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24 May 2012

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May 24, 2012

Abstract

In the post-WWII era, most developing economies had decent economic growth, but, with current growth trends, the great majority of them are unlikely to transform into developed economies in near future. In these economies, the dual economic structure, the coexistence of the modern/formal sector and the traditional/informal sector, is persistent. The educational level of the population increased greatly, but the growth of the *skill* level, especially when measured by the share of high-skill workers, is relatively modest. Wage inequality between workers with basic skills and with advanced skills rose over time, while the inequality between workers with and without basic skills fell greatly.

In order to understand these facts, this paper develops a dynamic dual-economy model and examines how the long-run outcome of the economy depends on the initial distribution of wealth and sectoral productivity. It is shown that, for fast transformation into a developed economy, the initial distribution must be such that extreme poverty is not prevalent and the size of "middle class" is enough. If the former is satisfied but the latter is not, which would be the case for many developing economies falling into "middle income trap", the fraction of workers with basic skills and the share of the modern sector rise, but inequality between workers with advanced skills and with basic skills worsens and the traditional sector remains, consistent with the above-mentioned facts.

JEL Classification Numbers: I25, J31, O15, O17

Keyword: dual economy, modernization, education, wealth distribution

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1 Introduction

In the post-WWII era, most developing economies had decent economic growth and raised standards of living reasonably. However, except some oil-rich economies, only a small number of economies in East Asia had persistent high growth and evolved into developed economies. With current income levels and growth trends, the great majority of developing economies are unlikely to achieve such transformation in near future.

In these economies, the dual economic structure, that is, the coexistence of the modern/formal sector characterized by advanced technology, large establishment sizes, skilled jobs, and high wages, and the traditional/informal sector with the contrasting features, is persistent (La Porta and Shleifer, 2008; OECD, 2009).[1,2] The educational level of the population increased greatly, but the growth of the *skill* level, especially when measured by the share of high-skill individuals, seems to be relatively modest, considering that enormous gaps in cognitive skills with developed economies remain (Hanushek and Woessmann, 2008).[3] Further, wage inequality between workers with basic skills (those taught in mandatory education) and with advanced skills rose over time, while the inequality between workers with and without basic skills fell greatly (Colclough, Kingdon, and Patrinos, 2010).[4] This might indicate that basic education has become less effective in mitigating poverty but taking further education, especially of good quality, is increasingly difficult for the poor.

In order to understand these facts, this paper develops a dynamic dual-economy model

[1] To be exact, the modern-traditional classification is mainly based on technologies, while the formal-informal one is mainly based on official registrations of businesses, so they are distinct. Firms with modern technology may choose the informal sector due to heavy regulations or taxation (OECD, 2009).

[2] The traditional/informal sector can be divided into the urban informal sector, traditional agriculture, and the household production sector (see footnote 6). Rapid urbanization lowered the share of agricultural employment significantly, but it did not raise the share of the modern/formal sector greatly in many countries. According to OECD (2009), informal employment, defined as the sum of urban informal-sector employment and formal-sector one without social protection (such as social security benefits) accounts for the majority of non-agricultural employment in developing economies.

[3] According to Hanushek and Woessmann (2008), the share of students without basic literacy in cognitive skills is more than 30% (as high as 82%) in most developing nations, while it is less than 10% (as low as 3%) in developed nations. Further, the share of high-performing students in the skills is more than 10% (as high as 22%) in most developed nations, while it is less than 1% (as low as 0.1%) in many developing nations. Reviewing the literature, they conclude that there is compelling evidence that cognitive skills, rather than mere school attainment, are strongly related to individual earnings and economic growth.

[4] Colclough, Kingdon, and Patrinos (2010) combine estimated returns to education in developing nations from recent cross-section studies (32 studies for 35 countries) with those from earlier studies (more than 100 studies using data from the 1960s to early 1990s), and find that, on average, the return to primary education fell rapidly over time and became lower than returns to post-primary education, which, particularly the one to tertiary education, fell very moderately. Since quality of education deteriorated over time in most developing nations due to rising enrollment under harsh budget, *quality-adjusted* returns to advanced education seem to have risen. They also review a limited number of country studies using time-series data after the 1980s, which find that the return to tertiary education rose greatly and the one to primary education fell.

and examines how the long-run outcome of the economy depends on the initial distribution of wealth and sectoral productivity. It is shown that, for fast transformation into a developed economy, the initial distribution must be such that extreme poverty is not prevalent and the size of "middle class" is enough. Both conditions seem to have held in the successful East Asian economies, where, as in the model economy undergoing such transformation, inequalities between workers with advanced education and others fell over time (Wood, 1994). In contrast, if the former is satisfied but the latter is not, which would be the case for many developing nations falling into "middle income trap", the fraction of workers with basic skills and the share of the modern sector rise, but inequality between workers with advanced skills and with basic skills worsens and the traditional sector remains for long periods, consistent with the above facts.[5] If the former condition does not hold, which would be true for poorest economies, the dual structure and large inequality between workers without basic skills and others (especially, those with advanced skills) last for very long periods.

The analysis is based on a deterministic, discrete-time, and small-open OLG model. The economy is inhabited by a continuum of two-period-lived individuals who are homogeneous in innate ability. In childhood, an individual receives a transfer from her parent and spends it on assets and education to maximize future income. Basic education, which corresponds to acquiring essential skills taught in primary and lower secondary education, is needed to become a middle-skill worker, and more costly advanced education is needed to become a high-skill worker. No credit market for the educational investment exists, so she cannot invest more than the received transfer. Since she can spend wealth on assets as well, she invests in education only if it is financially accessible *and* profitable. In adulthood, she obtains income from assets and work and spends it on *basic consumption*, *non-basic consumption*, and a transfer to her single child.

The economy is composed of *up to* two sectors, the *modern sector* producing good M and the *traditional sector* producing good T. The modern sector using advanced technology employs high-skill and middle-skill workers, and the traditional sector employs low-skill workers. Both goods can be used for basic consumption, while only good M can be used for non-basic consumption. In other words, goods for basic needs, such as clothing, food, and shelter, can be produced using either technology, while the advanced technology is required to produce goods such as electric appliances and IT gadgets. It is assumed that good M is tradable and good T is nontradable. The traditional sector produces goods for basic needs using primitive technology, thus it corresponds to the urban informal sector,

[5] Although skill-biased technical change is a possible contributor to the increasing inequality in recent years, particularly in middle-income economies, Colclough, Kingdon, and Patrinos (2010) find that this trend started well before IT technologies became economically important (see footnote 4).

traditional agriculture, and the household production sector in real economy, all of which supply goods mainly for domestic markets.[6] By contrast, the modern sector corresponds to modern manufacturing and commercial agriculture, which compete more directly with foreign producers. If good T is relatively cheap, only the traditional sector supplies goods for basic consumption, otherwise, the modern sector too or only the sector does.

Because the distribution of wealth in the initial period is unequal and the inequality is transmitted intergenerationally through transfers, generally, individuals are heterogeneous in accessibility to two types of education. Hence, those without enough wealth cannot take basic or advanced education even if the return to the education net of its cost is positive. Their descendants, however, may become accessible to it if enough wealth is accumulated. (Opposite is true for descendants of relatively wealthy individuals.)

Main results, which are concerned with the situation where sectoral productivities are not very low, are summarized as follows. First, the model has four types of steady states, which are different in proportions of the *poor* (those who cannot access advanced education) and the *very poor* (those who cannot access basic education), wage inequality, the size of the traditional sector, etc. The best steady state (in terms of aggregate output, aggregate net income, and average utility) has features of a typical developed economy: no poverty (universal access to advanced education), low wage inequality (wages net of education costs are equal), and no traditional sector (goods for basic consumption are totally supplied by the modern sector).[7] Other three types of steady states share the contrasting features, but differ in characteristics of poverty and wage inequality: in one type, no extreme poverty (universal access to basic education) but prevalent mild poverty, and high inequality between high-skill workers and others and low inequality between middle-skill and low-skill workers; in another type, no mild poverty (those who can access basic education can afford advanced education) but widespread extreme poverty, and high inequality between low-skill workers and others and low inequality between high-skill and middle-skill workers; in yet another type, pervasive extreme and mild poverty and typically high inequalities among three types of workers.

Second, to which type of steady states the economy converges depends on the initial distribution of wealth. In particular, for the best steady state to be realized, the initial distribution must be such that the very poor are not large in number and the non-poor must be enough relative to the poor.[8] If the initial size of the very poor is large, the dual

[6]The urban informal sector supplies basic nontradable services, such as petty trading of commodities and basic meals, and basic manufacturing goods mostly for domestic markets. Traditional agriculture is operated on a small scale by family farms and produces agricultural products mainly for basic needs of domestic consumers. And, the household sector produces basic goods and services mostly for self-consumption.

[7]Since net returns of two types of education are equal, some individuals just take basic education.

[8]Note, however, that the economy *can* converge to the second and third types of steady states too,

structure and large inequality between low-skill workers and others (especially, high-skill workers) remain in the long run, i.e. the economy converges to either of the last two types of steady states. If its size is not large but the non-poor are scarce relative to the poor, the fraction of middle-skill workers and the share of the modern sector rise, and inequality between middle-skill and low-skill workers shrinks over time. However, inequality between high-skill and middle-skill workers worsens, and typically the traditional sector remains in the long run, i.e. the economy converges to the second type.

These results are obtained from the model with time-invariant sectoral productivities. When the productivity of the modern sector grows continuously over time, ultimately, the economy converges to the best steady state from any initial condition, but the speed of convergence depends critically on the initial condition and thus the qualitative results of the constant productivity case remain to hold approximately. Hence, as stated earlier, the model can explain the facts described at the beginning.[9]

The main implication of the model is that, for fast modernization of an economy, the initial distribution of wealth must be such that extreme poverty is not prevalent so that most people can acquire basic skills and the size of "middle class" is enough so that an adequate number of workers possess advanced skills. Consistent with this and the above results, Hanushek and Woessmann (2009), using data on international student achievement tests for 50 countries, find that both the share of students with basic skills and that of top performance have significant effects on economic growth that are *complementary* each other. The model provides a sectoral-shift-based explanation for their finding.[10]

This paper is related to the theoretical literature on dual economy models, such as Galor and Zeira (1993), Banerjee and Newman (1998), Lucas (2004), Wang and Xie (2004), Proto (2007), Yuki (2007, 2008), and Vollrath (2009). Banerjee and Newman (1998) examine implications of differences in technological and institutional conditions between rural traditional and urban modern sectors for development and urbanization. Lucas (2004) examines

depending on details of the initial distribution. The best steady state is more likely to be reached as the size of the very poor is smaller and the proportion of the non-poor to the poor is higher.

[9]The paper also examines the situation where sectoral productivities are very low initially and grow continuously over time. When the modern sector's productivity is very low, the best steady state does not exist and, even with a good initial condition, the fraction of high-skill workers remains constant (that of middle-skill workers rises) and inequality between high-skill and middle-skill workers (low-skill workers too after some point) worsens over time. After the productivity reaches a certain level, however, the fraction rises, the inequality falls, and the economy converges to the best steady state. The dynamics may resemble experiences of many developed economies.

[10]As for relations among inequality, education, and growth, Deininger and Olinto (2000) find that growth is affected negatively by initial land inequality (a proxy for initial asset inequality) and positively by mean years of schooling per worker, which in turn is negatively affected by the initial inequality. Easterly (2001) finds that a greater size of middle class, measured as the share of income held by second through fourth quintiles of the distribution, is associated with more education, higher income, and higher growth.

rural-urban migration in a model where urban workers allocate time between human capital accumulation and production. Wang and Xie (2004) explore factors affecting the activation of a modern industry using a static two-sector model with non-homothetic preferences and uncompensated spillovers in the IRS modern sector. Based on a three-sector (agrarian, manufacturing, and informal) model, Proto (2007) analyzes how the initial number of unskilled landless workers, through its effect on their bargaining power against landlords and land rents, determines wealth and human capital accumulations and development. Vollrath (2009) shows that the marginal product of labor in the modern sector can be higher than in the traditional sector and such allocation is welfare-maximizing based on a model in which individuals allocate time between market and non-market activities.

The more closely related are Galor and Zeira (1993) and Yuki (2007, 2008), which develop dual economy models where, as in this paper, lumpy skill investment is constrained by intergenerational transfers motivated by impure altruism and examine the relationship between initial distribution and long-run outcome. Unlike the present paper, however, the type of education (skill investment) is single, and either the traditional sector produces the same good as the modern sector (Galor and Zeira) or only the sector can produce goods for basic education (Yuki). Hence, their models cannot analyze how proportions of workers with basic education and with advanced education, their wages, and wage inequality between them change over time, thereby exploring what roles different types of education play in development. Further, they cannot capture the process where the production of goods for basic consumption shifts from the traditional sector to the modern sector with development, which is universally observed in real economy, thus, in the models of Yuki (2007, 2008), the traditional sector remains even in the best steady state.

The paper is also somewhat related to the empirical literature showing the existence of multiple growth paths based on statistical methods. van Paap, Franses, and Dijk (2005) and Owen, Videras, and Davis (2009) find that countries can be clustered into multiple groups with distinct growth regimes. Alfo, Trovato, and Waldman (2008) show that countries can be clustered into many groups with different levels of per capita GDP and with no sign of convergence across groups.

The paper is organized as follows. Since the model can be considered as a sequence of quasi-static economies in which single generations make decisions, for ease of presentation, Section 2 presents and analyzes the model without taking into account intergenerational linkages, then Section 3 considers the linkages. Section 4 analyzes the model and derives main results, and Section 5 concludes. Appendix contains proofs of lemmas and propositions.

2 Model

Although the model is dynamic, it can be considered as a sequence of quasi-static economies in which single generations make decisions. For ease of presentation, this section presents and analyzes the model without taking into account intergenerational linkages, then the next section considers the linkages.[11]

2.1 Setup

Consider a deterministic, discrete-time, and small-open OLG economy. The economy is inhabited by a continuum of two-period-lived individuals who are homogeneous in innate ability. Each adult has a single child and thus the population is constant over time. The adult population is normalized to be 1.

Lifetime of an individual: In childhood, individual i receives a transfer b^i from her parent and spends it on assets a^i and education in order to maximize future income. Basic education (costs e_m), which corresponds to acquiring essential skills taught in primary and lower secondary education, is needed to become a middle-skill worker, and advanced education (costs $e_h > e_m$) is needed to become a high-skill worker.[12] Thus, if she spends e_j $(j = h, m)$ on education, $a^i = b^i - e_j$, and $a^i = b^i$ if she does not take education. Since no credit market exists for the educational investment, she cannot invest more than b^i, i.e. $a^i \geq 0$.

In adulthood, she obtains income from assets and work and spends it on *basic consumption* c^i_B, *non-basic consumption* c^i_N, and a transfer to her single child $(b^i)'$. A unit of non-basic consumption is a numeraire. Characteristics of the two types of consumption are explained later. She maximizes the Cobb-Douglas utility subject to the budget constraint:

$$\max\ U = (c^i_B)^{\gamma_B}(c^i_N)^{\gamma_N}[(b^i)']^{\gamma_b},\ \ \gamma_i \in (0,1),\ \ \gamma_B + \gamma_N + \gamma_b = 1, \tag{1}$$

$$s.t. \quad Pc^i_B + c^i_N + (b^i)' = w^i + (1+r)a^i, \tag{2}$$

where P is the relative price of basic consumption and w^i is her gross wage. By solving the maximization problem, the following consumption and transfer rules are obtained.

$$Pc^i_B = \gamma_B[w^i + (1+r)a^i], \tag{3}$$

$$c^i_N = \gamma_N[w^i + (1+r)a^i], \tag{4}$$

$$(b^i)' = \gamma_b[w^i + (1+r)a^i]. \tag{5}$$

Production: The small open economy (thus interest rate r is exogenous) is composed of *up to* two sectors, the modern sector producing good M and the traditional sector producing

[11] All variables are presented without time subscripts in this section.

[12] The cost of advanced education includes the cost of acquiring basic skill.

good T. The modern sector, which utilizes advanced technology, employs high-skill and middle-skill workers, and the traditional sector using primitive technology employs low-skill workers for production. Production functions of the two sectors are:

$$Y_M = A_M (L_h)^{\alpha} (L_m)^{1-\alpha}, \quad \alpha \in (0,1), \tag{6}$$

$$Y_T = A_T L_l, \tag{7}$$

where L_h, L_m, and L_l are numbers of high-skill, middle-skill, and low-skill workers respectively, and A_i $(i = M, T)$ is the exogenous productivity of sector i.[13]

Characteristics of goods and consumption: Both good M and good T can be used for basic consumption, while only good M can be used for non-basic consumption. In other words, goods for basic needs, such as clothing, food, and shelter, can be produced using either technology, while goods such as electric appliances and IT gadgets can be produced using the advanced technology only. Specifically, a unit of basic consumption can be fulfilled by the consumption of either a unit of good T or θ units of good M. The unit of measurement of non-basic consumption is good M, so $P \leq \theta$ must hold.[14]

Assume that good M is tradable and good T is nontradable. The assumption would be better understood by associating the two sectors with sectors in real economy. The traditional sector produces consumption goods for basic needs using primitive technology, thus it corresponds to the urban informal sector, traditional agriculture, and the household production sector. The urban informal sector supplies basic nontradable services, such as small-scale trades of commodities and meals, and basic manufacturing goods mostly for domestic markets, whose size is substantial in developing economies, in many cases, accounting for the majority of non-agricultural employment (OECD, 2009). Traditional agriculture is operated by family farms and supplies products mainly for basic needs of domestic consumers.[15] And, the household sector produces basic goods and services mostly for self-consumption, whose importance is significant in developing economies. By contrast, the modern sector corresponds to modern manufacturing and commercial agriculture, which compete more directly with foreign producers.[16]

[13]Because free international capital mobility is assumed, the production function of the modern sector may be considered as a reduced form of the function that includes physical capital K as an input:

$$Y_M = \widetilde{A_M} (L_h)^{\beta} (L_m)^{\gamma} (K)^{1-\beta-\gamma}, \quad \beta, \gamma \in (0,1). \tag{8}$$

When (6) is the reduced-form function, A_M depends positively on $\widetilde{A_M}$ and negatively on r.

[14]Good M is used for education too; the education cost is that of purchasing a fixed amount of the good.

[15]As in Yuki (2007), traditional agriculture may be introduced as a separate tradable sector operated by low-skill farmers. The analysis would be much more complicated without affecting most qualitative results.

[16]In real economy, there exist skill-intensive modern sectors supplying nontradables. However, in developing countries, most of skill-intensive nontradables are public services, health services, and education, where market forces have limited roles, while sectors such as finance and consulting services are limited in size.

Determination of wages: Goods and labor markets are competitive, thus wages of high-skill, middle-skill, and low-skill workers are given by:

$$w_h = \alpha A_M \left(\frac{L_m}{L_h}\right)^{1-\alpha}, \tag{9}$$

$$w_m = (1-\alpha) A_M \left(\frac{L_h}{L_m}\right)^{\alpha}, \tag{10}$$

$$w_l = P A_T. \tag{11}$$

For later use, denote wages of high-skill and middle-skill workers *net of costs of education* by $\widetilde{w_j} = w_j - (1+r)e_j$ $(j = h, m)$, which are:

$$\widetilde{w_h} = \widetilde{w_h}\left(\frac{L_h}{L_m}\right) \equiv \alpha A_M \left(\frac{L_m}{L_h}\right)^{1-\alpha} - (1+r)e_h, \tag{12}$$

$$\widetilde{w_m} = \widetilde{w_m}\left(\frac{L_h}{L_m}\right) \equiv (1-\alpha) A_M \left(\frac{L_h}{L_m}\right)^{\alpha} - (1+r)e_m. \tag{13}$$

Determination of P: When the relative price of good T is low, only good T of the traditional sector is used for basic consumption and thus its market-clearing condition is:

$$P A_T L_l = \gamma_B [w_h L_h + w_m L_m + w_l L_l + (1+r) \textstyle\sum_i a^i], \tag{14}$$

where the right-hand side is obtained by aggregating (3) over the adult population. Denote aggregate intergenerational transfers by B. Then, $\sum_i a^i = B - (e_h L_h + e_m L_m)$ holds. By plugging this expression, $w_l = P A_T$, and $L_l = 1 - (L_h + L_m)$ into (14) and solving for P,

$$P = \frac{\gamma_B}{1-\gamma_B} \frac{[w_h - (1+r)e_h]L_h + [w_m - (1+r)e_m]L_m + (1+r)B}{A_T[1-(L_h+L_m)]}. \tag{15}$$

By substituting (9) and (10) into (15), P is expressed as a function of L_h, L_m, and B :

$$P = P(L_h, L_m, B) \equiv \frac{\gamma_B}{1-\gamma_B} \frac{A_M (L_h)^{\alpha} (L_m)^{1-\alpha} + (1+r)[B - e_h L_h - e_m L_m]}{A_T[1-(L_h+L_m)]}, \tag{16}$$

where $P(L_h, L_m, B)$ is increasing in L_h, L_m, and B (since, as seen later, $\widetilde{w_j} = w_j - (1+r)e_j > 0$, $j = h, m$, in equilibrium). $P(L_h, L_m, B) \leq \theta$ must hold for $P = P(L_h, L_m, B)$ to be true.

When L_h, L_m, and B are large, the demand for good T is high and its supply is low enough that $P(L_h, L_m, B) > \theta$ holds. In this case, good M too is used for basic consumption and $P = \theta$ is satisfied. The amount of good M used for basic consumption, C_{BM}, equals

$$C_{BM} = \gamma_B \{A_M (L_h)^{\alpha} (L_m)^{1-\alpha} + (1+r)[B - e_h L_h - e_m L_m]\} - (1-\gamma_B)\theta A_T [1-(L_h+L_m)], \tag{17}$$

which too is increasing in L_h, L_m, and B.

From these results, the low-skill wage equals:

$$w_l = w_l(L_h, L_m, B) \equiv \begin{cases} P(L_h, L_m, B) A_T & \text{when } P(L_h, L_m, B) \leq \theta \\ \theta A_T & \text{when } P(L_h, L_m, B) \geq \theta \end{cases}. \tag{18}$$

2.2 Equilibrium educational choices and wages

Individuals are heterogenous in received transfer b^i. Let F_h be the proportion of individuals who can afford e_h to become a high-skill worker, and let F_m be the proportion of those who *cannot* afford e_h but can afford e_m to become a middle-skill worker (thus $F_h + F_m \leq 1$). Since an individual can spend wealth on assets as well, she invests in education only if it is financially accessible *and* profitable. An individual with $b^i \geq e_h$ spends e_h only if $\widetilde{w_h} \geq \max\{\widetilde{w_m}, w_l\}$, and one with $b^i \geq e_m$ spends e_m only if $\widetilde{w_m} \geq w_l$. Thus, $L_h \leq F_h$ and $L_h + L_m \leq F_h + F_m$ must hold, but $L_h = F_h$ and $L_m = F_m$ do not necessarily hold. This section examines how L_h, L_m, and wages are determined depending on F_h, F_m, and B.

2.2.1 Critical equations determining educational choices and wages

From the above discussion, magnitude relations of $\widetilde{w_h}$ to $\widetilde{w_m}$ and of $\widetilde{w_m}$ to w_l at $L_h = F_h$ and $L_m = F_m$ are critical in determining L_h and L_m. For example, if $\widetilde{w_h} \geq \widetilde{w_m}$ and $\widetilde{w_m} \geq w_l$ at $L_h = F_h$ and $L_m = F_m$, $L_h = F_h$ and $L_m = F_m$ hold in equilibrium, i.e. if each level of education is profitable when all individuals take highest affordable education, they do take it. Hence, combinations of F_h and F_m satisfying $\widetilde{w_h}(\frac{F_h}{F_m}) = \widetilde{w_m}(\frac{F_h}{F_m})$ and the combinations satisfying $\widetilde{w_m}(\frac{F_h}{F_m}) = w_l(F_h, F_m, B)$ are examined next.

Denote $\frac{F_h}{F_m}$ satisfying $\widetilde{w_h}(\frac{F_h}{F_m}) = \widetilde{w_m}(\frac{F_h}{F_m})$ by $(\frac{F_h}{F_m})_{hm}$, which exists and is unique since $\widetilde{w_h}$ ($\widetilde{w_m}$) decreases (increases) with $\frac{F_h}{F_m}$ and $\widetilde{w_h} > (<) \widetilde{w_m}$ at $\frac{F_h}{F_m} = 0 (= +\infty)$ from (12) and (13). Denote $\frac{F_h}{F_m}$ satisfying $\widetilde{w_m}(\frac{F_h}{F_m}) = \theta A_T$ (w_l when $P = \theta$) by $(\frac{F_h}{F_m})_{ml,\theta}$, which equals, from (13):

$$(\frac{F_h}{F_m})_{ml,\theta} = \left[\frac{\theta A_T + (1+r)e_m}{(1-\alpha)A_M}\right]^{\frac{1}{\alpha}}. \tag{19}$$

Assumption 1 [17] $(\frac{F_h}{F_m})_{hm} > (\frac{F_h}{F_m})_{ml,\theta} \Leftrightarrow A_M(\alpha)^\alpha(1-\alpha)^{1-\alpha} > [\theta A_T + (1+r)e_h]^\alpha[\theta A_T + (1+r)e_m]^{1-\alpha}$.

The assumption implies $\widetilde{w_h} = \widetilde{w_m} > \theta A_T$ at $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$, that is, the highest (lowest) net middle-skill (high-skill) wage is strictly greater than the highest low-skill wage.

Lemma 1 shows the existence of F_h and F_m satisfying $\widetilde{w_m}(\frac{F_h}{F_m}) = P(F_h, F_m, B)A_T$ (w_l when $P < \theta$) and describes the shape of the equation and its relation with $(\frac{F_h}{F_m})_{hm}$ and $(\frac{F_h}{F_m})_{ml,\theta}$ when $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$. (When $\frac{\gamma_B}{1-\gamma_B}(1+r)B \geq \theta A_T$, $P(F_h, F_m, B) > \theta$ from (16) and thus $P = \theta$.)

Lemma 1 *Suppose $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$. Then, positive F_h and F_m satisfying $\widetilde{w_m}(\frac{F_h}{F_m}) = P(F_h, F_m, B)A_T$ exists and is expressed as $F_m = \phi(F_h, B)F_h$, where $\phi(\cdot)$ is a function satisfying $\lim_{F_h \to 0} \phi(F_h, B) = \overline{\phi}(B) \equiv \left[\frac{(1-\alpha)A_M}{(1+r)(\frac{\gamma_B}{1-\gamma_B}B + e_m)}\right]^{\frac{1}{\alpha}}$. When $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{hm}$, $\phi(F_h, B)$ is a decreasing function of its arguments, and, for given B, there exists a unique $F_h > 0$ satisfying $[\phi(F_h, B)]^{-1} = (\frac{F_h}{F_m})_{hm}$, denoted $F_h^{\ddagger}(B)$, and the one satisfying $[\phi(F_h, B)]^{-1} = (\frac{F_h}{F_m})_{ml,\theta}$,*

[17] The second equation is obtained since $\widetilde{w_h} - \widetilde{w_m}$ is decreasing in $\frac{L_h}{L_m}$ and thus $\widetilde{w_h} - \widetilde{w_m} > 0$ at $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{ml,\theta}$.

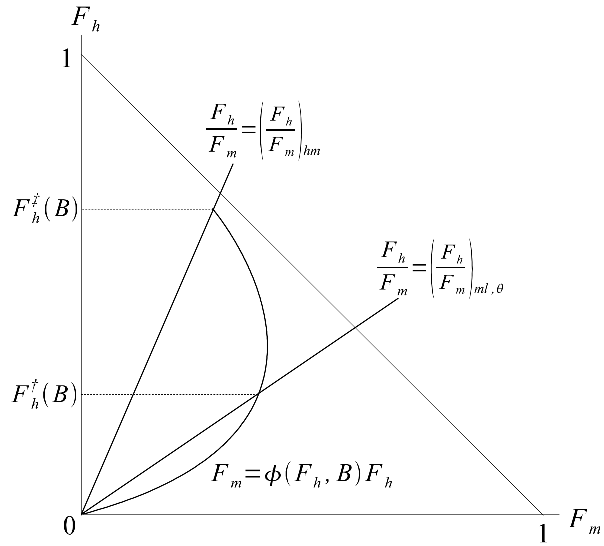

Figure 1: Lemma 1

denoted $F_h^{\dagger}(B)$*, where* $F_h^{\ddagger}(B)$ *and* $F_h^{\dagger}(B)$ *are decreasing functions and* $F_h^{\ddagger}(B)>F_h^{\dagger}(B)$*.*

Based on the lemma, Figure 1 illustrates $F_m=\phi(F_h,B)F_h$ ($\widetilde{w_m}(\frac{F_h}{F_m})=P(F_h,F_m,B)A_T$), $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$, and $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{ml,\theta}$ on the (F_m,F_h) plane. $F_h^{\ddagger}(B)$ and $F_h^{\dagger}(B)$ are unique intersections of $F_m=\phi(F_h,B)F_h$ with $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$ and $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{ml,\theta}$, respectively. As $F_h\to 0$, F_m satisfying $F_m=\phi(F_h,B)F_h$ approaches 0 (since $\lim_{F_h\to 0}\phi(F_h,B)=\overline{\phi}(B)<\infty$). The slope of the curve, $\frac{1}{\phi(F_h,B)}$, increases with F_h, thus F_m increases with F_h on the curve for low $\frac{1}{\phi(F_h,B)}$, but the relationship turns negative for high $\frac{1}{\phi(F_h,B)}$. As B increases, $\phi(F_h,B)$ decreases and thus the curve shifts leftward and $F_h^{\ddagger}(B)$ and $F_h^{\dagger}(B)$ decrease.

From (18), $F_m=\phi(F_h,B)F_h \Leftrightarrow \widetilde{w_m}(\frac{F_h}{F_m})=P(F_h,F_m,B)A_T$ affects educational choices only when $P(F_h,F_m,B)\leq\theta$, and $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{ml,\theta}\Leftrightarrow \widetilde{w_m}(\frac{F_h}{F_m})=\theta A_T$ affects the choices only when $P(F_h,F_m,B)\geq\theta$. Hence, relative positions of these loci and $P(F_h,F_m,B)=\theta$ must be examined. The next lemma, the final one needed to obtain equilibrium L_h and L_m and wages, describes the shape of $P(F_h,F_m,B)=\theta$ and its relation with $F_m=\phi(F_h,B)F_h$.

Lemma 2 *Suppose* $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$*. When* $\frac{F_h}{F_m}\in[[\overline{\phi}(0)]^{-1},(\frac{F_h}{F_m})_{hm}]$ ($[\overline{\phi}(0)]^{-1}$ *is the smallest* $\frac{F_h}{F_m}$ *satisfying* $F_m=\phi(F_h,0)F_h$), $P(F_h,F_m,B)$ *is an increasing function of its arguments. Given* B*, for any* $\frac{F_h}{F_m}\in[[\overline{\phi}(0)]^{-1},(\frac{F_h}{F_m})_{hm}]$*,* F_h *and* F_m *satisfying* $P(F_h,F_m,B)=\theta$ *exist and are unique, and for* $\frac{F_h}{F_m}>(<)(\frac{F_h}{F_m})_{ml,\theta}$*,* $F_m<(>)\phi(F_h,B)F_h$ *when* $P(F_h,F_m,B)=\theta$*.*

Figure 2 interposes $P(F_h,F_m,B)=\theta$ on Figure 1. On the figure, $P(F_h,F_m,B)=\theta$ is located below (above) $F_m=\phi(F_h,B)F_h$ for $\frac{F_h}{F_m}>(<)(\frac{F_h}{F_m})_{ml,\theta}$ and $P(F_h,F_m,B)<(>)\theta$ at the left (right) side of the locus. As B increases, the locus shifts leftward.

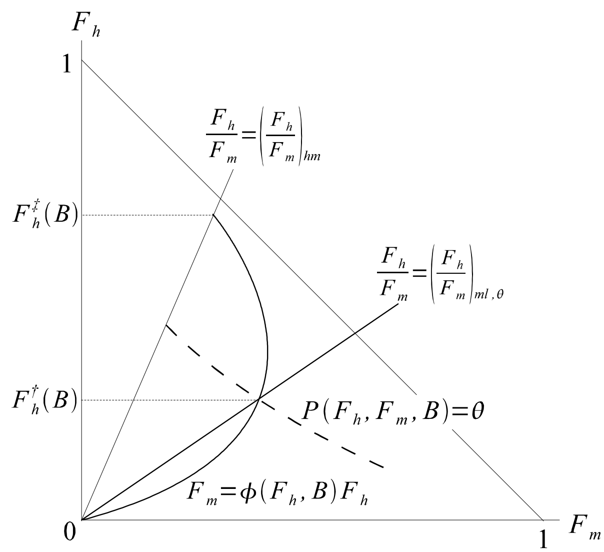

Figure 2: Lemma 2

2.2.2 Educational choices and wages

Based on the lemmas, educational choices and sectoral choices of individuals are presented in the following proposition. Henceforth, individuals with $b^i \geq e_h$, those with $b^i \in [e_m, e_h)$, and those with $b^i < e_m$ are named the *non-poor*, the *poor*, and the *very poor*, respectively.

Proposition 1 (Educational choices and sectoral choices) *Suppose* $F_h > 0$.

(*i*) *If* $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$, *the non-poor are indifferent between two types of education* $(\widetilde{w_h} = \widetilde{w_m})$ *and the poor strictly prefer basic education* $(\widetilde{w_m} > w_l)$, $L_h = \frac{(\frac{F_h}{F_m})_{hm}}{1+(\frac{F_h}{F_m})_{hm}}(F_h + F_m) \leq F_h$, $L_m = \frac{1}{1+(\frac{F_h}{F_m})_{hm}}(F_h + F_m) \geq F_m$, *and* $L_l = 1 - F_h - F_m$.

(*ii*) *If* $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$, *the non-poor strictly prefer advanced education* $(\widetilde{w_h} > \widetilde{w_m})$ *and* $L_h = F_h$.

(*a*) *If* $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$, *the poor strictly prefer basic education* $(\widetilde{w_m} > w_l)$, $L_m = F_m$, *and* $L_l = 1 - F_h - F_m$.

(*b*) *If* $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$,

1. *When* $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ *and* $F_h < F_h^{\dagger}(B)$, *if* $F_m \geq \phi(F_h, B)F_h$, *the poor are indifferent between basic education and no education* $(\widetilde{w_m} = w_l)$, $L_m = \phi(F_h, B)F_h \leq F_m$, *and* $L_l = 1 - (1 + \phi(F_h, B))F_h$; *otherwise, same as* (*a*).
2. *Or else, the poor are indifferent between basic education and no education* $(\widetilde{w_m} = w_l)$, $L_m = [(\frac{F_h}{F_m})_{ml,\theta}]^{-1} F_h \leq F_m$, *and* $L_l = 1 - \{1 + [(\frac{F_h}{F_m})_{ml,\theta}]^{-1}\}F_h$.

Figure 3 illustrates how L_h and L_m are determined depending on F_h and F_m when $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ on the (F_m, F_h) plane. As for $F_m = \phi(F_h, B)F_h$ and $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{ml,\theta}$, only portions of the loci that are *effective* (affect the determination of L_h and L_m) are drawn.

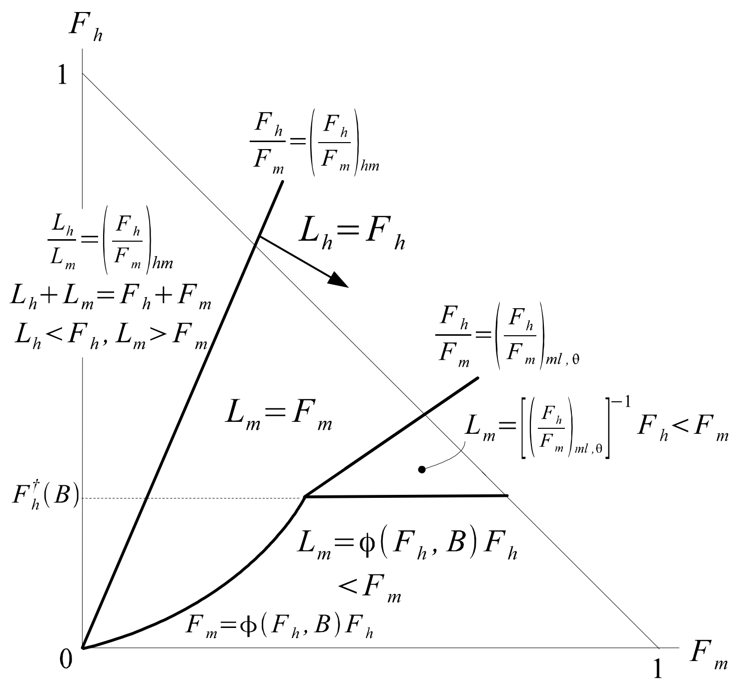

Figure 3: Educational and sectoral choices when $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ (Proposition 1)

When $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$, the relative number of the non-poor (those with $b^i \geq e_h$) to the poor (those with $b^i \in [e_m, e_h)$) is high enough that net wages of high-skill and middle-skill workers are equated. Thus, some of the non-poor do not take advanced education, while all the poor take basic education, i.e. $L_h = \frac{(\frac{F_h}{F_m})_{hm}}{1+(\frac{F_h}{F_m})_{hm}}(F_h+F_m) \leq F_h$ ($<$ when $\frac{F_h}{F_m} > (\frac{F_h}{F_m})_{hm}$) and $L_h+L_m = F_h+F_m$. Graphically, for any F_h and F_m on a straight line with slope -1, L_h and L_m are equal to F_h and F_m at the intersection of the line with $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{hm}$.

When $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$, the non-poor are scarce relative to the poor and thus the net wage of high-skill workers is strictly higher than the one for middle-skill workers and all the non-poor take advanced education, i.e. $L_h = F_h$. As for the poor, when $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$ and thus the scarcity is not so great, the net middle-skill wage is strictly higher than the low-skill wage and all of them take basic education, i.e. $L_m = F_m$. By contrast, when $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$ and thus the net middle-skill wage is lower, the educational choice of the poor depends on F_h as well as $\frac{F_h}{F_m}$. For given $\frac{F_h}{F_m}$, when F_h (thus F_m too) is small, i.e. $F_m < \phi(F_h,B)F_h \Leftrightarrow \frac{1}{\phi(F_h,B)} < \frac{F_h}{F_m}$ ($\phi(F_h,B)$ is a decreasing function), the demand for good T, its relative price, and the low-skill wage are low and thus $L_m = F_m$ holds, whereas when F_h (and F_m) is not small, the low-skill wage is equated with the net middle-skill wage, and if $F_m > \phi(F_h,B)F_h$, some of the poor do not take basic education. Specifically, when the non-poor are not abundant ($F_h < F_h^\dagger(B)$), the relative price of good T is less than θ and $L_m = \phi(F_h,B)F_h < F_m$, while

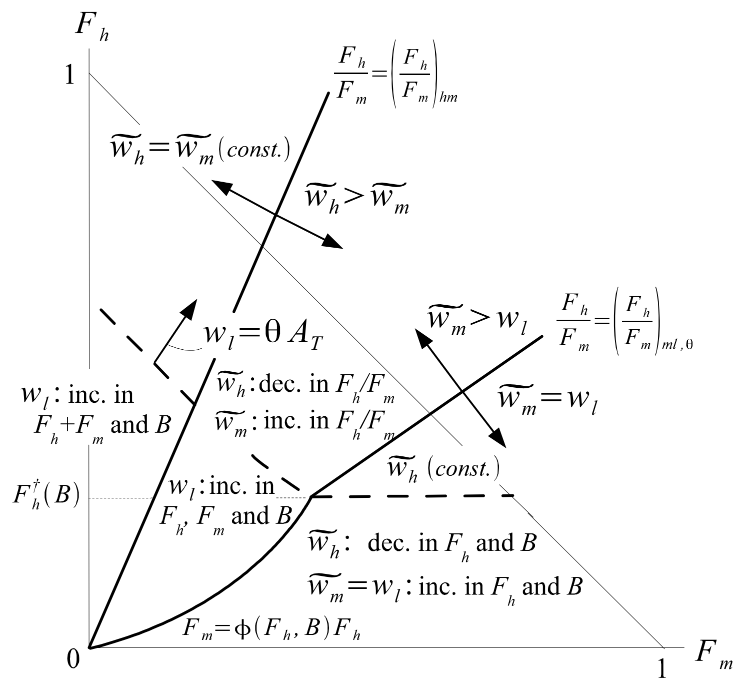

Figure 4: Wages when $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$ (Proposition 2)

when they are large in number ($F_h\geq F_h^\dagger(B)$), $P=\theta$ and $L_m=[(\frac{F_h}{F_m})_{ml,\theta}]^{-1}F_h<F_m$.

The loci in the figure are drawn for *given* B satisfying $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$. When B is higher, $F_m=\phi(F_h,B)F_h$ is located more left and $F_h^\dagger(B)$ is lower. Further, when $\frac{\gamma_B}{1-\gamma_B}(1+r)B\geq\theta A_T$, $P=\theta$ always holds and the region satisfying $F_h\leq F_h^\dagger(B)$ disappears.

Based on Proposition 1, Proposition 2 shows how (net) wages depend on F_h, F_m, and B.

Proposition 2 (Wages) *Suppose* $F_h>0$.

(*i*) *If* $\frac{F_h}{F_m}\geq(\frac{F_h}{F_m})_{hm}$, $\widetilde{w_h}=\widetilde{w_m}=\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(>w_l)$, *and* $w_l=\frac{\gamma_B}{1-\gamma_B}\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B}{1-(F_h+F_m)}$ *when* $F_h+F_m<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\left[\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T\right]}$, $w_l=\theta A_T$ *otherwise.*

(*ii*) *If* $\frac{F_h}{F_m}<(\frac{F_h}{F_m})_{hm}$,

(*a*) *If* $\frac{F_h}{F_m}\in((\frac{F_h}{F_m})_{ml,\theta},(\frac{F_h}{F_m})_{hm})$, $\widetilde{w_h}=\widetilde{w_h}(\frac{F_h}{F_m})$, $\widetilde{w_m}=\widetilde{w_m}(\frac{F_h}{F_m})$, *and* $w_l=P(F_h,F_m,B)A_T$ *when* $P(F_h,F_m,B)\leq\theta$ *(possible when* $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$*),* $w_l=\theta A_T$ *otherwise, where* $\widetilde{w_h}>\widetilde{w_m}>w_l$.

(*b*) *If* $\frac{F_h}{F_m}\leq(\frac{F_h}{F_m})_{ml,\theta}$,

1. *When* $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$ *and* $F_h<F_h^\dagger(B)$, *if* $F_m\geq\phi(F_h,B)F_h$, $\widetilde{w_h}=\widetilde{w_h}([\phi(F_h,B)]^{-1})$ *and* $\widetilde{w_m}=w_l=\widetilde{w_m}([\phi(F_h,B)]^{-1})$ $(<\theta A_T<\widetilde{w_h})$*; otherwise, same as* (*a*) *when* $P(F_h,F_m,B)\leq\theta$.
2. *Or else,* $\widetilde{w_h}=\widetilde{w_h}((\frac{F_h}{F_m})_{ml,\theta})$ *and* $\widetilde{w_m}=w_l=\theta A_T(<\widetilde{w_h})$.

Figure 4 illustrates magnitude relations of $\widetilde{w_h}$, $\widetilde{w_m}$, and w_l and how the wages depend on F_h, F_m, and B when $\frac{\gamma_B}{1-\gamma_B}(1+r)B<\theta A_T$ on the (F_m,F_h) plane. In the figure, the locus $P(F_h,F_m,B)=\theta$ is represented by a bold dashed line and $P=\theta$ on or above the line.

When $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$, the non-poor is abundant relative to the poor (those with $b^i \in [e_m, e_h)$) and $\widetilde{w_h} = \widetilde{w_m} = \widetilde{w_m}((\frac{F_h}{F_m})_{hm})$ (the same level for any F_h and F_m in this region). Since both the non-poor and the poor receive the same level of net wage, the demand for good T increases with $L_h + L_m = F_h + F_m$, not F_h and F_m separately, and thus w_l increases with $F_h + F_m$, unless $F_h + F_m$ is high enough that $P = \theta$ and $w_l = \theta A_T$ hold.

When $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$, the non-poor are scarce relative to the poor and thus $\widetilde{w_h} > \widetilde{w_m}$ and $L_h = F_h$. When the scarcity is not so great, i.e. $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$, the net middle-skill wage is not very low and $\widetilde{w_m} > w_l$ and $L_m = F_m$ hold. Hence, $\widetilde{w_h}$ decreases and $\widetilde{w_m}$ increases with $\frac{F_h}{F_m}$, while $w_l = P(F_h, F_m, B)A_T$ increases with F_h, F_m, and B, unless they are high enough that $P = \theta$. When the scarcity is greater, i.e. $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$, the result depends on F_h and $\frac{F_h}{F_m}$. For given $\frac{F_h}{F_m}$, if F_h (and thus F_m) is small, i.e. $F_m < \phi(F_h, B)F_h$, the result is same as the previous case, whereas if F_h is higher, the demand for good T (and thus P) is high enough that $\widetilde{w_m} < w_l$ with $L_m = F_m$ and hence $\widetilde{w_m} = w_l$ in equilibrium. When $F_h < F_h^{\dagger}(B)$ and thus $L_m = \phi(F_h, B)F_h$, $\widetilde{w_h} = \widetilde{w_h}([\phi(F_h, B)]^{-1})$ and $\widetilde{w_m} = w_l = \widetilde{w_m}([\phi(F_h, B)]^{-1})$, that is, $\widetilde{w_h}$ decreases and $\widetilde{w_m} = w_l$ increases with F_h and B, while when $F_h \geq F_h^{\dagger}(B)$ and thus $P = \theta$ and $L_m = [(\frac{F_h}{F_m})_{ml,\theta}]^{-1} F_h$, $\widetilde{w_m} = w_l = \theta A_T$ and $\widetilde{w_h} = \widetilde{w_h}((\frac{F_h}{F_m})_{ml,\theta})$, that is, the wages are constant.

To summarize, when $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$, $\widetilde{w_h} = \widetilde{w_m} > w_l$; when $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$ and either $\frac{F_h}{F_m} > (\frac{F_h}{F_m})_{ml,\theta}$ or $F_m < \phi(F_h, B)F_h$, $\widetilde{w_h} > \widetilde{w_m} > w_l$; and when $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$ and $F_m \geq \phi(F_h, B)F_h$, $\widetilde{w_h} > \widetilde{w_m} = w_l$. w_l is the highest when $P = \theta$ (the region on or above the bold dashed line) and reaches the infimum as $F_h, F_m \to 0$; while $\widetilde{w_h}(\widetilde{w_m})$ is lowest (highest) when $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$ and $\widetilde{w_h}(\widetilde{w_m})$ reaches the supremum (infimum) as $\frac{F_h}{F_m} \to [\overline{\phi}(B)]^{-1}$ ($[\overline{\phi}(B)]^{-1}$ is smallest $\frac{F_h}{F_m}$ satisfying $F_m = \phi(F_h, B)F_h$) when $F_m < \phi(F_h, B)F_h$ and as $F_h \to 0$ when $F_m \geq \phi(F_h, B)F_h$.

2.3 Net aggregate income and average utility

Proposition 1 shows that increases in F_h and F_m raise proportions of workers with higher net wages (except when $\widetilde{w_m} = w_l$, in which an increase in F_m has no effect), while Proposition 2 shows that increases in F_h, F_m, and B (when $\widetilde{w_m} = w_l < \theta A_T$) decrease either $\widetilde{w_h}$ or $\widetilde{w_m}$ except when $\widetilde{w_h} = \widetilde{w_m}$ and when $\widetilde{w_m} = w_l = \theta A_T$. So the effect of these variables on aggregate income net of education costs, $NI \equiv \widetilde{w_h} L_h + \widetilde{w_m} L_m + w_l(1 - L_h - L_m) + (1+r)B$, is not obvious. The effect on average utility of the population too is unclear, because it can be expressed as $(\gamma_B)^{\gamma_B}(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}\frac{NI}{(P)^{\gamma_B}}$. The next proposition examines the effects on these variables.

Proposition 3 (Net aggregate income and average utility) *Suppose $F_h > 0$.*

(i) If $\frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$, NI and average utility increase with $F_h + F_m$ and B.

(ii) If $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$,

(a) If $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$, they increase with F_h, F_m, and B.

(b) If $\frac{F_h}{F_m} \le (\frac{F_h}{F_m})_{ml,\theta}$*,*

1. *When* $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ *and* $F_h < F_h^{\dagger}(B)$*, if* $F_m \ge \phi(F_h,B)F_h$*, they increase with* F_h *and* B*; otherwise, same as (a).*
2. *Or else, they increase with* F_h *and* B*.*

Both net aggregate income and average utility increase with B and the proportion(s) of individuals accessible to education for jobs with higher net wages, i.e. F_h+F_m when $\widetilde{w_h}=\widetilde{w_m}$, F_h and F_m when $\widetilde{w_h}>\widetilde{w_m}>w_l$, and F_h when $\widetilde{w_m}=w_l$. As for NI and average utility when $P=\theta$, this is because the negative effect through $\widetilde{w_h}$ or $\widetilde{w_m}$ (except when $\widetilde{w_h}=\widetilde{w_m}>w_l=\theta A_T$ or $\widetilde{w_h}>\widetilde{w_m}=w_l=\theta A_T$) is dominated by positive effects through other wages (except when $\widetilde{w_h}=\widetilde{w_m}>w_l=\theta A_T$), proportions of workers with higher net wages, and B. When $P<\theta$, increases in these variables raise P and thus have a negative effect on average utility, but the positive effect through net aggregate income dominates.

2.4 Aggregate output and sectoral composition

The final analysis of the quasi-static economy is concerned with effects of F_h, F_m, and B on aggregate output, $Y=Y_M+PY_T$, the share of the modern sector in production, $\frac{Y_M}{Y}$, and the sector's share in basic consumption when $P=\theta$, $\frac{C_{BM}}{PC_B}$.

Proposition 4 (Aggregate output and sectoral composition) *Suppose* $F_h>0$*.*

(i) If $\frac{F_h}{F_m} \ge (\frac{F_h}{F_m})_{hm}$*, when* $F_h+F_m < \frac{(1-\gamma_B)\theta A_T - \gamma_B(1+r)B}{\left[\gamma_B \widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + (1-\gamma_B)\theta A_T\right]}$*,* Y *increases with* F_h+F_m *and* B*, and* $\frac{Y_M}{Y}$ *increases with* $\frac{F_h+F_m}{B}$*; otherwise, they increase with* F_h+F_m*, and* $\frac{C_{BM}}{PC_B}$ *increases with* F_h+F_m *and* B*.*

(ii) If $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$*,*

(a) If $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$*, when* $P(F_h,F_m,B) \le \theta$ *(possible only when* $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$*),* Y *increases with* F_h*,* F_m*, and* B*, and* $\frac{Y_M}{Y}$ *increases with* F_h *and* F_m *and decreases with* B*; otherwise, they increase with* F_h *and* F_m*, and* $\frac{C_{BM}}{PC_B}$ *increases with* F_h*,* F_m*, and* B*.*

(b) If $\frac{F_h}{F_m} \le (\frac{F_h}{F_m})_{ml,\theta}$*,*

1. *When* $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ *and* $F_h < F_h^{\dagger}(B)$*, if* $F_m \ge \phi(F_h,B)F_h$*,* Y *increases with* F_h *and* B*, and* $\frac{Y_M}{Y}$ *decreases with* B *(depends on* F_h *too); otherwise, same as (a) when* $P(F_h,F_m,B) \le \theta$*.*
2. *Or else,* Y *and* $\frac{Y_M}{Y}$ *increase with* F_h*, and* $\frac{C_{BM}}{PC_B}$ *increases with* F_h *and* B*.*

When $P < \theta$, aggregate output increases with B and the proportion(s) of individuals accessible to education for jobs with higher net wages, as net aggregate income and average utility do. In the case of $F_m<\phi(F_h,B)F_h$, this is because the increased proportion(s) raises L_h and L_m and shifts production to the more productive modern sector (an increase in Y_M is greater than a decrease in Y_T), plus they and B increase net aggregate income, thereby raising the demand for good T and thus P.[18] The share of the modern sector in total production increases with the proportion(s) (except the case $F_m \geq \phi(F_h,B)F_h$ of (b) 1, in which the effect is ambiguous) but *decreases* with B.

When $P=\theta$, by contrast, P does not depend on net aggregate income and thus Y and $\frac{Y_M}{Y}$ are independent of B (and increase with the proportion(s)). The modern sector too produces goods for basic consumption, i.e. $C_{BM}>0$, in this case. The proportion of basic consumption supplied by the modern sector increases with B as well as the proportion(s), because $\frac{C_{BM}}{PC_B} = \frac{PC_B - PY_T}{PC_B} = 1 - \frac{\theta Y_T}{\gamma_B NI}$ and thus it increases with net aggregate income and decreases with $Y_T = A_T(1-L_h-L_m)$.

3 Dynamics

As noted at the beginning of the previous section, the model can be considered as a sequence of quasi-static economies connected by intergenerational transfers. Based on results of the previous section, this section takes into account the intergenerational linkages.

3.1 Dynamics of individual transfers

Remember that the individual transfer rule is given by (now with time subscripts):

$$b^i_{t+1} = \gamma_b[w^i_t + (1+r)a^i_t], \tag{20}$$

where w^i_t and a^i_t are the wage and the asset of individual i born in period $t-1$ and spends period t as an adult, and b^i_{t+1} is the transfer to her child (whose adulthood is in period t+1).

Since a^i_t depends on b^i_t, the dynamic equation linking the received transfer b^i_t to the transfer given to the next generation b^i_{t+1} can be derived from the above equation. For a high-skill worker, by substituting $a^i_t = b^i_t - e_h$ into (20) and using $\widetilde{w_{ht}} = w_{ht} - (1+r)e_h$,

$$b^i_{t+1} = \gamma_b\{\widetilde{w_{ht}} + (1+r)b^i_t\}, \tag{21}$$

where $b^i_t \geq e_h$. $\gamma_b(1+r) < 1$ is assumed so that the fixed point of the equation for given $\widetilde{w_{ht}}$, $b^*(\widetilde{w_{ht}}) \equiv \frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_{ht}}$, exists. The fixed point becomes crucial in later analyses. For a middle-skill worker, a similar equation with the net wage $\widetilde{w_{mt}}$ and $b^i_t \geq e_m$ holds. Finally, for a low-skill worker, since $a^i_t = b^i_t$,

$$b^i_{t+1} = \gamma_b\{w_{lt} + (1+r)b^i_t\}. \tag{22}$$

The equations show that the dynamics of transfers within a lineage depend on the time evolution of wages, which in turn are determined by the dynamics of F_{ht}, F_{mt}, and B_t.

[18] In the case $F_m \geq \phi(F_h,B)F_h$ of (b) 1, the effect of F_h on Y_M is ambiguous and that of B is negative, but their effects on PY_T are positive and dominate.

3.2 Aggregate dynamics

Given the initial distribution of transfers over the population, F_{h0}, F_{m0}, and B_0 are determined directly, while levels of the aggregate variables in subsequent periods are determined by the dynamics of the distribution of transfers. However, information on the distributional dynamics is not required to derive main implications of the model. What is needed is information on *directions of motion* of the aggregate variables, which is examined in this subsection. For exposition, the dynamics of F_{ht} and F_{mt} and those of B_t are examined separately fixing the other variable(s) first, then their interactions are taken into account.

3.2.1 Dynamics of F_{ht} and F_{mt}

The dynamics of F_{ht} and F_{mt} are determined by the dynamics of individual transfers. As for the dynamics of F_{ht}, if children of some middle-skill workers become accessible to advanced education through wealth accumulation, $F_{ht+1} > F_{ht}$ holds.[19] This takes places iff there exist lineages satisfying $b_t^i < e_h$ and $b_{t+1}^i \geq e_h$. From (21) with $\widetilde{w_{ht}}$ replaced by $\widetilde{w_{mt}}$, the following condition must hold for such lineages to exist:

$$b^*(\widetilde{w_{mt}}) = \frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_{mt}} > e_h. \tag{23}$$

If the equation holds, $F_{ht+1} \geq F_{ht}$, otherwise, $F_{ht+1} = F_{ht}$. (When the equation holds, $F_{ht+1} = F_{ht}$ is possible depending on the distribution of transfers, but, if it continues to hold, F_{ht} does increase at some point.)

Regarding levels of $b^*(\widetilde{w_{ht}})$ and $b^*(\widetilde{w_{mt}})$, the following is assumed.

Assumption 2 $b^*(\widetilde{w_h}((\frac{F_h}{F_m})_{hm})) = b^*(\widetilde{w_m}((\frac{F_h}{F_m})_{hm})) = \frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) > e_h$.

The assumption implies that offspring of high-skill (middle-skill) workers can afford advanced education when their wage is lowest (highest) and thus F_{ht} never decreases. Assume that the initial distribution of transfers is such that $F_{h0} > 0$. Then, $F_{ht} > 0$ for any $t > 0$.

As for the dynamics of F_{mt}, since $F_{ht+1} \geq F_{ht}$ is true, if $b^*(w_{lt}) = \frac{\gamma_b}{1-\gamma_b(1+r)}w_{lt} > e_m$, $F_{ht+1} + F_{mt+1} \geq F_{ht} + F_{mt}$; if $b^*(\widetilde{w_{mt}}) = \frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_{mt}} < e_m$, $F_{ht+1} = F_{ht}$ and $F_{mt+1} \leq F_{mt}$; otherwise, $F_{ht+1} + F_{mt+1} = F_{ht} + F_{mt}$.

Hence, directions of motion of F_{ht} and F_{mt} can be known from relative values of $b^*(\widetilde{w_{mt}})$ to e_h and e_m and of $b^*(w_{lt})$ to e_m, except when $b^*(\widetilde{w_{mt}}) > e_h$ and $b^*(w_{lt}) > e_m$, in which the direction of motion of F_{mt} is ambiguous ($F_{ht+1} \geq F_{ht}$ and $F_{ht+1} + F_{mt+1} \geq F_{ht} + F_{mt}$).

Regarding the value of $b^*(w_{lt})$, the following is assumed.

Assumption 3 $\frac{\gamma_b}{1-\gamma_b(1+r)}\theta A_T \in (e_m, e_h)$.

[19] From Assumption 3 below, children of low-skill workers never become accessible to advanced education.

The assumption states that children of some low-skill workers can afford basic education but not advanced education when their wage is highest. The two assumptions are maintained until Section 4.3 where effects of productivity growth are examined.

From the two assumptions and Proposition 2, there exist combinations of F_h and F_m satisfying $b^*(\widetilde{w_m}) = e_h$, those satisfying $b^*(\widetilde{w_m}) = e_m$, and those satisfying $b^*(w_l) = e_m$ (see Figure 5). $b^*(\widetilde{w_m}) = e_h$ equals a $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$ such that $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}(\frac{F_h}{F_m}) = e_h$. If $F_h^\flat(B)$ (a decreasing function) is defined as F_h satisfying $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}(\frac{1}{\phi(F_h,B)}) = e_m$, $b^*(\widetilde{w_m}) = e_m$ equals a $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{ml,\theta}$ such that $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}(\frac{F_h}{F_m}) = e_m$ for $F_m < \phi(F_h^\flat(B),B)F_h^\flat(B)$ and equals $F_h = F_h^\flat(B)$ for $F_m \geq \phi(F_h^\flat(B),B)F_h^\flat(B)$. Finally, $b^*(w_l) = e_m$ equals:

$$\text{for } \frac{F_h}{F_m} \geq (\tfrac{F_h}{F_m})_{hm}, \qquad \frac{\gamma_b}{1-\gamma_b(1+r)}\frac{\gamma_B}{1-\gamma_B}\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B}{1-(F_h+F_m)} = e_m \tag{24}$$

$$\Leftrightarrow F_h+F_m = \frac{\frac{1-\gamma_b(1+r)}{\gamma_b}e_m - \frac{\gamma_B}{1-\gamma_B}(1+r)B}{\frac{\gamma_B}{1-\gamma_B}\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + \frac{1-\gamma_b(1+r)}{\gamma_b}e_m}; \tag{25}$$

$$\text{for } \frac{F_h}{F_m} \in \left(\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], (\tfrac{F_h}{F_m})_{hm}\right), \quad \frac{\gamma_b}{1-\gamma_b(1+r)}P(F_h,F_m,B)A_T = e_m; \tag{26}$$

$$\text{and for } \frac{F_h}{F_m} \leq \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], \qquad F_h = F_h^\flat(B). \tag{27}$$

Figure 5 illustrates the dynamics of F_{ht} and F_{mt} for given B by placing the three critical loci on the (F_m, F_h) plane. In the figure, $b^*(\widetilde{w_m}) > (<) e_h$ at the left (right) side of $b^*(\widetilde{w_m}) = e_h$ (the bold solid line), $b^*(\widetilde{w_m}) > (<) e_m$ above (below) $b^*(\widetilde{w_m}) = e_m$ (the bold dashed line), and $b^*(w_l) > (<) e_m$ above (below) $b^*(w_l) = e_m$ (the bold dotted line). Positions of F_{ht} and F_{mt} relative to the three loci determine directions of motion of the two variables. In regions with horizontal arrows only, only F_{mt} changes: for example, in the region below $b^*(\widetilde{w_m}) = e_m$, $b^*(\widetilde{w_m}) < e_m$ holds and thus F_{mt} decreases. Arrows with slope -1 are present in the region on or below $b^*(w_l) = e_m$ and above $b^*(\widetilde{w_m}) = e_h$, because $b^*(\widetilde{w_m}) > e_h$ and $b^*(w_l) \leq e_m$ and thus F_{ht} increases with $F_{ht}+F_{mt}$ constant. By contrast, in the region above $b^*(w_l) = e_m$ and $b^*(\widetilde{w_m}) = e_h$ (thus $b^*(w_l) > e_m$ and $b^*(\widetilde{w_m}) > e_h$) and below $F_h+F_m = 1$, arrows with slope -1 and horizontal arrows are drawn, since F_{ht} and $F_{ht}+F_{mt}$ increase but the direction of F_{mt} is ambiguous (the direction of motion of F_{ht} and F_{mt} is between the two arrows). Finally, both F_{ht} and F_{mt} are constant and thus no arrows are present in the region on or below $b^*(\widetilde{w_m}) = e_h$ and $b^*(w_l) = e_m$ and on or above $b^*(\widetilde{w_m}) = e_m$.

Note that positions of $b^*(\widetilde{w_m}) = e_m$ and $b^*(w_l) = e_m$ as well as those of $P(F_h,F_m,B) = \theta$ and $F_m = \phi(F_h,B)F_h$ *change with* B. Thus, the dynamics of F_{ht} and F_{mt} must be examined together with those of B_t. Before examining the joint dynamics, the dynamic equation of B_t is derived and the direction of motion of B_t for *given* F_{ht} and F_{mt} is examined next.

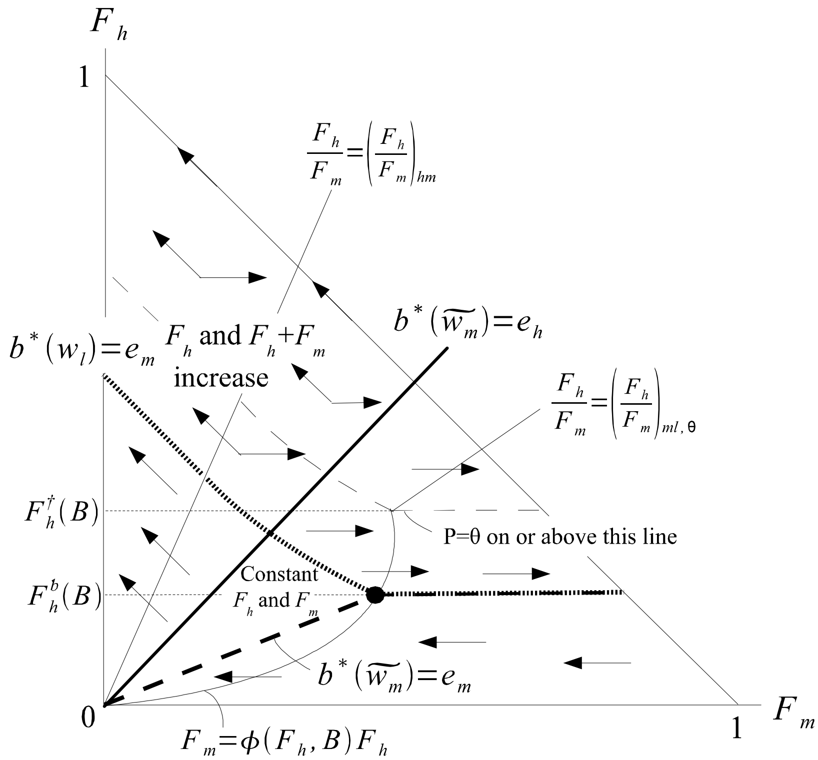

Figure 5: Dynamics of F_{ht} and F_{mt} for given B

3.2.2 Dynamics of aggregate transfers

The dynamic equation of aggregate transfers is obtained by aggregating the dynamic equations for individual transfers over the population:

$$B_{t+1}=\gamma_b\left\{\widetilde{w_{ht}}L_{ht}+\widetilde{w_{mt}}L_{mt}+w_{lt}(1-L_{ht}-L_{mt})+(1+r)B_t\right\}, \tag{28}$$

where the expression inside the curly bracket of the RHS is aggregate net income NI_t, which can be expressed as a function of F_{ht}, F_{mt}, and B_t using the proof of Proposition 3.

When $\frac{F_{ht}}{F_{mt}}\geq(\frac{F_h}{F_m})_{hm}$, if $F_{ht}+F_{mt}<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B_t}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$ and thus $P_t<\theta$, the equation is:

$$B_{t+1}=\frac{\gamma_b}{1-\gamma_B}\{\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})(F_{ht}+F_{mt})+(1+r)B_t\}. \tag{29}$$

$\frac{\gamma_b}{1-\gamma_B}(1+r)<1$ is assumed so that the fixed point for given $F_{ht}+F_{mt}$ exists, which equals:

$$\widehat{B}^*(F_{ht}+F_{mt})=\frac{\gamma_b}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})(F_{ht}+F_{mt}). \tag{30}$$

Clearly, when $B_t<(>)\widehat{B}^*(F_{ht}+F_{mt})$, $B_{t+1}>(<)B_t$. If $F_{ht}+F_{mt}\geq\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B_t}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$ and thus $P_t=\theta$, the dynamic equation and its fixed point equal:

$$B_{t+1} = \gamma_b \{\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})(F_{ht}+F_{mt}) + \theta A_T[1-(F_{ht}+F_{mt})] + (1+r)B_t\}, \tag{31}$$

$$\widehat{B}^*(F_{ht}+F_{mt}) = \frac{\gamma_b}{1-\gamma_b(1+r)}\{\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})(F_{ht}+F_{mt}) + \theta A_T[1-(F_{ht}+F_{mt})]\}, \tag{32}$$

where $\widehat{B}^*(F_{ht}+F_{mt})$ is an increasing function.

When $\frac{F_{ht}}{F_{mt}} \in ((\frac{F_h}{F_m})_{ml,\theta},(\frac{F_h}{F_m})_{hm})$, if $P_t = P(F_{ht},F_{mt},B_t) \leq \theta$, they equal:

$$B_{t+1} = \frac{\gamma_b}{1-\gamma_B}\{[A_M(F_{ht})^{\alpha}(F_{mt})^{1-\alpha} - (1+r)(e_h F_{ht} + e_m F_{mt})] + (1+r)B_t\}, \tag{33}$$

$$B^*(F_{ht},F_{mt}) = \frac{\gamma_b}{1-\gamma_B-\gamma_b(1+r)}\{A_M(F_{ht})^{\alpha}(F_{mt})^{1-\alpha} - (1+r)(e_h F_{ht} + e_m F_{mt})\}, \tag{34}$$

where $B^*(F_{ht},F_{mt})$ is an increasing function. If $P(F_{ht},F_{mt},B_t) > \theta$ (thus $P_t = \theta$), they are:

$$B_{t+1} = \gamma_b\{A_M(F_{ht})^{\alpha}(F_{mt})^{1-\alpha} - (1+r)(e_h F_{ht} + e_m F_{mt}) + \theta A_T(1-F_{ht}-F_{mt}) + (1+r)B_t\}, \tag{35}$$

$$B^*(F_{ht},F_{mt}) = \frac{\gamma_b}{1-\gamma_b(1+r)}\{A_M(F_{ht})^{\alpha}(F_{mt})^{1-\alpha} - (1+r)(e_h F_{ht} + e_m F_{mt}) + \theta A_T(1-F_{ht}-F_{mt})\}, \tag{36}$$

where $B^*(F_{ht},F_{mt})$ is an increasing function since $\widetilde{w_{ht}} > \widetilde{w_{mt}} > w_{lt} = \theta A_T$.

When $\frac{F_{ht}}{F_{mt}} \leq (\frac{F_h}{F_m})_{ml,\theta}$, $\frac{\gamma_B}{1-\gamma_B}(1+r)B_t < \theta A_T$, and $F_{ht} < F_h^{\dagger}(B_t)$, if $F_{mt} < \phi(F_{ht},B_t)F_{ht}$, the equations are (33) and (34) above. If $F_{mt} \geq \phi(F_{ht},B_t)F_{ht}$, the dynamic equation is:

$$B_{t+1} = \frac{\gamma_b}{1-\gamma_B}\left\{\left[A_M(\phi(F_{ht},B_t))^{1-\alpha} - (1+r)(e_h + \phi(F_{ht},B_t)e_m)\right]F_{ht} + (1+r)B_t\right\}. \tag{37}$$

The next lemma shows that, given F_{ht}, B_t converges monotonically to the unique fixed point of (37), $\overline{B}^*(F_{ht})$, and $\overline{B}^*(F_{ht})$ increases and $\phi(F_{ht},\overline{B}^*(F_{ht}))$ decreases with F_{ht}.

Lemma 3 *When the dynamics of B_t follow* (37), *given F_{ht}, B_t converges monotonically to unique $\overline{B}^*(F_{ht})$, which is a solution to*

$$\overline{B}^*(F_{ht}) = \frac{\gamma_b}{1-\gamma_B-\gamma_b(1+r)}\{A_M(\phi(F_{ht},\overline{B}^*(F_{ht})))^{1-\alpha} - (1+r)(e_h + \phi(F_{ht},\overline{B}^*(F_{ht}))e_m)F_{ht}\}, \tag{38}$$

and when $B_t < (>) \overline{B}^*(F_{ht})$, $B_{t+1} > (<) B_t$. $\overline{B}^*(F_{ht})$ is increasing and $\phi(F_{ht},\overline{B}^*(F_{ht}))$ is decreasing in F_{ht} and $\lim_{F_{ht}\to 0}\phi(F_{ht},\overline{B}^*(F_{ht})) = \overline{\phi}(0) \equiv \lim_{F_{ht}\to 0}\phi(F_{ht},0)$.

When $\frac{F_{ht}}{F_{mt}} \leq (\frac{F_h}{F_m})_{ml,\theta}$ and either $\frac{\gamma_B}{1-\gamma_B}(1+r)B_t < \theta A_T$ and $F_{ht} \geq F_h^{\dagger}(B_t)$ or $\frac{\gamma_B}{1-\gamma_B}(1+r)B_t \geq \theta A_T$,

$$B_{t+1} = \gamma_b\{\widetilde{w_h}((\tfrac{F_h}{F_m})_{ml,\theta})F_{ht} + \theta A_T(1-F_{ht}) + (1+r)B_t\}, \tag{39}$$

$$\overline{B}^*(F_{ht}) = \frac{\gamma_b}{1-\gamma_b(1+r)}\{\widetilde{w_h}((\tfrac{F_h}{F_m})_{ml,\theta})F_{ht} + \theta A_T(1-F_{ht})\}, \tag{40}$$

where $\overline{B}^*(F_{ht})$ is an increasing function.

To summarize, the dynamic equation of B_t differs depending on F_{ht} and F_{mt}, and for given F_{ht} and F_{mt}, the direction of motion of B_t is determined by the magnitude relation of B_t to the unique fixed point of the equation: B_t increases (decreases) when it is smaller (greater) than the value at the fixed point. Based on the results of this subsection, the joint dynamics of the three variables are considered in the next subsection.

3.3 Joint dynamics of the aggregate variables

As mentioned earlier, as B_t changes over time, positions of $P(F_h,F_m,B)=\theta$, $F_m=\phi(F_h,B)F_h$, $b^*(\widetilde{w_m})=e_m$, and $b^*(w_l)=e_m$ in Figure 5 of Section 3.2 change and thus directions of motion of F_{ht} and F_{mt} could be affected. Hence, in general, it is difficult to analyze the joint dynamics even using a diagram like Figure 5.

However, it turns out that under the following weak assumption on B_0, characteristics of the dynamics are mostly determined by relative positions of F_{ht} and F_{mt} to these loci when aggregate transfers are *at fixed points levels* (and relative positions of F_{ht} and F_{mt} to $b^*(\widetilde{w_m})=e_h$, $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$, and $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{ml,\theta}$).

Assumption 4 *The initial level of aggregate transfers is such that* $B_0\leq\overline{B}^*(F_{h0})$ *when* $\frac{F_{h0}}{F_{m0}}\leq\min\{[\phi(F_{h0},B_0)]^{-1},(\frac{F_h}{F_m})_{ml,\theta}\}$, $B_0\leq B^*(F_{h0},F_{m0})$ *when* $\frac{F_{h0}}{F_{m0}}\in\left(\min\{[\phi(F_{h0},B_0)]^{-1},(\frac{F_h}{F_m})_{ml,\theta}\},(\frac{F_h}{F_m})_{hm}\right)$, *and* $B_0\leq\widehat{B}^*(F_{h0}+F_{m0})$ *when* $\frac{F_{h0}}{F_{m0}}\geq(\frac{F_h}{F_m})_{hm}$.

The assumption states that the initial level of aggregate transfers is less than the fixed point level at $(F_h,F_m)=(F_{h0},F_{m0})$.

From (16) and (34), $P(F_h,F_m,B^*(F_h,F_m))=\theta$ equals:

$$\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\frac{A_M(F_h)^{\alpha}(F_m)^{1-\alpha}-(1+r)(e_hF_h+e_mF_m)}{A_T[1-(F_h+F_m)]}=\theta. \tag{41}$$

As for $F_m=\phi(F_h,\overline{B}^*(F_h))F_h$, Lemma 3 shows that $\phi(F_h,\overline{B}^*(F_h))$ is decreasing in F_h and $\lim_{F_h\to 0}\phi(F_h,\overline{B}^*(F_h))=\overline{\phi}(0)$. If $F_h^\flat$ is defined as F_h satisfying $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}(\frac{1}{\phi(F_h,\overline{B}^*(F_h))})=e_m$ (that is, F_h satisfying $F_h=F_h^\flat(\overline{B}^*(F_h))$), $b^*(\widetilde{w_m})=e_m$ equals a $\frac{F_h}{F_m}<(\frac{F_h}{F_m})_{ml,\theta}$ such that $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}(\frac{F_h}{F_m})=e_m$ for $F_m<\phi(F_h^\flat,\overline{B}^*(F_h^\flat))F_h^\flat$ and $F_h=F_h^\flat$ for $F_m\geq\phi(F_h^\flat,\overline{B}^*(F_h^\flat))F_h^\flat$. Finally, from (25) and (30), $b^*(w_l)=e_m$ equals:

$$\text{for } \frac{F_h}{F_m}\geq(\tfrac{F_h}{F_m})_{hm},\quad \frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\tfrac{F_h}{F_m})_{hm})\frac{F_h+F_m}{1-(F_h+F_m)}=\frac{1-\gamma_b(1+r)}{\gamma_b}e_m$$

$$\Leftrightarrow F_h+F_m=\frac{\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}{\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}; \tag{42}$$

$$\text{for } \frac{F_h}{F_m}\in\left(\widetilde{w_m}^{-1}\left[\tfrac{1-\gamma_b(1+r)}{\gamma_b}e_m\right],(\tfrac{F_h}{F_m})_{hm}\right),\quad \frac{\gamma_b}{1-\gamma_b(1+r)}P(F_h,F_m,B^*(F_h,F_m))A_T=e_m; \tag{43}$$

$$\text{and for } \frac{F_h}{F_m}\leq\widetilde{w_m}^{-1}\left[\tfrac{1-\gamma_b(1+r)}{\gamma_b}e_m\right],\qquad F_h=F_h^\flat. \tag{44}$$

Shapes of these loci are similar to the case of constant B and their positions on the (F_h,F_m) plane can be illustrated by a figure similar to Figure 5.

4 Main Results

4.1 Characteristics of steady states

Before the dynamics are examined, characteristics of steady states are investigated. The next proposition shows that there exist four types of steady states. Note that $F_h^{\dagger}$ is defined as F_h satisfying $[\phi(F_h,\overline{B}^*(F_h))]^{-1} = (\frac{F_h}{F_m})_{ml,\theta}$ (that is, F_h satisfying $F_h = F_h^{\dagger}(\overline{B}^*(F_h))$).

Proposition 5 (Steady states) *There exist the following four types of steady states.*[20]

1. $(F_h,F_m,B) = (1,0,\widehat{B}^*(1))$. *$L_h$ and L_m satisfy* $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$ *and* $L_h + L_m = 1$ *(thus* $L_l = 0$ *and* $Y_T = 0$*),* $P = \theta$*, and* $\widetilde{w_h} = \widetilde{w_m} = \widetilde{w_m}((\frac{F_h}{F_m})_{hm})$.
2. $F_h = L_h$ *satisfies* $F_h > F_h^{\flat}$ *and* $b^*(\widetilde{w_m}) \le e_h \Leftrightarrow \frac{F_h}{1-F_h} \le \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$*, and* $F_m = 1-F_h$.
   a. *When* $\frac{F_h}{1-F_h} \le (\frac{F_h}{F_m})_{ml,\theta}$*,* $B = \overline{B}^*(F_h)$*,* $L_m = \max\{\phi(F_h,\overline{B}^*(F_h)),[(\frac{F_h}{F_m})_{ml,\theta}]^{-1}\}F_h$ *(thus* $L_l = 1-F_h-L_m$ *and* $Y_T > 0$*),* $P = P(F_h,L_m,\overline{B}^*(F_h)) < \theta$ *if* $F_h < F_h^{\dagger}$ *and* $P = \theta$ *otherwise, and* $\widetilde{w_h} = \widetilde{w_h}(\min\{[\phi(F_h,\overline{B}^*(F_h))]^{-1},(\frac{F_h}{F_m})_{ml,\theta}\}) > \widetilde{w_m} = w_l = PA_T$.
   b. *When* $\frac{F_h}{1-F_h} > (\frac{F_h}{F_m})_{ml,\theta}$*,* $B = B^*(F_h,F_m)$*,* $L_m = F_m = 1-F_h$ *(thus* $L_l = 0$ *and* $Y_T = 0$*),* $P = \theta$*, and* $\widetilde{w_h} = \widetilde{w_h}(\frac{F_h}{F_m}) > \widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{F_m})$.
3. F_h *satisfies* $b^*(w_l) \le e_m \Leftrightarrow F_h \le \dfrac{\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}{\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + \frac{1-\gamma_b(1+r)}{\gamma_b}e_m}$ *and* $(F_m,B) = (0,\widehat{B}^*(F_h))$. *$L_h$ and L_m satisfy* $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$ *and* $L_h + L_m = F_h$ *(thus* $L_l = 1-F_h$ *and* $Y_T > 0$*),* $P = \frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})F_h}{A_T(1-F_h)} < \theta$*, and* $\widetilde{w_h} = \widetilde{w_m} = \widetilde{w_m}((\frac{F_h}{F_m})_{hm}) > w_l = PA_T$.
4. F_h *and* F_m *satisfy* $\frac{F_h}{F_m} \in \left[\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]\right]$ *and* $P(F_h,F_m,B^*(F_h,F_m))A_T \le \frac{1-\gamma_b(1+r)}{\gamma_b}e_m$*, and* $B = B^*(F_h,F_m)$. $L_h = F_h$*,* $L_m = F_m$*, and* $L_l = 1-F_h-F_m$ *(thus* $Y_T > 0$*),* $P = P(F_h,F_m,B^*(F_h,F_m)) < \theta$*, and* $\widetilde{w_h} = \widetilde{w_h}(\frac{F_h}{F_m}) > \widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{F_m}) > w_l = PA_T$.

Figure 6 illustrates four types of steady states, which differ in proportions of the poor and the very poor, wage inequality, the size of the traditional sector, on the (F_h,F_m) plane. In Steady state 1, all individuals are non-poor, i.e. they have enough wealth to take advanced education ($F_h = 1$), net wages of high-skill and middle-skill workers are equal ($\widetilde{w_h} = \widetilde{w_m}$), and the traditional sector does not exist (thus $P = \theta$ and $L_l = 0$). In Steady state 2, the very poor do not exist, i.e. everyone can access at least basic education ($F_h + F_m = 1$), but (net) wage inequality between high-skill workers and others exists ($\widetilde{w_h} > \widetilde{w_m}$). When $\frac{F_h}{1-F_h} \le (\frac{F_h}{F_m})_{ml,\theta}$, net wages of middle-skill and low-skill workers are equal ($\widetilde{w_m} = w_l$), thus some do not take basic education ($L_l > 0$) and the traditional sector exists, where $P < \theta$ if $F_h < F_h^{\dagger}$ and $P = \theta$ otherwise. When $\frac{F_h}{1-F_h} > (\frac{F_h}{F_m})_{ml,\theta}$, by contrast, everyone takes at least basic education

[20]Actually, there exists another type of steady states satisfying $F_h = F_h^{\flat}$, $F_m > \phi(F_h,\overline{B}^*(F_h))F_h$, and $B = \overline{B}^*(F_h)$, but, as shown in the proof of Proposition 7, this type of steady states can be reached only if the economy starts from the steady state values and thus is not considered.

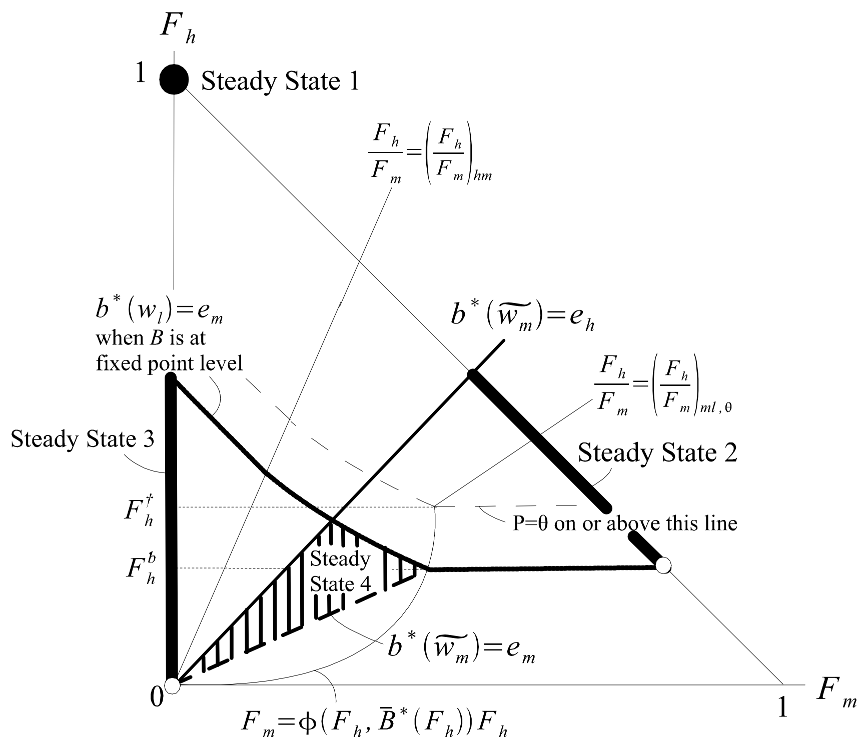

Figure 6: Steady states (Proposition 5)

($L_l\!=\!0$), thus only the modern sector exists and $P\!=\!\theta$. In Steady state 3, there are no poor people ($F_m\!=\!0$) and $\widetilde{w_h}\!=\!\widetilde{w_m}$ holds as in Steady state 1, but the very poor do exist ($F_h\!<\!1$) and become low-skill workers ($L_l > 0$), inequality between low-skill workers and others is high, and only the traditional sector supplies goods for basic consumption (thus $P < \theta$). Finally, in Steady state 4, there are both poor and very poor people, inequality among three types of workers exist ($\widetilde{w_h}\!>\!\widetilde{w_m}\!>\!w_l$), and the traditional sector is the sole supplier of goods for basic consumption (thus $P\!<\!\theta$).

Steady state 1 has features of a typical developed economy: no poverty, low wage inequality (wages net of education costs are equal), and no traditional sector (goods for basic consumption are totally supplied by the modern sector). Other types of steady states share the contrasting features (except the fact that the traditional sector does not exist when $\frac{F_h}{1-F_h}>(\frac{F_h}{F_m})_{ml,\theta}$ of Steady state 2), but differ in characteristics of poverty and wage inequality. In Steady state 2, extreme poverty does not exist but many cannot access advanced education, thus wage inequality between high-skill and other workers is high, while inequality between middle-skill and low-skill workers is low. In Steady state 3, those who can afford basic education can access advanced education as well, but many cannot afford even basic education, hence wage inequality between low-skill workers and others is high, while net wages of high-skill and middle-skill workers are equal. In Steady state 4, many cannot afford basic or advanced education, and typically inequality between middle-skill and low-skill workers as well as the one between high-skill and middle-skill workers are high.

The next proposition examines the steady states in terms of welfare, output, and sectoral composition, based on Propositions 3 and 4 of Section 2 and the previous proposition.

Proposition 6 (Welfare, output, and sectoral composition in steady states)

(i) *Aggregate net income and average utility are highest in Steady state 1. They increase with F_h in Steady states 2 and 3, and with F_h and F_m in Steady state 4. Their maxima in Steady states 2 and 3 are strictly higher than the ones in Steady state 4, and the infinima in Steady state 2 are strictly higher than the ones in Steady states 3 and 4.*

(ii) *The same result as* (i) *holds for aggregate output, except that the magnitude relation of the maxima in Steady states 3 and 4 is unclear. In Steady state 1, $\frac{Y_M}{Y} = \frac{C_{BM}}{PC_B} = 1$. In Steady state 2, if $F_h < F_h^\dagger$, $\frac{Y_M}{Y}$ increases (decreases) with $\frac{F_h}{F_m} = [\phi(F_h,\overline{B}^*(F_h))]^{-1}$ for $[\phi(F_h,\overline{B}^*(F_h))]^{-1} > (<) \frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$, where $\frac{\alpha}{1-\alpha}\frac{e_m}{e_h} > \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$; if $F_h \geq F_h^\dagger$ and $\frac{F_h}{1-F_h} \leq (\frac{F_h}{F_m})_{ml,\theta}$, $\frac{Y_M}{Y}$ and $\frac{C_{BM}}{PC_B}$ increase with F_h; otherwise, $\frac{Y_M}{Y} = \frac{C_{BM}}{PC_B} = 1$. In Steady state 3, $\frac{Y_M}{Y}$ is constant. In Steady state 4, $\frac{Y_M}{Y}$ increases (decreases) with $\frac{F_h}{F_m}$ for $\frac{F_h}{F_m} > (<) \frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$.*[21]

The proposition proves that Steady state 1 is the best in terms of aggregate net income, average utility, and aggregate output. Other steady states cannot be ranked definitely, but if they are to be ranked, Steady state 2 is the second best, Steady state 3 follows, and Steady state 4 is the worst: the maximum values of these variables in Steady states 2 and 3 (except aggregate output in Steady state 3) are strictly higher than the ones in Steady state 4, and the infinima in Steady state 2 are strictly higher than the ones in Steady states 3 and 4.

The proposition also shows that the three variables increase with the proportion(s) of individuals accessible to education for jobs with higher net wages, i.e. F_h in Steady states 2 and 3, and F_h and F_m in Steady state 4 (see Figure 6). As for the shares of the modern sector in production and in basic consumption, when $P < \theta$ (thus $\frac{C_{BM}}{PC_B} = 0$), $\frac{Y_M}{Y}$ depends on $\frac{F_h}{F_m}$ and the relation can be *non-monotonic*: in the case $F_h < F_h^\dagger$ of Steady state 2 and in Steady state 4, $\frac{Y_M}{Y}$ decreases with $\frac{F_h}{F_m}$ for $\frac{F_h}{F_m} < \frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$ (note $\frac{\alpha}{1-\alpha}\frac{e_m}{e_h} > \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$) and the relation turns positive for $\frac{F_h}{F_m} > \frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$ if $\frac{\alpha}{1-\alpha}\frac{e_m}{e_h} < \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$. That is, the production share of the modern sector *decreases* with $\frac{F_h}{F_m}$ when $\frac{F_h}{F_m}$ is relatively low. By contrast, when $P = \theta$, i.e. in the case $F_h \geq F_h^\dagger$ and $\frac{F_h}{1-F_h} \leq (\frac{F_h}{F_m})_{ml,\theta}$ of Steady state 2, $\frac{Y_M}{Y}$ and $\frac{C_{BM}}{PC_B}$ increase with F_h. (They equal 1 in Steady state 1 and in the case $\frac{F_h}{1-F_h} > (\frac{F_h}{F_m})_{ml,\theta}$ of Steady state 2; $\frac{Y_M}{Y}(<1)$ is constant and $\frac{C_{BM}}{PC_B} = 0$ in Steady state 3.)

[21] $C_{BM} = 0$ in the case $F_h < F_h^\dagger$ of Steady state 2 and in Steady states 3 and 4.

4.2 Relationship between initial conditions and steady states

Propositions 5 and 6 characterize four types of steady states in terms of poverty, wage inequality, sectoral composition, welfare, and aggregate output and net income etc. From a given initial distribution of wealth, to which type of steady states does the economy converge in the long run? The next proposition provides the answer to this question. Since the proof of the proposition requires the lengthy and complicated analysis of the dynamics, the proof is provided in a separate appendix posted on the author's website.[22]

Proposition 7 (Initial conditions and steady states)

(i) When $\frac{F_{h0}}{F_{m0}} < \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$

a. If $F_{h0} < F_h^\flat$, F_{ht} *is constant,* F_{mt} *falls, and the economy most likely converges to Steady state 4.*[23]

b. If $F_{h0} \geq F_h^\flat$, *when* $F_{h0} \geq F_h^\flat(B_0)$, F_{ht} *is constant,* F_{mt} *increases, and the economy converges to Steady state 2.*[24] *When* $F_{h0} < F_h^\flat(B_0)$, *at first,* F_{ht} *is constant and* F_{mt} *decreases, and it could converge to any type of steady states or cycle.*[25]

(ii) When $\frac{F_{h0}}{F_{m0}} \in \left[\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]\right]$

a. If $b^*(w_l) \leq e_m$ *at* $(F_h, F_m, B) = (F_{h0}, F_{m0}, B^*(F_{h0}, F_{m0}))$, F_{ht} *and* F_{mt} *are constant and the final state is Steady state 4.*

b. Otherwise, F_{ht} *is constant,* F_{mt} *rises, and the economy converges to Steady state 2.*

(iii) When $\frac{F_{h0}}{F_{m0}} > \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$, F_{ht} *increases and* $F_{ht}+F_{mt}$ *non-decreases at first.*

a. If $\frac{F_{h0}}{F_{m0}} \geq (\frac{F_h}{F_m})_{hm}$ *and* $b^*(w_l) \leq e_m$ *at* $(F_h, F_m) = (F_{h0}, F_{m0})$ *and* $B = \widehat{B}^*(F_{h0}+F_{m0})$, $F_{ht}+F_{mt}$ *is constant and the economy converges to Steady state 3.*

b. If $\frac{F_{h0}}{F_{m0}} < (\frac{F_h}{F_m})_{hm}$ *and* $b^*(w_l) \leq e_m$ *at* $(F_h, F_m) = (F_{h0}, F_{m0})$ *and* $B = B^*(F_{h0}, F_{m0})$, *the following three scenarios are possible depending on details of the initial distribution.*

1. *The more likely is the same scenario as a.*
2. $F_{ht}+F_{mt}$ *rises from the start or after some period and the final state is Steady state 1.*
3. *After* $F_{ht}+F_{mt}$ *increases for a while,* F_{ht} *becomes constant,* F_{mt} *increases, and the economy converges to Steady state 2.*

The first scenario is more likely as F_{h0} *and* F_{m0} *are lower, and the second one is more likely than the third one as* $\frac{F_{h0}}{F_{m0}}$ *is higher.*

c. Otherwise, the same scenarios as 2. *and* 3. *of b. are possible.*

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[22] The address is http://www.econ.kyoto-u.ac.jp/~yuki/english.html.

[23] F_{mt} could "jump over" the region $\frac{F_h}{F_m} \in \left[\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]\right]$ depending on the initial distribution, in which case it converges to another type of steady states, particularly Steady state 3.

[24] The exception is when $F_{h0} = F_h^\flat$ and $B_0 = \overline{B}^*(F_{h0})$, in which case both F_{mt} and B_t are constant.

[25] The economy possibly cycles between the region $\frac{F_h}{F_m} < \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$ and $F_h \in [F_h^\flat, F_h^\flat(B))$ and the region $\frac{F_h}{F_m} \in \left[\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right], \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]\right]$.

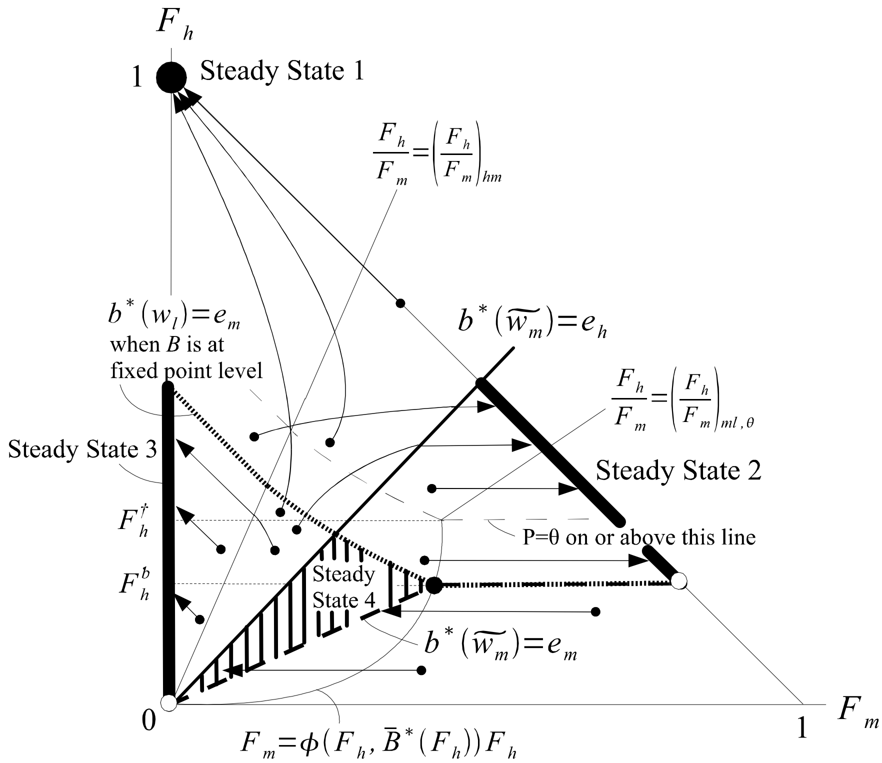

Figure 7: Initial conditions and steady states (Proposition 7)

Figure 7 presents illustrative trajectories of the dynamics based on the proposition. The position of $(F_h, F_m) = (F_{h0}, F_{m0})$ relative to $b^*(\widetilde{w_m}) = e_h$ essentially determines whether the economy can converge to Steady state 1 or not. When $\frac{F_{h0}}{F_{m0}} \leq \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$, Steady state 1 cannot be reached except rare possibilities described in (i) of the proposition. Because the ratio of high-skill workers to middle-skill workers is low, the middle-skill wage is not high enough for children of middle-skill workers to access advanced education, i.e. F_{ht} is constant. If F_{h0} and F_{m0} are relatively high, the low-skill wage is high enough that $b^*(w_l) > e_m$ holds initially (at the fixed point level of B), descendants of low-skill workers become accessible to basic education over time, i.e. F_{mt} increases, and the economy converges to Steady state 2. By contrast, if $b^*(w_l) \leq e_m$ holds initially, F_{mt} non-increases (F_{mt} decreases while $\frac{F_{ht}}{F_{mt}}$ is low enough that $b^*(\widetilde{w_m}) < e_m$ is satisfied), and the economy converges to Steady state 4.

When $\frac{F_{h0}}{F_{m0}} > \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$, the middle-skill wage is high enough that descendants of middle-skill workers become accessible to advanced education over time, i.e. F_{ht} increases. Unless $\frac{F_{h0}}{F_{m0}} \geq (\frac{F_h}{F_m})_{hm}$ and $b^*(w_l) \leq e_m \Leftrightarrow F_{h0}+F_{m0} \leq \frac{\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}{\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}$ (from eq. 42), in which case $F_{ht}+F_{mt}$ is constant and the final state is Steady state 3, the economy could converge to Steady state 1 through rises in $\frac{F_{ht}}{F_{mt}}$ and F_{ht} (thus inequalities between high-skill workers and others fall), although it could converge to Steady states 2 and 3 too depending on details of the initial distribution. Steady state 1 is more likely to be reached when wages of low-skill and middle-skill wages are high relative to the high-skill wage, i.e. when F_{h0}, F_{m0}, and $\frac{F_{h0}}{F_{m0}}$ are relatively high.

The result suggests that, for the best long-run outcome to be realized, the initial distribution of wealth must be such that the very poor (those without enough wealth to acquire basic skills) are not large in number and the non-poor (those with enough wealth to acquire advanced skills) must be sufficient relative to the poor. Both conditions seem to have held in East Asian economies evolving into developed economies, where, as in the model economy converging to the best steady state, inequalities between workers with advanced education and others fell over time (Wood, 1994). If the initial size of the very poor is large, i.e. $F_{h0} + F_{m0}$ is low, which would be true for poorest economies, the dual structure and large inequality between low-skill workers and others (particularly, high-skill workers) remain even in the long run. If the size of the very poor is not large but the non-poor are scarce relative to the poor, i.e. $F_{h0} + F_{m0}$ is not low but $\frac{F_{h0}}{F_{m0}}$ is low, which would be the case for many developing economies with decent but not spectacular growth, the proportion of middle-skill workers and the share of the modern sector rise, and inequality between middle-skill and low-skill workers shrinks over time.[26] However, inequality between high-skill and middle-skill workers worsens, and typically the traditional sector remains. These are what an average developing economy have experienced, as described at the beginning of the introduction.

The main implication is that, for the full modernization of an economy, the initial distribution of wealth must be such that extreme poverty is not prevalent so that most people can acquire basic skills and the size of "middle class" is enough so that an adequate number of workers possess advanced skills. Consistent with this and the above results, Hanushek and Woessmann (2009), using data on international student achievement tests for 50 countries, find that both the share of students with basic skills and that of top performance have significant effects on economic growth that are *complementary* each other. The model provides a sectoral-shift-based explanation for their finding.

4.3 Productivity Growth

So far, productivity levels of the two sectors, A_M and A_T, are assumed to be time-invariant. In real economy, they change over time, in particular, A_M usually grows persistently due to technological growth. What happens to the dynamics and steady states when A_M increases over time? From the equations for the critical loci in the previous section, an increase in

[26] To be precise, if the size of the non-poor is very small, i.e. $F_{h0} < F_h^\flat$, this description does not apply. As is clear from Figure 7, F_{mt} falls over time and the long-run state becomes same as the case of low $F_{h0} + F_{m0}$.

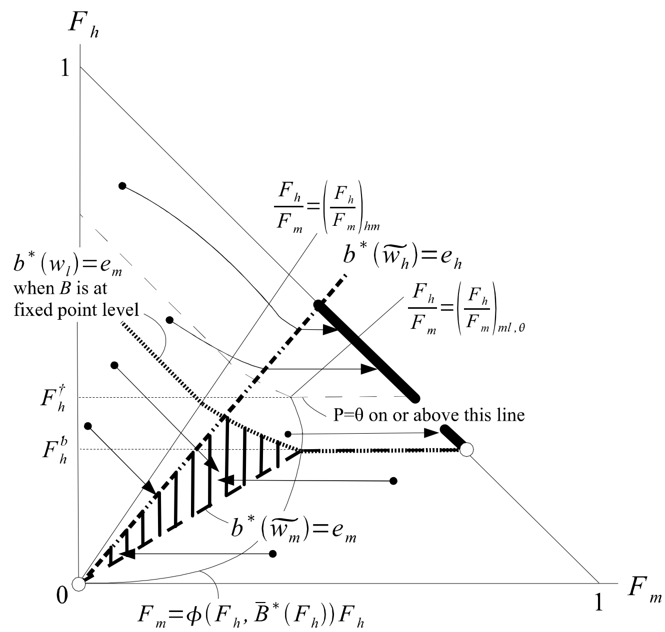

Figure 8: Case of low A_M, i.e. $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm})\leq e_h$

A_M shifts $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{hm}$ upward and shifts the remaining loci except $F_m = \phi(F_h,\overline{B}^*(F_h))F_h$ (the effect is ambiguous) downward on the (F_m,F_h) plane with relative positions of the loci unchanged (see Figure 7). Hence, over time, the economy becomes more likely to converge to Steady state 1 and the relative number of high-skill workers to middle-skill workers in Steady state 1 rises. With the continuous productivity growth, the economy converges to the best steady state from any initial condition ultimately, but the speed of convergence depends critically on the initial condition. Hence, qualitative results of the constant A_M case remain to hold approximately.

Another assumption maintained until now is Assumption 2, $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm}) > e_h$, which states that offspring of high-skill (middle-skill) workers can afford advanced education at $\widetilde{w_h}=\widetilde{w_m}$, that is, when their wage is lowest (highest). The assumption would apply to most economies in the present world except those with very bad institutions, but it may not in the past. If $\frac{\gamma_b}{1-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm})\leq e_h$ holds but A_M is not extremely low, the phase diagram looks like Figure 8.[27] Unlike Figure 7, $b^*(\widetilde{w_h})=e_h$, not $b^*(\widetilde{w_m})=e_h$, exists below $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$and above $b^*(\widetilde{w_m})=e_m$. Since F_{ht} decreases over time above $b^*(\widetilde{w_h})=e_h$, $F_h=F_m$ = 1 is not a steady state. There exist two types of steady states similar to Steady states 2 and 4 of the original economy, where the convergence to the former type of steady state is

[27]When A_M is extremely low, $b^*(\widetilde{w_h})=e_h$ is located below $b^*(\widetilde{w_m})=e_m$, and the economy converges to $F_h=F_m=0$ from any initial distribution, which is clearly not realistic in modern times.

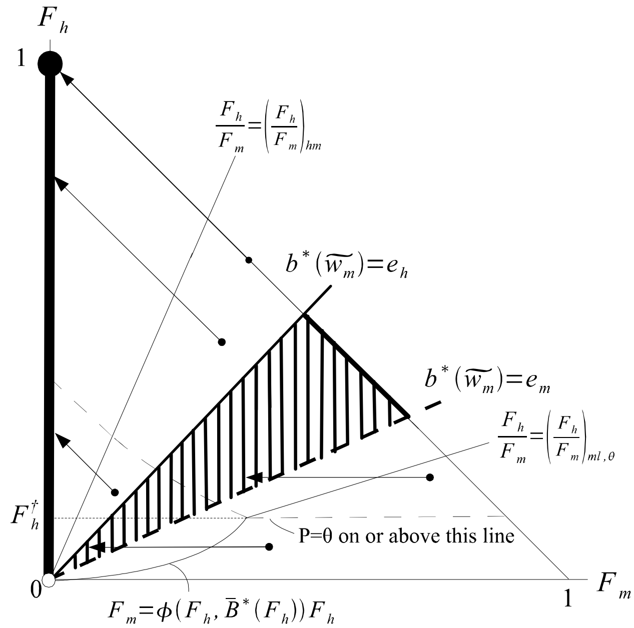

Figure 9: Case of low A_T, i.e. $\frac{\gamma_b}{1-\gamma_b(1+r)}\theta A_T \leq e_m$

more likely as F_{h0} and F_{m0} are higher.

The related assumption on A_T is Assumption 3, $\frac{\gamma_b}{1-\gamma_b(1+r)}\theta A_T \in (e_m, e_h)$. The productivity of the traditional sector is less affected by the advancement of science and technology, but it also would grow slowly over time in real economy, thus the assumption may not hold far in the past or in the future. (It may not hold for an economy with very bad land quality or climate too.) When $\frac{\gamma_b}{1-\gamma_b(1+r)}\theta A_T \leq e_m$, children of low-skill workers cannot access basic education even at $P=\theta$ and F_{mt} non-increases over time. Figure 9 illustrates this case. Unlike the original economy, $b^*(w_l) = e_m$ does not exist, $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{ml,\theta}$ is located below $b^*(\widetilde{w_m}) = e_m$, and the dividing locus between $P<\theta$ and $P=\theta$ (the locus with the broken line) is located at the lower position on the (F_m,F_h) plane. With constant A_T, there exist two kinds of steady states, one "combining" Steady states 1 and 3 of the original economy and the other "combining" Steady states 2 and 4, and if $b^*(\widetilde{w_m}) > e_h$ at $(F_h,F_m) = (F_{h0},F_{m0})$, the economy converges to the first type of steady state, otherwise, it converges to the other one. By contrast, when $\frac{\gamma_b}{1-\gamma_b(1+r)}\theta A_T > e_h$, that is, even children of low-skill workers can access advanced education at $P=\theta$, the result is somewhat similar to the original case, but the economy is more (less) likely to converge to Steady state 1 (Steady state 2).[28]

[28]In this case, $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{ml,\theta}$ is located above $b^*(\widetilde{w_m}) = e_h$; $b^*(w_l) = e_h$ exists and is located between $b^*(w_l) = e_m$ and the dividing locus between $P<\theta$ and $P=\theta$; and $b^*(w_l) = e_h$ and $b^*(\widetilde{w_m}) = e_h$ intersect on $F_m = \phi(F_h, \overline{B}^*(F_h))F_h$ (see Figure 7). If the initial economy is located above $b^*(w_l) = e_h$, it converges to Steady state 1 for certain, otherwise, the dynamics are qualitatively same as the original economy.

These results can be used to examine the dynamics from far in the past when the productivities of both sectors grow over time. For example, as for an economy whose initial A_M does not satisfy Assumption 2 but initial A_T satisfies Assumption 3, the dynamics are illustrated by Figure 8 at first and by Figure 7 after some point.[29] Hence, if F_{h0} and F_{m0} are relatively high, at first, F_{mt}, but not F_{ht}, rises and the inequality between high-skill and middle-skill workers (low-skill workers too when $P = \theta$) enlarges over time, but after A_M becomes high enough for the assumption to hold, F_{ht} rises, the inequality shrinks, and the economy converges to the best steady state. The dynamics may resemble experiences of many developed economies.

5 Conclusion

This paper has developed a dynamic dual-economy model and examined how the long-run outcome of the economy depends on the initial distribution of wealth and sectoral productivity. It is shown that, for fast transformation into a developed economy, the initial distribution must be such that extreme poverty is not prevalent so that most people can acquire basic skills and the size of "middle class" is enough so that an adequate number of workers possess advanced skills. Both conditions seem to have held in successful East Asian economies, where, as in the model economy undergoing such transformation, inequalities between workers with advanced education and others fell over time (Wood, 1994). In contrast, if the former condition is satisfied but the latter is not, which would be the case for many developing nations falling into "middle income trap", consistent with facts, the fraction of workers with basic skills and the share of the modern sector rise, but inequality between workers with advanced skills and with basic skills worsens and the traditional sector remains for long periods. If the former condition does not hold, which would be true for poorest economies, the dual structure and large inequality between workers without basic skills and others (especially, those with advanced skills) last for very long periods. Consistent with these results, Hanushek and Woessmann (2009) find that both the share of students with basic skills and that of top performance have significant effects on economic growth that are *complementary* each other.

[29] As mentioned before, the growth of A_M shifts $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{hm}$ and $b^*(\widetilde{w_h}) = e_h$ upward and the remaining loci except $F_m = \phi(F_h, \overline{B}^*(F_h))F_h$ (the effect is ambiguous) downward. The growth of A_T, by contrast, shifts $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{ml,\theta}$ and the dividing locus between $P < \theta$ and $P = \theta$ *upward*. If A_M grows faster than A_T, a realistic assumption, the two loci shift *downward*, so the transition from Figure 8 to Figure 7 takes place.

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6 Appendix: Proofs of lemmas and propositions

Proof of Lemma 1. (Existence of function $\phi(\cdot)$) Let $\phi = \frac{F_m}{F_h}$. Then, from (13) and (16), $\widetilde{w_m}(\frac{F_h}{F_m}) = P(F_h, F_m, B)A_T$ is expressed as:

$$(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B}\frac{A_M(\phi)^{1-\alpha}F_h + (1+r)[B - (e_h + \phi e_m)F_h]}{1-(1+\phi)F_h}, \tag{45}$$

where $F_h < \frac{1}{1+\phi} \Leftrightarrow \phi < \frac{1-F_h}{F_h}$ must be true. When $F_h \to 0$, the equation becomes:

$$(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B}(1+r)B, \tag{46}$$

whose solution $\phi = \overline{\phi}(B) \equiv \left[\frac{(1-\alpha)A_M}{(1+r)\left(\frac{\gamma_B}{1-\gamma_B}B + e_m\right)}\right]^{\frac{1}{\alpha}}$ satisfies $\overline{\phi}(B) \leq \overline{\phi} \equiv \overline{\phi}(0) = \left[\frac{(1-\alpha)A_M}{(1+r)e_m}\right]^{\frac{1}{\alpha}}$, where $\overline{\phi}$ is the solution to $\widetilde{w_m} = (1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = 0$. The LHS of (45) is decreasing and the RHS is increasing in ϕ for $\phi < \min\{\frac{1-F_h}{F_h}, \overline{\phi}\}$; as $\phi \to 0$, $LHS \to +\infty$ and thus $LHS > RHS$; and as $\phi \to \min\{\frac{1-F_h}{F_h}, \overline{\phi}\}$, $LHS < RHS$ because, when $\phi = \overline{\phi} < \frac{1-F_h}{F_h}$, $LHS = 0$ and $RHS > 0$ (since, from $\overline{\phi} > [(\frac{F_h}{F_m})_{ml,\theta}]^{-1} > [(\frac{F_h}{F_m})_{hm}]^{-1}$, $\widetilde{w_h} > \widetilde{w_m} = 0$ and thus $A_M(\phi)^{1-\alpha} - (1+r)(e_h + \phi e_m) = \widetilde{w_h} + \phi\widetilde{w_m} > 0$ at $\phi = \overline{\phi}$), and when $\frac{1-F_h}{F_h} \leq \overline{\phi}$, $RHS \to +\infty$ as $\phi \to \frac{1-F_h}{F_h}$. Hence, for given $F_h > 0$ and B, there exists a unique $\phi \in (0, \min\{\frac{1-F_h}{F_h}, \overline{\phi}\})$ satisfying the equation, which is denoted as $\phi = \phi(F_h, B)$ and $\lim_{F_h \to 0}\phi(F_h, B) = \overline{\phi}(B)$.

(Properties of $\phi(\cdot)$) The RHS of (45) is strictly increasing in F_h ($< \frac{1}{1+\phi}$) when $\phi \in [[(\frac{F_h}{F_m})_{hm}]^{-1}, \min\{\frac{1-F_h}{F_h}, \overline{\phi}\})$, because $A_M(\phi)^{1-\alpha} - (1+r)(e_h + \phi e_m) = \widetilde{w_h} + \phi\widetilde{w_m} > (1+\phi)\theta A_T > 0$ at $\phi = [(\frac{F_h}{F_m})_{hm}]^{-1}$ from Assumption 1. Thus, $\phi(F_h, B)$ is a decreasing function. $\overline{\phi}(B) > [(\frac{F_h}{F_m})_{hm}]^{-1}$ because $\widetilde{w_m} > \theta A_T$ at $\phi = [(\frac{F_h}{F_m})_{hm}]^{-1}$ from Assumption 1 and $\widetilde{w_m} = \frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$ at $\phi = \overline{\phi}(B)$ from (46). Then, since $\lim_{F_h \to 0}\phi(F_h, B) = \overline{\phi}(B) > [(\frac{F_h}{F_m})_{hm}]^{-1}$ and the limit of $\phi(F_h, B)$ when $F_h \to \frac{1}{1+[(\frac{F_h}{F_m})_{hm}]^{-1}}$ is strictly less than $[(\frac{F_h}{F_m})_{hm}]^{-1}$ (from eq. 45), for given B, there exists a unique $F_h > 0$ satisfying $\phi(F_h, B) = [(\frac{F_h}{F_m})_{hm}]^{-1}$, which is denoted as $F_h^{\ddagger}(B)$. The existence of $F_h^{\dagger}(B)$ can be proved similarly. $F_h^{\ddagger}(B) > F_h^{\dagger}(B)$ is from Assumption 1. ■

Proof of Lemma 2. As shown in the proof of Lemma 1, $\overline{\phi}(0) \geq \overline{\phi}(B) > [(\frac{F_h}{F_m})_{hm}]^{-1}$, $\widetilde{w_m} \geq (>) 0$ for $\frac{F_h}{F_m} \geq (>) [\overline{\phi}(0)]^{-1}$, and, from the definition of $(\frac{F_h}{F_m})_{hm}$, $\widetilde{w_h} \geq (>) \widetilde{w_m}$ for $\frac{F_h}{F_m} \leq (<) (\frac{F_h}{F_m})_{hm}$. Hence, the numerator of (16) and thus $P(F_h, F_m, B)$ are increasing in F_h and F_m for $\frac{F_h}{F_m} \in [[\overline{\phi}(0)]^{-1}, (\frac{F_h}{F_m})_{hm}]$.

From (16) and $\phi = \frac{F_m}{F_h}$, $P(F_h, F_m, B) = \theta$ is expressed as:

$$\frac{1}{A_T} \frac{\gamma_B}{1-\gamma_B} \frac{A_M(\phi)^{1-\alpha} F_h + (1+r)[B - (e_h + \phi e_m) F_h]}{1 - (1+\phi) F_h} = \theta, \tag{47}$$

where $F_h < \frac{1}{1+\phi} \Leftrightarrow \phi < \frac{1-F_h}{F_h}$. For given $\phi \in [[(\frac{F_h}{F_m})_{hm}]^{-1}, \overline{\phi}(0)]$, when $F_h = 0$, $LHS = \frac{1}{A_T} \frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta$; when $F_h \to \frac{1}{1+\phi}$, $LHS \to +\infty$; and the LHS is increasing in F_h (since $A_M(\phi)^{1-\alpha} - (1+r)(e_h + \phi e_m) = \widetilde{w_h} + \phi \widetilde{w_m} > 0$). Hence, given B, for any $\frac{F_h}{F_m} \in [[\overline{\phi}(0)]^{-1}, (\frac{F_h}{F_m})_{hm}]$, there exists a unique $F_h \in (0, \frac{1}{1+[\frac{F_h}{F_m}]^{-1}})$ satisfying $P(F_h, [\frac{F_h}{F_m}]^{-1} F_h, B) = P(F_h, F_m, B) = \theta$. When $\frac{F_h}{F_m} > (<) (\frac{F_h}{F_m})_{ml,\theta}$ and thus $\widetilde{w_m}(\frac{F_h}{F_m}) > (<) \theta A_T$, at $P(F_h, F_m, B) = \theta$, $\widetilde{w_m}(\frac{F_h}{F_m}) > (<) \theta A_T = P(F_h, F_m, B) A_T$, that is, $F_m < (>) \phi(F_h, B) F_h$. ∎

Proof of Proposition 1. Since $F_h > 0$, an equilibrium satisfying L_h, $L_m > 0$ always exists from the shape of the sector M production function. Thus, equilibrium L_h and L_m must satisfy $\widetilde{w_h} \geq \widetilde{w_m}$ (thus $\frac{L_h}{L_m} \leq (\frac{F_h}{F_m})_{hm}$) and $\widetilde{w_m} \geq w_l$. Since $\widetilde{w_h} = \widetilde{w_m} > \theta A_T \geq w_l$ at $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$ (from Assumption 1) and $\widetilde{w_h}(\widetilde{w_m})$ is decreasing (increasing) in $\frac{L_h}{L_m}$, there does not exist equilibrium $\frac{L_h}{L_m}$ satisfying $\widetilde{w_h} = \widetilde{w_m} = w_l$. Hence, when $\widetilde{w_h} = \widetilde{w_m}$, $\widetilde{w_m} > w_l$, while when $\widetilde{w_m} = w_l$, $\widetilde{w_h} > \widetilde{w_m}$ in equilibrium. In the former case, $L_h \leq F_h$, $L_h + L_m = F_h + F_m$, and $\frac{L_h}{L_m} = \frac{L_h}{F_h + F_m - L_h} \leq \frac{F_h}{F_m}$, and in the latter case, $L_h = F_h$, $L_m \leq F_m$, and $\frac{L_h}{L_m} = \frac{F_h}{L_m} \geq \frac{F_h}{F_m}$.

(i) $\widetilde{w_m} = w_l$ is not possible because $\widetilde{w_h} > \widetilde{w_m}$ and $\frac{L_h}{L_m} = \frac{F_h}{L_m} \geq \frac{F_h}{F_m} \geq (\frac{F_h}{F_m})_{hm}$ cannot hold simultaneously. Thus, $\widetilde{w_m} > w_l$, $L_h + L_m = F_h + F_m$ and $\frac{L_h}{L_m} = \frac{L_h}{F_h + F_m - L_h} \leq \frac{F_h}{F_m}$. When $\frac{F_h}{F_m} = (\frac{F_h}{F_m})_{hm}$, $\widetilde{w_h} > \widetilde{w_m}$ with $L_h < F_h$ (since $\frac{L_h}{L_m} < \frac{F_h}{F_m} = (\frac{F_h}{F_m})_{hm}$) and thus $L_h = F_h$, $L_m = F_m$, and $\widetilde{w_h} = \widetilde{w_m}$ in equilibrium. When $\frac{F_h}{F_m} > (\frac{F_h}{F_m})_{hm}$, $\widetilde{w_h} < \widetilde{w_m}$ with $L_h = F_h$ and thus $L_h < F_h$ and $\widetilde{w_h} = \widetilde{w_m}$ in equilibrium. Equilibrium values of L_h and L_m are obtained from $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$ and $L_h + L_m = F_h + F_m$.

(ii) If $\widetilde{w_h} = \widetilde{w_m}$, as shown above, $\frac{L_h}{L_m} = \frac{L_h}{F_h + F_m - L_h} \leq \frac{F_h}{F_m}$ must hold, which implies $\frac{L_h}{L_m} \leq \frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$ and thus $\widetilde{w_h} > \widetilde{w_m}$, a contradiction. Hence, $\widetilde{w_h} > \widetilde{w_m}$ and $L_h = F_h$ in equilibrium.

When $\frac{\gamma_B}{1-\gamma_B}(1+r)B \geq \theta A_T$, the RHS of (16) is greater than θ for any equilibrium L_h and L_m (since $\widetilde{w_i} > 0, i = h, m$, when $L_i > 0$), thus $P = \theta$ and $w_l = \theta A_T$ in equilibrium. Hence, when $\frac{F_h}{F_m} \in ((\frac{F_h}{F_m})_{ml,\theta}, (\frac{F_h}{F_m})_{hm})$, $\widetilde{w_m} > w_l$ and $L_m = F_m$, while when $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$, $\widetilde{w_m} = w_l$ and $\frac{L_h}{L_m} = \frac{F_h}{L_m} = (\frac{F_h}{F_m})_{ml,\theta}$.

When $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$, since $\frac{F_h}{F_m} < (\frac{F_h}{F_m})_{hm}$, from Lemma 1, positive F_h and F_m satisfying $\widetilde{w_m}(\frac{F_h}{F_m}) = P(F_h, F_m, B) A_T$ exist for any $\frac{F_h}{F_m} \geq [\overline{\phi}(B)]^{-1}$ and is expressed as $F_m = \phi(F_h, B) F_h$, where $\phi(F_h, B)$ is a decreasing function, and from Lemma 2, F_h and F_m

satisfying $P(F_h,F_m,B)=\theta$ exist for any $\frac{F_h}{F_m} \geq [\overline{\phi}(0)]^{-1}$, where $P(F_h,F_m,B)$ is an increasing function. Note that $(\frac{F_h}{F_m})_{ml,\theta} > [\overline{\phi}(B)]^{-1} \geq [\overline{\phi}(0)]^{-1}$ from (45) and (46) in the proof of Lemma 1 and $\frac{\gamma_B}{1-\gamma_B}(1+r)B < \theta A_T$.

(a) When $P(F_h,F_m,B) < \theta$, $\widetilde{w_m}(\frac{F_h}{F_m}) > \theta A_T > P(F_h,F_m,B)A_T$ from $\frac{F_h}{F_m} > (\frac{F_h}{F_m})_{ml,\theta}$. Hence, $L_m = F_m$ and $\widetilde{w_m} > \theta A_T > w_l = P(F_h,F_m,B)A_T$ in equilibrium. When $P(F_h,F_m,B) \geq \theta$, $\widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{L_m}) = P(F_h,L_m,B)A_T = w_l \geq \widetilde{w_m}(\frac{F_h}{F_m})$ cannot be true since $\widetilde{w_m}(\frac{F_h}{F_m}) > \theta A_T$ from $\frac{F_h}{F_m} > (\frac{F_h}{F_m})_{ml,\theta}$. Hence, $\widetilde{w_m} > w_l$, $L_m = F_m$, and $P=\theta$ in equilibrium.

(b) 1. From Lemma 1 (see Figure 1 too), for any $\frac{F_h}{F_m} \in [\,[\overline{\phi}(B)]^{-1}, (\frac{F_h}{F_m})_{ml,\theta})$, there exists $F_h < F_h^\dagger(B)$ satisfying $F_m = \phi(F_h,B)F_h$. When $P(F_h,F_m,B) \geq \theta$ (then, $F_m > \phi(F_h,B)F_h$ from Lemma 2) or when $P(F_h,F_m,B) < \theta$ and $F_m \geq \phi(F_h,B)F_h$, $\widetilde{w_m}(\frac{F_h}{F_m}) \leq P(F_h,F_m,B)A_T$ and thus $\widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{L_m}) = P(F_h,L_m,B)A_T = w_l$ and $L_m = \phi(F_h,B)F_h$ in equilibrium, where $\widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{L_m}) < \theta A_T$ from $\frac{F_h}{L_m} = \frac{1}{\phi(F_h,B)} < \frac{1}{\phi(F_h^\dagger(B),B)} = (\frac{F_h}{F_m})_{ml,\theta}$. When $P(F_h,F_m,B) < \theta$ and $F_m < \phi(F_h,B)F_h$, $\widetilde{w_m} = \widetilde{w_m}(\frac{F_h}{F_m}) > P(F_h,F_m,B)A_T = w_l$ and $L_m = F_m$ in equilibrium.

2. When $\frac{F_h}{F_m} \leq (\frac{F_h}{F_m})_{ml,\theta}$ and $F_h \geq F_h^\dagger(B)$, from Lemma 2 (see Figure 2 too), $P(F_h,F_m,B) = P(F_h,[\frac{F_h}{F_m}]^{-1}F_h,B) \geq P(F_h,[(\frac{F_h}{F_m})_{ml,\theta}]^{-1}F_h,B) \geq P(F_h^\dagger(B),[(\frac{F_h}{F_m})_{ml,\theta}]^{-1}F_h^\dagger(B),B) = \theta$. From Lemma 2, when $P(F_h,F_m,B) \geq \theta$, $F_m \geq \phi(F_h,B)F_h$ and thus $\widetilde{w_m}(\frac{F_h}{F_m}) \leq \theta A_T \leq P(F_h,F_m,B)A_T$. Hence, $\widetilde{w_m} = \theta A_T = w_l$, $P=\theta$, $L_m = [(\frac{F_h}{F_m})_{ml,\theta}]^{-1}F_h$, and $\widetilde{w_h} = \widetilde{w_h}([(\frac{F_h}{F_m})_{ml,\theta}]^{-1})$ in equilibrium. Note that $\widetilde{w_m} = w_l = P(F_h,L_m,B)A_T < \theta A_T$ (thus $\frac{L_h}{L_m} = \frac{F_h}{L_m} > (\frac{F_h}{F_m})_{ml,\theta}$) is not possible because, from Lemma 2, if $\frac{F_h}{L_m} > (\frac{F_h}{F_m})_{ml,\theta}$, $\widetilde{w_m}(\frac{F_h}{L_m}) > P(F_h,L_m,B)A_T$ when $P(F_h,L_m,B) < \theta$. ■

Proof of Proposition 2. (i) From Proposition 1 (i), $\frac{L_h}{L_m} = (\frac{F_h}{F_m})_{hm}$ and thus $\widetilde{w_h} = \widetilde{w_m} = \widetilde{w_m}((\frac{F_h}{F_m})_{hm})$, which is strictly greater than θA_T (thus w_l) from Assumption 1. By substituting $\widetilde{w_h} = \widetilde{w_m} = \widetilde{w_m}((\frac{F_h}{F_m})_{hm})$ and $L_h + L_m = F_h + F_m$ into P (eq. 15) and equating it with θ,

$$\frac{\gamma_B}{1-\gamma_B}\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B}{1-(F_h+F_m)} = \theta A_T \Leftrightarrow F_h+F_m = \frac{(1-\gamma_B)\theta A_T - \gamma_B(1+r)B}{\gamma_B \widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + (1-\gamma_B)\theta A_T}. \tag{48}$$

Thus, the result for w_l holds. (ii) Straightforward from proofs of Proposition 1 (ii). ■

Proof of Proposition 3. Net aggregate income is computed from L_h, L_m, and wages of Propositions 1 and 2 and (15), and average utility is from net aggregate income and (15).

(i) When $F_h + F_m < \frac{(1-\gamma_B)\theta A_T - \gamma_B(1+r)B}{\left[\gamma_B \widetilde{w_m}((\frac{F_h}{F_m})_{hm}) + (1-\gamma_B)\theta A_T\right]}$, $NI = \frac{1}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]$ and thus it increases with F_h+F_m and B. Average utility equals

$$(\gamma_B)^{\gamma_B}(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}\left\{\frac{\frac{\gamma_B}{1-\gamma_B}[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B]}{A_T(1-F_h-F_m)}\right\}^{-\gamma_B}\frac{1}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]$$

$$= \frac{(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}}{(1-\gamma_B)^{1-\gamma_B}}\left[A_T(1-F_h-F_m)\right]^{\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]^{1-\gamma_B}, \tag{49}$$

the derivative of which with respect to F_h+F_m equals the average utility times

$$-\frac{\gamma_B}{1-F_h-F_m}+\frac{(1-\gamma_B)\widetilde{w_m}((\frac{F_h}{F_m})_{hm})}{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B}=\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(1-\gamma_B-F_h-F_m)-\gamma_B(1+r)B}{(1-F_h-F_m)[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B]}, \quad (50)$$

where, from $F_h+F_m<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$, the numerator of the expression is greater than

$$\frac{[(1-\gamma_B)\widetilde{w_m}((\frac{F_h}{F_m})_{hm})-\gamma_B(1+r)B][\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T]-\widetilde{w_m}((\frac{F_h}{F_m})_{hm})[(1-\gamma_B)\theta A_T-\gamma_B(1+r)B]}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$$

$$=\frac{[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})-\theta A_T]\gamma_B(1-\gamma_B)[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1+r)B]}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}>0. \quad (51)$$

Hence the average utility too increases with F_h+F_m and B. When $F_h+F_m\geq\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$, $NI=\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+\theta A_T(1-F_h-F_m)+(1+r)B$ and average utility equals $\gamma_B{}^{\gamma_B}\gamma_N{}^{\gamma_N}\gamma_b{}^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h+F_m and B.

(ii) (a) When $P(F_h,F_m,B)\leq\theta$, $NI=\frac{1}{1-\gamma_B}[A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)]$ and thus it increases with F_h, F_m, and B. Average utility equals

$$\frac{(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}}{(1-\gamma_B)^{1-\gamma_B}}[A_T(1-F_h-F_m)]^{\gamma_B}\left[A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)\right]^{1-\gamma_B}, \quad (52)$$

the derivative of which with respect to $F_i(i=h,m)$ equals the average utility times

$$-\frac{\gamma_B}{1-F_h-F_m}+\frac{(1-\gamma_B)\widetilde{w}_i(\frac{F_h}{F_m})}{A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)}\geq\frac{\gamma_B}{1-F_h-F_m}[-1+\frac{\widetilde{w}_i(\frac{F_h}{F_m})}{\theta A_T}]>0, \quad (53)$$

where the first inequality is from $P(F_h,F_m,B)\leq\theta\Leftrightarrow\frac{\gamma_B}{1-\gamma_B}\frac{A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)}{A_T[1-F_h-F_m]}\leq\theta$. Hence, the average utility too increases with F_h, F_m, and B. When $P(F_h,F_m,B)>\theta$ and thus $P=\theta$, $NI=A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)+\theta A_T(1-F_h-F_m)$ and average utility equals $(\gamma_B)^{\gamma_B}(\gamma_N)^{\gamma_N}(\gamma_b)^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h, F_m, and B.

(b) 1. When $F_m\geq\phi(F_h,B)F_h$, $NI=\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h)+(1+r)B$. The derivative of NI with respect to F_h equals

$$\widetilde{w_h}([\phi(F_h,B)]^{-1})-\widetilde{w_m}([\phi(F_h,B)]^{-1})-\frac{\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)}{[\phi(F_h,B)]^2}\frac{\partial\phi}{\partial F_h}, \quad (54)$$

where $\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)=\alpha(1-\alpha)A_M\left([\phi(F_h,B)]^{-1}\right)^{\alpha-1}[1-F_h-\phi(F_h,B)F_h]>0$ (55)

and thus the derivative is positive. Similarly, the derivative of NI with respect to B equals $-\left[\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)\right][\phi(F_h,B)]^{-2}\frac{\partial\phi}{\partial B}+(1+r)>0$.

Since $P = \frac{\widetilde{w_m}([\phi(F_h,B)]^{-1})}{A_T}$, average utility equals

$$(\gamma_B A_T)^{\gamma_B}\gamma_N{}^{\gamma_N}\gamma_b{}^{\gamma_b}\left[\widetilde{w_m}([\phi(F_h,B)]^{-1})\right]^{-\gamma_B}\left\{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h)+(1+r)B\right\}. \tag{56}$$

The derivative with respect to F_h equals the average utility times

$$-\left[-\gamma_B\frac{\widetilde{w_m}'([\phi(F_h,B)]^{-1})}{\widetilde{w_m}([\phi(F_h,B)]^{-1})}+\frac{\widetilde{w_h}'([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}'([\phi(F_h,B)]^{-1})(1-F_h)}{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h)+(1+r)B}\right][\phi(F_h,B)]^{-2}\frac{\partial\phi}{\partial F_h}$$
$$+\frac{\widetilde{w_h}([\phi(F_h,B)]^{-1})-\widetilde{w_m}([\phi(F_h,B)]^{-1})}{\widetilde{w_h}([\phi(F_h,B)]^{-1})F_h+\widetilde{w_m}([\phi(F_h,B)]^{-1})(1-F_h)+(1+r)B}, \tag{57}$$

where the expression inside the big square bracket of the first term equals ($\phi \equiv \phi(F_h,B)$) $\frac{1}{\widetilde{w_m}(\phi^{-1})[\widetilde{w_h}(\phi^{-1})F_h+\widetilde{w_m}([\phi^{-1})(1-F_h)+(1+r)B]}$ times

$$-\gamma_B\widetilde{w_m}'(\phi^{-1})\left[\widetilde{w_h}(\phi^{-1})F_h+\widetilde{w_m}(\phi^{-1})(1-F_h)+(1+r)B\right]+\widetilde{w_m}(\phi^{-1})\left[\widetilde{w_h}'(\phi^{-1})F_h+\widetilde{w_m}'(\phi^{-1})(1-F_h)\right]$$
$$=-\widetilde{w_m}'(\phi^{-1})\left[1-(1+\phi)F_h\right]\widetilde{w_m}(\phi^{-1})+\left[\widetilde{w_h}'(\phi^{-1})F_h+\widetilde{w_m}'(\phi^{-1})(1-F_h)\right]\widetilde{w_m}(\phi^{-1}) \text{ (from eq. 14)}$$
$$=[\widetilde{w_h}'(\phi^{-1})+\widetilde{w_m}'(\phi^{-1})\phi]F_h\widetilde{w_m}(\phi^{-1})=0.$$

Hence, the derivative is positive. The derivative with respect to B can be proved to be positive similarly. When $F_m<\phi(F_h,B)F_h$, the proof of (ii)(a) when $P(F_h,F_m,B)\le\theta$ applies.

2. $NI=\widetilde{w_h}((\frac{F_h}{F_m})_{ml,\theta})F_h+\theta A_T(1-F_h)+(1+r)B$ and average utility equals $\gamma_B{}^{\gamma_B}\gamma_N{}^{\gamma_N}\gamma_b{}^{\gamma_b}(\theta)^{-\gamma_B}NI$. Thus, they increase with F_h and B. ■

Proof of Proposition 4. Y and Y_M are computed from equilibrium L_h and L_m (Proposition 1), (6), and (16). Since $PC_B=\gamma_B NI$ and $C_{BM}=\gamma_B NI-\theta A_T[1-(L_h+L_m)]$ (eq. 17), the result on $\frac{C_{BM}}{PC_B}=\gamma_B-\theta A_T\frac{1-(L_h+L_m)}{NI}$ is obtained from Propositions 1 and 3.

(i) When $F_h+F_m<\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\left[\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T\right]}$,

$Y=A_M\frac{((\frac{F_h}{F_m})_{hm})^\alpha}{1+(\frac{F_h}{F_m})_{hm}}(F_h+F_m)+\frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_m}((\frac{F_h}{F_m})_{hm})(F_h+F_m)+(1+r)B\right]$, where the first term is Y_M. Thus, Y increases with F_h+F_m and B, and $\frac{Y_M}{Y}$ increases with $\frac{F_h+F_m}{B}$. When $F_h+F_m\ge\frac{(1-\gamma_B)\theta A_T-\gamma_B(1+r)B}{\gamma_B\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+(1-\gamma_B)\theta A_T}$, $Y=A_M\frac{((\frac{F_h}{F_m})_{hm})^\alpha}{1+(\frac{F_h}{F_m})_{hm}}(F_h+F_m)+\theta A_T(1-F_h-F_m)$, where the first term is Y_M. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h+F_m. $\frac{C_{BM}}{PC_B}=\gamma_B-\theta A_T\frac{1-(F_h+F_m)}{NI}$ and thus it increases with F_h+F_m and B.

(ii)(a) When $P(F_h,F_m,B)\le\theta$, $Y=A_M(F_h)^\alpha(F_m)^{1-\alpha}+\frac{\gamma_B}{1-\gamma_B}[A_M(F_h)^\alpha(F_m)^{1-\alpha}+(1+r)(B-e_hF_h-e_mF_m)]$, where the first term is Y_M. Thus, Y increases with F_h, F_m, and B, and $\frac{Y_M}{Y}$ increases with F_h and F_m and decreases with B. When $P(F_h,F_m,B)>\theta$ and thus $P=\theta$, $Y=A_M(F_h)^\alpha(F_m)^{1-\alpha}+\theta A_T(1-F_h-F_m)$, where the first term is Y_M. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h and F_m. $\frac{C_{BM}}{PC_B}=\gamma_B-\theta A_T\frac{1-(F_h+F_m)}{NI}$ and thus it increases with F_h, F_m, and B.

(b) 1. $Y=A_M(\phi(F_h,B))^{1-\alpha}F_h+\frac{\gamma_B}{1-\gamma_B}\{A_M(\phi(F_h,B))^{1-\alpha}F_h+(1+r)[B-(e_h+\phi(F_h,B)e_m)F_h]\}$, where the first term is Y_M. The derivative of Y with respect to F_h equals ($\phi \equiv \phi(F_h,B)$)

$$\frac{1}{1-\gamma_B}[A_M(\phi)^{1-\alpha}-\gamma_B(1+r)(e_h+\phi e_m)]+\frac{1}{1-\gamma_B}[(1-\alpha)A_M(\phi)^{-\alpha}-\gamma_B(1+r)e_m]F_h\frac{\partial\phi}{\partial F_h}$$
$$=\frac{1}{1-\gamma_B}[(1-\alpha)A_M(\phi)^{-\alpha}-\gamma_B(1+r)e_m](\phi+F_h\frac{\partial\phi}{\partial F_h})+\frac{1}{1-\gamma_B}[\alpha A_M(\phi)^{1-\alpha}-\gamma_B(1+r)e_h]$$

$$>\frac{1}{1-\gamma_B}[\widetilde{w_m}(\phi^{-1})(\phi+F_h\frac{\partial\phi}{\partial F_h})+\widetilde{w_h}(\phi^{-1})]. \tag{58}$$

In the above equation, from $(1-\alpha)A_M(\phi)^{-\alpha}-(1+r)e_m=\frac{\gamma_B}{1-\gamma_B}\frac{A_M(\phi)^{1-\alpha}F_h+(1+r)[B-(e_h+\phi e_m)F_h]}{1-(1+\phi)F_h}$ (eq. 45 in the proof of Lemma 1),

$$\frac{\partial\phi}{\partial F_h}=-\frac{(1+\phi)\left[(1-\alpha)A_M(\phi)^{-\alpha}-(1+r)e_m\right]+\frac{\gamma_B}{1-\gamma_B}\left[A_M(\phi)^{1-\alpha}-(1+r)(e_h+\phi e_m)\right]}{\frac{1}{1-\gamma_B}\left[(1-\alpha)A_M(\phi)^{-\alpha}-(1+r)e_m\right]F_h+\left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]}$$
$$=-\frac{(1+\phi)\widetilde{w_m}(\phi^{-1})+\frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_h}(\phi^{-1})+\phi\widetilde{w_m}(\phi^{-1})\right]}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h+\left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]}. \tag{59}$$

Thus, the expression inside the square bracket of (58) equals $\frac{1}{\frac{\widetilde{w_m}(\phi^{-1})F_h}{1-\gamma_B}+[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]}$ times

$$\left[\widetilde{w_h}(\phi^{-1})+\phi\widetilde{w_m}(\phi^{-1})\right]\left\{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h+\left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]\right\}$$
$$-\left\{(1+\phi)\widetilde{w_m}(\phi^{-1})+\frac{\gamma_B}{1-\gamma_B}\left[\widetilde{w_h}(\phi^{-1})+\phi\widetilde{w_m}(\phi^{-1})\right]\right\}\widetilde{w_m}(\phi^{-1})F_h$$
$$=\left[\widetilde{w_h}(\phi^{-1})+\phi\widetilde{w_m}(\phi^{-1})\right]\left\{\widetilde{w_m}(\phi^{-1})F_h+\left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]\right\}-(1+\phi)\widetilde{w_m}(\phi^{-1})\widetilde{w_m}(\phi^{-1})F_h$$

$$=\left[\widetilde{w_h}(\phi^{-1})+\phi\widetilde{w_m}(\phi^{-1})\right]\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}[1-(1+\phi)F_h]+(\widetilde{w_h}(\phi^{-1})-\widetilde{w_m}(\phi^{-1}))\widetilde{w_m}(\phi^{-1})F_h>0.$$

The derivative of Y with respect to B equals

$$\frac{\gamma_B(1+r)}{1-\gamma_B}+\frac{1}{1-\gamma_B}[(1-\alpha)A_M\phi^{-\alpha}-\gamma_B(1+r)e_m]F_h\frac{\partial\phi}{\partial B}>\frac{1}{1-\gamma_B}\left[\widetilde{w_m}(\phi^{-1})F_h\frac{\partial\phi}{\partial B}+\gamma_B(1+r)\right]. \tag{60}$$

In the above equation, from $(1-\alpha)A_M(\phi)^{-\alpha}-(1+r)e_m=\frac{\gamma_B}{1-\gamma_B}\frac{A_M(\phi)^{1-\alpha}F_h+(1+r)[B-(e_h+\phi e_m)F_h]}{1-(1+\phi)F_h}$,

$$\frac{\partial\phi}{\partial B}=-\frac{\frac{\gamma_B}{1-\gamma_B}(1+r)}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h+\left[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}\right]\left[1-(1+\phi)F_h\right]}. \tag{61}$$

Thus,

$$\widetilde{w_m}(\phi^{-1})F_h\frac{\partial\phi}{\partial B}+\gamma_B(1+r)=\frac{\gamma_B(1+r)}{1-\gamma_B}\frac{\frac{\gamma_B}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h+[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]}{\frac{1}{1-\gamma_B}\widetilde{w_m}(\phi^{-1})F_h+[\alpha(1-\alpha)A_M(\phi)^{-\alpha-1}][1-(1+\phi)F_h]}>0. \tag{62}$$

Hence, Y increases with F_h and B. Since $\frac{Y_M}{Y}=\left(1+\frac{\gamma_B}{1-\gamma_B}\left\{1+(1+r)\frac{B-(e_h+\phi(F_h,B)e_m)F_h}{A_M(\phi(F_h,B))^{1-\alpha}F_h}\right\}\right)^{-1}$, $\frac{Y_M}{Y}$ decreases with B, but the effect of F_h is ambiguous.

2. $Y = A_M[(\frac{F_h}{F_m})_{ml,\theta}]^{\alpha-1}F_h + \theta A_T\left(1 - \left\{1 + [(\frac{F_h}{F_m})_{ml,\theta}]^{-1}\right\} F_h\right)$, where the first term is Y_M. Thus, Y and $\frac{Y_M}{Y}$ increase with F_h. $\frac{C_{BM}}{PC_B} = \gamma_B - \frac{\theta A_T}{NI}\left(1 - \left\{1 + [(\frac{F_h}{F_m})_{ml,\theta}]^{-1}\right\} F_h\right)$ and thus it increases with F_h and B. ■

Proof of Lemma 3. From the proof of Lemma 2, $\phi = \phi(F_{ht}, B_t)$ is a solution to

$$(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B}\frac{[A_M(\phi)^{1-\alpha} - (1+r)(e_h + \phi e_m)]F_{ht} + (1+r)B_t}{1-(1+\phi)F_{ht}}. \tag{63}$$

where the first term of the numerator of the RHS equals $\widetilde{w_{ht}} + \phi\widetilde{w_{mt}} > 0$ from (12) and (13)Since the LHS decreases with ϕ and the RHS and the denominator of the RHS increase with ϕ, the numerator of the RHS increases with B_t. Thus, the numerator of the RHS of (37) is positive at $B_t = 0$ and is increasing in B_t. Further, for any $B_t > 0$,

$$\frac{\partial RHS}{\partial B_t} = \frac{\gamma_b}{1-\gamma_B}\left\{\left[(1-\alpha)A_M(\phi(F_{ht}, B_t))^{-\alpha} - (1+r)e_m\right]F_{ht}\frac{\partial\phi(F_{ht}, B_t)}{\partial B_t} + (1+r)\right\} < \frac{\gamma_b(1+r)}{1-\gamma_B} < 1. \tag{64}$$

Hence, for given F_{ht}, B_t converges monotonically to the unique solution to (38), $\overline{B}^*(F_{ht})$, and when $B_t < (>)\overline{B}^*(F_{ht})$, $B_{t+1} > (<)B_t$. From (63) and (38), $\phi = \phi(F_{ht}, \overline{B}^*(F_{ht}))$ is a solution to:

$$(1-\alpha)A_M(\phi)^{-\alpha} - (1+r)e_m = \frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\frac{A_M(\phi)^{1-\alpha} - (1+r)(e_h + \phi e_m)}{1-(1+\phi)F_{ht}}F_{ht}. \tag{65}$$

Thus, $\phi(F_{ht}, \overline{B}^*(F_{ht}))$ is decreasing in F_{ht} and, as $F_{ht} \to 0$, $\phi(F_{ht}, \overline{B}^*(F_{ht})) \to \overline{\phi}(0) \equiv \left[\frac{(1-\alpha)A_M}{(1+r)e_m}\right]^{\frac{1}{\alpha}}$. Finally, $\frac{d\overline{B}^*(F_{ht})}{dF_{ht}} > 0$ is from (28) and Proposition 3 (ii)(b) 1. ■

Proof of Proposition 5. When the economy is in a steady state, relative positions of the critical loci determining the dynamics of F_h and F_m and the magnitude relation of P and θ are illustrated by Figure 6. In the region satisfying $b^*(\widetilde{w_m}) > e_h$ and $b^*(w_l) > e_m$ of the figure, F_h and $F_h + F_m$ increase when $F_h < 1$, thus $F_h < 1$ cannot be a steady state. Hence, $(F_h, F_m) = (1,0)$ is the only steady state (Steady state 1). Since $\frac{F_h}{F_m} = +\infty > (\frac{F_h}{F_m})_{hm}$ and $P = \theta$ from the figure, $B = \widehat{B}^*(1)$ holds from (32). In the region satisfying $b^*(\widetilde{w_m}) \leq e_h$ and $b^*(w_l) > e_m$, F_h is constant and F_m increases when $F_h + F_m < 1$, thus steady states are such that $F_m = 1 - F_h$ and F_h satisfies $b^*(\widetilde{w_m}) \leq e_h \Leftrightarrow \frac{F_h}{F_m} = \frac{F_h}{1-F_h} \leq \widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]$ (from the paragraph just after Assumption 3) and $b^*(w_l) > e_m \Leftrightarrow F_h > F_h^\flat$ (from eq. 44) [Steady state 2]. Since $L_m = \max\{\phi(F_h, \overline{B}^*(F_h)), [(\frac{F_h}{F_m})_{ml,\theta}]^{-1}\}F_h$ when $\frac{F_h}{F_m} = \frac{F_h}{1-F_h} \leq (\frac{F_h}{F_m})_{ml,\theta}$ and $L_m = F_m$ when $\frac{F_h}{1-F_h} > (\frac{F_h}{F_m})_{ml,\theta}$ from Proposition 1, $B = \overline{B}^*(F_h)$ when $\frac{F_h}{1-F_h} \leq (\frac{F_h}{F_m})_{ml,\theta}$ from (38) and (40), and $B = B^*(F_h, F_m)$ when $\frac{F_h}{1-F_h} > (\frac{F_h}{F_m})_{ml,\theta}$ from $P = \theta$ and (36). In the region satisfying $b^*(\widetilde{w_m}) > e_h$ and $b^*(w_l) \leq e_m$, F_h increases and F_m decreases when $F_m > 0$, thus steady states are such that $F_m = 0$ and F_h satisfies $b^*(w_l) \leq e_m \Leftrightarrow F_h \leq$

$\frac{\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}{\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\widetilde{w_m}((\frac{F_h}{F_m})_{hm})+\frac{1-\gamma_b(1+r)}{\gamma_b}e_m}$ (from eq. 42) [Steady state 3]. Since $P<\theta$ from the figure, $B=\widehat{B}^*(F_h)$ holds from (30). In the region satisfying $b^*(\widetilde{w_m})\le e_h$ and $b^*(w_l)\le e_m$, F_h is constant and F_m decreases (is constant) when $b^*(\widetilde{w_m})<(\ge)e_m$, thus steady states are: F_h and F_m satisfying $e_m\le b^*(\widetilde{w_m})\le e_h \Leftrightarrow \frac{F_h}{F_m}\in\left[\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right],\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right]\right]$ and $b^*(w_l)\le e_m \Leftrightarrow P(F_h,F_m,B^*(F_h,F_m))A_T\le\frac{1-\gamma_b(1+r)}{\gamma_b}e_m$ (from eq. 43), and $B=B^*(F_h,F_m)$ (from eq. 34) [Steady state 4]; and $F_h=F_h^\flat$, $F_m\ge\phi(F_h^\flat,\overline{B}^*(F_h^\flat))F_h^\flat$ (thus $\frac{F_h}{F_m}<\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$), and $B=\overline{B}^*(F_h)$ (see footnote 20).

In Steady state 2, from the figure and the result on B, $P=P(F_h,L_m,\overline{B}^*(F_h))<\theta$ if $F_h\le F_h^\dagger$ and $P=\theta$ otherwise. In Steady state 3, $P=P(L_h,L_m,\widehat{B}^*(F_h))=\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\frac{\widetilde{w_m}((\frac{F_h}{F_m})_{hm})F_h}{A_T(1-F_h)}$ from (16), (30), and $\widetilde{w_h}=\widetilde{w_m}=\widetilde{w_m}((\frac{F_h}{F_m})_{hm})$. Levels of L_h, L_m, and L_l, and wages are from Propositions 1 and 2 and the result on P. ■

Proof of Proposition 6. (i) From Proposition 3 (i), aggregate net income (NI) and average utility of Steady state 1 are strictly greater than those of Steady state 3, and they increase with F_h in Steady state 3 ($B=\widehat{B}^*(F_h)$ from Proposition 5 3.). In Steady state 2, when $\frac{F_h}{1-F_h}\le(\frac{F_h}{F_m})_{ml,\theta}$, NI and average utility increase with F_h from Propositions 3 (ii)(b) and 5 2. a. ($B=\overline{B}^*(F_h)$), while when $\frac{F_h}{1-F_h}>(\frac{F_h}{F_m})_{ml,\theta}$, they increase with F_h because $NI=\frac{1}{1-\gamma_b(1+r)}\{A_M(F_h)^\alpha(1-F_h)^{1-\alpha}-(1+r)[e_hF_h+e_m(1-F_h)]\}$ (note $\widetilde{w_h}>\widetilde{w_m}$) and average utility equals a constant times NI from the proof of Proposition 3 (ii)(a), Proposition 5 2. b. ($F_m=1-F_h$, $B=B^*(F_h,F_m)$, and $P=\theta$), and (36). Since NI and average utility of Steady state 1 equal those when $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$ and $F_m=1-F_h$, and the above proof of their being increasing in F_h when $\frac{F_h}{1-F_h}>(\frac{F_h}{F_m})_{ml,\theta}$ applies when $\frac{F_h}{1-F_h}\in\left(\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\right],(\frac{F_h}{F_m})_{hm}\right]$ as well, these variables of Steady state 2 are strictly small than those of Steady state 1. In Steady state 4, they increase with F_h and F_m from Propositions 3 (ii)(a) and 5 4. ($B=B^*(F_h,F_m)$). In Steady state 4, they are highest when $b^*(\widetilde{w_m})=e_h$ and $b^*(w_l)=e_m \Leftrightarrow P(F_h,F_m,B^*(F_h,F_m))A_T=\frac{1-\gamma_b(1+r)}{\gamma_b}e_m$, because they are highest on $b^*(w_l)=e_m$ from Figure 6 and increase with F_h among steady states on the locus from (41) and the expressions for these variables in the proof of Proposition 3 (ii)(a). (Note that the absolute value of the slope of the locus is less than 1.) The highest NI and average utility of Steady state 4 are strictly lower than those of Steady state 3, since the latter coincide with those when $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$ and $b^*(w_l)=e_m$. They are also strictly lower than those of Steady state 2, since they are highest at $b^*(\widetilde{w_m})=e_h$ in both types of steady states. They are at the infinimum when $F_h\to 0$ in Steady states 3, and when $\frac{F_h}{F_m}=\widetilde{w_m}^{-1}\left[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\right]$ and $F_h\to 0$ in Steady states 4, hence the infinima equal 0. The infinima of Steady state 2 are strictly higher than the ones in Steady states 3 and 4, since the former coincide with the NI and average utility at the intersection of $b^*(\widetilde{w_m})=e_m$ and $b^*(w_l)=e_m$ of Steady state 4.

(ii) In Steady state 3, Y increases with F_h from Propositions 4 (i) and 5 3. ($B=\widehat{B}^*(F_h)$), and $\frac{Y_M}{Y}$ is constant from the proof of Proposition 4 (i) and (30). Y is strictly lower than Y of Steady state 1, since Y increases with F_h when $b^*(w_l)>e_m$ too. In Steady state 2, when $F_h<F_h^\dagger$, Y increases with F_h from Propositions 4 (ii)(b) 1. and 5 2. a. ($B=\overline{B}^*(F_h)$). From the proof of Proposition 4 (ii)(b) 1. and (38), $Y=A_M(\phi(F_h,\overline{B}^*(F_h)))^{1-\alpha}F_h+\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\Big[A_M(\phi(F_h,\overline{B}^*(F_h)))^{1-\alpha}F_h-(1+r)(e_h+\phi(F_h,\overline{B}^*(F_h))e_m)F_h\Big]$(the first term is Y_M). Hence, $\frac{Y_M}{Y}=\Big\{1+\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\Big[1-\frac{1+r}{A_M}\Big(\frac{e_h}{(\phi(F_h,\overline{B}^*(F_h)))^{1-\alpha}}+e_m(\phi(F_h,\overline{B}^*(F_h)))^{\alpha}\Big)\Big]\Big\}^{-1}$ and $\frac{Y_M}{Y}$ increases (decreases) with $[\phi(F_h,\overline{B}^*(F_h))]^{-1}$ for $[\phi(F_h,\overline{B}^*(F_h))]^{-1}>(<)\frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$, where $\frac{\alpha}{1-\alpha}\frac{e_m}{e_h}>\widetilde{w_m}^{-1}\Big[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\Big]$ can be proved as follows. First, Assumption 2 implies $\alpha A_M((\frac{F_h}{F_m})_{hm})^{-(1-\alpha)}>\frac{e_h}{\gamma_b}\Leftrightarrow\alpha A_M(\frac{F_h}{F_m})^{-(1-\alpha)}-(1+r)e_h<(1-\alpha)A_M(\frac{F_h}{F_m})^{\alpha}-(1+r)e_m$ at $\frac{F_h}{F_m}=(\frac{\gamma_b\alpha A_M}{e_h})^{\frac{1}{1-\alpha}}\Leftrightarrow A_M\alpha^\alpha(1-\alpha)^{1-\alpha}>\frac{e_h^\alpha}{\gamma_b}[e_h-\gamma_b(1+r)(e_h-e_m)]^{1-\alpha}$. Then, the last equation proves $\frac{\alpha}{1-\alpha}\frac{e_m}{e_h}>\widetilde{w_m}^{-1}\Big[\frac{1-\gamma_b(1+r)}{\gamma_b}e_m\Big]\Leftrightarrow\gamma_b(1-\alpha)A_M(\frac{\alpha}{1-\alpha}\frac{e_m}{e_h})^\alpha>e_m\Leftrightarrow A_M\alpha^\alpha(1-\alpha)^{1-\alpha}>\frac{e_h^\alpha e_m^{1-\alpha}}{\gamma_b}$. When $F_h\geq F_h^\dagger$ and $\frac{F_h}{1-F_h}\leq(\frac{F_h}{F_m})_{ml,\theta}$, Y, $\frac{Y_M}{Y}$, and $\frac{C_{BM}}{PC_B}$ increase with F_h from Propositions 4 (ii)(b) 2. and 5 2. a. ($B=\overline{B}^*(F_h)$). When $\frac{F_h}{1-F_h}>(\frac{F_h}{F_m})_{ml,\theta}$, Y increases with F_h from Proposition 5 2. b. ($F_m=1-F_h$ and $P=\theta$) and the proof of Proposition 4 (ii)(a) ($Y=A_M(F_h)^\alpha(1-F_h)^{1-\alpha}$), and $\frac{Y_M}{Y}=1$ and $\frac{C_{BM}}{PC_B}=1$ from Proposition 5 2. b. ($Y_T=0$). The highest Y of Steady state 2 (at $b^*(\widetilde{w_m})=e_h$) is strictly lower than Y of Steady state 1, because the latter coincides with Y when $\frac{F_h}{F_m}=(\frac{F_h}{F_m})_{hm}$ and $F_m=1-F_h$, and the above proof of Y increasing with F_h applies when $\frac{F_h}{1-F_h}\in\Big(\widetilde{w_m}^{-1}\Big[\frac{1-\gamma_b(1+r)}{\gamma_b}e_h\Big],(\frac{F_h}{F_m})_{hm}\Big]$ as well. In Steady state 4, Y increases with F_h and F_M from Propositions 4 (ii)(a) and 5 4. ($B=B^*(F_h,F_m)$). Since $Y=A_M(F_h)^\alpha(F_m)^{1-\alpha}+\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}[A_M(F_h)^\alpha(F_m)^{1-\alpha}-(1+r)(e_hF_h+e_mF_m)]$ from the proof of Proposition 4 (ii)(a) and (34), $\frac{Y_M}{Y}=\Big\{1+\frac{\gamma_B}{1-\gamma_B-\gamma_b(1+r)}\Big[1-\frac{1+r}{A_M}\Big(e_h(\frac{F_h}{F_m})^{1-\alpha}+e_m(\frac{F_h}{F_m})^{-\alpha}\Big)\Big]\Big\}^{-1}$ and thus $\frac{Y_M}{Y}$ increases (decreases) with $\frac{F_h}{F_m}$ for $\frac{F_h}{F_m}>(<)\frac{\alpha}{1-\alpha}\frac{e_m}{e_h}$. From Figure 6, for given $\frac{F_h}{F_m}$, Y in this steady state is strictly lower than Y in Steady state 2. Thus, the highest Y in Steady state 4 is strictly lower than in Steady state 2. The infinimum in Steady state 2 can be proved to be strictly higher than in Steady states 3 and 4 in the same way as (i). ■