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May 2012

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May 1, 2012

Abstract

We formally define a concept of functional cointegration linking the dynamics of two time series via a functional coefficient. This is achieved through the use of a concept of summability as an alternative to I(1)'ness which is no longer suitable under nonlinear dynamics. We subsequently introduce a nonparametric approach for estimating the unknown functional coefficients. Our method is based on a piecewise local least squares principle and is computationally simple to implement. We establish its consistency properties and evaluate its performance in finite samples.

Keywords: Functional Coefficients, Unit Roots, Cointegration, Piecewise Local Linear Estimation.

JEL: C22, C50.

[1]We wish to thank seminar participants at the 2012 SNDE conference in Istanbul for very useful comments and suggestions. The second author also wishes to thank the ESRC for partial financial support. Address for Correspondence: Jean-Yves Pitarakis, University of Southampton, School of Social Sciences, Division of Economics. Highfield, Southampton, SO17 1BJ, United-Kingdom.

1 Introduction

A vast body of research in the recent time series econometrics literature has concentrated on developing methods of capturing nonlinear regime specific behaviour in the joint dynamics linking economic and financial variables. An important complication that arises when moving from simple linear structures with constant coefficients to such models with nonlinear dynamics has to do with the open ended nature of the functional forms one may want to adopt for describing the changing nature of the model parameters and underlying moments. Popular parametric specifications include the well known threshold models, Markov switching models, models with structural breaks among numerous others. Although such models can allow researchers to capture rich and economically meaningful nonlinearities the ad-hoc nature of the functional forms may also be seen as problematic. An alternative to having to take a stand on a particular functional form is to instead allow the changing coefficients to be described by some unknown function to be estimated from the data as for instance in $y = f(q)x + e$. Such semiparametric specifications are commonly referred to as varying or functional coefficient models and were introduced in the early work of Cleveland, Grosse and Shyu (1991), Hastie and Tibshirani (1993), Chen and Tsay (1993), Fan and Zhang (1999) amongst numerous others (see also Fan and Yao (2003) and references therein). An important motivation underlying this class of models is their ability to capture rich dynamics in a flexible way while at the same time avoiding the curse of dimensionality characterising fully nonparametric specifications.

Our initial objective in this paper is to formally define a novel concept of functional cointegration linking two highly persistent variables via functional coefficients. Our framework is analogous to the well known linear cointegration property linking I(1) variables except that in the present nonlinear framework I(1)'ness is no longer suitable for describing the stochastic properties of our variables. Our work also falls within the bounds of the very recent literature on nonlinear cointegration tackled from a purely nonparametric point of view (Karlsten, Myklebust and Tjostheim (2007), Wang and Phillips (2009), Kasparis and Phillips (2009) amongst others). Note that the idea of a nonlinear long run equilibrium relationship (attractor) was also put forward in the early work of Granger and Hallman (1989), Breitung (2001), Saikkonen and Choi (2004) amongst others.

The most common way of estimating the unknown functions of such varying coefficient models is through kernel smoothing and local polynomial techniques. These typically reduce to a weighted least squares type of objective function with the weights dictated by some chosen kernel function. Our subsequent objective in this paper is to propose an alternative estimation approach based on a piecewise linear least squares principle and to obtain its properties within our nonstandard context that allows for the presence of a unit root variable as in the recent work of Juhl (2009), Xiao (2009) and Cai, Li and Park (2009). Our method is different from kernel smoothing based methods, does not generally require the differentiability of the density of q and is shown to have good finite sample properties.

The plan of the paper is as follows. Section 2 introduces and motivates our model and formally defines the concept of functional cointegration. Section 3 describes our estimation methodology and derives its asymptotic properties. Section 4 explores its performance and finite sample. Section 5 concludes. All proofs are relegated to the appendix.

2 The Model and Motivations

We consider the following functional coefficient regression model

$$y_t = f_0(q_{t-d}) + f_1(q_{t-d})x_t + u_t \tag{1}$$
$$x_t = x_{t-1} + v_t \tag{2}$$

where u_t and v_t are stationary disturbance terms and $f_0(q_{t-d})$ and $f_1(q_{t-d})$ are unknown functions of the stationary scalar random variable q_{t-d} while x_t is taken as an I(1) process throughout. The particular choice of d is not essential for our analysis and will be set at $d = 1$ throughout. The generality of (1)-(2) can be seen by noting that it can easily be specialised to well known parametric specifications such as threshold effects as in $f_i(q_{t-1}) = \beta_{i1} I(q_{t-1} \leq \gamma) + \beta_{i2} I(q_{t-1} > \gamma)$ (see Gonzalo and Pitarakis (2006)) or ESTAR/LSTAR type of variants such as $f_i(q_{t-1}) = [1 + e^{-\gamma_i(q_{t-1}-c_i)}]^{-1}$ amongst others.

Before proceeding with the estimation of the unknown functions $f_0(q)$ and $f_1(q)$ it is important to motivate our model in (1)-(2) as a long run equilibrium relationship. As it stands (1) cannot be interpreted as a stationary nonlinear combination of I(1) variables in a traditional sense. Indeed, it is easy to see that although x_t is a standard $I(1)$ process, y_t can no longer be viewed as I(1) as it would have been the case for instance if $f_0(q)$ and $f_1(q)$ were constants. Differently put, the concept of integratedness of order 0 or 1 is mainly relevant within a linear framework while not being very helpful when dealing with nonlinear transformations of variables. In the context of our model in (1) for instance it is straightforward to see that first differencing y_t will not result in a stationary process because of the time varying nature of the functional coefficients.

To gain further insight into this phenomenon consider a simplified version of (1) which we compactly write as $y_t = f_t x_t + u_t$ and with f_t denoting some stationary process. It is now clear that $\Delta y_t = f_t \Delta x_t + x_{t-1}\Delta f_t + \Delta u_t$ making it difficult to view Δy_t as a stationary process due to the presence of the term $x_{t-1}\Delta f_t$ which has a variance that grows with t. Instead, cointegration in the context of (1) is understood in the sense that although y_t and x_t have variances that grow with t, the *functional combination* given by u_t is stationary.

Because of these conceptual difficulties and for the purpose of motivating (1)-(2) we propose to use the concept of *Summability* as an alternative to the concept of I(1)'ness and which was proposed in Gonzalo and Pitarakis (2006) and more recently refined and formalised in Berenguer-Rico (2010) and Berenguer-Rico and Gonzalo (2011). A time series y_t is said to be summable of order δ, symbolically represented as $S_y(\delta)$, if the sum $S_y = \sum_{t=1}^{T}(y_t - d_t)$ is such that $S_y/T^{\frac{1}{2}+\delta} = O_p(1)$ as $T \to \infty$ and where d_t denotes a deterministic sequence. Note that in the context of this definition, a process that is I(d) can be referred to as $S_y(d)$ and the functional process introduced in (1) is clearly $S_y(1)$ as discussed further below. Using this concept of summability of order δ we can now provide a formal definition of the concept of functional cointegration as follows

Definition (Functional Cointegration): Let y_t and x_t be $S_y(\delta_1)$ and $S_y(\delta_2)$ respectively. They are functionally cointegrated if there exists a functional combination $(1, -f_1(q_{t-1}))$ such that $z_t = y_t - f_1(q_{t-1})x_t$ is $S_y(\delta_0)$ with $\delta_0 < \min(\delta_1, \delta_2)$.

Given the formal definition of functional cointegration presented above it is now clear that within our specification in (1), y_t and x_t are functionally cointegrated with $\delta_0 = 0$ and $\delta_1 = \delta_2 = 1$. This follows from the fact that taking u_t and q_t to be stationary processes ensures that $\sum y_t/T^{3/2} = O_p(1)$ while u_t is such that $\sum u_t/\sqrt{T} = O_p(1)$ as clarified further below. It is also worth highlighting the fact that within our specification in (1) we have $z_t = f_0(q_{t-1}) + u_t$ which is of the same order of magnitude as u_t since under our assumptions we will have $\sum f_0(q_{t-1})/T \xrightarrow{p} E[f_0(q_{t-1})]$ and $\sum f_0(q_{t-1})/T^{3/2} = o_p(1)$.

Having provided a rationale for our specification in (1)-(2) we next concentrate on obtaining reliable estimates of the unknown functional coefficients $f_0(q)$ and $f_1(q)$ and exploring their consistency properties. For this purpose we introduce a piecewise linear estimation approach as developed in Banerjee (1994, 2007) in the context of average derivative estimation and adapt it to the nonstationary functional coefficient setting given by (1)-(2). This will also allow us to compare our approach with the more commonly used kernel smoothing approaches.

3 Piecewise Local Linear Estimation

We now concentrate on the estimation of the unknown functional coefficients linking y_t and x_t. We propose to do that through a piecewise local linear procedure recently used in Banerjee (1994, 2007) in the context of average derivative estimation. We partition the support of q_{t-1} into k disjoint bins of equal length $|H_r| = h$, $r = 1, \ldots, k$ (note that q_{t-1} is not sorted in any particular order). For every q_{t-1} falling in the r^{th} bin we then fit the least squares line $y_t = \beta_{0r} + \beta_{1r}x_t + u_t$ connecting the $\{y_t, x_t\}$ data within the bin. More specifically, letting $\tilde{x}_t = (1, x_t)'$ and $I_r(q_{t-1}) \equiv I(q_{t-1} \in H_r) = 1$ if q_{t-1} falls within the r^{th} bin and zero otherwise and $\beta_r = (\beta_{0r}, \beta_{1r})'$ we write

$$\hat{\beta}_r = S_{xx}^{(r)-1} S_{xy}^{(r)} \tag{3}$$

where $S_{xx}^{(r)} = \sum_{t=1}^T \tilde{x}_t \tilde{x}_t' I_{rt-1}$ and $S_{xy}^{(r)} = \sum_{t=1}^T \tilde{x}_t y_t I_{rt-1}$ with $I_{rt-1} \equiv I_r(q_{t-1})$. Note that $\hat{\beta}_r$ provides the least squares estimators of the intercept and slope parameters characterising the linear regression line within each bin. Interestingly, in a series of recent papers, Senturk and Mueller (2005, 2006) also used an estimation technique similar to what we consider below in an unobserved variable setting under iid'ness and in which observed and unobserved variables are linked through functional coefficients.

Once the $\hat{\beta}_r$'s have been estimated within each bin, our estimator of the functional coefficients is then given by

$$(\hat{f}_0(q), \hat{f}_1(q)) = \left(\sum_{r=1}^k \hat{\beta}_{0r} I_r(q), \sum_{r=1}^k \hat{\beta}_{1r} I_r(q) \right) \tag{4}$$

with $I_r(q) = I(q \in H_r)$.

Having introduced the mechanics behind our estimator our main goal is to establish its consistency. Since in this nonstationary setting consistency typically holds under minimally restrictive assumptions that can accomodate serial correlation and/or endogeneity we proceed and operate under a broad set of assumptions. The following baseline assumptions will be maintained throughout the entire paper where we let $q_t = \mu + u_{qt}$.

Assumptions A. (i) $w_t = \{u_t, v_t, u_{qt}\}$ is such that $E[w_t] = 0$, $E||w_0||^{\rho+\epsilon} < \infty$ for some $\rho > 2$ and the sequence $\{w_t\}$ is strictly stationary, strong mixing with mixing coefficients α_m such that $\sum \alpha_m^{1-2/\rho} < \infty$. (ii) The density of q denoted $g_q(q)$ is strictly positive and satisfies $\sup_q g_q(q) < \underline{c} < \infty$ and $\inf_q g_q(q) > \underline{c} > 0$. (iii) $g_q(q)$ has compact support. (iv) The functional coefficients are twice continuously differentiable in q.

Assumptions A above impose a very standard set of restrictions on the dynamics driving (1)-(2) leaving all random disturbances to be flexible enough to display rich dynamics such as ARMA process. Their joint interactions is also left to be very flexible allowing u_t and v_t to be correlated at all leads and lags and similarly for the interactions bwteen q_t and the remaining variables. It is naturally understood that the associated long run variances of those processes are positive. In this sense the above setting is at least as flexible as the well known linear cointegration model formulated in triangular form allowing for both serial correlation and endogeneity. Note also that the strictly stationary and strong mixing nature of u_{qt} also implies that the indicator function series I_{rt} are strictly stationary and strong mixing with the same mixing coefficients. Assumption A(ii) is concerned with the density of q_t and is required so as to ensure that there are observations in each bin. Since our estimation methodology requires fitting a least squares line within each bin of length $|H_r| = h$ it is understood throughout this paper that for estimability purposes there are enough observations falling within each bin. Note however that we do not impose any smoothness conditions on the density of q. This is in contrast with other methods that have been used in the literature (e.g. kernel smoothing via local linear regression). Assumption (iii) requires the support of q to be compact. More specifically we require q to be bounded from below and above. In practice and throughout our simulations we form the support of q_t by taking the range of a top (say 0.9) and bottom (say 0.1) quantile. Finally, the differentiability of the $f_i(q)'s$ will allow us to use their local Taylor expansions at a point q within each bin.

We are now in a position to state our main result which establishes the consistency of our piecewise local linear estimator. It is summarised in the following Proposition.

Proposition 1. *Under Assumptions A and B, as $T \to \infty$ and if $Th \to \infty$ and $Th^{3/2} \to 0$ as $h \to 0$ we have $(\hat{f}_0(q) - f_0(q)) = O_p(1/\sqrt{Th})$ and $(\hat{f}_1(q) - f_1(q)) = O_p(1/T\sqrt{h})$.*

The above proposition has focused on the consistency of our proposed estimator under a setting that allows a great degree of generality in the dynamics linking (1) and (2). We note that the slope function converges at a faster rate than the intercept function (i.e. $T\sqrt{h}$ versus $\sqrt{Th}$). This is directly analogous to the standard linear cointegration setting in which the slope converges at rate T while the intercept converges at the slower $\sqrt{T}$ rate. Our convergence rates conform with related studies that explored the use of functional coefficients in unit root settings using kernel smoothing techniques (Juhl (2006), Xiao (2009)).

4 Finite Sample Analysis

Our goal here is to illustrate the behaviour of our piecewise local linear estimators via a series of simulation experiments. We will consider five functional forms including one that violates our differentiability assumption in A(iv). The stochastic structure of our DGPs will be sufficiently general to allow for the presence of endogeneity and a rich dynamic structure for the errors driving x_t. Specifically, our DGP is given by

$$\begin{aligned}
y_t &= f_0(q_{t-1}) + f_1(q_{t-1})\; x_t + u_t \\
x_t &= x_{t-1} + v_t \\
u_t &= \rho_u u_{t-1} + eu_t \\
v_t &= \rho_v v_{t-1} + ev_t \\
q_t &= \rho_q q_{t-1} + eq_{qt}. \qquad (5)
\end{aligned}$$

Letting $z_t = (eu_t, ev_t, eq_t)'$ and $\mathbf{\Sigma}_z = E[z_t z_t']$, we use

$$\mathbf{\Sigma}_z = \begin{pmatrix} 1 & \sigma_{uv} & \sigma_{uq} \\ \sigma_{uv} & 1 & \sigma_{vq} \\ \sigma_{uq} & \sigma_{qv} & 1 \end{pmatrix}$$

for the covariance structure of the random disturbances. Our chosen covariance matrix parameterisation allows q_t to be contemporaneously correlated with the shocks to y_t and throughout all our experiments we set $\{\sigma_{uv}, \sigma_{uq}, \sigma_{vq}\} = \{-0.5, 0.5, 0.5\}$.

The range of possible functional coefficients we consider for either the intercept or the slope functions is given by

$$\begin{aligned} A : f(q) &= 0.3 - 0.5\ e^{-1.25q^2} \\ B : f(q) &= \frac{0.5}{1 + e^{-4q}} - 0.75 \\ C : f(q) &= 0.25\ e^{-q^2} \\ D : f(q) &= 1 + 2(q > 0.5) \\ E : f(q) &= (1.5 + 0.6q)\ e^{-0.5(0.5q-1.5)^2} \end{aligned} \tag{6}$$

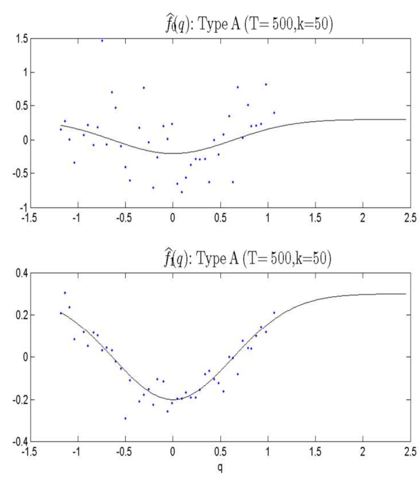
thus covering a very wide range of shapes including for illustration purposes a threshold type function which violates our differentiability assumption. Following standard practice in the functional coefficient literature, the quality of our estimators will be assessed via the computation of the root MSE defined as follows

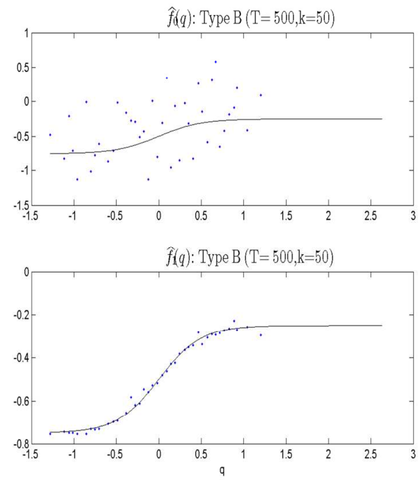
$$RMSE_i = \sqrt{\frac{1}{k}\sum_{r=1}^{k}(\hat{f}_i(q_r) - f_i(q_r))^2} \quad i = 0, 1 \tag{7}$$

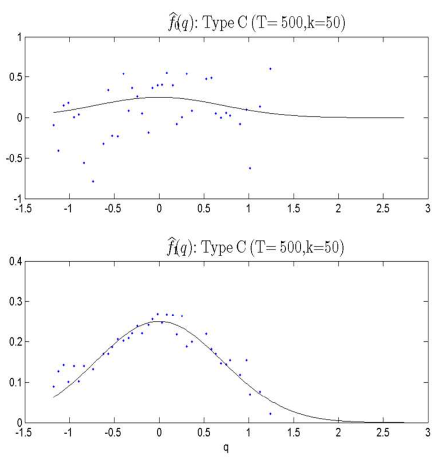
for some q_r falling within each bin, say the midpoint (note that since we operate under piecewise linearity the location at which we evaluate the function within the bin does not affect its value). All our experiments use $NID(0, 1)$ variables for the random disturbances z_t while setting $\{\rho_u, \rho_v, \rho_q\} = \{0.25, 0.25, 0.25\}$ thus allowing both serial correlation and endogeneity.

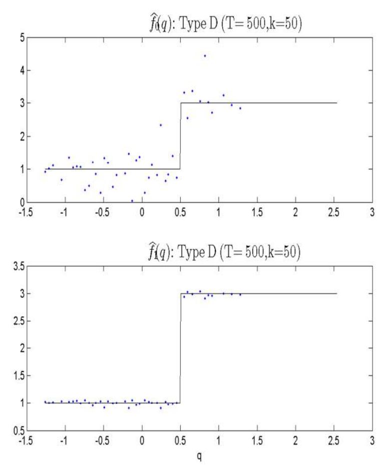
Before proceeding with our simulations we give a snapshot of the performance of our estimators by displaying plots of single realisation based $\hat{f}_i(q)'s$ for $i = 0, 1$ together with their true counterparts. Figure 1 below presents the plots of the functions corresponding to our formulations in A-E across samples of size T=500 and T=2000. The corresponding choice for the number of bins was k=50 and k=100.

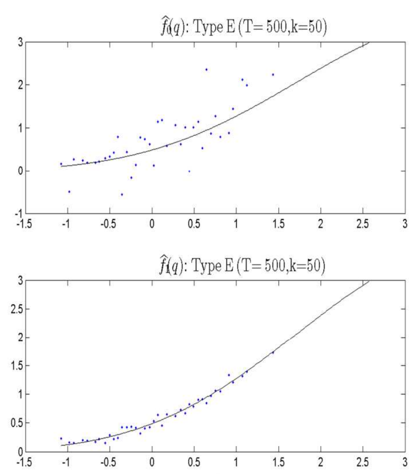
Figure 1: Piecewise Local Linear Estimation under T=500 and T=2000

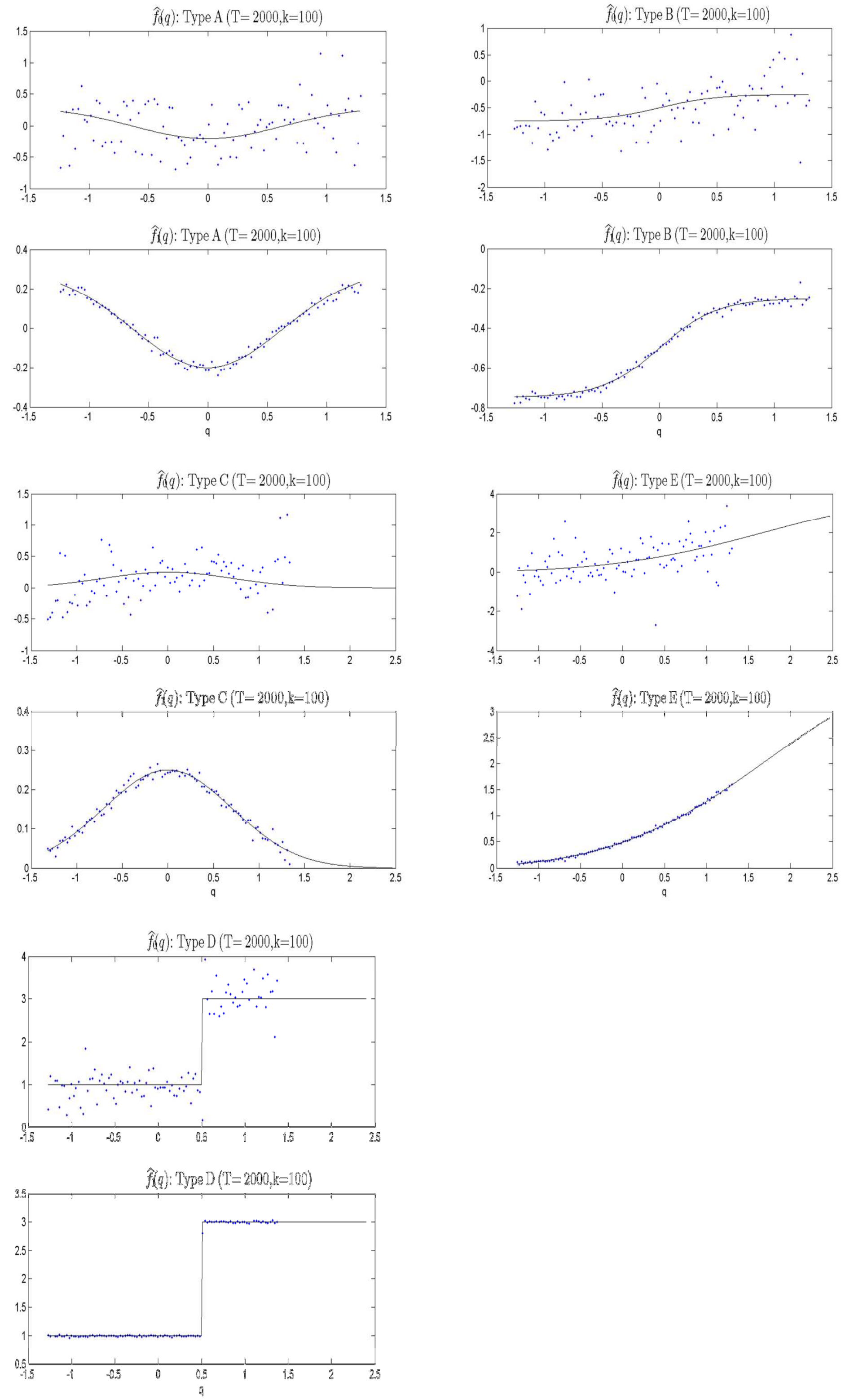

$\hat{f}(q)$: Type A (T= 500,k=50)
$\hat{f}(q)$: Type A (T= 500,k=50)
q


$\hat{f}(q)$: Type B (T= 500,k=50)
$\hat{f}(q)$: Type B (T= 500,k=50)
q


$\hat{f}(q)$: Type C (T= 500,k=50)
$\hat{f}(q)$: Type C (T= 500,k=50)
q


$\hat{f}(q)$: Type D (T= 500,k=50)
$\hat{f}(q)$: Type D (T= 500,k=50)
q


$\hat{f}(q)$: Type E (T= 500,k=50)
$\hat{f}(q)$: Type E (T= 500,k=50)
q



f̂0(q): Type A (T= 2000,k=100)
f̂0(q): Type B (T= 2000,k=100)
f̂1(q): Type A (T= 2000,k=100)
f̂1(q): Type B (T= 2000,k=100)
f̂0(q): Type C (T= 2000,k=100)
f̂0(q): Type E (T= 2000,k=100)
f̂1(q): Type C (T= 2000,k=100)
f̂1(q): Type E (T= 2000,k=100)
f̂0(q): Type D (T= 2000,k=100)
f̂1(q): Type D (T= 2000,k=100)
q

The above plots suggest that $\hat{f}_1(q)$ displays a good ability to trace its true counterpart $f_1(q)$ along the chosen domain. Interestingly, our estimator also appears to match its true counterpart closely under scenario D when the chosen functional form has a kink. At this stage it is worth recalling that these figures have been obtained allowing for both serial correlation and endogeneity in the underlying dynamics.

Unlike $\hat{f}_1(q)$ however, the estimator of $f_0(q)$ appears to perform poorly overall especially when the sample size is small. This is not unexpected and stems from the slow convergence of the estimator relative to that of $\hat{f}_1(q)$ as well as its large variance. Regardless of the sample size the plots make clear the fact that the variance of $\hat{f}_0(q)$ is substantially larger than that of $\hat{f}_1(q)$.

We next aim to highlight more formally the consistency properties of our estimators by documenting the progression of the corresponding RMSEs as the sample size and associated bin number is allowed to increase. Results across a selective set of scenarios are summarised in Table 1 below which displays simulated averages of (7) across N=2000 Monte-Carlo replications. The rows labelled PLL correspond to our piecewise local linear estimator while the rows labelled KER are based on a Kernel estimation as described in Xiao (2009) and using a Gaussian Kernel with $h = 1/k$ (the number of bins associated with each sample size is denoted k).

Table 1. RMSE of Estimators under Serial Correlation and Endogeneity

		$T = 250$ $k = 40$	$T = 500$ $k = 70$	$T = 1000$ $k = 110$	$T = 2000$ $k = 160$	$T = 250$ $k = 40$	$T = 500$ $k = 70$	$T = 1000$ $k = 110$	$T = 2000$ $k = 160$
		$\hat{f}_0(q)$				$\hat{f}_1(q)$			
A	PLL	0.879	0.893	0.799	0.710	0.081	0.056	0.036	0.023
	KER	9.838	2.442	1.317	0.626	0.840	1.463	0.071	0.021
B	PLL	0.893	0.860	0.800	0.694	0.079	0.054	0.037	0.025
	KER	2.607	7.092	1.368	0.583	0.301	1.132	0.085	0.021
C	PLL	0.895	0.843	0.769	0.696	0.081	0.055	0.035	0.022
	KER	3.086	8.205	1.417	0.886	0.284	0.408	0.058	0.039
D	PLL	2.010	2.624	2.683	1.883	0.177	0.139	0.098	0.069
	KER	4.866	3.737	1.808	1.791	0.401	0.297	0.146	0.101
E	PLL	0.927	0.914	0.805	0.701	0.082	0.055	0.036	0.022
	KER	8.692	7.462	5.348	2.156	0.520	0.385	0.141	0.039

Across all functional forms we note a clear decline in the PLL based RMSEs corresponding to $\hat{f}_1(q)$ as T and k are allowed to increase. Interestingly and with the exception of scenario D which is ruled out by our assumptions the average RMSE figures are also very similar across T and k. A suitable choice for h or k is an important topic in its own right and merits further research. For our purpose our choice was guided by the requirement that $Th^{3/2} \to 0$ which gave us a rough benchmark for setting k but we have also repeated the above experiment across different choices of k and results remained very much similar. As expected from Proposition 1, the slope functions see their RMSEs decline substantially faster than their intercept counterparts. Looking at the RMSE figures corresponing to $\hat{f}_0(q)$ we note their tendency to decline very slowly.

Our comparisons with an alternative Kernel based estimator also suggest that our method works well. Naturally, since alternative Kernels or functional forms may produce different finite sample outcomes it would be misleading to argue that our PLL approach dominates alternative approaches. Indeed our key goal here was to introduce a simple approach to estimating functional coefficients that displays good finite sample properties rather than proposing an alternative methodology that aims to dominate existing approaches.

5 Conclusions

This paper introduced the concept of functional cointegration and proposed a novel method of estimating the unknown functional coefficients linking the variables of interest under a nonstationary unit root setting. Our method is based on a simple binning idea and is shown to perform well asymptotically as well as in finite samples. Operating within a highly general probabilistic setting that allows for both serial correlation and endogeneity we established the consistency of our function estimators. Since developing formal inferences was beyond the scope of this paper, in future work it will be interesting to use our results to obtain the properties of test statistics that could be used to tests hypotheses such as the null of a linearly cointegrated model versus our functional specification.

APPENDIX

LEMMA 1: As $h \to 0$ (i) $E[I_{rt-1}]/h \to g_q(q)$, (ii) $E[I_{rt-1}(q_{t-1}-q)^m] = o(h^{m+1})$.

PROOF: We focus on (ii) and evaluate the expression at some $q = q_r$. We have

$$\begin{aligned} |E[(q_{t-1}-q_r)^m I_{rt-1}]| &= \left| \int_{H_r} (q-q_r)^m g_q(q) dq \right| \\ &\leq \int_{H_r} |q-q_r|^m g_q(q) dq \\ &\leq h^m \int_{H_r} g_q(q) dq = const *\ h^{m+1} \end{aligned} \tag{8}$$

and the result follows.

PROOF OF PROPOSITION 1: Given x_t, y_t, q_t and the known bin cutoff locations the least squares estimators of the intercept β_{0r} and slope parameter β_{1r} of the regression line within each bin can be formulated as

$$\begin{aligned} \hat{\beta}_{0r} &= \overline{y}_r - \hat{\beta}_{1r}\overline{x}_r \\ \hat{\beta}_{1r} &= \frac{\sum(x_t - \overline{x}_r) I_{rt-1} y_t}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}} \end{aligned} \tag{9}$$

with $\overline{x}_r = \sum x_t I_{rt-1} / \sum I_{rt-1}$ and $\overline{y}_r = \sum y_t I_{rt-1} / \sum I_{rt-1}$. Next, using $y_t = f_0(q_{t-1}) + f_1(q_{t-1})x_t + u_t$, taking a first order Taylor expansion of the unknown coefficients around some $q \in H_r$

$$f_i(q_{t-1}) \approx f_i(q) + f_i'(q)(q_{t-1} - q) + o(h^2)$$

for $i = 0, 1$ and ignoring terms that are $o(h^2)$ we can rewrite $\hat{\beta}_{1r}$ as

$$\begin{aligned} \hat{\beta}_{1r} - f_1(q) &= \frac{\sum(x_t - \overline{x}_r) I_{rt-1}[f_0(q_{t-1}) + f_1(q_{t-1})x_t]}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}} + \frac{\sum(x_t - \overline{x}_r) I_{rt-1} u_t}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}} \\ &= f_0'(q) \frac{\sum(x_t - \overline{x}_r)(q_{t-1} - q) I_{rt-1}}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}} + f_1'(q) \frac{\sum x_t (x_t - \overline{x}_r)(q_{t-1} - q) I_{rt-1}}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}} \\ &+ \frac{\sum(x_t - \overline{x}_r) I_{rt-1} u_t}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}}. \end{aligned} \tag{10}$$

It is now also convenient to reformulate the above as

$$\begin{aligned} T\sqrt{h}(\hat{\beta}_{1r} - f_1(q)) &= f_0'(q) \left(\frac{\sum(x_t - \overline{x}_r)(q_{t-1} - q) I_{rt-1}/T^2 h}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}/T^2 h} \right) T\sqrt{h} + \\ & f_1'(q) \left(\frac{\sum x_t (x_t - \overline{x}_r)(q_{t-1} - q) I_{rt-1}/T^2 h}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}/T^2 h} \right) T\sqrt{h} + \\ & \frac{\sum(x_t - \overline{x}_r) I_{rt-1} u_t / Th}{\sum(x_t - \overline{x}_r)^2 I_{rt-1}/T^2 h} \\ &\equiv T\sqrt{h} f_0'(q) A_r + T\sqrt{h}\ f_1'(q) B_r + C_r \end{aligned} \tag{11}$$

and the result follows by showing that $T\sqrt{h}\ A_r$ and $T\sqrt{h}\ B_r$ are asymptotically negligible when $Th^{3/2} \to 0$ while C_r is $O_p(1)$. Note that the denominators of the above are always bounded in distribution as $Th \to \infty$, since

$$\begin{aligned}
& \left| \sum x_t{}^2 I_{rt-1}/T^2h - g_q(q) \int B_v^2(s) \right| \\
\leq & \left| \sum x_t{}^2 I_{rt-1}/T^2h - \sum B_v^2(t/T) I_{rt-1}/Th \right| + \left| \sum B_v^2(t/T) I_{rt-1}/Th - g_q(q) \int B_v^2(s) \right| \\
\leq & \sup_t |I_{rt-1}/h| \left| \sum x_t^2/T^2 - \sum B_v^2(t/T)/T \right| + \left(\sup_{s\in[0,1]} B_v(s) + 1 \right)^2 \left| \sum I_{rt-1}/Th - g_q(q) \right|. \qquad (12)
\end{aligned}$$

Using Markov inequality $\Pr\left(\sup_t |I_{rt-1}/h| > M\right) \leq \sup_t E\left(I_{rt-1}\right)/Mh \leq \sup g_q(q)/M \to 0$ as $M \to \infty$ therefore I_{rt-1}/h is uniformly bounded. Our assumptions also ensure that $\sum x_t{}^2/T^2 \Rightarrow \int_0^1 B_v^2$ (see Phillips (1987)) and finally the asymptotic negligibility of the last term in (12) as $Th \to \infty$ follows from a suitable law of large numbers for strong mixing processes (e.g Hansen (1991, Corollary 4). See also Hansen (2008, Theorem 1)). Similarly for $\overline{x}_r$.

We have for $q \in H_r$, $|q_{t-1} - q| < h$ and $f_1'(q)$ bounded,

$$\begin{aligned}
T\sqrt{h}\,|B_r| &\leq T\sqrt{h} \frac{\sum |x_t(x_t - \overline{x}_r)(q_{t-1} - q)|\, I_{rt-1}}{\sum (x_t - \overline{x}_r)^2 I_{rt-1}} \\
&\leq Th^{3/2} \frac{\sum |x_t(x_t - \overline{x}_r)|\, I_{rt-1}}{\sum (x_t - \overline{x}_r)^2 I_{rt-1}} \to 0 \qquad (13)
\end{aligned}$$

since $Th^{3/2} \to 0$. The asymptotic negligibility of $T\sqrt{h}\ A_r$ follows along identical lines using the fact that

$$\begin{aligned}
T\sqrt{h} \left| \sum (x_t - \overline{x}_r)(q_{t-1} - q) I_{rt-1}/T^2h \right| &\leq \sqrt{T}h^{3/2} \max_{t\leq T} \left| \frac{x_t}{\sqrt{T}} \right| \sum I_{rt-1}/Th \\
&\leq \sqrt{T}h^{3/2} \left(\sup_{s\in[0,1]} B_v(s) + 1 \right) \sum I_{rt-1}/Th \quad (14)
\end{aligned}$$

since as before $\left(\sup_{s\in[0,1]} B_v(s) + 1\right) \sum I_{rt-1}/Th$ is bounded $T\sqrt{h}\ A_r \to 0$.

Finally, for C_r, using $x_t = x_{t-1} + v_t$ we write

$$\frac{\sum (x_t - \overline{x}_r) I_{rt-1} u_t}{T\sqrt{h}} = \frac{\sum x_{t-1} I_{rt-1} u_t}{T\sqrt{h}} + \frac{\sum u_t v_t I_{rt-1}}{T\sqrt{h}} - \overline{x}_r \frac{\sum u_t I_{rt-1}}{\sqrt{Th}}. \qquad (15)$$

Notice that $\Pr\left(\left|\sum u_t v_t I_{rt-1}/T\sqrt{h}\right| > \varepsilon\right) \leq \frac{1}{Th} E[u_t^2 v_t^2 I_{rt-1}] \to 0$. Same goes for the term $\sum u_t I_{rt-1}/\sqrt{Th}$ and $\overline{x}_r$ is bounded by $\left(\sup_{s\in[0,1]} B_v(s) + 1\right)$ hence the third term is $O_p(1)$. So we can concentrate on $\sum x_{t-1} u_t I_{rt-1}/T\sqrt{h}$. We write as before

$$\left| \frac{1}{T\sqrt{h}} \sum x_{t-1} u_t I_{rt-1} \right| \leq \left(\sup_{s\in[0,1]} B_v(s) + 1 \right) \frac{1}{\sqrt{Th}} \sum |u_t| I_{rt-1} = O_p(1) \qquad (16)$$

and hence leading to the required result.

Proceeding along the same lines for $\hat{\beta}_{0r}$ and using $\hat{\beta}_{1r} = f_1(q) + O_p(1/T\sqrt{h})$ we write

$$\begin{aligned}
\hat{\beta}_{0r} - f_0(q) &= f_0'(q)\frac{\sum(q_{t-1}-q)I_{rt-1}}{\sum I_{rt-1}} + f_1'(q)\frac{\sum(q_{t-1}-q)x_t I_{rt-1}}{\sum I_{rt-1}} + \\
&\quad \frac{\sum u_t I_{rt-1}}{\sum I_{rt-1}} - \overline{x}_r O_p(\frac{1}{T\sqrt{h}}). \qquad (17)
\end{aligned}$$

Applying suitable normalisations we reformulate (17) as

$$\begin{aligned}
\sqrt{Th}(\hat{\beta}_{0r} - f_0(q)) &= f_0'(q)\left(\frac{\sum(q_{t-1}-q)I_{rt-1}}{\sum I_{rt-1}}\right)\sqrt{Th} + \left(f_1'(q)\frac{\sum(q_{t-1}-q)x_t I_{rt-1}}{\sum I_{rt-1}}\right)\sqrt{Th} + \\
&\quad \frac{\sum u_t I_{rt-1}/\sqrt{Th}}{\sum I_{rt-1}/Th} + O_p(1). \qquad (18)
\end{aligned}$$

Proceeding as above it is again straightforward to observe that under $\sqrt{T}h^{3/2} \to 0$ the first two terms in the right hand side of (18) are asymptotically negligible while the third term is $O_p(1)$ by our Assumptions A.

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