A first introduction to S-Transitional Lotteries

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12 June 2012

A FIRST INTRODUCTION TO $\mathcal{S}$-TRANSITIONAL LOTTERIES

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Abstract. In this paper I shall introduce a new method by which it is possible to study the dynamical decision maker's behaviour. It can be tought of as an application of the $\mathcal{S}$-Linear Algebra of Professor David Carfì, thus this theory it is assumed to be known. I shall focus on the Feynman's propagator and thus the Feynman-Strati propagator. The latter stems form the former. It will be of utmost importance so as to give a meaning to both the evolution and the H-operator by which I shall derive the probability density of this kind of tempered distribution γ^p. Then I shall define the $\mathcal{S}$-transitional lottery's meaning as the γ^p's motion.

1. Introduction

In this section I shall introduce the concept of *$\mathcal{S}$-transitional lotteries*, its meaning and the tool we have to use so as to compute this kind of lottery, the *Feynman propagator*. But the Feynman's transition amplitude theorem is intended to be a tool which stems from the $\mathcal{S}$-Linear Algebra rather than from Hilbert spaces, in Carfì's words [1]:

> Of this theorem there is no a rigorous proof, neither in the context of Hilbert spaces, but this is not the worse problem; the very critical point is that there is not a clear, univocal and unambiguous statement of the result; this affect badly on the use of this indubitably good result, firstly because it is not clear what is the precise meaning of the symbols and operations that Feynman presents, and secondly because the context proposed (the Hilbert spaces) appears, at a deep view, inadequate indeed for its application. Nevertheless, its efficiency in the applications, thanks to the good intuitions of physicists, made it a basic instrument in many questions of experimental, computational and theoretical analysis of dynamical systems.

The intresting point is that the $\mathcal{S}$-Linear Algebra (SA, henceforth) is a brilliant tool because of its ability to simplifies the tedious framework that stems from the "canonical" functional analysis without any loss of power. In this paper we have a particular kind of transition amplitude, the *tempered distribution* and we shall depict the passage from a pure state u to the family of eigenstates of an $\mathcal{S}$-observable ω. Surely we are acquainted with the $A\omega = \lambda\omega$, where λ are the eigenvalues, hence we know the formulation

$$A(u) = \int_{\mathbb{R}^n} \lambda[u|\omega]\omega. \tag{1.1}$$

We denote by ω the eigenbasis of an $\mathcal{S}$-observable, the operator $A(u) \in \mathscr{L}(S_n')$ and u is a tempered distribution. Given that we are able to compute the product

FIGURE 1.1. decision paths

u with the family ω, the result is a tempered distribution, but in general not a regular one [1].

In order to grasp transitional lotteries' (TS, henceforth) meaning we have to define what kind of lottery is and to what kind of *time* we are handle with. When we are talking about TS, we are talking about dynamic objects, rather if we use them we want to study all of the possible paths which can lead to a certain decision. In a more precise fashion: a man called $\mathbb{A}$, has to make a decision on something at time t_0. At that point in time $\mathbb{A}$ is going to make the decision a, and it is pretty sure ([Fig.1.1] on the left hand side). We can accept this simplification if and only if we were living in a *ceteris paribus*-world, but it is not so. Since our world is difficult to be *a priori*-determined we have to use some methods which are able to give us a probability that can depict a probable outcome. Thus, the a is no more so sure, because the decision could be a b or a c and so forth ([Fig.1.1] on the right hand side).

It is straightforward that we want to understand what happens inside the grey-circle [Fig.1.1], and we can do that by the TS. We do not care about long span of time, we want to dwell on a short span of time $t_0 \to t_1$. Of course the choice of the span's lenght is arbitrary, but the more the time goes by, the more is difficult to be aware of a future outcome. We have to study the possible mind-pahts throughout $[t_0 \leftrightsquigarrow t_1]$, this means that we do observe the flow of "*information-time*" (denoted by $\leftrightsquigarrow$) which goes from t_0 to t_1. This time-span can be thought of as a closed set, rather we could denote an infimum t_0 and a supremum t_1 which belong to the information-time set $[t_0 \leftrightsquigarrow t_1]$.

2. THE FEYNMAN-STRATI $\mathcal{S}$-PROPAGATORS

In this section we shall study a particular kind of propagator which stems from the Feynman one, the so called "Feynman-Strati $\mathcal{S}$-propagators" (F-S or FS henceforth). We have to modify the concept of Feynman propagatos because of its strong link with Quantum Mechanics, but when we are talking about mind-paths we can not talk "quanta-like". I shall use the Feynman propagators developed by Carfì [1] [2], thus, let me recall some useful definitions.

Definition 1 (Feynman propagator)**.** We call a function

$$P : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n) \tag{2.1}$$

a Feynman $\mathcal{S}$-propagator if:

F_1 $P(t,t) = \delta \ \forall t \in \mathbb{R}$;
F_2 $P(t_0,t)$ is invertible and $P(t_0,t) = P(t,t_0)^{-1} \ \forall t_0, t \in \mathbb{R}$;
F_3 $P(t_0,t_2) = P(t_0,t_1) \times P(t_1,t_2)$.

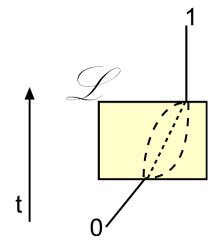

FIGURE 2.1. simple F-S propagator

Definition 2 (Propagator of a process)**.** Let $u : \mathbb{R} \to \mathcal{S}'_n$ be a dynamic process. We say that a propagator (2.1) is an $\mathcal{S}$-Green function for the process u if

$$u(t) = \int_{\mathbb{R}} u(t_0)P(t_0,t) \quad \forall t_0, t \in T. \tag{2.2}$$

In [DEF2] we used the concept of superposition [2], in particular the state of u at time t is the superposition (2.2) of the family $P(t_0,t)$ w.r.t. the system of coefficients coinciding with the state of u at t_0. Thus, the $\mathcal{S}$-propagator can be thought of as a ground upon which a state can walk. Now we can look at another important definition:

Definition 3 (F-S propagator)**.** Given (2.1) we define an $\mathcal{S}$-propagator the Feynman-Strati propagator iff for every $P(t_0,\mathscr{L},t)$ $\exists \mathscr{L} : (t_0,\mathscr{L}) \to (\mathscr{L},t)$ gives probabilistic paths $c(t)_\Omega$.

It is obvious that [DEF3] plays an important role in $[t_0 \leftrightsquigarrow t_1]$, but what does [DEF3] mean?

In few words, we say that when we move from an abstract place j at t_0 toward another abstract place j_1, in which we shall come along at time t_1, "something happens". This is an intuitive way to think about FS propagator, albeit it is not so precise, and we have to notice that we should focus on the $\mathscr{L}$ operator. In [Fig.2.1] it is straightforward that we face "something" while we are walking on the path $[t_0 \to t_1]$ and this "something" is called *probabilistic box* and we denote it by $\mathscr{L}$. In [Fig.2.1] I stressed the role of $\mathscr{L}$, rather we have only one path-outcome and three $c(t)_\Omega$; this kind of FS propagator is called *simple*. When we have to face a decision-matter it would be very strange to solve it by quanta-like methods because there is no link from the mental-paths and particles-paths. In order to depict mental-paths we have to use something that permits to observe an abstract situation given an information bundle. Now that we have tasted the FS propagator, we need to define its tools:

- FS_1 $P(t_0,t)$ is the time which goes from a point t_0 to a point t ,
- FS_2 $\mathscr{L}$ is the probability box wherein we compute some possible paths given an information bundle $\mathbb{I}^\alpha$ where the α's are the information sets with $\alpha \in \mathbb{N}$,
- FS_3 $c(t)_\Omega$ are the probable paths which belong to Ω-type of distribution.
- FS_4 Ω is a set of distribution defined in $\mathcal{S}(\mathbb{R}^n,\mathcal{S}'_n)$ which has a compact support K in $\mathscr{L}$. It is importat to notice that $\int_{\mathbb{R}^n} c(t)_\Omega < \infty$

Now we have to define in a deeper fashion the probabilistic box.

3. The F-S propagator and the $\mathscr{L}$-spaces

In [§2] I defined the FS in an intuitive way, and I listed some useful concepts which do need a further study. In this section we focus on a particular kind of space that is the core of the FS propagator, without it FS was simply the Feynman propagator. Thus, as you know, we call it $\mathscr{L}$-space. It is worth to notice that from the FS_2-FS_4 follow $\Omega \subset \mathscr{L} \subset \mathcal{S}'_n \subset \mathcal{S}_n$.

In $\mathscr{L}$-space we have a distribution $h(x,t)$, where $x = x_1,\dots,x_n$ and $t = t_0,\dots,t_m$ with $m \le n-1$. The $h(x,t) \in \mathbb{I}^{\alpha_1}$ and $h(x,t)_2 \in \mathbb{I}^{\alpha_2}$ and so forth; thus the h's are the information bundles. They are Hausdorff topological spaces and inner regular in a σ-algerbra defined on Borel sets $\mathcal{B}$, that is to say the h's are of Radon measure. But what does it mean in a mind-path study? The first step is to compute the Feynman-propagator of a process (2.2) we want to study an we shall obtain $u(t)$. We call this first result the *benchmark path*, it is important because it tells us information about the most probable path that a "state" can pursue. One has to keep in mind that "state" is a mental path in a certain (x,t). In order to answer the question it is needed an example:

Example 1. (first part) John needs to go out of his home, the sky is cloudy and weather forecast is not so comforting. Thus he has two macro-information boundles $\mathbb{I}^{\alpha_1}$ (clouds) and $\mathbb{I}^{\alpha_2}$ (metheorology). It is important to notice who John is and what John has to do. If he has to go to his university for i.e. an examination it was quite sure that he is going to go out, whereas if he has to buy i.e. a newspaper it is not so sure that he would goes out. Thus $\mathscr{L}$-spaces are strongly related to what John has to do, we can say that if John could do two different things there will be two different $\mathscr{L}$-spaces. Thus, we have to compute the Feynman propagator to the process "John goes out" $[u_0]$ and given "John has to buy a newspaper" $[f(x,t)] \in \mathscr{L}$ and it has not started raining yet $[f(x_0,t_0)]$ we infer that John is just went out and will be buy a newspaper given that at $\int_{\mathbb{R}^n} f(t_0,t)u(t_0)$ his clothes were dry. This is the benchmark path.

It is natural to ask how we could measure proposition like ["it is cloudy" then "it is going to rain"]. In the case of $\mathscr{L}$-space we define an $\mathbb{I}^{\alpha}$ in this way:

Definition 4 (Information basis)**.** Given $A : \mathcal{S}'_n \to \mathcal{S}'_n$ indexed by $\mathbb{R}^n$, for a given distribution f and an orthonormal eigenbasis $\mathcal{B} = o_1,\dots,o_n$ we define "Information base" the integral:

$$A^n(f_n) = \int_{\mathbb{R}^n} \lambda_n[f|o]o \tag{3.1}$$

Given that (in a tempered space) we could define a direction in a stable fashion. Now we have to choose some "drift" operator which might define more paths. The drift is a particular evolution operators defined as

$$e^{([\int_{\mathbb{R}^n} A^n(f_n)u(t)]\circ H)}. \tag{3.2}$$

The (3.2) embedded H which is an operator defined in this way:

$$H = (\mathbb{I}^{-\alpha}), (\mathbb{I}^{-\alpha_2}), \dots, (\mathbb{I}^{-\alpha_m}).$$

There are negative information bundles, but the sign "$-$" is an abuse of notation, rather it does not mean that it is embedded with negative numbers, but that this kind of bundle could trigger off a drif to the banchmark.

Definition 5 (H-operator)**.** Given $H \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ we define a drift operator

$$H(\mathbb{I}^{-\alpha}) : \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n) \to \mathscr{L}$$

an H-operator.

At first glance [DEF5] could seem pretty obscure, but let me show you some intuitions. From the $\mathbb{R}^n$ tempered distribution space we map given functional into the $\mathscr{L}$-space. Accordingly, we map $H(\mathbb{I}^{-\alpha}) : \int_{\mathbb{R}^n} g(x,t)\psi(x) \to (\int_{\mathbb{R}^n} \lambda_n[f|o]o) \times u(t)$. The eigenbasis of $g(x,t)$ could arrange a different path which might drift from the banchmark. It is important the following example.

Example 2. (second part [EX1]) We have already computed the benchmark path and we are ready to study the drifts which can occur. Thus, it could happen "rain" and so $g(x,t)$ may be a functional by which (like [EX1]) John decides to stay home. Thus, by superposition $\int_{\mathbb{R}^n} g(t_0,t)u_0 = u(t)_1$ it is going to rain. But we have to compare this result to what we have computed in [EX1], rather in one case John comes out whereas in the other one John stay home.

It is obvious that given (3.2) we can define several paths, as much as information bundle one wants to include, and if we call E the integral in (3.2) we say

$$e^{EH} = [\varphi(x_n, t_n)] \tag{3.3a}$$

$$[\varphi(x_n, t_n)]^2 = \gamma^p \tag{3.3b}$$

The (3.3b) is the probability density γ^p of all the possible paths $c(t)_\Omega$. We have to notice that the H-operator is crucial in γ^p changing and thus for the definition of the following section.

4. $\mathcal{S}$-TRANSITIONAL LOTTERIES

I have defined a way *ad hoc* so as to study the decisional path in a more practical fashion. The formulæ are thought of as everyday events in order to understand the importance of concrete studies. In this section I shall talk about the $\mathcal{S}$-transitional lotteries ($\mathcal{S}$TL, henceforth), and why we should use them.

As I have said, we need a more concrete fashion so as to study the real world, therefore, the aim of the $\mathcal{S}$TL is that of studies a dynamical mind-path process when one faces decision-matter. The term dynamical it is intended much as the above definitions and the reader should be aware of this term at this point, still I have to clarify something. The dynamic is given by e^{EH} which embedded $A(f)$, the latter integral is of utmost importance, rather I can say that in (3.2) the information basis "climbs" (through E) the $u(t)$ process by a given *homogeneous* direction. This is a dynamical motion [Fig.4.1] of the information bundle which defines a path (benchmark) in a more precise way. We could interpret that "climb" as something gives a certain swell along the $u(t)$. For each of λ's there is a definite direction.

The $\mathcal{S}$TL is a *motion* of a particular $\mathcal{S}$-family made of all the tools in [§3].

Definition 6 ($\mathcal{S}$-transitional lotteries)**.** We define a motion of $\mathcal{S}$-family through a path $P(t_0,t)$ given by $\Delta\gamma^p$ an $\mathcal{S}$-transitional lottery and we denote it by

$$\mathcal{S}TL = [(t_0, x_0) \leftrightsquigarrow (t, x_{(s)})] \tag{4.1}$$

where (s) denotes the possible different paths-arrivals.

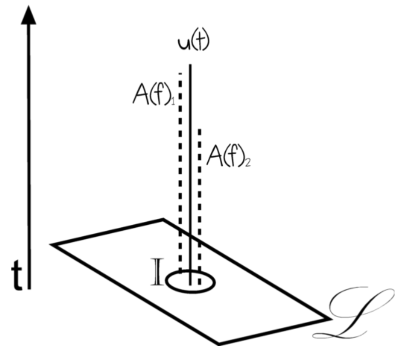

FIGURE 4.1. The A(f)'s climb

Hence, the $[\leftrightsquigarrow]$ means the evolution between the two time spans. Thus $\Delta\gamma^p$ is a motion in probability density so as to observe what could happen if... and the role of H-operator becomes crucial.

By H-operator we can foresee a possible drift in $u(t)$, I say "possible" because the H-operator's existence depends on the acquaintance of information bundles which *could* occur. Of course the choice in using a bundle or not is a subjective one. If it was not so, we would put our trust in a wintry algorithm.

5. APPENDIX

5.1. H-operator and the eigenspectrum. Given that

$$
\begin{aligned}
&H : \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n) \to \mathscr{L} \\
&\quad : \int_{\mathbb{R}^n} g(x,t)\psi(x) \to (\int_{R^n} \lambda_n([f|o]o) \times u(t)
\end{aligned}
\tag{5.1}
$$

it is worth to notice that $g(x,t) \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ and $\psi(x) \in \mathcal{O}_M$, thus the $hull^{\mathcal{S}}$ span $g(x,t)$ contains the $hull^{\mathcal{S}}$ span $(g(x,t)\psi(x))$, from this follows that $g(x,t)$ is $\mathcal{S}$-linearly indipendent and it is so for $(g(x,t)\psi(x))$ as well.

We have to stress the importance of eigenvalues and eigenvectors. Given an $A \in \mathcal{L}(\mathcal{S}'_n)$ an $\mathcal{S}$-linear endomorphism, $\lambda \in \mathcal{O}_M$ and $o \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ an $\mathcal{S}$-linearly indipendent eigenfamily of A w.r.t. the system of eigenvales λ. We have seen [DEF4] the spectral $\mathcal{S}$-expansion

$$A(f) = \int_{\mathbb{R}^n} \lambda[f|o]o.$$

We know that

$$
\begin{aligned}
A(o_p) &= A(o_p) \\
&= \lambda(p)o_p \\
&= (\lambda o)(p),
\end{aligned}
$$

thus

$$
\begin{aligned}
A(f) &= A(\int_{\mathbb{R}^n} [f|o]o) \\
&= \int_{\mathbb{R}^n} [f|o]A(o) \\
&= \int_{\mathbb{R}^n} [f|o](\lambda o) \\
&= \int_{\mathbb{R}^n} (\lambda [f|o])o.
\end{aligned} \tag{5.2}
$$

As we have noticed, the eigenspectrum is of outmost importance in studying the $\mathcal{S}$TL, therefore we state what it is in a more precise fashion. The eigenspectrum of an $\mathcal{S}$-diagonalizable operator i.e. A is the $im\lambda$ of the function λ, that is to say $im\lambda = \sigma^e(A) = (\lambda \mathbb{R}^n)$ (where σ^e is the eigenspectrum). Hence σ^e, being an image, it is straightforward the meaning of projections. This is a very important tool so as to define the different decision-pahts, rather in the H-operator we have to define the eigenspectrum of the left hand side and that of the right hand side. Given that we shall obtain a *locus* wherein the decisions are formed.

5.2. **The $u(t)$'s dynamics.** We know (from [DEF2]) that $u : \mathbb{R} \to \mathcal{S}'_n$ is a dynamical process, we say that it is a $\sigma^{(\mathcal{S}'_n)}$-differentiable curve and $P(\cdot,\cdot) \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$. We define the curve t

$$
\int_{\mathbb{R}^n} u(\cdot)P(\cdot,\cdot) : \mathbb{R} \to \mathcal{S}'_n : t \to \int_{\mathbb{R}^n} u(\cdot)(t)P(\cdot,\cdot) \tag{5.3}
$$

and we have that

$$
\Big(\int_{\mathbb{R}^n} u(\cdot)P(\cdot,\cdot)\Big)'_{\sigma(\mathcal{S}'_n)} = \int_{\mathbb{R}^n} u(\cdot)'_{\sigma(\mathcal{S}'_n)} P(\cdot,\cdot).
$$

The (5.3) ensures a dynamics of u, these curves are very important in the $\mathcal{S}$TL's framework because of the property of build the $\mathcal{S}$-propagator and thus the banchmark path. If u was undefined we could not be able to have rigorous decision paths.

5.3. **The $c(t)_\Omega$'s dynamics.** The $c(t)_\Omega$'s have been only mentioned so far, we have already used them but maybe we did not know that. The $\mathbb{I}^\alpha$ is an Hausdorff topological vector space $\in \Omega \subset \mathscr{L}$. The evolution of them are the "$c(t)_\Omega$'s dynamics". Therefore, given the restriction Ω we can define a curve in it $t : \mathbb{R} \to \Omega$, thus the curve t is said to be differentiable at the point $x \in \mathbb{R}$ iff the map

$$
w_x : \mathbb{R}^{\neq} \to \Omega : h \to \frac{t(x+h) - t(x)}{h},
$$

thus we define the differential

$$
(t)'_\tau(x) := \lim_{h \to 0} \frac{t(x+h) - t(x)}{h}.
$$

Where $\mathbb{R}^{\neq} = \mathbb{R} \setminus 0$ and τ is intended to stress the topology of the space. And $(t)'_\tau(x)$ is the derivative of the curve t at x. The curves are different for every different $\mathbb{I}^\alpha$, they are of utmost importance so as to define possible differences of these paths from the benchmark one.

5.4. **The $\circ H$ mystery.** This mystery has to be solved because of its importance. The $\circ H$ is the operative way by which the H-operator triggers off a drift to the benchmark. We have seen it in (3.2) but I have not clarified the reasons behind this operation. The "$\circ$" is a way to say that there is an operation there, not a defined one, that is for the nature of the H-opeartor. We cannot say that this operator enters into the E ((3.3)) by a simple multiplication. We say that the H-operator defines some drifts without annihilates the banchmark "paths"[1], but it defines a divergence, with a particular meaning of the term "divergence". We do not have to define a measure from the banchmark to the H-states in a strict way, because of the H-operator is built on a different $\mathscr{L}$-space, although very close to it. Given that $H \in \mathscr{L} \subset \mathcal{S}'_n$ we have to find the answers to our mystery in the $\mathcal{S}'_n$-space which contains each of $\mathscr{L}$-space. The square of the *sum* of the exponential of our tempered distributions will be the probability density of this bunch of $\mathcal{S}$-functionals.

References

[1] Carfí, David, "Feynman's transition amplitudes in the space $\mathcal{S}'_n$" AAPP, Vol.85, Issue 1 (2007).
[2] Carfí, David "Foundations of Superposition Theory, Vol.1" Il Gabbiano, 2010

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[1]we say path(s) because, given the $A(f)'s$ climber there could be the possibility to have more than one banchmark.