A mathematical introduction to transitional lotteries

Strati, Francesco

Università di Messina

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FRANCESCO STRATI

Abstract. When we face a decision matter we do not face a frozen-time where all keep still while we are making a decision, but the time goes by and the probability distribution keeps moving by new available information. In this paper I want to build up the mathematical framework of a special kind of lottery: the *transitional lotteries*. This theory could be helpful to give to the decision theory a new key so as to define a more accurate mental path. In orther to do that we will need a mathematical framework based upon the Kolmogorov operator which will be our *transitional* object, the core of this kind of lottery.

1. Lotteries and the Fokker-Planck equation

In this paper I shall describe the mathematical framework so as to rich a very powerful method in computing lotteries. It would be difficult to study a truthful decision's path without consider some intuitions with regard to stochastic processes. Rather, we shall do that because of the randomness soakes up everything concerns decisions. We have to talk about the possible outcomes to which a decision could lead to and thus the risky alternatives; I am talking about the lotteries. A simple lottery $\mathscr{L}$ is defined as $\mathscr{L} = (p_1 \dots p_n)$ with $p \geq 0$ and $\sum_n p_n = 1$ where p is a probability that something happens.

We can treat the lotteries in this fashion: given a probability space $(\Omega, \mathscr{F}, \mathcal{P})$, a random variable (lottery) $\mathscr{L} : \Omega \to \mathbb{R}^+$ and the measure μ; we define the measure on a density f $(\upsilon = f.\mu)$ with respect to μ

$$\upsilon(A) = \int_A f(x) d\mu(x),$$

if υ, μ are σ-finite and $\upsilon \ll \mu$ (absolutely continuos) there exist the density f (Radon-Nikodym). We can assert that the probability density remains unchanged, but it could be difficult to state it in many cases. In decision theory it would be very likely to have a trasformation in processes, therefore it is of utmost importance to treat this subject in changing "probability". But, what does it mean "change"? An intuitive way to think about the meaning of transitional lotteries is this: If we have some information about a guy A and his utility function U_A. We know that two objects (c, b) lie on the same indifference curve, but the budget constraint of A allows him to buy either c or b. Thus, A at time t_0 is in the place x_0, but he wants b that is in the place x_1. He will be in the place x_1 at time t_1, hence if we know that he is already going to x_1 so as to take b it is pretty sure that he will satisfies his b-wish. But while is walking he see a store where it is selling c, thus A change his wish and goes toward that store so as to buy c. He changed his mind

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path because c was closer than b and we know they lie on the same indifference curve. The probability at time t_0 was right, for he really wanted to buy b at t_0, the probability distribution had changed when he sow the store of $c \sim b$. We had to observe the change in probability distribution from $(t_0, x_0) \to (t_1, x_1)$ and not at the point A has changed his mind, but as a flow in the time span $[t_1, dt, d_2]$. Of course dt is not a derivative, it means that we do not have to observe a point in the midst of the time span, but we have to be careful to the flow throughout that span. We have to use a method which admits a change, and in decision theory I rekon it would be of utmost importance to foresee a movement in probability distribution when a decision maker has to decide.

In what follows we shall see the mathematical framework which will admit a change in probability distribution, and in the last section we shall see the transitional lotteries made up that framework. In [§1.1] and [§1.2] we shall see the fundamental theory underpins every rational decision path. Through [§1.3] to [§1.6] we shall define the mathematical framework of the [§1.7] wherein there will be a definition of transitional lotteries.

1.1. **Orders and preorders**[1]**.** This section is by no means a complete treatment of the order theory, what follows is to indent so as to give a hint about this topic. We have to rationalize the decision maker's action, that is to say, we have to give some rules in order to have a benchmaker by which one can study how a decision maker should act.

A relation satisfying these property

P_1**:** $x \leq x$, Reflexivity
P_2**:** $x \leq y$ and $y \leq x$ imply $x = y$, Antisymmetry
P_3**:** $x \leq y$ and $y \leq z$ imply $x \leq z$, Transitivity

is called an *ordering* and a non-empty set which holds such a relations is called an *order*. We call *relation* $\mathcal{R}$ from a set A to a set B a subset $\mathcal{R} \subset A \times B$. Given $a \in A$ and $b \in B$, we write $a\mathcal{R}b$ (a is related to b) iff $(a, b) \in \mathcal{R}$. If $A = B$, we write $A^2 = A \times A$, then a *binary relation* $\mathcal{R}$ on A is a subset of A^2. If $\mathcal{R}$ satisfies the properties (P_1, P_2, P_3), then, $\mathcal{R}$ is an ordering and we can denote it by $\leq$. If $a \leq b$ we say that a is majorized by b or that b majorizes a [8], thus

1. $a \leq b$ means that $b \leq a$
2. $a < b$ means that $a \leq b$ and $a \neq b$
3. $a > b$ means $b < a$.

If $\mathcal{R}$ satisfies transitivity, reflexivity and symmetry[2] (rather than antisymmetry) we have an *equivalence relation* $\mathcal{R}^\epsilon$, denoted $a \equiv b(\text{mod}\mathcal{R}^\epsilon)$ as well. If $\mathcal{R}$ is transitive and reflexive, we named such relations *preorders*. If $\mathcal{R}^\epsilon$ is a preordering on A, we can define $\mathcal{R}^\epsilon$ on A: if $(a, b)(b, a) \in \mathcal{R}$ [8].

An $(A, \mathcal{R})$ is an ordered space and, in decision theory, we can define A a non-empty strategy set and $\mathcal{R}$ a preorders on A. A preorder allows us to compare several strategies rationally (under a constraint S), so we say that a preorder is the decision maker's rationality by which he can face a problem. We define $(A, \mathcal{R}, S)$ the decision matter where A is the matter-strategic set, $\mathcal{R}$ the rationality and S the feasible set.

[1]This section is based upon [2]
[2]if $a\mathcal{R}b \to b\mathcal{R}a$.

We have $(A, \leq)$ with $a, b \in A$, if $a \sim b$ then a is indifferent to b and satisfies both $a \leq b$ and $b \leq a$. If we have $a < b$ then a is strictly less than b or for $a > b$ we say that a is strictly dominant on b if $a \geq b$ (is weakly dominant) but it is not indifferent.

There are subjective and objective preorders, of course the meaning is that one cannot use the subjective one in lieu of the objective preorder, but that the former has to be a refinemet of the latter. For instance, if we have a $\vec{p} \in \mathbb{R}^n \to \exists x, y \in (\vec{p}, \leq, \mathbb{R}^n) \to x \leq_p y \leftrightarrow p_i x_i \leq p_i y_i \, \forall i$, it is reflexive and transitive and the $\leq_p \equiv \leq_s$ where $s_i = sgn(p_i)$. In decision theory we can say that $(\mathcal{R}, \neq)$ means that $\exists$ good-choices, bad-choices and unimportant-choices $(p_i = 0)$, furthermore if $-p_i$ in $p_i x_i \leq p_i y_i$ means $x_j \geq y_j$ it would be a bad-choice whereas a $+p_i \to x_j \leq y_j$ means that it would be a good-choice. The weak preference $\leq_p$ of the y-quality means that it is not less than the x-quality, and the y-bad quality must be not greater than the x-bad quality. Now we have to know an important propery. Given two lattices $\mathscr{X} = (X, \leq)$ and $\mathscr{Y} = (Y, \leq)$ we say that they are isomorphic $(\mathscr{X} \cong \mathscr{Y})$ and the map $\phi : \mathscr{X} \to \mathscr{Y}$ is an isomorphism, if ϕ is a bijection and $a \leq b \in \mathscr{X} \leftrightarrow \phi(a) \leq \phi(b) \in \mathscr{Y}$. We call homomorphism of a semilattice $(A_1, \circ)$ into $(A_2, \circ)$ if $\phi : A_1 \to A_2 \Longrightarrow \psi(a \circ b) = \phi(a) \circ \phi(b)$. In the decision theory, if a funcition f has an isomorphism, we say that $x \leq_f y$, then y is weakly preferred to x by means of the function $f : X \to \mathbb{R}$. It is worth to notice that the preorder induced by the function is still a preorder in its domain.

Given X_1 and X_2 two non-empty sets and $\mathcal{R}_1$ $\mathcal{R}_2$ on X_1 X_2, the binary relation $\mathcal{R}$ defined on $X_1 \times X_2 \, \forall x, y \in X_1 \times X_2$, $x\mathcal{R}y \leftrightarrow x_1 \mathcal{R}_1 y_1$, $x_2 \mathcal{R}_2 y_2$; we call it the product relation of $\mathcal{R}_1$ $\mathcal{R}_2$. The preorder $\leq$ defined on the cartesian product is called product-preorder of $\leq_1, \leq_2$. The preordered space $(X_1 \times X_2, \leq)$ is called the product-preordered space of $(X_1, \leq_1), (X_2, \leq_2)$. Given the product, an $x \ll y$ is a strong preference (of minorizing) induced by the product-preorder of the preferences $\leq_1, \leq_2$.

Given $X \neq \varnothing$ and $\mathcal{R}_1$ $\mathcal{R}_2$ on X. Thus $x\mathcal{R}y \leftrightarrow x\mathcal{R}_1 y$ and $x\mathcal{R}_2 y \ \forall \ x, y \in X$, it is called the conjoined relation of $\mathcal{R}_1$ $\mathcal{R}_2$. We say that the preorder $\leq$ defined on X by $x \leq y$ is a conjoined preorder $\leftrightarrow x \leq_1 y$ and $x \leq_2 y$ whereas the space $(X, \leq)$ is called the conjoined preorered space of $(X_1, \leq_1)$ and $(X_2, \leq_2)$.

1.2. **Optimal boundaries for decision**[3]**.** In this section I want to discuss the optimal boundaries in the field of preordered spaces. Remember that in $(X, \leq)$ if $x_0 < y$ and the relation $x_0 \geq y$ does not hold we say that y is strictly preferered to x_0. Bearing this in mind, let $(X, \leq)$ be a preordered space and S a part of X, $x_0 \in S$. The element x_0 is called *Pareto maximum of S* (or maximal) with respect to the preorder $\leq$, if $\nexists y \in S : y > x_0$. We call *Pareto maxima of S*, those points $x_0 \in S : \nexists y < x_0$ and we define them by $max^P_{\leq}(S)$ meanwhile the set of *Pareto minima of S* is denoted by $min^P_{\leq}(S)$. These two sets are called maximal Pareto-boundary of S and minimal Pareto boundary of S respectively. It is as plain as day that $x_0 \in (X, \leq)$ is a maximal element of the space iff $\forall \, y_i \geq x_0 \to y_i \sim x_0$.

Now we have to state that the maximal elements are not comparable among them, therefore let S be a part of a preordered set $(X, \leq)$. Then

1. if $max^{P_1}_{\leq}(S) \nleftrightarrow max^{P_2}_{\leq}(S) \ \therefore \ \to / \leftarrow$[4]

[3]This section is based on [3].

[4]In this section $\wedge$ has the logic meaning of "and" rather than "minimum" as above section and by $\to / \leftarrow$ I mean "not comparable".

2. if $(X, \leq) \wedge max_{\leq}^{P_1}(S) \neq max_{\leq}^{P_2}(S) \therefore \;\to / \leftarrow$
3. if $(X, \leq)$ is a totally preordered space then every Pareto maximum is a maximum of S
4. if $(X, \leq)$ is a totally preordered space a Pareto maximum of S, if it exists, is unique and it is the (unique) maximum of S.

Every maximum of S is a maximal element of S and it is that iff it is a maxima of S, thus two maximal elements are $\sim$ among them and $\exists!$ Pareto maximum.

We have to introduce another important concept, a subset C of a preordered set X is said to be *cofinal* (resp. *coinitial*) in X if $\forall\, x \in X\; \exists\, y \in C : x \leq y$ (resp. $y \leq x$). To say that an ordered set X has the greatest (resp. least) element, therefore, means that X has a cofinal (resp. coinitial) subset consisting of single element[1]. We could say that a cofinal part "the smaller is the more is good", hence a preordered space has a maximum iff $\exists$ a cofinal part whose elements are in the same class of indifference. We call C the smallest cofinal of X w.r.t. $\leq$ if it is cofinal and it is contained in each cofinal part of the space therefore it coincides with the intersection of all cofinal parts of the space.

Given the notion of cofinal, it is obvious that each cofinal part of an ordered space contains the maximal boundary of the space and it is the infimum of the set of cofinal parts of the ordered space w.r.t. the set inclusion in the power-set $\mathcal{P}(x)$, thus the intersection of cofinal parts has to be non-empty (even if it could be). It is trivial to say that if $\exists$ the smallest cofinal part of $\leq$, then it is coincides with the maximal boundary of the space. Moreover, an ordered space has the smallest cofinal part iff the maximal boundary of the space is cofinal.

There is an important concept that we have to use when we face preordered spaces which are not ordered "the saturated part w.r.t. an equivalence relation". We already know that every relation which satisfies reflexivity, symmetry and trasitivity is called an equivalence relation on X [1]. The partition $\mathfrak{P} \subset X$ is called the quotient space X by the relation $\mathcal{R}(x)$, we denoted by $X/\mathcal{R}$ its equivalence classes w.r.t. $\mathcal{R}$ (the notation $x \equiv y(mod\mathcal{R})$ is sometimes a synonym for $\mathcal{R}\{x, y\}$) [1]. The mapping $x \mapsto H(x)$ of X onto $X/\mathcal{R}$ is called the canonical map. Thus, let $\mathcal{R}$ be an equivalence relation on X, and $A \subset X$. Then $x \equiv y(mod\mathcal{R}) \in A$ is called the relation induced by $\mathcal{R}$ on A ($\mathcal{R}_A$). An $A \subset X$ is said to be saturated w.r.t. the equivalence relation $\mathcal{R}$ if for each $x \in A$ the equivalence class of x w.r.t. $\mathcal{R}$ is contained in A, thus are the unions of equivalence classes w.r.t. $\mathcal{R}$. If f is the canonical mapping of X onto $X/\mathcal{R}$ then a set is saturated if it is of the form $f^{-1}(X)$ where $X \subset X/\mathcal{R}$ [1]. Given that we say that the maximal boundary of a preordered space is saturated, for if $x \sim x_0$ where x_0 is the maximal element and by contradiction $\exists\, y : y > x \to y > x_0$, but it is impossible by definition. So x belongs to the maximal boundary. Now we can state that all $\cap S_c^n$ (intersection of saturated cofinal parts) coincides with the maximal boundary of the space, hence, if there is a maximal element of the space, such intersection is non-empty.

If $\exists$ a strictly increasing real functional on $\mathbb{R}^n$, then if K is a compact part of $\mathbb{R}^n$, the maximal boundary of $K \neq \varnothing$ and cofinal for K w.r.t. $(\mathbb{R}^n, \leq) \therefore \exists$ the smallest part of K.

A triple $(X, \leq, \tau)$ is said topological preorder space with X non-void set, $\leq$ a preorder on X and τ a topology on X. If the sets of upper (lower) bounds of each element of X w.r.t. $\leq$, are closed in τ, the $\leq$ is said upper (lower) compatible with the topology τ. So that assume $\exists$ on X at least a real upper semicontinuos strictly increasing functional, then the maximal boundary of a compact S is $S \neq \varnothing$ and cofinal for S. Let $(X, \leq, \tau)$, if every point of $X \in \tau$-closure of the set of its own strict upper (lower) bounds, then the maximal (minimal) boundary of a $S \subset X$ is contained in the τ-boundary of S.

1.3. **A mathematical introduction**[5]**.** We are given a real separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$) [7]. We denote by $L(H)$ the set of all bounded linear operator $T : H \to H$ endowed with the operator norm $\|\cdot\|$. The set of all symmetric and nonnegative operators of $L(H)$ is denoted by $L^+(H)$ and we denote by $\mathcal{B}(H)$ the σ-algebra of all Borel subsets of H [7]. We have already seen the measure μ, thus given the set of all probability measures $\mathscr{M}(H)$ on $(H, \mathcal{B}(H))$, if $\mu \in \mathscr{M}(H)$ then its Fourier transform is defined by

$$\hat{\mu}(\gamma) = \int_H e^{i\langle\gamma,x\rangle}\mu(x),\ \gamma \in H; \tag{1.1}$$

obviously $\hat{\mu}$ is the characteristic function of μ, furthermore $\hat{\mu}$ uniquely identifies μ.

Given $c \in \mathbb{R}$, we define the Gaussian measure on $\mathbb{R}$

$$N_{c,0}(dx) = \delta_c(dx) \tag{1.2}$$

where δ_c is the Dirac measure at c. If $\gamma > 0$

$$N_{c,\gamma}(dx) = \frac{1}{\sqrt{2\pi\gamma}}\, e^{-\frac{(x-c)^2}{2\gamma}}\, dx. \tag{1.3}$$

The Fourier transform of $N_{c,\gamma}$

$$\widehat{N_{c,\gamma}}(dx) = \int_{\mathbb{R}} e^{ihx} N_{c,\gamma}(dx) = e^{ich - \frac{1}{2}\gamma h^2},\ h \in \mathbb{R}. \tag{1.4}$$

Now we set $L_1^+(H) = L^+(H) \cap L_1(H)$ where $L_1(H)$ is the set of all operators of $L(H)$ of trace class [7]. Thus, given $Q \in L_1^+(H)$ in an arbitrary separable Hilbert space, then $\exists$ a unique measure $N_{c,Q}$ such that

$$\widehat{N_{c,Q}}(dx) = \int_{\mathbb{R}} e^{i\langle hx\rangle} N_{c,Q}(dx) = e^{i\langle hx\rangle - \frac{1}{2}Qh,h},\ h \in \mathbb{R}. \tag{1.5}$$

Thus given $c \in H$ and $Q \in L_1^+(H)$ $\exists$ a unique measure μ on $(H, \mathcal{B}(H))$ such that [7]

$$\int_{\mathbb{R}} e^{i\langle hx\rangle}\mu(dx) = e^{i\langle c,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle},\ h \in \mathbb{R} \tag{1.6}$$

μ is a Gaussian measure $\mu = N_{c,Q}$. Now we define the mean c and the covariance Q

$$\langle c, h\rangle = \int_H \langle x, h\rangle \mu(dx),\ h \in H \tag{1.7}$$

$$\int_H \langle x - c, y\rangle\langle x - c, z\rangle N_{c,Q}(dx) = \langle Qy, z\rangle,\quad y, z \in H. \tag{1.8}$$

We set $L^2(H, N_{c,Q}) = L^2(H, \mathcal{B}(H), N_{c,Q})$, following [6] we say that for any $h \in H$, the exponential function E_h, defined as

$$E_h(x) = e^{\langle h,x\rangle},\quad x \in H, \tag{1.9}$$

[5]This section is based on [4] and [5]

belongs to $L^p(H, N_{c,Q})$, $p \geq 1$, and

$$\int_H e^{i\langle h,x\rangle} N_{c,Q}(dx) = e^{\langle a,h\rangle} e^{\frac{1}{2}Qh,h}. \tag{1.10}$$

Furthermore the subspace of $L^2(H, N_{c,Q})$ spanned by all E_h, $h \in H$, is dense on $L^2(H, N_{c,Q})$ [7]. Now, given $Q \in L_1^+(H)$, we denote by (e_k) a complete orthonormal system in H and by (λ_k) a sequence of positive numbers (eigenvalues) such that $Qe_k = \lambda_k e_k$, $\forall k \in \mathbb{N}$ [7]. We define the subspace $Q^{\frac{1}{2}}(H)$ the reproducing kernel of the measure N_Q, and if $\mathrm{Ker}Q = \{0\}$ then $Q^{\frac{1}{2}}(H)$ is dense on H. We define the operator $Q^{\frac{1}{2}}$ as

$$Q^{\frac{1}{2}}x = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle x, e_k\rangle e_k, \quad x \in H. \tag{1.11}$$

Its range $Q^{\frac{1}{2}}(H)$ is the reproducing kernel (as we have just seen) of the measure N_Q [6]. We have said that $Q^{\frac{1}{2}}(H)$ is a proper subspace of H and that it is dense in H

$$Q^{\frac{1}{2}}(H) = \left\{ y \in H : \sum_{k=1}^{\infty} y_k^2 \lambda_k^{-1} < +\infty \right\}. \tag{1.12}$$

An interesting feature of $Q^{\frac{1}{2}}(H)$ is that $\mu(Q^{\frac{1}{2}}(H)) = 0$. Thus given $f \in Q^{\frac{1}{2}}(H)$ we could consider the function $W_f \in L^2(H, \mu)$ defined as [6]

$$W_f(x) = \langle Q^{\frac{1}{2}} f, x\rangle, \quad x \in H. \tag{1.13}$$

Following [6] it would be important to define $W_f \forall f \in H$, but we have to be careful because of its zero measure. Thus given [6]

$$W : Q^{\frac{1}{2}}(H) \subset H \to L^2(H, \mu), f \to W_f, \tag{1.14}$$

where $W_f(x) = \langle x, Q^{\frac{1}{2}} f\rangle, x \in H$. Then we can extend W to all H since it is an isomorphism. We have in fact for any $f, g \in Q^{\frac{1}{2}}(H)$

$$\int_H W_f(x) W_g(x) \mu(dx) = \langle QQ^{-\frac{1}{2}} f, Q^{-\frac{1}{2}} g\rangle = \langle f, g\rangle. \tag{1.15}$$

The function W is very important, it is the white noise function.

1.4. **The Brownian motion and the Wiener integral.** We can define a Brownian motion $\mathscr{B}$ considering the probability space $(H, \mathcal{B}(H), \mu)$[6]. Let $\mathscr{B}(t) = W_{\vartheta_{[0,t]}}$, $t \geq 0$, where $\vartheta_{[0,t]}$ is the characteristic function of the interval $[0, t]$. Then $\mathscr{B}$ is a real Brownian motion on $(H, \mathcal{B}(H), \mu)$ [7].

We can define an H-valued Brownian motion as a pair of (e_k), $(W_k(t)$ where the latter is a sequence of mutually independent real Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Thus we define

$$AW(t) = \sum_{k=1}^{\infty} W_k(t) A e_k, t \geq 0, \tag{1.16}$$

that is convergent in $L^2(\Omega; H)$ $\forall A \in L_2(H)$ (Hilbert-Schmidt operators) and so we define the expectation

$$\mathbb{E}|AW(t)|^2 = t\mathrm{Tr}[AA^\star] = t\|A\|_{HS}^2 \tag{1.17}$$

[6] $H = L^2(0, +\infty)$, $\mu = N_Q$ where $Q \in L_1^+ : \mathrm{Ker}Q = \{0\}$.

given that, we shall consider Wiener integrals of real functions with values in $L_2(H)$. Let $F \in L^2([0,T]; L_2(H))$ and $T > 0$

$$\int_0^T F(s)dW(s) = \sum_{k=1}^{\infty} \int_0^T F(s)e_k dW_k(s), \tag{1.18}$$

thus

$$\mathbb{E}\left|\int_0^T F(s)dW(s)\right|_H^2 = \int_0^T \|F(s)\|_{HS}^2 ds; \tag{1.19}$$

and the series is convergent in $L^2(\Omega; H)$.

1.5. **Spaces of continuos function.** We define $C_b(H)$ the Banach space consisting of all mappings $\varphi : H \to \mathbb{R}$ which are continuous and bounded, endowed with the norm

$$\|\varphi\|_0 := \sup_{x \in H} |\varphi(x)|, \ \varphi \in C_b(H). \tag{1.20}$$

We denote by $UC_b(H)$ the mappings $\varphi : H \to \mathbb{R}$ that are *uniformly* continuos and bounded.; it is not a separable space albeit $H = \mathbb{R}$. Following [6] we define some important subspaces.

(i) C_b^1 (resp.$UC_b^1(H)$) is the space of all continuos (resp. uniformly continuos) and bounded functions $\varphi : H \to \mathbb{R}$ which are Fréchet differentiable on H with a continuous (resp.uniformly continuous) and bounded derivative D_φ.

$$[\varphi]_1 := \sup_{x \in H} |D_\varphi(x)|, \ \ \|\varphi\|_1 = \|\varphi\|_0 + [\varphi]_1, \ \ \forall \varphi \in C_b^1(H), \tag{1.21}$$

if $\varphi \in C_b^1(H)$ and $x \in H$, we identify $D_\varphi(x)$ with the unique element h of H: $D_\varphi(x)y = \langle h, y\rangle$, $\forall y \in H$.

(ii) In general, for any $k \in \mathbb{N}$, $C_b^k(H)$ (resp.$UC_b^k(H)$) $\subset UC_b(H)$ of $\varphi : H \to \mathbb{R}$ which are k-times Fréchet differentiable on H with continuous (resp. uniformly) and bounded derivatives D_φ^h with $h \le k$.

$$[\varphi]_k := \sup_{x \in H} |D_\varphi(x)^k|, \ \ \|\varphi\|_k = \|\varphi\|_0 + \|\varphi\|_k, \ \ \forall \varphi \in C_b^k(H). \tag{1.22}$$

Furthermore

$$C_b^\infty(H) = \bigcap_{k=1}^{\infty} C_b^k(H) \tag{1.23}$$

1.6. **Toward the Fokker-Planck equation.** There would be a lot of things about this fascinating subject, but for the sake of brevity we have to do a sort of summary. We donote by $K(t)$ the Kolmogorov operator

$$K(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}[AA^\star D_x^2\varphi(x)] + \langle x, S^\star D_x\varphi(x)\rangle + \langle f(t), D_x\varphi(x)\rangle. \tag{1.24}$$

This stems from the finite-dimensional Ornstein-Uhlenbeck process

$$\begin{cases} dX = SXdt + AdW(t) \\ X(0) = x \in \mathbb{R}^d \end{cases}, \tag{1.25}$$

where $S : D(S) \subset H \to H$ generates a C_0 semigroup $e^{tS} : \|e^{tS}\| \le e^{-\omega t}$, $\omega > 0$, $A \in L(H)$ and we consider a H-valued brownian motion $((e_k),(W_k))$. The unique solution of the process is

$$X(t,x) = e^{tS}x + W_S(t), \quad t \ge 0, \tag{1.26}$$

where the stochastic convolution $W_S(t)$ is

$$W_S(t) = \int_0^t e^{(t-l)S} A dW(l). \tag{1.27}$$

Now (in order to extend the $X(t,x)$) we do have to give a meaning to the stochastic convolution as a Wiener integral for each fixed t

$$[0,t] \to L(H), \quad l \mapsto e^{(t-l)S} A \tag{1.28}$$

this mapping must belong to $L^2(0,t;L_2(H))$, so that

$$\int_0^t \mathrm{Tr}[e^{lS} AA^\star e^{lS^\star}] dl < \infty. \tag{1.29}$$

By it for any $t > 0$ the operator Q_t is of trace class

$$Q_t x = \int_0^t e^{lS} AA^\star e^{lS^\star} x dl, \quad x \in H, \tag{1.30}$$

if it is fulfilled, setting

$$W_S(t) := \sum_{h=1}^{\infty} \int_0^t e^{(t-l)S} A e_h dW_h(l), \tag{1.31}$$

we have a convergent series in $L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$ and $\mathbb{E}[|W_S(t)|^2] = \mathrm{Tr} Q_t$, furthermore $W_S(t)$ is a Gaussian random variable with values in H with mean 0 and covariance Q_t (that is to say of N_Q law). Thus by the $X(t,x)$ and the $W_S(t)$ we find the corresponding transition semigroup, called the Ornstein-Uhlenbeck semigroup

$$R_t \varphi(x) = \int_H \varphi(e^{tS}x + y) N_{Q_t}(dy), \quad t \ge 0, \quad x \in H, \quad \forall \varphi \in B_b(H), \tag{1.32}$$

where $B_b(H)$ is the space of all mappings $H \to \mathbb{R}$ bounded and Borel, so it is a semigroup of linear bounded operators on $UC_b(H)$, and it is strongly continuos only if $S = 0$. Now (without proof) we state that the space of exponential functions $\Theta(H)$ is stable for R_t.

We know (2.1) that $E_h(x) = e^{i\langle x,h\rangle}$ $\forall h \in H$, $x \in H$, we have just said that $\Theta(H)$ is the linear span of all real parts of E_h for $h \in H$. Albeit $\Theta(H)$ is not dense in $C_b(H)$, any function can be pointwise approximated from it by functions of $\Theta(H)$. We have to notice that, in several cases, the strong continuity fails to hold in $C_b(H)$ and that is the case of Ornestein-Uhlenbeck semigroup. Thus, the semigroup R_t belongs to a special class of semigroups on $UC_b(H)$, the so called $\pi-$*semigroups*. A sequence $(\varphi_n) \subset C_b(H)$ is said to be $\pi-$*convergent* to a map φ and we write $\varphi_n \xrightarrow{\pi} \varphi$ as $n \to \infty$ if the following conditions hold [9]

(i) $\quad \varphi \in C_b(H), \quad \sup_{n\ge1} \|\varphi_n\|_0 < \infty;$

(ii) $\quad \lim_{x\to\infty} \varphi_n(x) = \varphi(x), \quad x \in H.$

A subset Ξ is called π-dense if for any $\varphi \in UC_b(H)$ $\exists$ $(\varphi_n) \subset \Xi : \varphi_n \xrightarrow{\pi} \varphi$.

Following [7], for any $\varphi \in UC_b(H)$ $\exists$ a multi-sequence $(\varphi_{k,n,j})$ in $\Theta_S(H)$ such that

(i) $\quad \lim_{k\to\infty} \lim_{n\to\infty} \lim_{j\to\infty} \varphi_{k,n,j}(x) = \varphi(x), \ \forall x \in H$

(ii) $\quad \|\varphi_{k,n,j}\|_0 \le \|\varphi\|_0 + \frac{1}{n}, \ \forall n,k,j \in \mathbb{N}.$

If $\varphi \in \Theta(H)$ then $u(t,x) := R_t\varphi$, fulfills the Kolmogorov equation

$$\begin{cases} D_t X(t,x) = \frac{1}{2}\mathrm{Tr}[AA^\star D_x^2 u(t,x)] + \langle x, S^\star D_x u(t,x)\rangle, \\ u(0,x) = \varphi(x) \end{cases} \tag{1.33}$$

given that , the operator

$$\mathscr{K}_0\varphi(x) := \frac{1}{2}\mathrm{Tr}[AA^\star D_x^2 u(t,x)] + \langle x, S^\star D_x u(t,x)\rangle,\ x \in H, \tag{1.34}$$

is the Kolmogorov operator corresponding to O.U (1.25).

We need that the Kolmogorov operator (1.34) becomes contingent on a time dependent Ornstein-Uhlenbeck process in this form

$$\begin{cases} dX = (SXdt + f(t)) + AdW(t) \\ X(l) = x, \quad l \in [0,T) \end{cases}, \tag{1.35}$$

where $f \in C([0,T];H)$. Now we have $X(t,l,x)$ and $R_{l,t}\varphi(x) = \mathbb{E}[\varphi(X(t,l,x))]$, $x \in H$ and $0 \le l \le t \le T$. So adapting what we have done above to the this new form, we obtain

$$K(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}[AA^\star D_x^2\varphi(x)] + \langle x, S^\star D_x\varphi(x)\rangle + \langle f(t), D_x\varphi(x)\rangle, \tag{1.36}$$

which we have seen in the beginning

$$D_t R_{l,t}\varphi(x) = R_{l,t}K(t)\varphi(x), \quad 0 \le l \le t \le T \quad \forall \varphi \in \Theta_S(H). \tag{1.37}$$

Following [5] we denote by $\mathscr{P}_k(H)$, $k \in \mathbb{N}$ the set of all Borel probability measures μ in H and by $\mathscr{P}_k([0,T] \times H)$ the set of all probability kernels such that

$$\sup_{t\in[0,T]} \int_H |x|^k \mu_t(dx) < \infty. \tag{1.38}$$

Thus, given $\eta \in \mathscr{P}_1(H)$ we are looking for a probability kernel $\mu \in \mathscr{P}_k([0,T]\times H)$: $\mu_0 = \eta$ and

$$\frac{d}{dt}\int_H \varphi d\mu_t = \int_H K(t)\varphi d\mu_t, \quad t \in [0,T], \ \forall \varphi \in \Theta_S(H). \tag{1.39}$$

This is the Fokker-Planck (from now on F-P) equation and if $(\mu_t)_{t\in[0,T]}$ exists it is the F-P solution.

The F-P equation (1.39) describes the dynamics of stochastic systems, it models the time evolution of the probability distribution in a system under uncertainty by describing generic drift-diffusion processes. The F-P equation (1.39) is also known as Kolmogorov forward equation. This can give us a hint on the meaning of this equation, thus we could say, informally, that given a certain information about the state χ of the system Δ_χ at time s, we have a probability disrtibution (as we have seen above) $\mathscr{P}_s(\chi)$. Thus, we look for the change in probability distribution at time $t > s$, so that, the initial condition is integrated forward in time.

Given simple (or not) lotteries it would be intresting to study their in changing, in order to manage the decision theory in a more formal fashion. Hence, by the F-P equation (1.39) we can study the motion of the probability distribution and thus we could name this kind of lottery the “transitional-lotteries”.

1.7. **Transitional lotteries: an intuitive introduction.** Given the definition of preordered space [§1.1] by $[\leq; P_1, P_3]$, we define χ-preorder, a preorder where

P_2^χ If $x \leq y$ and $y \leq x \quad \rightarrow \quad \neg y \vee \neg x$

P_4^χ If $x \leq \diamond \leq y$, where $\diamond$ is a lack of knowledge we have a $[\chi]_\mu^\diamond$ such that given $x \leq [\chi]_\mu^\diamond \leq y$, $f : [\chi]_\mu^\diamond \rightarrow \mathbb{R}$ we define $[\chi]_\mu^\diamond$ as the measure of the rectangle $x \leq \chi \leq y$ given by f.

Corollary 1. *If in an ordered space there exist at least one undefined χ at a point-function $g \mapsto \mathbb{R}$, then it has to be approximated by the rectangle measure given from the difference $\|\{y\}_{ci} - \{x\}_{cf}\|$. The x_{cf} is the smallest cofinal part of x and y_{ci} is the smallest coinitial part of y.*

Thus, given P_2^χ, the $\{y\}_{ci}$ (resp. $\{x\}_{cf}$) is made up of one and only one element. This statment is true for if we have $y_0 \sim y \in \{y\}_{ci}$, then the $\neg y_0 \vee \neg y$.

We have to use the Corollary 1 because of the lack of knowledge that sometimes occurs, rather if we use the rectangle measure onto $\mathbb{R}$ we can obtain a geometric zone wherein the probability density function (p.d.f) surely passes through. For the sake of brevity, we say that given a set X of all elements, we have a σ-algebra, ξ, of subsets of X. Thus, we say that (X, ξ) is a measurable space and $\mu : \xi \rightarrow [0, +\infty)$ is a σ-additive function, hence μ is a measure on (X, ξ) and (X, ξ, μ) is a measure space.

We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space (X, ξ). We already know that $\mathcal{F}$ are the events, therefore we have a map $U : \Omega \rightarrow X$ such that $I \in \xi \rightarrow U^{-1}(I) \in \mathcal{F}$. We call it a random variable on (X, ξ) with law $U_\# \mathbb{P}$[7]. We know that $\mathcal{F}_U := \{U^{-1}(I) : I \in \xi\}$, thus $\mathcal{F}_U$ is the smallest σ-algebra in $\mathcal{F}$ such that U is measurable. We have to consider the $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ rather than (X, ξ) for justify the independence of the random variables

$$U_\# \mathbb{P} = \bigtimes_{i=1}^{k} (U_i)_\# \mathbb{P}. \tag{1.40}$$

We define a χ-lottery (χ_L), a lottery embedded a χ-preorder and one can define it on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by $(\Omega, \mathcal{F}, \mathbb{P})$ with $U_\# \mathbb{P}$.

Of course we have several χ_L, so we denote them $\chi_L^n \rightarrow (\mathbb{R}^n, \mathcal{B}^n(\mathbb{R}))$ embedded with the gaussian measure $N_{a,Q}$. We can set $x_h = \langle x, e_h \rangle$, $h \in \mathbb{N}$ and $\mathscr{P}_n x = \sum_{k=1}^n x_k e_k, x \in H, n \in \mathbb{N}$. Thus, the χ_L^n have an isomorphism γ, $H \rightarrow \ell^2$, furthermore we have to notice that $\mathscr{P}_n x, \quad \forall\, x \in \ell^2$. We have to state that Q is of trace class and $\det Q > 0$; these statements are needed for the independence and strongly continuity properties of the gaussian measure respectively.

We have built a χ_L^n within a gaussian measure is embedded, so that we define $\chi_{\mathcal{N}L}^n$ the gaussian lotteries. We have to say another definition, a $I \subset h$ where $I = \{x \in H : (x_1, \ldots, x_n) \in B\}, B \in \mathcal{B}(\mathbb{R})$, is called cylindrical. Now, we have defined (albeit in general) the $\chi_{\mathcal{N}L}^n$, hence we have to understand (in an intuitive way) the link between $\chi_{\mathcal{N}L}^n$ and its transition. We have seen the Ornstein-Uhlenbeck semigroup, we have defined [§1.6]

$$Q_t x = \int_0^t e^{lS} A A^\star e^{lS^\star}, \tag{1.41}$$

[7]The image measure of $\mathbb{P}$ through U.

where e^{lS} is a strongly continuous semigroup on H, $AA^\star$ the self-adjoint and nonegative bounded operator on H, Q_t a trace class operator and $e^{lS^\star}$ the adjoint semigroup of e^{lS}, thus by a N_Q law the O-U semigroup

$$R_t\varphi(x) = \int_H \varphi(e^{tS}x + y)N_Q(dy). \tag{1.42}$$

We have said [§1.6] that unless $e^{tS} = I$ (identy matrix) for any $t \geq 0$, R_t is not a strongly continuos semigroup on $UC_b(H)$, thus we have introduced [§1.6] the π-semigroup so as to define R_t in it. There is something to say, following [7], we call the infinitesimal generator (of the π-semigroup),

$$\begin{cases} D(L_\pi) = \{\varphi \in UC_b(H) : \exists\psi \in UC_b(H) : \Delta_h\varphi \xrightarrow{\pi} \psi,\ h \to 0\}, \\ L_{\pi(\varphi)} = \psi, \end{cases} \tag{1.43}$$

where $\Delta_h = \frac{1}{h}(R_h - I)$ with $h > 0$. Thus, $D(L_\pi)$ is dense in $UC_b(H)$, hence if $\varphi \in D(L_\pi)$ then $R_t\varphi \in D(L_\pi)$ $\forall t \geq 0$. An important thing is given that $\varphi \in D(L_\pi)$ then $R_t\varphi(x)$ is differentiable $\forall t \geq 0$. Thus, the resolvent $\rho(L_\pi) = R_t(z, L_\pi) := (z - L_\pi)^{-1}$ contains $(0, +\infty)$, by Hille–Yosida theorem

$$R(z, L_\pi)\phi(x) = \int_0^\infty e^{-zt}R_t\phi(x)dt,\ \phi \in UC_b(H),\ z > 0,\ x \in H. \tag{1.44}$$

This is the integral representation of the resolvent, and we have to keep in mind that every strongly continuos semigroup can be rescaled to becomes bounded. We find that if $\varphi_k \xrightarrow{\pi} \varphi$, hence $R(z, L_\pi)\varphi_k \xrightarrow{\pi} R(z, L_pi)\varphi$, $\forall z > 0$.

The definition of the C_0-semigroup is of utmost importance so as to give a strong framework for the transition process. Given an O-U process (1.35) $\exists$ a unique solution $X(\cdot, l, x)$ on $[0, T]$ given by

$$X(t, l, x) = e^{(t-l)S}x + \int_l^t e^{(t-r)S}f(r)dr + \int_l^t e^{(t-r)S}AdW(r), \tag{1.45}$$

with mean and covariance $e^{(t-l)S}x + m_{l,t}$ and Q_{t-l} respectively. Thus, $\forall t, l : 0 \leq l \leq t \leq T$ and $\varphi \in B_{b,1}(H)$ we gain the transition evolution operator[8]

$$R_{l,t}\varphi(x) = \mathbb{E}[\varphi(X(t, l, x))],\ x \in H. \tag{1.46}$$

Given the law of Gaussian measure

$$R_{l,t}\varphi(x) = \int_H \varphi(y)N_{e^{(t-l)S}x+m_{l,t},Q_{t-l}}(dy). \tag{1.47}$$

We can define the transition, thus through the forward Kolmogorov equation we can observe the distribution in changing from $l \to t$. We have understood the transition semigroup importance, and how a change can happen. But, I have passed over some topics that are beyond the aim of this introduction, for example the definition of the probability kernel and its smoothness and the study of semigroup in a deep fashion that I rekon it is needed in order to define the $\chi^n_{NL_\pi}$. These and other sujects will be developed in another paper that goes in for this topic.

For now we know that so as to handle with decision matter, we have to study not only the photograph of the distribution at a moment in time, but also the possibly evolution from t_0 to t_1. In the microeconomic theory we obtain a "transitional

[8] $B_{b,1}(H)$ is the space of all Borel functions with linear growth, $\varphi \in B_{b,1}(H)$ iff $\|\varphi\|_{b,1} := sup_{x\in H}\frac{|\varphi(x)|}{1+|x|} < \infty$.

expected utility function" where (by the e.u.f. theorem) the decision maker's preferences over lotteries have to satisfy the independence axioms and the continuity. Our lottery satisfies these property so we can extend the reasoning of χ-lotteries to the $\chi_{\mathbb{E}}$-utility function.

1.8. **Conclusions.** In this paper we have seen the mathematical foundation of the transitional lotteries' theory. We have used a *Brownian*-approach and by several objects we have built an integral equation by which the *transition* is well-defined. This kind of lottery has to be seen as a ground upon which the decision theory may lie, each decision is embedded with uncertainty and the transitional lotteries admit a stochastic world, but they also admit a change in a probability distribution as time goes by. The aim of this paper is to define the technical refinements of the theory, thus it is not need a deep knowledge about transitional lotteries application, a hint about it is enough. For now we want to aks: what are the bricks by which the transitional lotteries' theory is made of? This paper is intended to give an answer to this question.

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University of Messina, Department of Statistics

E-mail address: strati.francesco@gmail.com