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Intesa Sanpaolo

June 2012

Determining marginal contributions of the economic capital of credit risk portfolio: an analytical approach

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June 2012, first draft

We address the problem of decomposing the risk of a multi-factor credit portfolio into marginal contributions through a fast analytical approach: it is based on Taylor polynomial expansion of the overall risk and on the subsequent partial derivatives with respect to the single exposures, exploiting the Euler principle. The proposed approximation, which also accommodates for an efficient treatment of obligors with similar risk profile, is suitable for large and complex bank portfolios; furthermore, it proves to perform quite well if tested against numerical techniques, among which we chose the Harrel-Davis estimator. The latter, aside from representing a benchmark measure, should however be applied in the case of very small and concentrated portfolios. In addition, a comparison with the most usual variance-covariance approach is drawn, emphasising its drawbacks in the correct representation of risk allocation.

1 INTRODUCTION

The economic capital of a credit portfolio is a global measure that allows banks to understand the true risk of the lending activity, and on that subject a whole literature exists and a big effort is still on-going in order to improve the methodologies, which are becoming more and more sophisticated in their aim to reach the most accurate representation of risk. For the purposes of

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internal risk management, portfolio economic capital is not enough, as the process of capital allocation requires total risk to be decomposed into individual risk contributions (at transaction or counterparty level).

Theoretically, most standard risk measures can be decomposed as a sum of sensitivities or risk contributions that can be interpreted as marginal impacts on total portfolio risk. The key property allowing for this decomposition is the risk measure being positive homogeneous, meaning that the risk or capital of a portfolio scales in proportion to the size of the portfolio. This is the case for instance of Expected Shortfall and Value at Risk (VaR), the latter being the focus of the present document.

While in the context of a single factor infinitely granular model, like the model underlying the regulatory capital formulas, the decomposition is straightforward, in the framework of a multifactor credit portfolio model with a finite number of obligors there is no explicit analytical solution for deriving marginal contributions of the single exposures to the economic capital. The most widespread approach to capital allocation is thus to adopt the marginal contribution of each exposure to the volatility of the portfolio loss distribution. This is the so called variance-covariance approach, which is considered to be intuitive and easy to compute, but, being based on the strong hypothesis that losses are Normally distributed, becomes misleading and can produce inconsistent results in the most realistic case of non-Normal credit losses. On the other hand, simulation-based approaches could be quite burdensome in particular for large portfolios: in fact, the marginal risks can be interpreted as expectations of losses on exposures, conditional on events in the tail of the loss distribution for the full portfolio. Each contribution depends on the probability of a rare event (a default), which in turn depends on an even rarer event (an extreme loss for the portfolio). Ordinary Monte Carlo estimation is thus impractical, so that specific numerical techniques have to be used to overcome this problem.

We present here an analytical approximation of portfolio VaR, based on the second order Taylor expansion of losses as suggested by Pykhtin (2004). VaR can thus be split into three components: the VaR of a calibrated asymptotic single risk factor model; the adjustment for industry concentration, casting a bridge between single and multi-factor structure; the granularity adjustment. While Pykthin's method refers directly to the "actual" number of borrowers in the portfolio regardless of their similarities, our approach introduces some changes in order to facilitate the treatment of very large and complex portfolios. This is done first by clustering borrowers belonging to the same economic sector and sharing similar risk characteristics (e.g. the probability of default) and then by considering the name concentration inside the cluster, through the Hirschmann-Herfindahl index which measures the "effective" number of borrowers (see Gordy 2003). Furthermore, the analytically approximated capital can be used in order to determine the marginal credit VaR of different borrowers and/or transactions. In fact, under some conditions that are respected in the case of the analytical formula, we can adopt the Euler allocation principle, stating that, if risk contributions are computed as first-order derivatives of total risk with respect to portfolio exposures, their sum amounts to the total risk. We developed a solution for those derivatives and, in order to understand to what extent the analytical approximation was acceptable, results have been compared with a measure of the "true" contributions. Many numerical methods can be applied for this purpose, as shown in the literature (see Glasserman, 2005, and Mausser, 2008). As a benchmark we chose here the H-D (Harrel-Davis, 1982) estimator, belonging to the general class of L-estimators, since it provides good performances in terms of both accuracy and stability.

Some examples were drawn on test portfolios with different concentration characteristics, pointing out that the analytical approximation performs optimally, not only in terms of portfolio capital, which is close to the Monte Carlo outcome, but also for its allocation on single clusters: the contributions match in fact quite well the ones which are calculated through the HD estimator. On the other hand, the variance-covariance approach shows significant differences in the risk allocation profile if compared with the simulated benchmark.

The convergence of the analytical approach towards simulated contributions could not be satisfied under certain circumstances. For instance, if there is a non-negligible concentration in the portfolio, the analytical allocation becomes less accurate, failing to match the true risk as estimated through numerical techniques. The identification of the boundaries inside which it is safely applicable is beyond the scope of this paper, as they cannot be detected in absolute terms because of their dependence on a plurality of factors like credit quality, asset correlations and so on. Still, whenever the amount of one or a few exposures is predominant in the portfolio, the application of numerical techniques for determining the correct risk contributions is advisable.

A trade off does in fact exist between accuracy and computational effort (in terms of computing time and memory constraints), which must be evaluated taking into account both the complexity of the portfolio and the degree of concentration on single exposures. The analytical approximation we chose represents an effective trade off in the case of a large bank portfolio, but not necessarily for smaller or highly concentrated portfolios or sub-portfolios.

This paper idea, which is to propose a fast analytical solution for determining risk contributions and compare it to a numerical benchmark, is not new in the literature. For instance, the seminal paper of Emmer and Tasche (2003) applies this concept, in the context of the one-factor credit risk setting: formulae are developed for both the granularity adjustment approach and for a semi-asymptotic approach. Dullmann and Puzanova (2011) compute analytical marginal contributions in a multi-factor framework, comparing the results with those obtained through the importance sampling algorithm: however, this research was developed in a completely different context, with the aim of assessing individual banks' weight on systemic risk.

More recently, a similar philosophy has been adopted by Lutz and When (2012), who use the Cornish-Fisher expansion to derive analytical formulas for market risk contributions, pointing out the advantages in terms of computational efficiency of such a method if compared to simulation-based approaches.

The rest of the paper proceeds as follows. Section 2 describes all the assumptions underlying the multi-factor credit risk model and how Value At Risk approximation works. Section 3 explains the core derivatives involved in marginal risk allocation and the benchmark numerical solution adopted, while section 4 is devoted to some empirical examples. The last section concludes the paper.

2 STRUCTURE OF THE MULTI-FACTOR MODEL AND ANALYTICAL *VALUE AT RISK* SPECIFICATION

In this section a typical multi-factor model is introduced. In such a model, the dependence structure among obligors is driven by a number of systematic risk factors related to sector dynamics. Let's assume additionally that obligors who belong to the same sector with similar PDs and $LGDs$ are grouped into specific clusters c. Thus, the asset return of obligor i in sector s mapped onto cluster c can be represented by:

$$X_{i,c} = r_c Y_s + \sqrt{1 - r_c^2}\, \varepsilon_i \tag{2.1}$$

where Y_s is the sector risk factor, ε_i the idiosyncratic factor and r_c the loading which measures the borrower *i*'s sensitivity to the systematic risk (its square defines the intra-sector correlation, that is the correlation between two firms sharing the same cluster). Variables Y_s and ε_i follow Standard Normal distributions and are independent from each other. Furthermore, Y_s can be seen as a linear combination of independent and normally distributed factors Z_k:

$$Y_s = \sum_{k=1}^{S} \alpha_{s,k} Z_k \qquad \text{with} \quad \sum_{k=1}^{S} \alpha_{s,k}^2 = 1$$

in which the weights $\alpha_{s,k}$ come from the inter-sector correlation matrix decomposition (i.e. by means of Cholesky or eigenvalues/eigenvectors decomposition of asset returns correlation). Hence, the asset correlation between obligor *i* in sector *s* associated for instance to cluster a, and obligor *j* in sector *t* related to cluster b is given by:

$$Corr\left(X_{i,a}, X_{j,b}\right) = \begin{cases} 1 & if\ s = t\ and\ i = j \\ r_a r_b & if\ s = t\ and\ i \neq j \\ r_a r_b \sum_{k=1}^{S} \alpha_{s,k} \alpha_{t,k} & if\ s \neq t \end{cases}$$

The dependence structure in the multi-factor model is then fully described through the intra- and inter-sector correlation. The portfolio loss distribution, considering deterministic EAD and LGD parameters, can be written as

$$L(Z) = \sum_{c=1}^{C} \sum_{i=1}^{N_c} \omega_{i,c} LGD_c 1_{\{X_{i,c} < \Phi^{-1}(PD_c)\}} \tag{2.2}$$

where C is the total number of clusters, N_c represents the *actual* number of borrowers in each cluster, LGD_c is the loss given default per cluster c and $\omega_{i,c}$ stands for the share of total exposure at default (EAD) referred to obligor i. If $X_{i,c}$ falls below the threshold $\Phi^{-1}(PD_c)$[1], obtained by inverting the cumulative Standard Normal distribution corresponding to the probability of default PD_c, obligor i defaults and the third summand of L becomes 1. A suitable transformation of equation (2.2), aimed at embedding the concentration effect in case of large and heterogeneous portfolios, relies on the *effective* number of obligors N_c^*, computed via Herfindahl–Hirschmann Index (HHI)[2]:

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[1] due to an adverse realization of vector Z containing the systemic factors.

[2] N_c^* can be interpreted as the number of credits in a homogeneous portfolio with the equivalent degree of name concentration risk expressed by N_c.

$$L(Z) = \sum_{c=1}^{C} \omega_c LGD_c HHI_c \sum_{i=1}^{N_c^*} 1_{\{X_{i,c} < \Phi^{-1}(PD_c)\}} \tag{2.3}$$

with

$$\omega_c = \frac{EAD_c}{EAD} \quad \text{and} \quad HHI_c = \sum_{i=1}^{N_c} \left(\frac{EAD_{i,c}}{EAD_c}\right)^2 = \frac{1}{N_c^*}$$

In addition, conditional on risk factors Z, the number of defaults is binomially distributed, that is

$$\sum_{i=1}^{N_c^*} 1_{\{X_{i,c} < \Phi^{-1}(PD_c)\}} \sim \mathcal{B}\big(N_c^*, p_c(Z)\big)$$

where

$$p_c(Z) = \Phi\left(\frac{\Phi^{-1}(PD_c) - r_c Y_s}{\sqrt{1 - r_c^2}}\right)$$ [3]

It is worth noticing that if the portfolio is large enough (meaning that N_c^* tends to infinite), most of the idiosyncratic risk is diversified away and losses are mainly driven by systematic factors: in such a case, equation (2.3), by applying the law of large numbers (see Gordy 2003), degenerates to the limiting loss distribution of an infinitely fine-grained portfolio that is:

$$L^\infty(Z) = \sum_{c=1}^{C} \omega_c LGD_c \, p_c(Z) \tag{2.4}$$

The portfolio loss distribution (2.3) can be determined numerically through Monte Carlo simulations: in each scenario, the sector factors Z as well as the idiosyncratic component are randomly generated. After running the simulation and sorting the loss outcomes, the VaR measure at a given confidence level α is calculated in the following conventional manner:

$$VaR_\alpha(L) := q_\alpha(L) = inf\{l \in \mathbb{R} | \mathbb{P}[L \le l] \ge \alpha\}$$

Since the main drawback of this kind of models is the time-consuming calculation in case of large portfolios, an analytical approximation method as the one suggested by Pykhtin (2004) could be adopted. The basic idea behind the approach is to approximate the portfolio loss L of the multi-factor framework with the loss $\tilde{L}^\infty$ of an appropriately adjusted single factor model (ASRF): this is achieved by mapping the correlation structure into a single factor, let's define it $\tilde{Y}$, whose correlations with respect to the original composite drivers (Y_s) are maximized[4]. Based on this, a Taylor series expansion is performed around the single-factor model. Entering into details, the approximated loss $\tilde{L}^\infty$ is expressed as[5]:

$$\tilde{L}^\infty(\tilde{Y}) = \sum_{c=1}^{C} \omega_c LGD_c \, p_c(\tilde{Y}) \tag{2.5}$$

[3] Defined as conditional probability of default, obtained by simply quantifying the probability of the Gaussian process [1] being lower than threshold $\Phi^{-1}(PD_c)$, where Φ stands for the cumulative Normal distribution.
[4] $\tilde{Y}$ can in fact be viewed as a linear combination of the Z independent factors, that is $\tilde{Y} = \sum_{k=1}^{S} b_k Z_k$. In a nutshell, b_k is the outcome of the following optimization problem: $\max_{b_k}\left(\sum_{c=1}^{C} \gamma_c \sum_{k=1}^{S} b_k \alpha_{s,k}\right)$ such that $\sum_{k=1}^{S} b_k^2 = 1$. Pykhtin sets the weight γ_c as stand-alone VaR_α of obligor (cluster) c, using ASRF model with correlation parameter equal to r_c.
[5] where $a_c = r_c \sum_{k=1}^{S} \alpha_{s,k} b_k$ measures the maximizing correlation parameter.

$$p_c(\tilde{Y}) = \Phi\left(\frac{\Phi^{-1}(PD_c) - a_c\tilde{Y}}{\sqrt{1-a_c^2}}\right)$$

Being $\tilde{L}^\infty$ a deterministic monotonically decreasing function of $\tilde{Y}$, the quantile at level α is:

$$VaR_\alpha(\tilde{L}^\infty) := q_\alpha(\tilde{L}^\infty) = \tilde{L}^\infty(\Phi^{-1}(1-\alpha)) \tag{2.6}$$

Next step consists in calculating a Taylor series expansion up to quadratic term in order to reduce the approximation error of $\tilde{L}^\infty$: let's think the true portfolio loss as the variable $\tilde{L}^\infty$ plus a stochastic perturbation $U = L - \tilde{L}^\infty$ such that $L_\varepsilon = \tilde{L}^\infty + \varepsilon U$ with ε indicating the extent of the perturbation. Setting $\Delta\varepsilon = 1$ to move from L_0 to L_1 the portfolio loss quantile becomes

$$VaR_\alpha(L) \cong VaR_\alpha^{prox}(L) = VaR_\alpha(L_\varepsilon)|_{\varepsilon=0} + \left.\frac{dVaR_\alpha(L_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2}\left.\frac{d^2VaR_\alpha}{d\varepsilon^2}\right|_{\varepsilon=0} \tag{2.7}$$

It can be shown that the first derivative, referred to the second hand-side term, is equal to zero[6] due essentially to the maximizing choice of a_c. Furthermore, thanks to the work of Gourieroux et al. (2000), the second derivative applied to this framework can be written as

$$\frac{1}{2}\left.\frac{d^2VaR_\alpha}{d\varepsilon^2}\right|_{\varepsilon=0} = -\frac{1}{2\tilde{L}^\infty(\tilde{Y})'}\left[\mathbb{V}(\tilde{Y})' - \mathbb{V}(\tilde{Y})\left(\tilde{Y} + \frac{\tilde{L}^\infty(\tilde{Y})''}{\tilde{L}^\infty(\tilde{Y})'}\right)\right]\Bigg|_{\tilde{Y}=\Phi^{-1}(1-\alpha)} \tag{2.8}$$

Expressions $\tilde{L}^\infty(\tilde{Y})'$ and $\tilde{L}^\infty(\tilde{Y})''$are the first and the second derivatives of equation (2.5), evaluated in $\tilde{Y}$, whose calculation is straightforward, in fact:

$$\tilde{L}^\infty(\tilde{Y})' = \sum_{c=1}^{C} \omega_c\, LGD_c\, p_c(\tilde{Y})' \tag{2.9}$$

and

$$\tilde{L}^\infty(\tilde{Y})'' = \sum_{c=1}^{C} \omega_c\, LGD_c\, p_c(\tilde{Y})'' \tag{2.10}$$

with

$$p_c(\tilde{Y})' = -\frac{a_c}{\sqrt{1-a_c^2}}\phi\left(\frac{\Phi^{-1}(PD_c)-a_c\tilde{Y}}{\sqrt{1-a_c^2}}\right) \quad \text{and} \quad p_c(\tilde{Y})'' = -\frac{a_c^2}{1-a_c^2}\frac{\Phi^{-1}(PD_c)-a_c\tilde{Y}}{\sqrt{1-a_c^2}}\phi\left(\frac{\Phi^{-1}(PD_c)-a_c\tilde{Y}}{\sqrt{1-a_c^2}}\right)$$

where ϕ identifies the Normal density function.

$\mathbb{V}(\tilde{Y})$ and $\mathbb{V}(\tilde{Y})'$ are respectively the variance of L conditional on $\tilde{Y}$ and its first derivative still calculated in $\tilde{Y}$[7]. A useful insight to be noticed is the standard decomposition in terms of systematic and idiosyncratic components of the latter two functions.

In particular, $\mathbb{V}(\tilde{Y}) = \mathbb{V}_\infty(\tilde{Y}) + \mathbb{V}_{GA}(\tilde{Y})$ and $\mathbb{V}(\tilde{Y})' = \mathbb{V}_\infty(\tilde{Y})' + \mathbb{V}_{GA}(\tilde{Y})'$ where:

$$\mathbb{V}_\infty(\tilde{Y}) = var[\mathbb{E}(L|Z)|\tilde{Y}] = \tag{2.11}$$
$$= \sum_{i=1}^{C}\sum_{j=1}^{C} \omega_i\omega_j LGD_i LGD_j \left[\Phi_2\left(\Phi^{-1}[p_i(\tilde{Y})], \Phi^{-1}[p_j(\tilde{Y})]; \rho_{i,j}^{\tilde{Y}}\right) - p_i(\tilde{Y})p_j(\tilde{Y})\right]$$

[6] See Pykhtin (2004) for proof.

[7] By introducing some simple adjustments, the approach also accommodates for stochastic LGD setting.

$$\mathbb{V}_{GA}(\tilde{Y}) = \mathbb{E}[var(L|Z)|\tilde{Y}] = \tag{2.12}$$
$$= \sum_{i=1}^{C} \omega_i^2 LGD_i^2 HHI_i \left[p_i(\tilde{Y}) - \Phi_2\left(\Phi^{-1}[p_i(\tilde{Y})], \Phi^{-1}[p_i(\tilde{Y})]; \rho_{i,i}^{\tilde{Y}}\right)\right]$$

Equation (2.11) represents the variance of loss L^∞ conditional on the single comparable factor $\tilde{Y}$, accounting for the correction due to the multi-factor setting; equation (2.12), which is the expected value applied on the conditional variance of L, describes the effect of a finite number of loans, also known as granularity adjustment. Φ_2 is the normal bivariate cumulative distribution, whilst $\rho_{i,j}^{\tilde{Y}}$ has the meaning of conditional asset correlation between cluster i and j. Equation (2.1), referred to obligor i given a realization of the effective risk factor $\tilde{Y}$ (which shifts the mean of $X_{i,c}$), can be rewritten in the general form with ξ_i standard normally distributed,

$$X_{i,c} = a_c\tilde{Y} + \sum_{k=1}^{S}\left(r_c\alpha_{s,k} - a_c b_k\right) Z_k + \sqrt{1 - r_c^2}\, \xi_i \tag{2.13}$$

which entails

$$\rho_{i,j}^{\tilde{Y}} = \frac{r_i r_j \sum_{k=1}^{S} \alpha_{s,k}\alpha_{t,k} - a_i a_j}{\sqrt{1 - a_i^2}\sqrt{1 - a_j^2}} \tag{2.14}$$

Finally, as last computational step, first derivatives of (11) and (12) yield[8]:

$$\mathbb{V}_\infty(\tilde{Y})' = 2\sum_{i=1}^{C}\sum_{j=1}^{C} \omega_i\omega_j LGD_i LGD_j\, p_i(\tilde{Y})' \left[\Phi\left(\frac{\Phi^{-1}[p_j(\tilde{Y})] - \rho_{i,j}^{\tilde{Y}}\Phi^{-1}[p_i(\tilde{Y})]}{\sqrt{1 - \left(\rho_{i,j}^{\tilde{Y}}\right)^2}}\right) - p_j(\tilde{Y})\right] \tag{2.15}$$

$$\mathbb{V}_{GA}(\tilde{Y})' = \sum_{i=1}^{C} \omega_i^2 LGD_i^2 HHI_i\, p_i(\tilde{Y})' \left[1 - 2\Phi\left(\frac{\Phi^{-1}[p_i(\tilde{Y})] - \rho_{i,i}^{\tilde{Y}}\Phi^{-1}[p_i(\tilde{Y})]}{\sqrt{1 - \left(\rho_{i,i}^{\tilde{Y}}\right)^2}}\right)\right] \tag{2.16}$$

After having combined all the equations involved in the analytical approximation (2.7), VaR can be split into three components: the first concerns the VaR of a calibrated asymptotic single risk factor model (ASRF) $\tilde{L}^\infty$, the second provides the adjustment for industry-concentration casting a bridge between single- and multi-factor structure (by inserting equations (2.11) and (2.15) into (2.8)), the third, also named granularity adjustment, measures the name concentration effect (by feeding equation (2.8) with (2.12) and (2.16)). This is summarized as follows:

$$VaR_\alpha(L) \cong VaR_\alpha^{prox}(L) = VaR_\alpha(\tilde{L}^\infty) + \underbrace{\Delta VaR_\alpha(L)^\infty + \Delta VaR_\alpha(L)^{GA}}_{\Delta VaR_\alpha(L)^{\infty+GA}} \tag{2.17}$$

3 ANALYTICAL AND NUMERICAL CREDIT *VAR* CONTRIBUTION

In addition to computing the total VaR, it is important to develop methodologies that properly attribute the overall risk to sub-portfolios, counterparties or even individual transactions. An

[8] For complete and rigorous proofs see Hibbeln (2010), appendix 5.5, paragraph 5.5.4.

appealing approach that could be adopted, in particular for analytical frameworks, is based on the Euler allocation principle: it suggests that risk contributions are computed as first-order derivative of total risk with respect to portfolio exposures, and that their sum amounts to the total risk (full allocation property). Following Tasche (2007), this allocation scheme is applicable when the risk measure function ρ_α is continuously differentiable and homogeneous of degree one that is, $\rho_\alpha(hL) = h\rho_\alpha(L),\ h > 0$; furthermore, it turns out that first derivatives are equivalent to the expected value of specific losses conditional on total risk (see Gourieroux et al. (2000)).

In the case of VaR approximation, it can be easily shown that the two aforementioned properties hold. In fact, $VaR_\alpha^{prox}(L)$ is by definition a continuously differentiable function and if all the exposures ω_c involved in equations (2.6) and (2.8) scale by the quantity $h > 0$, $VaR_\alpha^{prox}(L)$ scales by the same magnitude too.

Thus, in the light of previous findings, the following representation is carried out:

$$VaR_\alpha^{prox}(L) = \sum_{c=1}^{C} \frac{\partial}{\partial \omega_c} VaR_\alpha^{prox}(L)$$

and

$$VaR_\alpha^{prox}(L_c|L) = \frac{\partial}{\partial \omega_c} VaR_\alpha^{prox}(L) = \mathbb{E}\left[L_c | L = VaR_\alpha^{prox}(L)\right] \tag{3.1}$$

Besides, the contributions in terms of industry and granularity effects read

$$\frac{\partial}{\partial \omega_c} VaR_\alpha^{prox}(L) = \frac{\partial}{\partial \omega_c} VaR_\alpha\left(\tilde{L}^\infty\right) + \frac{\partial}{\partial \omega_c} \Delta VaR_\alpha(L)^\infty + \frac{\partial}{\partial \omega_c} \Delta VaR_\alpha(L)^{GA} \tag{3.2}$$

As a next stage, in order to get closed form contributions, it is necessary to calculate the derivatives in ω_c of (2.6) and (2.8), which are respectively the first and the last two summands of (3.2).

As far as loss (2.6) is concerned, the first derivative leads to

$$\frac{\partial}{\partial \omega_c} VaR_\alpha\left(\tilde{L}^\infty\right) = LGD_c\, p_c\left(\tilde{Y}\right)\Big|_{\tilde{Y} = \Phi^{-1}(1-\alpha)} \tag{3.3}$$

Derivative of (2.8) is more burdensome, but basically it relies on quotient rule:

$$\begin{aligned}
\frac{\partial}{\partial \omega_c} \Delta VaR_\alpha(L)^{\infty+GA} = {} & \left\{2\left[\tilde{L}^\infty\left(\tilde{Y}\right)'\right]^2\right\}^{-1} \\
& \times \Bigg\{-\frac{\partial}{\partial \omega_c} \mathbb{V}\left(\tilde{Y}\right)' \tilde{L}^\infty\left(\tilde{Y}\right)' + \mathbb{V}\left(\tilde{Y}\right)' \frac{\partial}{\partial \omega_c} \tilde{L}^\infty\left(\tilde{Y}\right)' \\
& + \left[\frac{\partial}{\partial \omega_c} \mathbb{V}\left(\tilde{Y}\right) \tilde{L}^\infty\left(\tilde{Y}\right)' - \mathbb{V}\left(\tilde{Y}\right) \frac{\partial}{\partial \omega_c} \tilde{L}^\infty\left(\tilde{Y}\right)'\right] \\
& \times \left(\frac{\tilde{L}^\infty\left(\tilde{Y}\right)''}{\tilde{L}^\infty\left(\tilde{Y}\right)'} + \tilde{Y}\right) + \mathbb{V}\left(\tilde{Y}\right) \tilde{L}^\infty\left(\tilde{Y}\right)' \frac{\partial}{\partial \omega_c} \left(\frac{\tilde{L}^\infty\left(\tilde{Y}\right)''}{\tilde{L}^\infty\left(\tilde{Y}\right)'}\right)\Bigg\}\Bigg|_{\tilde{Y} = \Phi^{-1}(1-\alpha)}
\end{aligned} \tag{3.4}$$

What emerges from (3.4) is that the derivatives to be still evaluated concern $\tilde{L}^\infty\left(\tilde{Y}\right)'$, $\frac{\tilde{L}^\infty(\tilde{Y})''}{\tilde{L}^\infty(\tilde{Y})'}$, $\mathbb{V}\left(\tilde{Y}\right)$ and $\mathbb{V}\left(\tilde{Y}\right)'$. The first two present a simple solution that is

$$\frac{\partial}{\partial \omega_c}\tilde{L}^{\infty}(\tilde{Y})' = LGD_c\, p_c(\tilde{Y})' \tag{3.5}$$

and

$$\frac{\partial}{\partial \omega_c}\left(\frac{\tilde{L}^{\infty}(\tilde{Y})''}{\tilde{L}^{\infty}(\tilde{Y})'}\right) = \frac{LGD_c p_c(\tilde{Y})'' \sum_{i=1}^{C}\left[\omega_i LGD_i p_i(\tilde{Y})'\right] - LGD_c\, p_c(\tilde{Y})' \sum_{i=1}^{C}\left[\omega_i LGD_i p_i(\tilde{Y})''\right]}{\left[\sum_{i=1}^{C}\omega_i LGD_i p_i(\tilde{Y})'\right]^2} \tag{3.6}$$

Then, the derivatives of both variance components with respect to the individual exposure weight ω_c are

$$\frac{\partial}{\partial \omega_c}\mathbb{V}_{\infty}(\tilde{Y}) = 2LGD_c \sum_{i=1}^{C} \omega_i LGD_i\left[\Phi_2\left(\Phi^{-1}\left[p_c(\tilde{Y})\right], \Phi^{-1}\left[p_i(\tilde{Y})\right]; \rho_{c,i}^{\tilde{Y}}\right) - p_c(\tilde{Y}) p_i(\tilde{Y})\right] \tag{3.7}$$

and

$$\frac{\partial}{\partial \omega_c}\mathbb{V}_{GA}(\tilde{Y}) = 2\omega_c HHI_c LGD_c^2\left[p_c(\tilde{Y}) - \Phi_2\left(\Phi^{-1}\left[p_c(\tilde{Y})\right], \Phi^{-1}\left[p_c(\tilde{Y})\right]; \rho_{c,c}^{Y}\right)\right] \tag{3.8}$$

whilst the corresponding derivatives of $\mathbb{V}(\tilde{Y})'$, needed to complete formula (3.4), yield to:

$$\begin{aligned}\frac{\partial}{\partial \omega_c}\mathbb{V}_{\infty}(\tilde{Y})' = {} & 2LGD_c \sum_{i=1}^{C} \omega_i LGD_i p_i(\tilde{Y})' \left[\Phi\left(\frac{\Phi^{-1}\left[p_c(\tilde{Y})\right] - \rho_{i,c}^{\tilde{Y}}\Phi^{-1}\left[p_i(\tilde{Y})\right]}{\sqrt{1-\left(\rho_{i,c}^{\tilde{Y}}\right)^2}}\right) - p_c(\tilde{Y})\right] \\ & + 2LGD_c \sum_{i=1}^{C} \omega_i LGD_i p_c(\tilde{Y})' \left[\Phi\left(\frac{\Phi^{-1}\left[p_i(\tilde{Y})\right] - \rho_{i,c}^{\tilde{Y}}\Phi^{-1}\left[p_c(\tilde{Y})\right]}{\sqrt{1-\left(\rho_{i,c}^{\tilde{Y}}\right)^2}}\right) - p_i(\tilde{Y})\right]\end{aligned} \tag{3.9}$$

and finally[9]

$$\frac{\partial}{\partial \omega_c}\mathbb{V}_{GA}(\tilde{Y})' = 2\omega_c HHI_c LGD_c^2 p_c(\tilde{Y})' \left[1 - 2\Phi\left(\frac{\Phi^{-1}\left[p_c(\tilde{Y})\right] - \rho_{c,c}^{\tilde{Y}}\Phi^{-1}\left[p_c(\tilde{Y})\right]}{\sqrt{1-\left(\rho_{c,c}^{\tilde{Y}}\right)^2}}\right)\right] \tag{3.10}$$

So far, it has been proved that by applying Euler's theorem on a closed-form and homogeneous VaR representation, it is possible to obtain capital charges at cluster (transaction or sub-portfolio as well) level which add up to the total VaR itself. However, as outlined by Tasche (2007), the accuracy of this kind of approach becomes questionable if there are very large exposures in the portfolio. In that case, numerical methods, despite their computational burden, might be more appropriate: for instance, a technique that has been found to perform well in practice in terms of both accuracy and stability relies on the Harrell-Davis (1982) estimator[10], which belongs to the general class of *L*-estimators. For clarity sake, *L*-estimators compute a specific quantile estimate as

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[9] A useful hint for the calculation of expression (3.9) relies on matrix algebra. In fact, the variance $\mathbb{V}_{\infty}(\tilde{Y})'$ could be expressed through the quadratic form $\mathbb{V}_{\infty}(\tilde{Y})' = 2T(\omega)H\omega$, with ω being the vector of exposure, H the asymmetrical matrix which properly collects all the other terms in (2.15), T the transpose operator. Thus, the derivative (3.9) follows the rule $2[H + T(H)]\omega$.

[10] Other approaches, e.g. based on conditional independence framework coupled with importance sampling technique, exist. For a complete and exhaustive review see Glasserman (2005), Mausser and Rosen (2008).

weighted average of multiple order statistics $L^{(k)}$: in a sample of size S generated by means of Monte Carlo simulation, VaR_a can be calculated as

$$\overline{VaR}_{\alpha}(L) = \sum_{k=1}^{S} \omega_{\alpha,S,k}\, L^{(k)} \tag{3.11}$$

with $\sum_{k=1}^{S} \omega_{\alpha,S,k} = 1$. The HD estimator (see Mausser (2003)) is based on the fact that increasing the sample size S, the expected value of order statistics $\mathbb{E}\left[L^{((S+1)\alpha)}\right]$ converges to the true $VaR_{\alpha}(L)$. Thus, an intuitive interpretation of weights $\omega_{\alpha,S,k}$ is that they reflect somehow the probability of $L^{(k)}$ being equal to the actual $VaR_{\alpha}(L)$: more specifically, Harrell-Davis find analytical weights in the form

$$\omega_{\alpha,S,k} = I_{k/S}[(S+1)\alpha, (S+1)(1-\alpha)] - I_{(k-1)/S}[(S+1)\alpha, (S+1)(1-\alpha)] \tag{3.12}$$

where $I_X[a, b]$ is the Incomplete Beta function. From another point of view, HD is then a bootstrap estimator of $\mathbb{E}\left[L^{((S+1)\alpha)}\right]$ according to which the expectation is computed analytically rather than by resampling. Since $L^{(k)} = \sum_{c=1}^{C} L_c^{(k)}$, it becomes easy to show that the contribution of each risk c is

$$\overline{VaR}_{\alpha}(L_c|L) = \mathbb{E}[L_c|L = \overline{VaR}_{\alpha}(L)] = \sum_{k=1}^{S} \omega_{\alpha,S,k}\, L_c^{(k)} \tag{3.13}$$

In the next paragraph, through some real life examples, analytical VaR contributions will be analysed, also in terms of industry and name concentration effects. Results will be compared with Harrell-Davis allocation, treated as a benchmark.

In common practices, the most widespread approach for assigning economic capital (usually meant as VaR minus the expected loss) to each exposure is the marginal contribution to the volatility of portfolio loss distribution; this is the so called variance-covariance approach, considered by many financial institutions to be an intuitive, simple and easy to compute risk measure. It works well provided that losses are Normally distributed, since in this case economic capital is a constant multiple of portfolio volatility. Due to the non-normality of credit loss distribution L and L_c, volatility allocation becomes misleading and can produce inconsistent capital charges. The main shortcomings are that it misses the diversification property and that the contribution might exceed the loan exposure (Kalkbrener et al., 2004), in the sense that the capital allocated to a sub-portfolio (or transaction or counterparty) might exceed its risk capital calculated on a stand-alone basis. In following exercise, the variance-covariance and analytical VaR approaches will be compared, pointing out significant differences in the risk allocation profile.

4 APPLICATION TO REAL-LIFE CREDIT PORTFOLIOS

For the purpose of illustrating the previous sections, some numerical examples are provided. They ground on a credit portfolio grouped into 10 clusters (c), with PD, LGD (fixed), r_c (intra-sector correlation) and EAD as indicated at the top of Table 1.

Table 1 Characteristics of the four test-portfolios.

Parameter	**c1**	**c2**	**c3**	**c4**	**c5**	**c6**	**c7**	**c8**	**c9**	**c10**
EAD (mln €)	500	1000	1700	2000	1800	1100	900	600	300	100
PD (bps)	1	2	5	10	25	70	170	450	1200	3000
LGD	0,45	0,45	0,45	0,45	0,45	0,45	0,45	0,45	0,45	0,45
r_c	0,65	0,63	0,61	0,59	0,57	0,55	0,53	0,51	0,49	0,47
Portfolio 1										
Sector	1	1	1	2	2	2	3	3	3	3
$1/HHI_c$	20	200	250	300	250	180	100	80	60	40
a_c	0,52	0,50	0,48	0,45	0,43	0,42	0,48	0,46	0,44	0,42
Portfolio 2										
Sector	1	1	1	2	2	2	3	3	3	3
$1/HHI_c$	1	30	50	30	20	100	40	10	5	1
a_c	0,52	0,50	0,48	0,45	0,43	0,42	0,48	0,46	0,44	0,42
Portfolio 3										
Sector	1	1	1	1	1	1	2	3	3	3
$1/HHI_c$	20	200	250	300	250	180	100	80	60	40
a_c	0,60	0,58	0,56	0,54	0,52	0,51	0,42	0,42	0,40	0,38
Portfolio 4										
Sector	1	1	1	1	1	1	2	3	3	3
$1/HHI_c$	1	30	50	30	20	100	40	10	5	1
a_c	0,60	0,58	0,56	0,54	0,52	0,51	0,42	0,42	0,40	0,38

Total exposure is assumed to amount to 10 bln € and it is distributed among clusters so that the average PD equals 120 bps: the degree of name concentration is then expressed by HHI_c index and the "effective" number of counterparties can be retrieved through $1/HHI_c$. Only three economic sectors are involved, implying a three factors model set-up whose correlation matrix is displayed in Table 2.

Table 2 Correlation matrix among sectors.

	Sector 1	**Sector 2**	**Sector 3**
Sector 1	100	80	55
Sector 2	80	100	40
Sector 3	55	40	100

Percentage values.

Using the aforementioned parameters but with different name/sector concentration characteristics, four portfolios are introduced; this analysis allows us to assess the performance of closed-form contributions against the HD benchmark and the variance-covariance approach, also disentangling concentration effects on capital allocation. The four portfolios characteristics are here described:

- Portfolio 1 shows a granular distribution of exposures across clusters, as stated by the effective number of counterparties $1/HHI_c$. Sectors 1, 2 and 3 present a quite homogeneous dispersion, accounting for 32%, 49% and 19% of total exposure respectively.
- Portfolio 2 holds the same sectorial distribution but it shows a higher name concentration resulting in the number of “effective” counterparties being lower; in particular, the first, fifth and the last three clusters contains a very few homogenous positions.
- Portfolio 3 is characterized by a prominent sector concentration, since sector 1 covers around 80% of total exposure. The degree of name concentration is low and it is designed like Portfolio 1.
- Portfolio 4 combines the features of portfolio 2 and 3 thus representing an example of high name/sector concentration setting.

As a first stage, Table 3 focuses on the overall economic capital (EC_α)[11], computed by using a 99.9% confidence interval α. The analytical approach result is broken down into its one-factor, sectorial and name concentration components (second, third and fourth columns) and it is compared with the Monte Carlo outcome shown in the last column; the latter comes from the simulation algorithm briefly described in paragraph 2, for which 100 million scenarios have been applied.

What emerges is that the economic capital calculated through the analytical approach represents a very good approximation of the “true” result from the Monte Carlo framework; thus, the estimation error-gap between $EC_\alpha(L)$ and $EC_\alpha^{prox}(L)$ appears to be very small.

Table 3 Comparison of closed-form approximation (forth column) and Monte-Carlo simulation (last column) for the sequence of illustrative portfolios.

	$E(L)$	$EC_\alpha(\tilde{L}^\infty)$	$\Delta VaR_\alpha(L)^\infty$	$\Delta VaR_\alpha(L)^{GA}$	$EC_\alpha^{prox}(L)$	$EC_\alpha(L)$
Portfolio 1	56	393	14	5	411	413
Portfolio 2	56	393	14	34	440	440
Portfolio 3	56	426	12	4	443	441
Portfolio 4	56	426	12	32	471	469

All figures are related to the total exposure and expressed in terms of bps, α-percentile is equal to 99.9%.

ASRF component in the second column depends essentially on the sector concentration, which is more pronounced for portfolio 3 and 4, whilst the granularity adjustment increases substantially, as expected, for portfolio 2 and 4 (34 and 32 bps respectively).

The marginal allocation at cluster level brings the analysis to equation (3.2), which allows us to compute analytical risk contributions and their decomposition into industry/name concentration components; Table 4 exhibits these results:

[11] obtained by subtracting the expected loss $E(L)$ from the *VaR* percentile that is $EC_\alpha(L) = VaR_\alpha(L) - E(L)$.

Table 4 Analytical marginal contributions among clusters.

	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	Total
Portfolio 1											
$\partial/\partial\omega_c\, EC_\alpha(\tilde{L}^\infty)$	1.5	4.7	15.1	24.6	40.0	46.1	86.2	89.4	62.3	22.7	392.5
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^\infty$	0.2	0.7	2.0	7.5	9.6	8.1	-4.2	-5.4	-3.9	-1.2	13.6
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^{GA}$	0.1	0.0	0.1	0.1	0.5	0.6	1.8	1.5	0.4	0.0	5.0
$\partial/\partial\omega_c\, EC_\alpha^{prox}(L)$	1.8	5.4	17.1	32.3	50.1	54.8	83.8	85.5	58.8	21.5	411.1
Portfolio 2											
$\partial/\partial\omega_c\, EC_\alpha(\tilde{L}^\infty)$	1.5	4.7	15.1	24.6	40.0	46.1	86.2	89.4	62.3	22.7	392.5
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^\infty$	0.2	0.7	2.0	7.5	9.6	8.1	-4.2	-5.4	-3.9	-1.2	13.6
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^{GA}$	2.0	-0.1	-0.2	3.1	9.8	-2.5	-0.7	12.2	7.3	3.5	34.3
$\partial/\partial\omega_c\, EC_\alpha^{prox}(L)$	3.7	5.3	16.9	35.3	59.4	51.7	81.4	96.2	65.7	25.0	440.4
Portfolio 3											
$\partial/\partial\omega_c\, EC_\alpha(\tilde{L}^\infty)$	2.2	7.1	22.5	40.6	64.0	70.6	67.3	76.1	54.9	20.8	426.1
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^\infty$	0.4	1.3	3.9	6.7	9.7	9.3	6.1	-13.8	-8.7	-2.5	12.3
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^{GA}$	0.1	0.0	0.1	0.1	0.6	0.7	1.4	1.2	0.4	0.0	4.5
$\partial/\partial\omega_c\, EC_\alpha^{prox}(L)$	2.6	8.3	26.5	47.5	74.2	80.6	74.8	63.5	46.7	18.3	443.0
Portfolio 4											
$\partial/\partial\omega_c\, EC_\alpha(\tilde{L}^\infty)$	2.2	7.1	22.5	40.6	64.0	70.6	67.3	76.1	54.9	20.8	426.1
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^\infty$	0.4	1.3	3.9	6.7	9.7	9.3	6.1	-13.8	-8.7	-2.5	12.3
$\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^{GA}$	2.1	-0.2	-0.4	3.5	11.6	-3.6	0.3	9.7	6.2	3.2	32.5
$\partial/\partial\omega_c\, EC_\alpha^{prox}(L)$	4.7	8.1	26.1	50.9	85.3	76.3	73.7	72.0	52.5	21.5	471.0

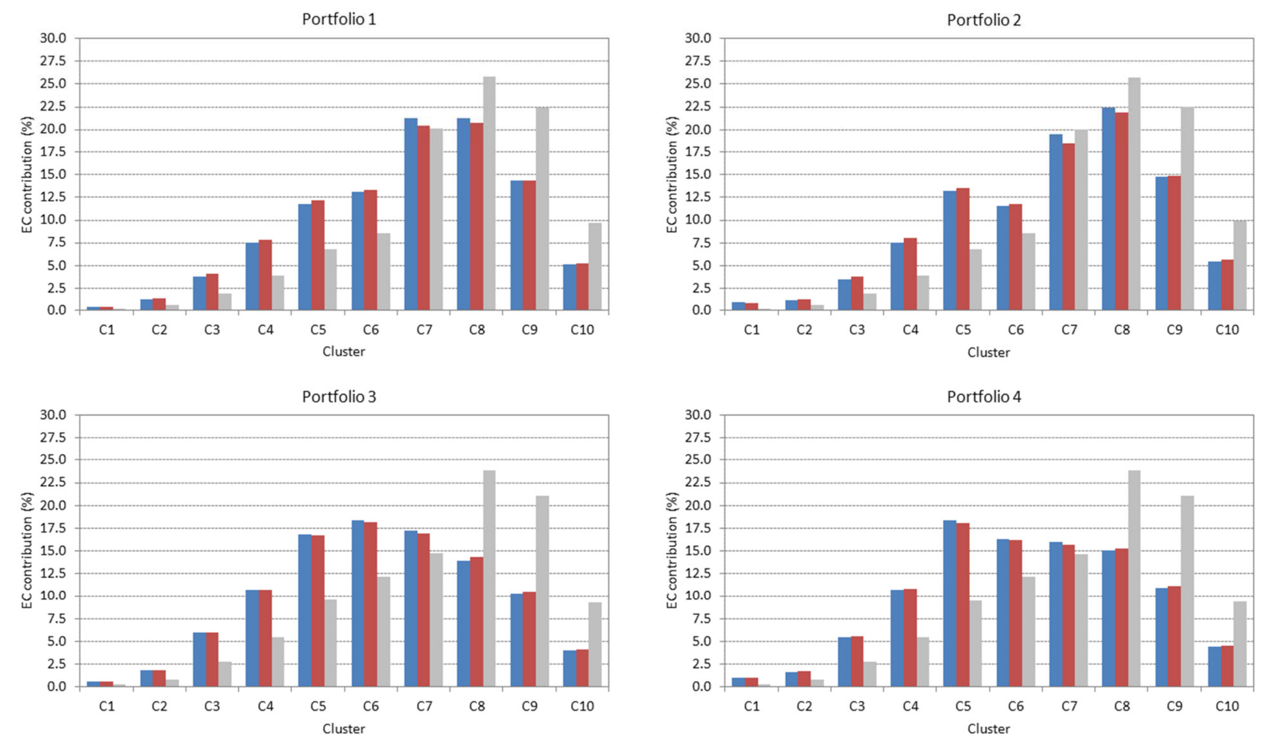
All figures are related to the total exposure and expressed in terms of bps, α-percentile is equal to 99.9%. The first addendum of each portfolio is the ASRF *VaR* percentile contribution minus the expected loss.

Some useful insights can be outlined. Regarding portfolios 1 and 2, the multifactor adjustment $\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^\infty$ recognises a diversification benefit[12] starting from c7 up to c10 and a capital burden for the other clusters. The last four belong in fact to sector 3, which shows a lower degree of asset returns dependency (see Table 2). Due to the different sector distribution, a more relevant impact can be detected for clusters 8, 9, 10 of portfolios 3 and 4. Furthermore, a significant name concentration adjustment ($\partial/\partial\omega_c\, \Delta VaR_\alpha(L)^{GA}$) emerges, as expected, for loans included in clusters 1, 4, 5, 8, 9, 10 where HHI_c indexes have been chosen in order to generate concentration risk; the magnitude slightly changes as far as portfolios 2 or 4 are concerned.

The first three components of each portfolio sum then up to the economic capital at cluster level, whose percentage contribution is compared with HD and variance/covariance approaches in Figure 1.

[12] that applies to the calibrated one-factor model $\partial/\partial\omega_c\, EC_\alpha(\tilde{L}^\infty)$.

Figure 1 Comparison among HD (blue bars), analytical (red bars) and variance/covariance (grey bars) - based contributions for economic capital at 99.9% confidence level.



Risk contributions of each approach sum up to 100%, since they are computed on the overall economic capital.

The main finding is that the analytical model appears very close to the numerical HD method that serves as a benchmark, whilst a complete different picture is found for variance/covariance approach. These figures confirm that assigning capital on the basis of the latter method might fail to properly represent the tail of the distributions and thus, as a consequence, their marginal risk allocation. The graph depicts in fact the under- or the over-estimation of risks along clusters.

Another interesting issue to be noticed is that the accuracy of the analytical contribution improves as the degree of industry correlation increases; in this respect, since portfolios 3 and 4 are dominated by sector 1, which represents the main driver for around 80% of the total exposure, they display more accurate contributions. This occurs because in the limit situations of the whole exposure belonging to one single sector or of a 100% asset correlation matrix (perfect dependency among asset returns), the model reduces to the one-factor framework, thus easing the approximation of the "true" risks.

5 CONCLUSION

In this paper we have studied the modeling of the economic capital of a credit risk portfolio. In this context the main issue that is faced in a real world implementation is the problem of deriving the marginal contribution of a subcomponent of the portfolio (transaction, name, cluster of names, etc.) to its overall amount.

Following the work of Pykhtin (2004), we have expressed the total economic capital using an analytical approximation based on a single factor model adjusted in order to account for multifactor dependence and name concentration. This is achieved by approximating the true economic capital through a Taylor expansion up to the quadratic term.

In this framework we applied the Euler rule to determine the marginal contribution to the total portfolio risk of a sub-portfolio component. This requires to compute the first order derivatives of the analytical approximation with respect to the sub-portfolio fraction of the total exposure.

To test the effectiveness of the approach we defined four different portfolios, each one composed of credit exposures divided into ten clusters. We then computed the total economic capital as well as the cluster contributions to the total capital and compared the results obtained through three different methods: a full Monte Carlo with an Harrel-Davis estimator which we consider the “relevant” benchmark, a Monte Carlo with the commonly used variance-covariance approach for determining the marginal contributions and the analytical approximation we developed in this paper.

The outcomes of the test show that the analytical approximation works extremely well in comparison with the “true” results while the variance-covariance approach fails badly to attribute risks to most of the portfolio’s clusters.

We consider the results obtained in this paper extremely interesting in the real world application of a portfolio model. In a commercial bank the typical size of the portfolio and the complexity of the reporting structure make the implementation of a full Monte Carlo very demanding in terms of computing time and hence impossible to use in practice. We believe this to be the main reason for the use of simplified methods such as the variance-covariance model; this choice, although permitting a practical use of economic capital models in a day-by-day risk management activity, nevertheless comes at the price of an inaccurate description of the risk contributions. On the contrary, the analytical formula we obtained represents correctly, as shown, the risk contributions, could be implemented in a rather straightforward way and does not require, even for very large portfolios, any special computing power.

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