sim N\left[(w-c_1)(1+r_S)+\gamma w(r_i-r_S),\gamma^2 w^2 \sigma_i^{\ 2}\right] \end{cases}$$

where $E[U(c_1,c_2)]$ is the expected lifetime utility function that depends on consumption in the two periods (c_1 and c_2 respectively). The constraint in equation (1) represents the budget constraint: second period consumption is given only by the returns from compulsory (i.e. pension) and voluntary saving done in the first period. In the case of the *S* plan we have assumed that σ_S^2 is equal to zero and so this constraint is given by $c_2=(w-c_1)(1+r_S)$. The important point in the latter case is that, under complete markets, individuals can borrow at the rate r_S up to the point of completely off-setting the compulsory saving so that, in the case of the *S* plan, γ does not even enter the budget constraint.

A closed form solution can be obtained assuming the following utility function

$$2) \quad U=-e^{-ac_1}-\rho e^{-ac_2}$$

where ρ is the rate of time preference and *a* is the Arrow-Pratt measure of absolute risk aversion.

Following Makarov and Schornick (2010) we assume that *a* depends on wage with $a=k\cdot w^{-\alpha}$, where α is a parameter representing the elasticity of risk aversion to wage and *k* is a positive scale factor. This assumption allows us to obtain absolute risk aversion that is decreasing in income, a property that is usually considered the most realistic one.

Under the above utility function we can solve[1] problem (1) for both plans and obtain the following solution in terms of optimal consumption $c_{1,i}^{*}$ and indirect expected utility $E\left(U_i^*\right)$:

3) $$c_{1,i}^{*} = \left[(1+d_i)wx - \frac{\log \rho x}{a}\right]\frac{1}{(1+x)}$$

4) $$E\left(U_i^*\right) = -(1+x)xe^{-k(1+d_i)w^{1-\alpha}\frac{x}{1+x}}\rho^{\frac{1}{1+x}}$$

where $x = 1 + r_S$, $d_i = \gamma \dfrac{r_i - r_S - \gamma k w^{1-\alpha}\sigma_i^{\,2}/2}{x}$ and clearly $d_S = 0$.

Individuals choose plan *R (S)* if and only if[2] $E\left(U_R^*\right) > (\leq) E\left(U_S^*\right)$ and according to equation (4) this inequality is verified if and only if

5) $$d_R = \gamma \frac{r_R - r_S - \gamma \cdot k \cdot w^{1-\alpha} \cdot \sigma_R^{\,2}/2}{x} > 0$$

thus the sign of d_R completely determines agents decisions in terms of pension plans. To better understand the above condition we plot[3] in Figure 1 the pattern of d_R with respect to w for different values of α.

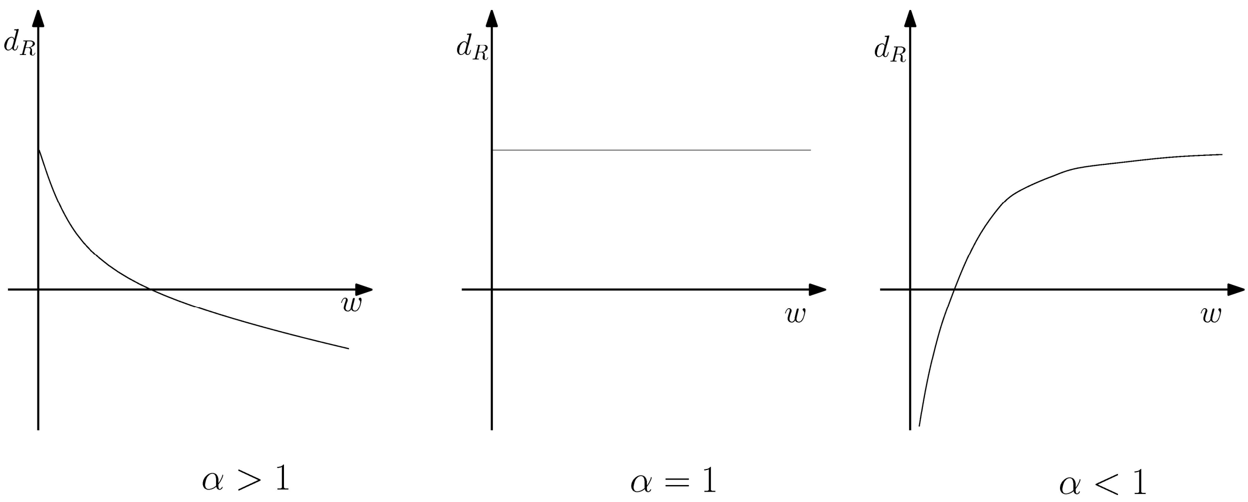


Figure 1

[1] The problem is solved using the property whereby, for any given stochastic variable z_j distributed normally with mean z and variance σ_z^2 we have $E(e^{-az_j}) = e^{-a\left(z - a\sigma_z^2/2\right)}$. See chapter 11 in Varian (1992) for details.

[2] By convention, we assume that for $E(U_R^*) = E(U_S^*)$ individuals choose plan *S*.

[3] We draw the figure for $r_R - r_S > 0$, as otherwise the problem becomes trivial and all (non-risk prone) individuals would simply choose *S*.

As it is clear from Figure 1, individuals choose their plans, among over things, on the base of their wage and the exact relationship between choice and income depends on the value of the elasticity of risk aversion to wage. However, this result is obtained in the absence of liquidity constraints: in the next section we explore the case of liquidity constraints.

3. The role of liquidity constraints

If individuals, due to incomplete financial markets, cannot borrow during their working period the problem (1) can be restated as

$$6)\quad \begin{cases} \max\limits_{c_1} E\left[U\left(c_1, c_2\right)\right] \\ s.t.\ c_2 \sim N\left[\left(w-c_1\right)\left(1+r_S\right)+\gamma w\left(r_i - r_S\right), \gamma^2 w^2 \sigma_i^{\,2}\right] \\ s.t.\ c_1 \leq \left(1-\gamma\right)\cdot w \end{cases}$$

where the second constraint exactly represents the non-borrowing condition and implies that consumption in the first period cannot exceed income net of the compulsory contribution. The addition of the second constraint implies that equations (3) and (4) represent now the inner solution of the problem. In particular (6) has an inner solution for

$$7)\quad \gamma \leq \left(1+\frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} - d_i \frac{x}{1+x}.$$

When the above condition does not hold constraints become binding. Interestingly, condition (7) depends on the plan chosen so that, in principle, constraints might be binding under a plan but not under the other. In particular, for plan *S*, we have $d_s = 0$ and equation (7) becomes:

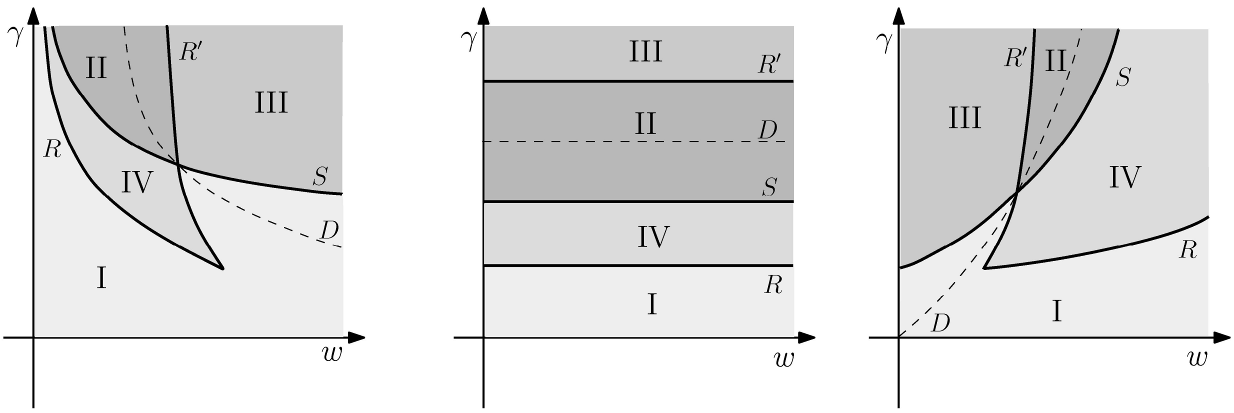
$$8)\quad \gamma \leq \left(1+\frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x}$$

so that when the compulsory contribution rate γ increases the above condition becomes more stringent. For plan *R* we can insert (5) in equation (7) and we obtain the following condition for inner solutions:

$$9)\quad kw^{1-\alpha}\sigma^2\gamma^2 - \left(2+r_R\right)\gamma + \left(1+\frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} \geq 0.$$

The above is a second order equation in γ and its roots are $\gamma_j = \frac{2+r_R \pm \sqrt{\left(2+r_R\right)^2 - 4\sigma^2\left(kw^{1-\alpha}+\log \rho x\right)}}{2w^{1-\alpha}\sigma^2}$.

We can use conditions (8) and (9) to define the values of wage and compulsory contribution for which liquidity constraints are binding. In particular, we draw in Figure 2 the couples $\left(w, \gamma\right)$ for which optimal consumption in period one is exactly equal to income. We depict three possible cases[4] depending on the value of the parameter α. In all cases, curve S represents the safe plan and is obtained from condition (8), such that above curve S, liquidity constraints are binding under the plan *S*. Curves R and R' represent the risky plan and are obtained from condition (9) so that within these two curves liquidity constraints are binding under plan *R*. In Figure 2, we also draw curve D that represents the couples (w,γ) for which $d_R = 0$: above *D* we have $d_R < 0$ while below it we have $d_R > 0$.



$\alpha < 1$ $\alpha = 1$ $\alpha > 1$

Figure 2

Summarizing, Figure 2 shows that curves S, R and R' define, for different values of α, four regions: in region I constraints are not binding for either plans; in area II, constraints are binding under both plans; in region III constraints are binding under plan *S* but not under plan *R* and, in region IV constraints are binding under plan *R* but not under plan *S*. Note that region III completely lies above curve D and thus within this region we have $d_R > 0$, while region IV fully lies below curve D and thus we have $d_R > 0$: we will show below that this result is particularly important for our analysis.

From an analytical point of view, whenever we are outside region I, condition (7) does not hold for at least a plan and, therefore, optimal consumption and indirect utilities are no longer described by equations (3) and (4) but instead the following corner solutions becomes emerge (the *C* index denotes the solution when constraints are binding):

10) $c_{1,i}^{C} = (1-\gamma)w \;\; \forall i$

11) $E\left(U_S^C\right) = -e^{-k(1-\gamma)w^{1-\alpha}} - \rho^{\frac{1}{1+x}} e^{-k\gamma x w^{1-\alpha}}$

12) $E\left(U_R^C\right) = -e^{-k(1-\gamma)w^{1-\alpha}} - \rho^{\frac{1}{1+x}} e^{-kw^{1-\alpha}x(d_R+\gamma)}$.

[4] Figure 2 is drawn for illustrative purposes only as the exact shape of the curves depends on the values of several parameters. In any case, while the shapes may qualitatively change, all properties we unveil in our analysis hold true for any value of the parameters.

The above corner solutions[5] show that, as expected and in line with previous literature, the amount of resources consumed and saved is affected by liquidity constraints. However, it is still arguable whether such constraints affect the decision on the destination of the compulsory saving, a point that is the heart of our analysis. To assess this point we present now five lemmas that characterize individuals' decisions on pension plans when constraints are potentially binding. We will then use the lemmas to formulate a proposition that fully describes individuals' decision on pension plans.

LEMMA 1

If constrains are binding under both the *S* and *R* plans, then *R* (*S*) is chosen if and only if $d_R > (\leq) 0$.

PROOF

If constraints are binding under both plans, then the plan R is chosen if and only if $E\left(U_R^C\right) - E\left(U_S^C\right) > 0$. *We can obtain the exact equation for this difference starting from eqs. (11) and (12) and, going through computations, we have:* $E\left(U_R^C\right) - E\left(U_S^C\right) = \rho^{\frac{1}{1+x}}\left(1 - e^{-kw^{1-\alpha}xd_R}\right)e^{-kw^{1-\alpha}\gamma x}$. *The latter is greater than zero if and only if* $d_R>0$. ■

Lemma 1 tells us that if constraints are binding under both plans, then individuals' choice is still solely and univocally determined by the value of d_R.

LEMMA 2

If constraints are binding under the *S* plan but not under the *R* plan, then we necessarily have $d_R<0$.

PROOF

If constrains are binding under the S but not under the R plan then condition (7) must be false for i=S but true for i=R. Given condition (7) this happens only if $\left(1 + \frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} > \left(1 + \frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} - d_R \frac{x}{1+x}$ *which can only be true if* $d_R<0$. ■

LEMMA 3

If constraints are binding under the *R* plan but not under the *S* plan, then we necessarily have $d_R > 0$.

PROOF

If constrains are binding under the R but not under the S plan then condition (7) must be true for i=S but false for i=R. Given condition (7) this happens only if $\left(1 + \frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} < \left(1 + \frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x} - d_R \frac{x}{1+x}$ *which can only be true if* $d_R>0$. ■

[5] To obtain (12) consider that $E(U_R^C) = -e^{-k(1-\gamma)w^{1-\alpha}} - \rho^{\frac{1}{1+x}} e^{-kw^{-\alpha}\left[\gamma w(1+r_R) - kw^{-\alpha}(w\gamma\sigma_R)^2/2\right]}$ and $\gamma w\left(1+r_R\right) - kw^{-\alpha}\left(w\gamma\sigma_R\right)^2/2 = wx\left(d_R+\gamma\right)$.

Lemma 2 and Lemma 3 allow us to state that if in the absence of constraints an individual prefers a given plan then, when constraints are present, it is not possible for the constraints to be binding in the non-chosen plan and not in the chosen one. This property was also illustrated by Figure 2 where, in fact, the region where constraints resulted binding under *S* but and not under *R* (region III in the figure) completely falls in the zone where $d_R < 0$ and an analogue property is true in region IV. There is an economic interpretation to this result: individuals choose the plan that makes them "richer" and therefore, due to a wealth effect, their (expected) consumption should be higher in both periods under the chosen plan than under the non-chosen plan. Therefore it is not possible that individuals find their consumption to be effectively constrained under the non-chosen plan but not under the chosen plan.

LEMMA 4

If constrains are binding under the *S* plan but not under the *R* plan, then $d_R \le 0$ implies the choice of *S*.

PROOF

Suppose that constraints are binding under plan S but not under plan R, then the indirect expected utilities under the two plans are, respectively,

$$13)\quad E\left(U_S^C\right) = -(1+x)xe^{-k(1-\gamma)w^{1-\alpha}} - \rho^{\frac{1}{1+x}}e^{-k\gamma xw^{1-\alpha}}$$

and

$$14)\quad E\left(U_R^*\right) = -(1+x)xe^{-kw^{-\alpha}c_{1,R}^*} - \rho^{\frac{1}{1+x}}e^{-kw^{-\alpha}\left\{\left[w(1-\gamma)-c_{1,R}^*\right]x+\gamma w\left(1+r_R-\gamma kw^{1-\alpha}\sigma^2/2\right)\right\}}.$$

We want to prove that $d_R \le 0$ implies $E\left(U_S^C\right) > E\left(U_R^*\right)$. To do that we show that for $d_R \le 0$ both the first and second terms in (13) are larger than their counterparts in (14). We start comparing the first term $E\left(U_S^C\right)$ as given by (13) with the first term of $E\left(U_R^*\right)$ as given by (14): since by assumption the constraint is not binding under scheme R, it must be $c_{1,R}^* < (1-\gamma)w$ and thus we strictly have $-(1+x)xe^{-k(1-\gamma)w^{1-\alpha}} > -(1+x)xe^{-kw^{-\alpha}c_{1,R}^*}$, that is, the first term of $E\left(U_S^C\right)$ is always larger than the first term of $E\left(U_R^*\right)$.

We now compare the second term of $E\left(U_S^C\right)$, as given in equation (13), with the second term of $E\left(U_R^*\right)$, as given in equation (14). If we compute the difference D of the second terms we have

$$15)\quad D = -\rho^{\frac{1}{1+x}}e^{-k\gamma xw^{1-\alpha}} - \rho^{\frac{1}{1+x}}e^{-kw^{-\alpha}\left\{\left[w(1-\gamma)-c_{1,R}^*\right]x+\gamma w\left(1+r_R-\gamma kw^{1-\alpha}\sigma^2/2\right)\right\}}$$

and inserting equation (3) in the above and rearranging we obtain

$$16)\quad D = -\rho^{\frac{1}{1+x}}\left[e^{-k\gamma xw^{1-\alpha}} - e^{-kw^{1-\alpha}\frac{x}{1+x}\left(1+d_R+\frac{\log \rho x}{kw^{1-\alpha}}\right)}\right].$$

The above is positive for $\gamma > \frac{1}{1+x}\left(1 + d_R + \frac{\log \rho x}{kw^{1-\alpha}}\right)$*. Since by assumption constraints are binding under plan S, we know from equation (9) that* $\gamma > \left(1 + \frac{\log \rho x}{kw^{1-\alpha}}\right)\frac{1}{1+x}$ *and therefore, whenever* $d_R \leq 0$ *we have that* $\gamma > \frac{1}{1+x}\left(1 + d_R + \frac{\log \rho x}{kw^{1-\alpha}}\right)$*.Thus* $d_R \leq 0$ *implies that D>0, i.e. the second term of* $E\left(U_S^C\right)$ *is larger than the second term of* $E\left(U_R^*\right)$*.*

Since the first term of $E\left(U_S^C\right)$ *is always larger than the first term of* $E\left(U_R^*\right)$ *and since* $d_R \leq 0$ *implies that second term of* $E\left(U_S^C\right)$ *is larger than the second term of* $E\left(U_R^*\right)$*, it follows that* $d_R \leq 0$ *implies that* $E\left(U_S^C\right) \geq E\left(U_R^*\right)$*.*■

LEMMA 5

If constraints are binding under the *R* plan but not under the *S* plan, then $d_R>0$ implies the choice of *R*.

PROOF

The proof follows the same arguments used to prove Lemma 2.

Lemma 4 and Lemma 5 are extremely important because they tell us that if an agent prefers a given plan when constraints are not binding, then that plan it is still preferred even if the constraints are binding only under the chosen plan.

We can use the five lemmas to formulate the following proposition

PROPOSITION 1

When individuals face liquidity constraints, they chose plan *R* (*S*) if and only if $d_R > (\leq) 0$.

PROOF

Once liquidity constraints are present, four possible cases may arise: (i) constraints are not binding under either plan, (ii) constraints are binding under both plans, (iii) constraints are binding under plan S but not under plan R and (iv), constraints are binding under plan R but not under plan S.
In case (i) we are back to the inner solutions and thus individuals choose the plan R (S) if and only if $d_R > (\leq) 0$*. In case (ii) we know from lemma 1 that individuals chose plan R (S) if and only if* $d_R > (\leq) 0$*. In case (iii) we know from lemma 2 that* d_R *is necessarily negative and thus, from lemma 4, plan S is chosen. In case (iv) we know from lemma 3 that* d_R *is necessarily positive and thus, from lemma 5, plan R is chosen. Therefore in all the possible four cases the value* d_R *determines univocally the choice of the plan and plan R (S) is chosen if and only if* $d_R > (\leq) 0$*.*■

Proposition 1 shows that the introduction of liquidity constraints does not affect the decision on which retirement plan to invest into.

4. Conclusions

Our analysis explores what happens to saving for retirement when liquidity constraints become binding. Our results show that liquidity constraints still affect the decisions on how much to consume and to save for retirement but they do not affect the destination of compulsory savings. In fact we prove that the very conditions for which constraints are binding also guarantee that the choice on pension plans remains the same as in the unconstrained case.

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