Time Varying Fractional Cointegration

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Abstract

According to Engle and Granger (1987), the concept of fractional cointegration was introduced to generalize the traditional cointegration to the long memory framework. In this paper, we extend the fractional cointegration model in Johansen (2008) and propose a time-varying framework, in which the fractional cointegrating relationship varies over time. In this case, the Johansen (2008) fractional cointegration setup is treated as a special case of our model.

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1.0 Introduction

Fractional cointegration has attracted interest in time series econometrics in recent years (see among others, Dittmann 2004). Fractional cointegration analysis has emerged based on the view that cointegrating relationships between non-stationary economic variables may exist without observable processes necessarily being unit root $I(1)$ processes or cointegrating errors necessarily $I(0)$ processes.

Both fractional and standard cointegration were originally defined simultaneously in Engle and Granger (1987), but standard cointegration has received extensive coverage. The standard cointegration allows only integer values for the memory parameter, and tests for the existence of cointegration rely on unit root theory. The fractional cointegration framework is more general since it allows the memory parameter to take fractional values, and to be any positive real number. In their standard approach, Engle and Granger (1987) and Johansen (1988) assumed that the cointegrating vector(s) do not change over time. However, when one takes into account such phenomenon as structural breaks and regime shifts, the assumption of fixed cointegrating vector(s) becomes quite restrictive.

In this paper, we extend this analysis by examining the fractional cointegration case using time-varying vector autoregression model. We specify the vector error correction model (VECM) with a cointegrating vector that varies with time and we approximate this vector by a linear combination of orthogonal Chebyshev time polynomials.

1.1 Fractional Cointegration

Following Granger (1986), a set of $I(d)$ variables are said to be cointegrated, or $CI(d,b)$, if there exists a linear combination that is $CI(d-b)$ for $b>0$. To define fractional cointegration, let x_t by n-dimensional vector $I(1)$ process. Then x_t is fractionally cointegrated if there is an $a \in R^n$, $a \neq 0$, such that $a^{'} x_t \sim I(d)$ with $0<d<1$. In this case, d is called the *equilibrium long-memory parameter* and write $x_t \sim I(d)$. Compared to classical cointegration, where $d=0$, defining the cointegration rank is more difficult for fractionally cointegrated systems, because different cointegrating relationship need not have the same long-memory parameter.

The fractional cointegration setup that we consider in this paper is based on an extension of the Johansen's (2008) Error Correction Mechanism (ECM) framework which is specified as follows:

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Phi D_t + \varepsilon_t \qquad (1.1)$$

where X_t is a vector of $I(1)$ series of order *k* x 1, D_t are deterministic terms, ε_t is a *k* x 1 vector of Gaussian errors with variance-covariance matrix Ω, and Π, $\Gamma_1,...,\Gamma_{k-1}, \Phi$ are freely varying parameters. When the vector X_t is cointegrated, we have the reduced rank condition $\Pi = \alpha\beta'$, where α and β are *N* x *r* constant parameter matrices, having rank *r*, representing the error correction and cointegrating coefficients, respectively.

Granger (1986) proposed the first generalization of the VECM model to the fractional case with the following form:

$$A^*(L)\Delta^d X_t = (1-\Delta^b)\Delta^{d-b}\alpha\beta' X_{t-1} + d(L)\varepsilon_t \qquad (1.2)$$

Where $A^*(L)$ is a lag polynomial, X_t and ε_t are *N* x 1, ε_t~i.i.d(o,Σ); α and β are as defined in (1.1) above; and b and d are real values, with d representing order of fractional integration and $d-b$ representing order of co-fractional order. The process X_t is a fractional order of d and co-fractional order of, $d-b$. In other words, that is there exists β vectors for which $\beta' X_t$ is fractional of order $d-b$. L represents lag operator, and (Δ^d) represents fractional difference parameter. Note that equation (1.2) has the conventional error correction representation when $d=1$ and $d-b=0$, i.e. $I(1)$ variables cointegrate to $I(0)$.

Dittman (2004) attempts to derive this model from a moving average form but, according to Johansen 2008, the results are not correctly proved. In this paper, we follow the formulation suggested by Johansen (2008):

$$\Delta^d X_t = \alpha\beta' \Delta^{d-b} L_b X_t + \sum_{i=1}^{k-1} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t \qquad (1.3)$$

This formulation implies the following changes from (1.2): $(1-\Delta^b)X_{t-1}$ is changed to $L_b X_t$; the lag polynomial $A^*(L)$ is changed to $A(L_b)$; i.e. the latter is lag polynomial in L_b (and not L_b). $L_b = 1-(1-L)^b$. The lag polynomial $d(L)$ is ignored.

When $d=1$ and $d-b=0$, i.e. $I(1)$ variables cointegrate to $I(0)$.

$$\Delta X = \alpha\beta' X_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta X_{t-i} + \varepsilon_t \tag{1.4}$$

However, from (1.2) also note that the condition

$$(1-L)^{d-b}\beta' x_t \sim 0 \tag{1.5}$$

is required so that the equation balances, having both sides *I*(0). $d-b$ represents cointegrating rank. Setting $d=b=1$ yields to the usual Johansen (1988, 1991) style VECM, but d and b can be real values with $d>0$ and $0<b\le d$. In this model, all elements of x_t exhibit the sane order of integration, not necessarily unit, and similarly, the cointegrating residuals $\beta' x_t$ are all of order $d-b$. It should be noted that in fractional cointegration, the cointegrating residual is long memory and possibly even non-stationary, but has a lower order of integration than its constituent variables.

1.2 Time-Varying Fractional Cointegration Representation

In this model, we extend the Johansen (2008) Fractional VECM(p) framework to a time-varying framework as follows:

$$\Delta^d X_t = \Pi_t' \Delta^{d-b} L_b X_t + \sum_{i=1}^{k-1} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t \tag{1.6}$$

where $\Pi_t' = \alpha_t \beta_t$, and β_t is time-varying cointegrating vector of coefficients. Thus one can test the null hypothesis of time-invariant cointegration, $\Pi_t' = \alpha\beta$, where α and β are fixed *k* and *r* matrices with rank r, against the time varying parameter of the type

$$\Pi_t' = \alpha_t \beta_t', \tag{1.7}$$

Where α_t and β_t's are time varying *k* x *r* matrices, with constant rank *r* and t represents time, where $t \ge 0$. In this case, both α_t's and β_t's are assumed to be time dependent.

Equation (1.7) is governed by the following assumptions:

Assumption 1: $\beta_t = \beta_{t/T}$, where each element of $\beta_\tau, \beta_{i,\tau}$, $i = 1,...,k, \tau \in (0,1)$ is a function of time, *t*. *Assumption* 2: u_t is a stationary martingale difference sequence with finite 4-th moments, which is independent of X_t at all leads and lags. *Assumption 3*: X_t is a vector of non-stationary variables.

Assumption 1 is quite essential. It specifies that β is a deterministic function of time. It is interesting to note that it depends not only on the point in time t, but also on the sample size *T*. This is necessary as one needs the sample size that relates to that parameter to tend to infinity, for one to estimate consistently a particular parameter. This is achieved by allowing an increasing number of neighbouring observations in order to obtain more information about β at time *t*. In other words, we have to assume that as the sample size grows, the function β_τ will extend to cover the whole period of the sample. This kind of setup has examples in the statistical literature. Assumptions 2 and 3 are standard conditions in cointegration analysis for the error term and X_t.

1.3 Chebyshev Time Polynomials

Making use of a theorem due to Halbert White, Granger (2002) claimed that any linear model can be estimated using a time-varying parameter linear model. Furthermore, he argued that time-varying coefficients could be deterministic function of time. This principle was implicitly introduced by Bierens and Martins (2010) in time varying relationships. In Bierens and Martins (2010), the time-varying cointegrating vector was approximated by a linear combination of orthogonal Chebyshev time polynomials so that the resulting vector error correction model had time invariant coefficients.

In this paper, we follow Halbert White and Granger's (2002) principle and model a time varying fractional cointegration using Chebyshev polynomials.

Chebyshev time polynomials $P_i T(t)$ are defined by

$$P_{0,T} = 1, P_{i,T}(t) = \sqrt{\cos}(i\pi(t-0.5)/T) \qquad (1.8)$$

` $t = 1,2,...,T$ t= $i = 1,2,...$

Bierens (1997) makes use of these polynomials in his unit root test against nonlinear trend stationarity. The polynomials are orthonormal, since for all integers $i, j, \frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)P_{j,T}(t) = 1(i = j)$, where 1(.) is indicator function. Because of this this orthonormality property, any function $g(t)$ of discrete time $t = 1,...,T$ can be specified as:

$$g(t) = \sum_{i=0}^{T-1} \xi_{i,T} P_{i,T}(t), \text{ where } \xi_{i,T} = \frac{1}{T}\sum_{t=1}^{T} g(t)P_{i,T}(t) \qquad (1.9)$$

Assume that in equation (1.9), $g(t)$ is linearly decomposed into parts $\xi_{i,T} P_{i,T}(t)$. Thus $g(t)$ can be estimated as follows:

$$g_{n,T}(t) = \sum_{i=0}^{T-1} \xi_{i,T} P_{i,T}(t),$$

for some fixed natural number $n < T - 1$

Lemma 1: *Assume* $g(t) = \lambda(t/T)$*, where* $\lambda(x)$ *is a square real function on* $[0,1]$. *Then*

$$T \overset{\lim}{\rightarrow} \infty \frac{1}{T}\sum_{t=1}^{T} g((t) - g_{n,T}(t))^2 = 0$$

Furthermore, if $\lambda(x)$ *is* $q \geq 2$ *times differentiable, where* q *is even, with* $\lambda^{(q)}(x) = d^q \lambda(x)/(dx)^q$ *satisfying* $\int_0^1 (\lambda^{(q)}(x))^2 dx < \infty$*, then for* $n \geq 1$

$$T \overset{\lim}{\rightarrow} \infty \frac{1}{T}\sum_{t=1}^{T} (g(t) - g_{n,T}(t))^2 \leq \frac{\int_0^1 (\lambda^{(q)}(x))^2 dx}{\pi^{2q}(m+1)^{2q}}$$

Proof: See Bierens and Martins (2010).

Thus we may specify β_t for t=1,….,T as $\beta_t = \sum_{i=0}^{T-1} \xi_{i,T} P_{i,T}(t)$, where $\xi_{i,T} = \frac{1}{T}\sum_{t=1}^{T} \beta_t P_{i,T}(t)$, i = 0,….,T-1 are unknown k x r matrices.

1.4 Modelling Time-varying Fractional Cointegration using Chebyshev Time Polynomials

Substituting $\alpha_t \beta_t = \Pi_t = \alpha[\sum_{i=0}^{m} \xi_i P_{i,T}(t)]'$ in 1.8 yields

$$\Delta^d X_t = \alpha[\sum_{i=0}^{m} \xi_i P_{i,T}(t)]' \Delta^{d-b} L_b X_t + \sum_{i=1}^{k-1} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t \qquad (1.10)$$

for some k x r matrices ξ_i.

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