Intergenerational links, taxation, and wealth distribution

Wan, Jing and Zhu, Shenghao

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Jing Wan
Natinal University of Singapore

Shenghao Zhu
National University of Singapore

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Abstract

We extend one of the main findings in Bossmann et al. (2007)("Bequests, taxation and the distribution of wealth in a general equilibrium model," *Journal of Public Economics*, 91, 1247-1271). Bequest motives per se reduce wealth inequality. We show that the result holds for a stronger criterion of inequality comparison between distributions. Bossmann et al. (2007) use the coefficient of varation as the inequality measure. Our Lorenz dominance result implies their result. We also strengthen two other conclusions in Bossmann et al. (2007). Earnings ability

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inheritance could increase wealth inequality and estate taxes could decrease wealth inequality.

JEL classification: D31; E21; H23

1 Introduction

We extend one of the main findings in Bossmann et al. (2007): Bequest motives per se reduce wealth inequality. We show that the result holds for a stronger criterion of inequality comparison between distributions. Bossmann et al. (2007) use the coefficient of variation as the inequality measure. Our Lorenz dominance result implies their result.

Following Bossmann et al. (2007), we investigate the impacts of intergenerational links on wealth inequality in a simple two-period overlapping generations (OLG) heterogeneous agents model. Each agent lives for two periods: young period and old period. Each young agent supplies 1 unit of labor inelastically and has idiosyncratic labor efficiency risk l_t. Old agents do not have labor earnings. Agents have "joy-of-giving" bequest motives. Government collects an estate tax and redistributes the tax revenue to all young agents in the economy as a lump-sum transfer.

As in Bossmann et al. (2007), we find that an economy with bequest motives has a more equal wealth distribution than an economy without bequest motives. Our result extends Bossmann et al. (2007) in three respects. First, we only assume that $\{l_t\}$ is a stationary and ergodic while Bossmann et al. (2007) assume that $\{l_t\}$ is either $i.i.d.$ or a linear mean-reverting process.[1] Second, we do not assume that $var(l_t) < \infty$. Bossmann et al. (2007) uses the coefficient of variation as their inequality measure. Thus they need the finite variance of wealth distribution. Our inequality measures are Lorenz curve and Gini coefficient, which only require the existence of the mean of wealth distribution. Third, our result is stronger than that of Bossmann et al. (2007). Our Lorenz dominance result implies the coefficient of variation result in Bossmann et al. (2007).

We then investigate the impacts of ability inheritance on wealth distribution. We find that impacts of earnings ability inheritance on wealth distribution is contrary to those of bequest inheritance. An economy with the inheritance of earnings ability has higher wealth inequality than an economy without inheritance of ability. De Nardi (2004) shows that ability inheritance can increase wealth concentration by simulating a general equilibrium OLG model with bequest motives. Finally, we investigate the effect of estate taxes on wealth distributions and find that estate taxes reduce wealth inequality.

Our theoretical results of the impacts of bequest motives, ability inheritance, and estate taxes on wealth distribution are about the Lorenz ordering. We do not incorporate precautionary savings motives into our model and we have the explicit forms of individual policy functions. This is different from the large literature of incomplete markets heterogeneous agents models, such as Aiyagari (1994), Castaneda et al. (2003) and De Nardi (2004). These papers usually incorporate precautionary savings motives, solve

[1] We use $\{x_t\}$ to represent a squence in this paper.

agent's policy functions numerically, and simulate the stationary wealth distribution.[2]

Lorenz dominance is widely used in the literatures of income and wealth inequality. For example Chatterjee (1994) uses Lorenz dominance to discuss wealth distribution in a neoclassical growth model.[3] And Zilcha (2003) uses Lorenz dominance to study the income distribution in an economy with two types of intergenerational transfers: investment of parents in the education of their offspring, and capital transfer. Early literatures include, among others, Atkinson (1970) and Rothschild and Stiglitz (1973). For a recent brief review on this topic see Gajdos and Weymark (2012). We also find that the convex order is a convenient tool for our linear model. For example the convex order is closed under convolutions.[4] We use this property in our proof for the conclusion that estate taxes reduce wealth inequality.[5]

The rest of this paper is organized as follows. Section 2 presents the basic structure of our model. Section 3 discusses the stationary wealth distribution of our model. We introduce different inequality measures in section 4. We show the effect of bequest motive on wealth distribution in section 5. Section 6 quests the impacts of ability inheritance on

[2]Benhabib, Bisin, and Zhu (2011) also find explicit forms of indivudual policy functions and study a stationary wealth distribution. They emphasize the role of idiosyncratic investment rates of return on producing the observed fat tail of the wealth distribution in the U.S.

[3]There is a minor difference about "Lorenz dominance" between Chatterjee (1994) and our paper. In Chatterjee (1994) "X Lorenz-dominates Y" means that X is more unequal than Y, while in our paper it means that X is more equal than Y.

[4]Let X_1 and X_2 be two independent random variables and let Y_1 and Y_2 be two other independent random variables. If $X_1 \preceq_{cx} Y_1$ and $X_2 \preceq_{cx} Y_2$, then $X_1 + X_2 \preceq_{cx} Y_1 + Y_2$. See the definition of $\preceq_{cx}$ in section 4.2. For this property of the convex order see page 120 of Shaked and Shanthikumar (2010).

[5]Zhu (2012) studies the impacts of income risk on wealth distributions by using Lorenz dominance and the convex order.

wealth distribution. Section 7 investigates the effect of estate tax on wealth distribution. Section 8 concludes the paper. Appendix contains most of the proofs.

2 The model

Our model is an overlapping generations heterogeneous agents economy. Each agent lives for two periods: young period and old period. Each old agent gives birth to one child. Each family consists of one parent and one child. There is a continuum of measure 1 families in the economy. The population of the economy keeps constant.

2.1 Agent's problem

Young agents work and earn labor earnings. Old agents do not have labor income. They consume their savings and leave bequests to their children. Each young agent supplies 1 unit of labor inelastically. But young agents have idiosyncratic labor efficiency risk l_t. We assume

Assumption 1: $\{l_t\}$ is stationary and ergodic.

Assumption 2: $l_t > 0$ has a finite mean. Without loss of generality,

$$E(l_t) = 1.$$

Note that we do not need the finiteness of $var(l_t)$.

Agents have "joy-of-giving" bequest motives. c_t^y is the consumption in young period for an agent born at period t, and c_{t+1}^o is his consumption in old period. s_t denotes his savings. He leaves bequest b_{t+1} to his child. The wage rate per efficiency unit is w_t. g_t is the lump-sum transfer from the government, and $\zeta \in [0, 1)$ is the estate tax rate. The interest rate is r_{t+1}.

The following figure shows the timing of the model.

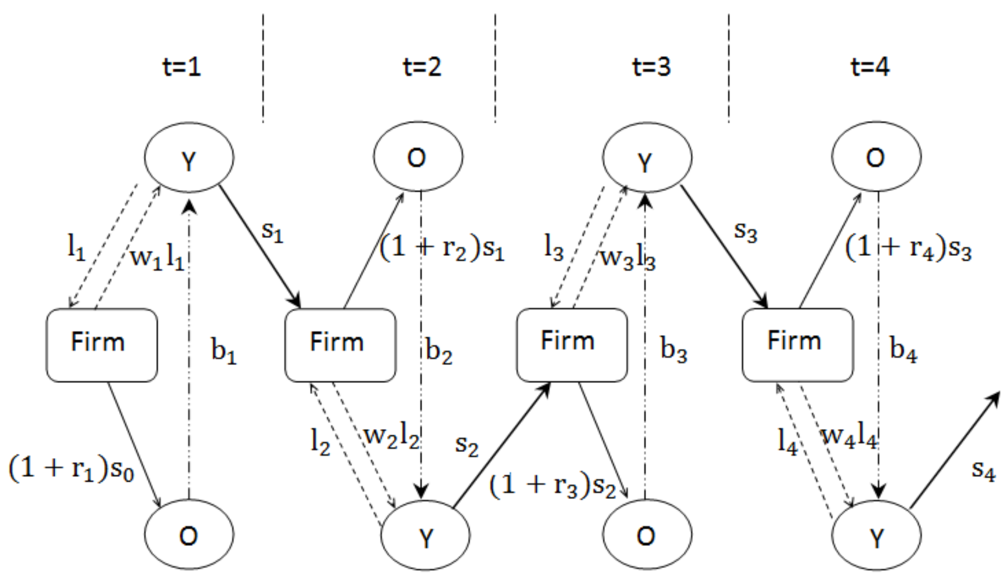

Figure 1: The timing of the model

At the beginning of period t, an young agent receives bequests b_t, and pays the estate tax ζb_t to the government. He draws his labor efficiency l_t, and receives government transfer g_t. Then the agent makes consumption and savings decisions. Thus the agent's problem is a deterministic optimization problem

$$\max_{c_t^y, s_t, c_{t+1}^o, b_{t+1}} \frac{(c_t^y)^{1-\eta}-1}{1-\eta} + \beta\left[\frac{(c_{t+1}^o)^{1-\eta}-1}{1-\eta} + \chi\frac{[(1-\zeta)\,b_{t+1}]^{1-\eta}-1}{1-\eta}\right] \tag{1}$$

$$\begin{aligned} s.t. \quad c_t^y + s_t &= w_t l_t + (1-\zeta)\,b_t + g_t \\ c_{t+1}^o + b_{t+1} &= (1+r_{t+1})s_t \end{aligned}$$

where $\eta \geq 1$ is the coefficient of relative risk aversion. $\beta \in (0,1)$ is the time discount factor. χ represents the bequest motive.

The agent's optimal policy functions are

$$c_{t+1}^o = \frac{1}{1+\chi^{\frac{1}{\eta}}(1-\zeta)^{\frac{1-\eta}{\eta}}}(1+r_{t+1})s_t$$

$$b_{t+1} = \frac{1}{1+\chi^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-1}{\eta}}}(1+r_{t+1})s_t$$

$$c_t^y = \frac{1}{1+\tilde{\beta}_{t+1}^{\frac{1}{\eta}}}\left[w_t l_t + (1-\zeta)\,b_t + g_t\right]$$

and

$$s_t = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\zeta)\,b_t + g_t\right]$$

where $\tilde{\beta}_{t+1} = \beta\left[1+\chi^{\frac{1}{\eta}}(1-\zeta)^{\frac{1-\eta}{\eta}}\right]^{\eta}(1+r_{t+1})^{1-\eta}$.

The agent's policy functions are linear. These linear policies brings us the linear relationship of our main equation (8) as in Bossmann et al. (2007).

From optimal policy functions of b_{t+1} and s_t, we derive the individual's wealth accumulation equation

$$s_t = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left[w_t l_t + (1-\zeta)\, \varphi (1+r_t) s_{t-1} + g_t \right] \tag{2}$$

where $\varphi = \frac{1}{1+\chi^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-1}{\eta}}}$.

2.2 Firm's problem

There is an aggregate production firm in the economy. The firm has a Cobb-Douglas production function $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where A is the technology level, K_t is capital, and L_t is labor. The firm chooses K_t and L_t to maximize its profit

$$\max_{K_t, L_t} \ \{AK_t^{\alpha} L_t^{1-\alpha} - w_t L_t - (r_t + \delta) K_t\}$$

where δ is the depreciation rate of capital.

The first order conditions of the firm's problem are

$$r_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} - \delta$$

and

$$w_t = (1-\alpha) A K_t^{\alpha} L_t^{-\alpha}.$$

2.3 Government

The government collects the estate tax revenue and gives a lump-sum transfer to young generation. Each young agent receives the same subsidy g_t. The government has a balanced budget in every period

$$g_t = \zeta \int b_t di \tag{3}$$

where $\int di$ means the aggregation of young agents.

2.4 General equilibrium

The aggregate population of young agents who are the workers in the economy is 1, and $E(l_t) = 1$. Thus the labor market clearing condition is

$$L_t = \int l_t di = 1 \tag{4}$$

where $\int di$ means the aggregation of young population. The capital market clearing condition is

$$K_{t+1} = \int s_t di \tag{5}$$

where $\int di$ means the aggregation of young agents.

Aggregating the savings of the equation (2) across young agents and using equations

(4), (5), and government budget constraint (3), we have

$$K_{t+1} = \frac{1}{1 + \tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left[w_t + \varphi(1 + r_t) K_t \right].$$

From labor market clearing condition we have $r_t = \alpha A K_t^{\alpha-1} - \delta$, and $w_t = (1-\alpha) A K_t^{\alpha}$. Thus the law of motion of the aggregate capital is

$$K_{t+1} = \frac{1}{1 + \tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left[(1 - \alpha + \alpha\varphi) A K_t^{\alpha} + \varphi(1 - \delta) K_t \right]. \tag{6}$$

We will concentrate on the steady-state aggregate economy in which the aggregate capital K, the wage rate w, and the interest rate r are constant.

Proposition 1 *The economy has a unique aggregate steady state. An economy with a higher bequest motive χ has a higher steady-state aggregate capital K.*

The higher the agent's bequest motive, the higher the agent's saving incentive. Thus there are more wealth accumulation. In one extreme case there is no bequest motive, i.e. $\chi = 0$. The steady-state aggregate economy with bequest motives of $\chi > 0$ has a higher aggregate wealth level than the economy without bequest motives. This plays an important role in our analysis of the impacts of bequest motives on wealth distribution in section 5.

3 Wealth distribution

We investigate the stationary distribution of individual wealth accumulation process in the steady-state aggregate economy. Following Bossmann et al. (2007) we use a_{t+1} to denote the individual wealth (before interest) in period $t+1$. Thus $a_{t+1} = s_t$.

From government's budget constraint, we have

$$g_t = g = \zeta\varphi(1+r)K \tag{7}$$

in the steady-state aggregate economy.

Substituting equation (7) into equation (2) we have the agent's wealth accumulation equation in the steady-state aggregate economy

$$a_{t+1} = c_3 l_t + c_4 a_t + c_5 \tag{8}$$

where $c_3 = \frac{1}{1+\tilde{\beta}^{-\frac{1}{\eta}}} w$, $c_4 = \frac{(1-\zeta)\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}}$ and $c_5 = \frac{\zeta\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}} K$.

Equation (8) is the main equation of our paper. Our aim is to investigate the stationary distribution of process $\{a_t\}$ in the steady-state aggregate economy. We will study the stationary distribution of $\{a_t\}$, especially the comparisons of different economies in sections of sections 5, 6, and 7, by using the linear relationship of equation (8).[6]

[6] Our main equation (8) has the same form as equation (20) of Bossmann et al. (2007). But the expressions of c_3, c_4, and c_5 are different from those in Bossmann et al. (2007), because we use different utility functions from Bossmann et al. (2007). However that difference is irrelavent to the new findings of our paper. It is the linar relationship that permits us to establish the results about Lorenze domninance

We first establish the ergodicity of the process $\{a_t\}$.

Proposition 2 $0 \leq c_4 < 1$

In the steady-state aggregate economy aggregate capital K is finite as shown in proposition 1 and the aggregate savings equal K. Suppose that $c_4 \geq 1$, then $a_t \to \infty$ almost surely. And $K = \infty$. Thus in the steady-state aggregate economy we must have $c_4 < 1$. Proposition 2 plays an important role when we characterize the stationary distribution of process $\{a_t\}$.

Proposition 3 *The unique stationary distribution of* $\{a_t\}$ *is*

$$a_\infty = c_3 \sum_{s=1}^{+\infty} c_4^{s-1} l_s + \frac{c_5}{1 - c_4}. \tag{9}$$

And a_t *converges to* a_∞ *almost surely.*

As in Bossmann et al. (2007) we show that a_t converges to a_∞ almost surely. Bossmann et al. (2007) establish this result by the two-series theorem. Thus they need the finiteness of $var(l_t)$. We use a different mathematical theorem in Brandt (1986) and we do not need the finiteness of $var(l_t)$. But we still obtain the ergodicity of $\{a_t\}$. Thus a_t converges to a_∞ in distribution, denoted by $a_t \to_{st} a_\infty$. We use this important property of convergence in distribution when we investigate the impacts of bequest motives, ability inheritance, and estate taxes on stationary wealth distributions. In these analyses our

relationship between different economies in our analyses.

strategy is that we first establish the intuition in static situations, then we extend the results to stationary wealth distributions by observing that they still hold when processes approach limiting distributions.

4 Inequality measures

We introduce Lorenz dominance and the convex order in this section. The concepts of Lorenz dominance and the convex order are our basic tools to investigate the impacts of bequest motives, ability inheritance, and estate taxes on wealth distribution.[7]

4.1 Lorenz dominance

Following Gastwirth (1971), we define the Lorenz curve for a non-negative random variable X with a positive finite mean.

Definition 4 *Let $F_X(x)$ be the cumulative distribution function of a non-negative random variable X with a positive finite mean. The Lorenz curve of X, $L_X(p)$, is defined as*

$$L_X(p) = \frac{1}{E(X)} \int_0^p F_X^{\leftarrow}(t)dt, \quad p \in [0, 1],$$

where $F_X^{\leftarrow}(t) = \inf\{x \in [0, +\infty) : F_X(x) \geq t\}$.

[7] Our main reference book is Shaked and Shanthikumar (2010). A good reference of Lorenz dominance is Arnold (1987). Another book devoted to the parametric statistical distribution of economic sizes is Kleiber and Kotz (2003).

A Lorenz curve satisfies the scale invariance axiom, i.e. random variables X and cX share the same Lorenz curve for any constant $c > 0$. Thus X and $\frac{X}{E(X)}$ share the same Lorenz curve.

By the Lorenz curve, we define the Lorenz ordering

Definition 5 *For two random variables X and Y, X Lorenz dominates Y if and only if $L_X(p) \geq L_Y(p)$, $p \in [0, 1]$, denoted by $X \succeq_L Y$.*

Obviously the Lorenz ordering is transitive, i.e. $X \succeq_L Y$ and $Y \succeq_L Z$ imply $X \succeq_L Z$. Note that $X \succeq_L Y$ implies that the distribution X is more equal than the distribution Y and the Gini coefficient of X is smaller than that of Y.

Another commonly used inequality measure is the coefficient of variation (CV). For a random variable X with a finite variance

$$CV(X) = \frac{\sqrt{var(X)}}{E(X)}.$$

Lorenz dominance implies a relation between two random variables with finite variances.

Lemma 6 *If both X and Y have finite variances, then $X \succeq_L Y$ implies $CV(Y) \geq CV(X)$.*

Proof: See page 68-69 of Marshall and Olkin (2007). ■

4.2 The convex order

Following Shaked and Shanthikumar (2010), we define the convex order of two random variables.

Definition 7 *For two random variables X and Y, X is smaller than Y in the convex order, denoted by $X \preceq_{cx} Y$, if and only if*

$$E[\phi(X)] \leq E[\phi(Y)]$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist.

Roughly speaking, $X \preceq_{cx} Y$ means that Y is more likely to take on "extreme" values than X. That is, Y is "more variable" than X.[8]

Note that the functions ϕ_1 and ϕ_2, defined by $\phi_1(x) = x$ and $\phi_2(x) = -x$, are both convex. Thus $X \preceq_{cx} Y$ implies $E(X) = E(Y)$, provided the expectations exist.

From the definition of the convex order, we see that the convex order is transitive, i.e. $X \preceq_{cx} Y$ and $Y \preceq_{cx} Z$ imply that $X \preceq_{cx} Z$.

For two non-negative random variables the convex order is closely related to the Lorenz ordering.[9] Theorem 3.A.10 of Shaked and Shanthikumar (2010) states that

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[8] See page 109 of Shaked and Shanthikumar (2010).

[9] For two random variables X and Y with equal means, a sufficient and necessary condition for $X \preceq_{cx} Y$ is

$$\int_{-\infty}^{x} F(u)du \leq \int_{-\infty}^{x} G(u)du \text{ for all } x,$$

Lemma 8 *Let X and Y be two non-negative random variables with equal means. Then $X \preceq_{cx} Y$ if and only if $L_X(p) \geq L_Y(p)$ for all $p \in [0,1]$.*

5 Bequest motive and wealth inequality

In order to emphasize the impacts of bequest motives on wealth distribution, following Bossmann et al. (2007), we set estate tax rate $\zeta = 0$. Thus $c_5 = 0$. The agent's wealth accumulation equation (8) becomes

$$a_{t+1} = c_3 l_t + c_4 a_t.$$

Following Bossmann et al. (2007), we assume that there are two economies: economy A and economy B. Agents in economy A do not have bequest motive, i.e. $\chi = 0$. Agents in economy B have bequest motive, i.e. $\chi > 0$ (B for bequest). Let a_∞^A be the stationary wealth distribution of economy A, and a_∞^B be the stationary wealth distribution of economy B.

In economy A there is no bequest motive and $c_4 = 0$. Thus

$$a_{t+1} = c_3 l_t$$

provided the integrals exist, where $F(\cdot)$ and $G(\cdot)$ are the cumulative distribution functions of X and Y, respectively. See Theorem 3.A.1 of Shaked and Shanthikumar (2010). This is a practical way to establish $X \preceq_{cx} Y$. If $E(X) = E(Y)$, then $X \preceq_{cx} Y$ is equivalent to that X second-order stochastically dominates Y. Thus the following lemma 8 is essentially the result of Atkinson (1970).

which has the same Lorenz curve as l_t. Thus a_∞^A has the same Lorenz curve as l_t.

In economy B there are bequest motives. By proposition 2 we have $0 < c_4 < 1$. Plugging $c_5 = 0$ into equation (9) we have

$$\begin{aligned} a_\infty^B &= c_3 \sum_{s=1}^{+\infty} c_4^{s-1} l_s \\ &= \frac{c_3}{1-c_4} \sum_{s=1}^{+\infty} (1-c_4) c_4^{s-1} l_s. \end{aligned}$$

Thus a_∞^B has the same Lorez curve as $Z \equiv \sum_{s=1}^{+\infty}(1-c_4)c_4^{s-1}l_s$. Note that the random variable Z is a weighted average of random variables, l_1, l_2, l_3, $\cdots$. Our analysis of the impacts of bequests on wealth distribution starts from this observation.

Theorem 3.A.36 of Shaked and Shanthikumar (2010) shows that

Lemma 9 *Let X_1, X_2, $\cdots$, X_n and Y be $n+1$ random variables. If $X_i \preceq_{cx} Y$, $i = 1$, 2, $\cdots$, n, then*

$$\sum_{i=1}^{n} a_i X_i \preceq_{cx} Y,$$

whenever $a_i \geq 0$, $i = 1$, 2, $\cdots$, n, and $\sum_{i=1}^{n} a_i = 1$.

Lemma (9) shows that the weighted average would not increase inequality. We extend this intuition to the comparison of stationary wealth distributions a_∞^A and a_∞^B.

Theorem 10 *Under assumptions 1 and 2, $a_\infty^B \succeq_L a_\infty^A$.*

Note that $a_\infty^B \succeq_L a_\infty^A$ also implies that the Gini coefficient of a_∞^B is smaller than that of a_∞^A. An economy in which agents have bequest motives has a more equal wealth distribution than an economy in which agents do not have bequest motives. Our result extends that of Bossmann et al. (2007) in three respects:

First, we only assume that $\{l_t\}$ is a stationary and ergodic. Bossmann et al. (2007) assume that $\{l_t\}$ is either *i.i.d.* or a linear process as in assumption 3 of section 6 in our paper.

Second, we do not assume that $var(l_t) < \infty$. Bossmann et al. (2007) uses the coefficient of variation as their inequality measure. Thus they need the finite variance of wealth distribution. Our inequality measures are Lorenz curve and Gini coefficient, which only require the existence of the mean of wealth distribution.

Third, our result is stronger than that of Bossmann et al. (2007). Bossmann et al. (2007) derive the coefficient of variation of wealth, the inequality measure, by calculating mean and variance of the wealth distribution. By lemma 6, theorem 10 implies that $CV(a_\infty^A) \geq CV(a_\infty^B)$, as shown in Bossmann et al. (2007).

6 Ability inheritance and wealth inequality

Solon (1992) and Zimmerman (1992) use different data sets in the United States to study the intergenerational mobility and find that the elasticity of child's earnings with respect to parent's earnings is about 0.4. We study the impacts of bequest inheritance on wealth distribution in last section. In this section we study the impacts of ability inheritance on wealth distribution. To that end we, following Davies and Kuhn (1991) and Bossmann et al. (2007), use a mean-reverting process as the labor efficiency process.

Assumption 3:

$$l_{t+1} = \bar{l} + v(l_t - \bar{l}) + \varepsilon_{t+1} \tag{10}$$

where $\bar{l} = 1$ and $0 < v < 1$. $\{\varepsilon_t\}$ is $i.i.d.$ with a zero mean, a finite variance, and a lower bound sufficient to keep $l_{t+1} > 0$.

In this section we permit $\zeta \in [0, 1)$. Thus there could be government transfer. The agent's wealth accumulation equation has the general form

$$a_{t+1} = c_3 l_t + c_4 a_t + c_5$$

as in equation (8).

Let l_1 starts from the unique stationary solution of equation (10).[10] Then $\{l_t\}$ is stationary and ergodic.

Proposition 11 $\bar{l} + \varepsilon_1 \preceq_{cx} l_t$, $\forall t \geq 1$. *Thus the distribution of* $\bar{l} + \varepsilon_1$ *Lorenz-dominates* l_t, $\forall t \geq 1$.

Following Esary et al. (1967) and Shaked and Shanthikumar (2010), we introduce the concept of positive association.

[10] For the existence and uniqueness of the stationary solution of equations $Y_n = A_n Y_{n-1} + B_n$, $n = 1, 2, \cdots$, with independent pairs $\{(A_n, B_n)\}$, see Vervaat (1979). The conditions in assumption 3 guarantees that there exists a unique stationary solution of equation (10) by part (b) of Theorem 1.6 of Vervaat (1979).

Definition 12 *Random variables $X_1,\ X_2,\cdots,\ X_n$ are said to be positively associated if*

$$Cov\left(h_1\left(X_1,\ X_2,\ \cdots,\ X_n\right), h_2\left(X_1,\ X_2,\ \cdots,\ X_n\right)\ \right) \geq 0$$

for all increasing functions h_1 and h_2 for which the above covariance is defined.

By Proposition 20.I.13 of Marshall and Olkin (2007), the ability inheritance implies that $\{l_t\}$ are positively associated. Also from equation (10) we have

Proposition 13 *Let $a_1 = 1$. Then a_t and l_t are positively associated for $t \geq 1$.*

Actually, from the proof of proposition 13, we see that if a_1 is independent of l_1 and $\{\varepsilon_t\}$, then a_t and l_t are positively associated for $t \geq 1$.

We assume that there are two economies: economy H and economy I. Agents in economy H do not have ability inheritance, i.e. $v = 0$. Agents in economy I have ability inheritance, i.e. $0 < v < 1$ (I for inheritance of ability). Let a_∞^H be the stationary wealth distribution of economy H, and a_∞^I be the stationary wealth distribution of economy I.

Theorem 3.A.39 of Shaked and Shanthikumar (2010) states that

Lemma 14 *Let $X_1,\ X_2,\cdots,\ X_n$ be positively associated random variables, and let Y_1, $Y_2,\cdots,\ Y_n$ be independent random variables such that $X_i =_{st} Y_i$, $i = 1,\ 2,\cdots,\ n$. Then*

$$\sum_{i=1}^{n} Y_i \preceq_{cx} \sum_{i=1}^{n} X_i.$$

Lemma 14 is intuitive. For two random variables X and Y with positive correlation their sum is more unequal than that of two independent random variables with the same marginal distributions as X and Y respectively, because there is a sorting mechanism with the positive correlation. We extend this intuition to the comparison of stationary wealth distributions a_∞^H and a_∞^I.

Theorem 15 *Under assumption 3,* $a_\infty^H \succeq_L a_\infty^I$.

Contrary to inheritance of bequests, the inheritance of ability causes higher wealth inequality. The difference between inheritance of bequests and inheritance of ability on wealth distribution as shown in theorems 10 and 15 is due to the different impacts on the mean of wealth. The bequest motives increase the mean of wealth in stationary distribution, i.e.

$$E(a_\infty^B) > E(a_\infty^A)$$

since $0 < c_4 < 1$ in economy B. However, the inheritance of ability only introduces correlation of earnings ability into the economy and does not influence the mean of wealth, i.e.

$$E(a_\infty^I) = E(a_\infty^H).$$

7 Estate tax and wealth inequality

When investigating the impacts of estate taxes on wealth distribution, we concentrate on the logarithmic utility as in Bossmann et al. (2007).

Assumption 4: Utility functions are logarithmic.

We need an additional assumption.

Assumption 5: $\{l_t\}$ is *i.i.d.*

Let $\eta = 1$, the CRRA utility of section 2 reduces to logarithmic utility. The agent's problem is

$$\max_{c_t^y, s_t, c_{t+1}^o, b_{t+1}} \ln c_t^y + \beta(\ln c_{t+1}^o + \chi \ln[(1-\zeta)\, b_{t+1}])$$

$$\begin{aligned} s.t. \quad c_t^y + s_t &= w_t l_t + (1-\zeta)\, b_t + g_t \\ c_{t+1}^o + b_{t+1} &= (1 + r_{t+1}) s_t. \end{aligned}$$

The household's optimal behaviours are

$$c_{t+1}^o = \frac{1}{1+\chi}(1 + r_{t+1}) s_t$$

$$b_{t+1} = \frac{1}{1+\frac{1}{\chi}}(1 + r_{t+1}) s_t$$

$$c_t^y = \frac{1}{1+\beta\,(1+\chi)}\left[w_t l_t + (1-\zeta) b_t + g_t\right]$$

and

$$s_t = \frac{1}{1 + \frac{1}{\beta(1+\chi)}} \left[w_t l_t + (1-\zeta) b_t + g_t \right].$$

There is no general equilibrium effect of estate tax on the economy. The estate tax here has only the role of redistribution. The estate tax does not affect aggregate capital as well as the interest rate and the wage rate.

The individual wealth accumulation equation is

$$a_{t+1} = c_6 l_t + c_7 \left[(1-\zeta) a_t + \zeta \bar{K} \right]$$

with

$$c_6 = \frac{1}{1 + \frac{1}{\beta(1+\chi)}} \bar{w}$$

and

$$c_7 = \frac{1}{\left(1 + \frac{1}{\beta(1+\chi)}\right)\left(1 + \frac{1}{\chi}\right)} (1 + \bar{r})$$

where $\bar{w} = (1-\alpha) A \left(\bar{K}\right)^{\alpha}$, $\bar{r} = \alpha A \left(\bar{K}\right)^{\alpha-1} - \delta$, and $\bar{K} = \left(\frac{1-\alpha+\chi}{1+\frac{1}{\beta}+\delta\chi} A\right)^{\frac{1}{1-\alpha}}$.

Again we starts from a static case.

Lemma 16 *For a non-negative random variable X with a positive finite mean, if $0 \leq \hat{\zeta} \leq \zeta < 1$, then $(1-\zeta)X + \zeta E(X) \succeq_L (1-\hat{\zeta})X + \hat{\zeta} E(X)$. Thus $(1-\zeta)X + \zeta E(X) \preceq_{cx} (1-\hat{\zeta})X + \hat{\zeta} E(X)$.*

A flat estate tax plus a lump-sum transfer is equivalent to a progressive tax since the effective average tax rate is increasing in bequests.[11] Lemma (16) implies that the higher the flat tax rate and thus the higher the lump-sum transfer, the lower the bequest inequality.[12]

Let a_∞^ζ be the stationary wealth distribution of an economy with estate tax ζ, and $a_\infty^{\hat{\zeta}}$ be the stationary wealth distribution of an economy with estate tax $\hat{\zeta}$.

Theorem 17 *Under assumptions 2, 4, and 5, if* $\zeta \geq \hat{\zeta}$*, then* $a_\infty^\zeta \succeq_L a_\infty^{\hat{\zeta}}$.

Theorem 17 extends the intuition of lemma 16 to stationary wealth distributions. Our theoretical result of theorem 17 supports the simulation results about the impacts of estate taxes on wealth distributions by Bossmann et al. (2007). Bossmann et al. (2007) employ *i.i.d.* labor efficiency with a two-parameter gamma distribution and find that estate taxes reduces the Gini coefficient of the wealth distribution by simulations.

As noted by Bossmann et al. (2007), this result depends on the assumption of utility functions. For example utility functions with $\eta > 1$ in section 2.1 of our paper and CES utility in Appendix B.1 of Bossmann et al. (2007) may not work, since estate tax rate influences the mean of wealth in these cases.

Using the coefficient of variation as their inequality measure, Bossmann et al. (2007) show that estate tax reduces wealth inequality for both *i.i.d.* labor efficiency and a linear

[11] For an individual with before-tax bequest x, the effective average tax rate is

$$\frac{x - [(1-\zeta)\,x + \zeta E(X)]}{x} = \zeta\left[1 - \frac{E(X)}{x}\right].$$

[12] See Fellman (1976) for a study on the effect of progressive taxes on income distributions.

process as in assumption 3 of our paper. We only show this result for the *i.i.d.* case. Future work could study whether the Lorenz dominance result also holds for a linear process as in assumption 3.

8 Conclusion

We extends three main findings in Bossmann et al. (2007). An economy with bequest motives has a more equal wealth distribution than an economy without bequest motives. However an economy with inheritance of earnings ability has higher wealth inequality than an economy without ability inheritance. An estate tax plus a lump-sum transfer reduces wealth inequality. Bossmann et al. (2007) establish these results by calculating the coefficient of variation. We strengthen these results by using Lorenz dominance.

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9 Appendix

9.1 Proof of proposition 1

Proof: Let $K_{t+1} = K_t = K$ in equation (6) we have

$$K = \left(\frac{1 - \alpha + \varphi\alpha}{1 + \tilde{\beta}^{-\frac{1}{\eta}} - \varphi(1-\delta)} A \right)^{\frac{1}{1-\alpha}} \tag{A.1}$$

where $\tilde{\beta} = \beta \left[1 + \chi^{\frac{1}{\eta}} (1-\zeta)^{\frac{1-\eta}{\eta}} \right]^{\eta} (1+r)^{1-\eta}$ and $r = \alpha A K^{\alpha-1} - \delta$.

Plugging equation (A.1) into $r = \alpha A K^{\alpha-1} - \delta$ we have

$$\frac{r+\delta}{\alpha} = \frac{1 + \tilde{\beta}^{-\frac{1}{\eta}} - \varphi(1-\delta)}{1 - \alpha + \varphi\alpha}. \tag{A.2}$$

Plugging $\tilde{\beta} = \beta \left[1 + \chi^{\frac{1}{\eta}} (1-\zeta)^{\frac{1-\eta}{\eta}} \right]^{\eta} (1+r)^{1-\eta} = \frac{\beta}{(1-\varphi)^{\eta}} (1+r)^{1-\eta}$ into equation (A.2) we have

$$\frac{1-\alpha}{\alpha}(r+\delta) + \varphi(1+r) - (1-\varphi)\beta^{-\frac{1}{\eta}} (1+r)^{1-\frac{1}{\eta}} = 1.$$

We show proposition 1 in two cases:

Case (i) $\eta > 1$

Note that $0 < \varphi < 1$. Define

$$h(\varphi, r) = \frac{1-\alpha}{\alpha}(r+\delta) + \varphi(1+r) - (1-\varphi)\beta^{-\frac{1}{\eta}} (1+r)^{1-\frac{1}{\eta}}.$$

The equilibrium r is determined by

$$h(\varphi, r) = 1.$$

Note that $h(\varphi, r)$ is a continuous function of r, with

$$h(\varphi, -\delta) = \varphi(1-\delta) - (1-\varphi)\beta^{-\frac{1}{\eta}}(1-\delta)^{1-\frac{1}{\eta}} < \varphi(1-\delta) < 1$$

and

$$\lim_{r\to\infty} h(\varphi, r) = \infty$$

Also $h_{22}(\varphi, r) = \left(1 - \frac{1}{\eta}\right)\frac{1}{\eta}(1-\varphi)\beta^{-\frac{1}{\eta}}(1+r)^{-\frac{1}{\eta}-1} > 0$ due to $\eta > 1$. Thus $h(\varphi, r)$ is a strictly convex function of r. Therefore there must exist a unique equilibrium $r > -\delta$.[13]

Note that $h(\varphi, r)$ is strictly increasing in φ. For $\varphi_1 < \varphi_2 < 1$, suppose that

$$h(\varphi_1, r_1) = 1 \quad \text{and} \quad h(\varphi_2, r_2) = 1.$$

We have

$$h(\varphi_2, r_1) > h(\varphi_1, r_1) = 1.$$

Thus $r_2 < r_1$ since $h(\varphi_2, -\delta) < 1$ and $h(\varphi_2, r)$ is a continuous function of r. A higher χ

[13] In the equilibrium r could be negative. Since saving is the only way to bring wealth to the next period, even if r is negative, the agent still saves.

implies a higher φ. Thus a higher χ implies a lower r and a higher K.

Case (ii) $\eta = 1$

In the equilibrium[14]

$$\begin{aligned} K &= \left(\frac{1-\alpha+\chi}{1+\frac{1}{\beta}+\delta\chi} A \right)^{\frac{1}{1-\alpha}} \\ &= \left(\left[\frac{1}{\delta} - \frac{\frac{1}{\delta}\left(1+\frac{1}{\beta}\right) - (1-\alpha)}{1+\frac{1}{\beta}+\delta\chi} \right] A \right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Thus a higher χ implies a higher K. ■

9.2 Proof of proposition 2

Proof: Obviously $c_4 \geq 0$. From equation (A.2) we have

$$1 + r = \frac{\left(1 + \tilde{\beta}^{-\frac{1}{\eta}}\right)\alpha + (1-\delta)(1-\alpha)}{(1-\alpha) + \varphi\alpha}$$

Thus

$$c_4 = \frac{(1-\zeta)\,\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}} = (1-\zeta)\,\frac{\alpha + \frac{1-\delta}{1+\tilde{\beta}^{-\frac{1}{\eta}}}(1-\alpha)}{\alpha + \frac{1}{\varphi}(1-\alpha)} < 1$$

since $\tilde{\beta} = \beta\left[1 + \chi^{\frac{1}{\eta}}(1-\zeta)^{\frac{1-\eta}{\eta}}\right]^{\eta}(1+r)^{1-\eta} > 0$ and $0 < \varphi < 1$. ■

[14] In this case $\tilde{\beta} = \beta(1+\chi)$.

9.3 Proof of proposition 3

Proof: Let $B_t = c_5 + c_3 l_t$. Thus $\{B_t\}$ is stationary and ergodic since $\{l_t\}$ is a stationary and ergodic by assumption 1. We have $-\infty \leq \log c_4 < 0$. Also $E(B_t) = c_5 + c_3 < \infty$, since $E(l_t) = 1$ by assumption 2. Thus $E(\log B_t)^+ \leq E(B_t) < \infty$. By Theorem 1 of Brandt (1986) we obtain the results of proposition 3. ■

9.4 Proof of theorem 10

Proof: Note that a_∞^A has the same Lorenz curve as l_1. We only need to show that $a_\infty^B \succeq_L l_1$.

In economy B, pick $a_1 = \frac{c_3}{1-c_4}$.[15] Thus

$$a_1 \preceq_{cx} \frac{c_3}{1-c_4} l_1$$

since $a_1 = E\left(\frac{c_3}{1-c_4} l_1\right)$.[16]

Suppose that

$$a_t \preceq_{cx} \frac{c_3}{1-c_4} l_1.$$

Thus $\frac{1-c_4}{c_3} a_t \preceq_{cx} l_1$.[17]

[15] We abuse notations a little bit. We use a_t instead of a_t^B without confusions.

[16] Let X be a random variable with a finte mean. $E(X) \preceq_{cx} X$ can be established by applying Jensen's Inequality and the definition of the convex order.

[17] $X \preceq_{cx} Y$ implies $bX \preceq_{cx} bY$ for any $b \in \mathbb{R}$. Note that $\phi(bx)$ is a convex function of $x \in \mathbb{R}$ if $\phi(x)$ is a convex function of $x \in \mathbb{R}$.

And

$$\begin{aligned} a_{t+1} &= c_3 l_t + c_4 a_t \\ &= \frac{c_3}{1-c_4}\left((1-c_4)l_t + c_4\frac{1-c_4}{c_3}a_t\right). \end{aligned}$$

Note that $(1-c_4)l_t + c_4\frac{1-c_4}{c_3}a_t$ is a weighted average of l_t and $\frac{1-c_4}{c_3}a_t$. For $\forall t \geq 1$, l_t and l_1 have the same distribution. We have $l_t \preceq_{cx} l_1$, $\forall t \geq 1$. By lemma 9 we have

$$(1-c_4)l_t + c_4\frac{1-c_4}{c_3}a_t \preceq_{cx} l_1.$$

Thus

$$a_{t+1} \preceq_{cx} \frac{c_3}{1-c_4}l_1.$$

By mathematical induction we have

$$a_t \preceq_{cx} \frac{c_3}{1-c_4}l_1, \quad \forall t \geq 1.$$

Since $a_t \rightarrow_{st} a_\infty^B$ we have

$$a_\infty^B \preceq_{cx} \frac{c_3}{1-c_4}l_1$$

by part (c) of Theorem 3.A.12 of Shaked and Shanthikumar (2010). By lemma 8 we have $a_\infty^B \succeq_L \frac{c_3}{1-c_4}l_1$ since $E\left(a_\infty^B\right) = E\left(\frac{c_3}{1-c_4}l_1\right) = \frac{c_3}{1-c_4}$. Thus $a_\infty^B \succeq_L l_1$. ■

9.5 Proof of proposition 11

Proof: From equation (10) we have

$$l_{t+1} = \bar{l} + \varepsilon_{t+1} + v\left(l_t - \bar{l}\right).$$

Let l_1 starts from the unique stationary solution of equation (10). Then $\{l_t\}$ is stationary and ergodic. We only need to show that $\bar{l} + \varepsilon_1 \preceq_{cx} l_{t+1}$.

We know that $\bar{l} + \varepsilon_{t+1}$ and $v\left(l_t - \bar{l}\right)$ are independent,

$$E\left(\bar{l} + \varepsilon_{t+1}\right) = \bar{l} = 1$$

and

$$E\left[v\left(l_t - \bar{l}\right)\right] = vE\left(l_t - \bar{l}\right) = 0.$$

By Theorem 3.A.34 of Shaked and Shanthikumar (2010) we have

$$\bar{l} + \varepsilon_{t+1} \preceq_{cx} \bar{l} + \varepsilon_{t+1} + v\left(l_t - \bar{l}\right)$$

i.e.

$$\bar{l} + \varepsilon_{t+1} \preceq_{cx} l_{t+1}.$$

Note that $\bar{l}+\varepsilon_1$ and $\bar{l}+\varepsilon_{t+1}, \forall t \geq 1$ have the same distribution. Thus

$$\bar{l}+\varepsilon_1 \preceq_{cx} l_{t+1}.$$

By lemma 8 we have

$$\bar{l}+\varepsilon_1 \succeq_L l_{t+1}$$

since $E\left(\bar{l}+\varepsilon_1\right)=E\left(l_{t+1}\right)=\bar{l}$. ■

9.6 Proof of proposition 13

Proof: We know that

$$\begin{aligned} a_t \;=\; & \frac{1-c_4^{t-1}}{1-c_4}c_5+\frac{c_3}{v-c_4}(1-v)\bar{l}\frac{c_4^{t-1}-v^{t-1}+v-c_4+c_4v^{t-1}-vc_4^{t-1}}{(1-v)(1-c_4)} \\ & +c_4^{t-1}a_1+c_3\frac{v^{t-1}-c_4^{t-1}}{v-c_4}l_1 \\ & +c_3\sum_{s=1}^{t-1}\frac{v^{t-1-s}-c_4^{t-1-s}}{v-c_4}\varepsilon_{s+1} \end{aligned}$$

and

$$l_t=\left(1-v^{t-1}\right)\bar{l}+v^{t-1}l_1+\sum_{s=1}^{t-1}v^{t-1-s}\varepsilon_{s+1}.$$

Note that

$$\frac{v^s - c_4^s}{v - c_4} = \begin{cases} 0 & if \quad s = 0 \\ 1 & if \quad s = 1 \\ c_4^{s-1} + c_4^{s-2}v + c_4^{s-3}v^2 + \cdots + c_4 v^{s-2} + v^{s-1} & if \quad s \geq 2 \end{cases}$$

Thus $\frac{v^s - c_4^s}{v - c_4} \geq 0$ for $s \geq 0$. For $t \geq 1$, a_t is an increasing function of a_1, l_1, and ε_2, ε_3, $\cdots$, ε_t. Also l_t is an increasing function of a_1, l_1, and ε_2, ε_3, $\cdots$, ε_t. And a_1, l_1, and ε_2, ε_3, $\cdots$, ε_t are independent. By Proposition 20.I.13 of Marshall and Olkin (2007), we know that for $t \geq 1$, a_t and l_t are positively associated. ■

9.7 Proof of theorem 15

Proof: In economy H, $v = 0$. Thus

$$a_{t+1}^H = c_3 \left(\bar{l} + \varepsilon_t\right) + c_4 a_t^H + c_5.$$

Note that $\bar{l} + \varepsilon_t$ and a_t^H are independent.

In economy I, $0 < v < 1$. From proposition 13 we know that a_t^I and l_t are positively associated.

Let $a_1^I = 1$ and $a_1^H = 1$. Thus

$$a_2^H = c_3 \left(\bar{l} + \varepsilon_1\right) + c_4 a_1^H + c_5 \preceq_{cx} c_3 l_1 + c_4 a_1^I + c_5 = a_2$$

since $\bar{l} + \varepsilon_1 \preceq_{cx} l_1$ by proposition 11.[18]

Now suppose that $a_t^H \preceq_{cx} a_t^I$. Find two independent random variables U and V such that

$$U =_{st} l_t$$

and

$$V =_{st} a_t^I$$

Thus $\bar{l} + \varepsilon_t \preceq_{cx} U$ and $a_t^H \preceq_{cx} V$. Thus

$$c_3\left(\bar{l} + \varepsilon_t\right) + c_4 a_t^H \preceq_{cx} c_3 U + c_4 V$$

by the property of the convex order in footnote 17 and part (d) of Theorem 3.A.12 of Shaked and Shanthikumar (2010).

By lemma 14 we have

$$c_3 U + c_4 V \preceq_{cx} c_3 l_t + c_4 a_t^I$$

since a_t^I and l_t are positively associated.

By the transitivity of the convex order we have

$$c_3\left(\bar{l} + \varepsilon_t\right) + c_4 a_t^H \preceq_{cx} c_3 l_t + c_4 a_t^I.$$

[18] $X \preceq_{cx} Y$ implies $bX + c \preceq_{cx} bY + c$ for any $b, c \in \mathbb{R}$. Note that $\phi(bx + c)$ is a convex function of $x \in \mathbb{R}$ if $\phi(x)$ is a convex function of $x \in \mathbb{R}$.

Thus

$$a_{t+1}^H = c_3\left(\bar{l}+\varepsilon_t\right) + c_4 a_t^H + c_5 \preceq_{cx} c_3 l_t + c_4 a_t^I + c_5 = a_{t+1}^I$$

by the property of the convex order in footnote 18.

By mathematical induction we have

$$a_t^H \preceq_{cx} a_t^I, \ \forall t \geq 1.$$

Since $a_t^H \rightarrow_{st} a_\infty^H$ and $a_t^I \rightarrow_{st} a_\infty^I$, we have

$$a_\infty^H \preceq_{cx} a_\infty^I$$

by part (c) of Theorem 3.A.12 of Shaked and Shanthikumar (2010). By lemma 8 we have

$$a_\infty^H \succeq_L a_\infty^I$$

since $E\left(a_\infty^H\right) = E\left(a_\infty^I\right) = \frac{c_5+c_3}{1-c_4}$. ■

9.8 Proof of lemma 16

Proof: Let

$$g(x) = (1-\zeta)x + \zeta E\left(X\right), \quad x \in [0,+\infty)$$

and

$$h(x) = (1-\hat{\zeta})x + \hat{\zeta}E\left(X\right), \quad x \in [0, +\infty)$$

Note that $g(\cdot)$ and $h(\cdot)$ are non-negative increasing functions defined on $[0, +\infty)$, since $0 \leq \hat{\zeta} \leq \zeta < 1$. Also $g(x) > 0$ and $h(x) > 0$ for $x > 0$. Note that $\frac{h(x)}{g(x)}$ is increasing in $x \in (0, +\infty)$, since

$$\begin{aligned}
\frac{h(x)}{g(x)} &= \frac{(1-\hat{\zeta})x + \hat{\zeta}E\left(X\right)}{(1-\zeta)x + \zeta E\left(X\right)} \\
&= \frac{1-\hat{\zeta}}{1-\zeta}\left[1 - \frac{\zeta-\hat{\zeta}}{\left(1-\hat{\zeta}\right)(1-\zeta)}\frac{E(X)}{x + \frac{\zeta}{1-\zeta}E(X)}\right].
\end{aligned}$$

By Theorem 3.A.26 of Shaked and Shanthikumar (2010) we have $g(X) \succeq_L h(X)$, i.e. $(1-\zeta)X + \zeta E\left(X\right) \succeq_L (1-\hat{\zeta})X + \hat{\zeta}E\left(X\right)$. By lemma 8 we have $(1-\zeta)X + \zeta E\left(X\right) \preceq_{cx} (1-\hat{\zeta})X + \hat{\zeta}E\left(X\right)$ since $E\left[(1-\zeta)X + \zeta E\left(X\right)\right] = E(X) = E\left[(1-\hat{\zeta})X + \hat{\zeta}E\left(X\right)\right]$. ■

9.9 Proof of theorem 17

Proof: Note that a^{ζ}_{∞} is the stationary distribution of the stochastic process $\{a^{\zeta}_t\}$ which is generated by

$$a^{\zeta}_{t+1} = c_6 l_t + c_7\left[(1-\zeta)a^{\zeta}_t + \zeta\bar{K}\right]$$

and a given a^{ζ}_1. And $a^{\hat{\zeta}}_{\infty}$ is the stationary distribution of the stochastic process $\{a^{\hat{\zeta}}_t\}$ which is generated by

$$a^{\hat{\zeta}}_{t+1} = c_6 l_t + c_7\left[(1-\hat{\zeta})a^{\hat{\zeta}}_t + \hat{\zeta}\bar{K}\right]$$

and a given $a_1^{\hat{\zeta}}$.

Let $a_1^{\zeta} =_{st} a_1^{\hat{\zeta}}$. Thus $a_1^{\zeta} \preceq_{cx} a_1^{\hat{\zeta}}$ by the definition of the convex order.

Now suppose that $a_t^{\zeta} \preceq_{cx} a_t^{\hat{\zeta}}$. By lemma 16 we have

$$(1-\zeta)a_t^{\zeta} + \zeta\bar{K} \preceq_{cx} (1-\hat{\zeta})a_t^{\zeta} + \hat{\zeta}\bar{K}$$

since $E\left(a_t^{\zeta}\right) = \bar{K}$.

By Corollary 3.A.22 of Shaked and Shanthikumar (2010) we have $(1-\hat{\zeta})a_t^{\zeta} \preceq_{cx} (1-\hat{\zeta})a_t^{\hat{\zeta}}$ since $\left(1-\hat{\zeta}\right)$ is independent of a_t^{ζ} and $a_t^{\hat{\zeta}}$. By Part (d) of Theorem 3.A.12 of Shaked and Shanthikumar (2010) we have

$$(1-\hat{\zeta})a_t^{\zeta} + \hat{\zeta}\bar{K} \preceq_{cx} (1-\hat{\zeta})a_t^{\hat{\zeta}} + \hat{\zeta}\bar{K}$$

since $\hat{\zeta}\bar{K}$ is independent of $(1-\hat{\zeta})a_t^{\zeta}$ and $(1-\hat{\zeta})a_t^{\hat{\zeta}}$. By the transitivity of the convex order we have

$$(1-\zeta)a_t^{\zeta} + \zeta\bar{K} \preceq_{cx} (1-\hat{\zeta})a_t^{\hat{\zeta}} + \hat{\zeta}\bar{K}.$$

By Corollary 3.A.22 of Shaked and Shanthikumar (2010) we have $c_7\left[(1-\zeta)a_t^{\zeta} + \zeta\bar{K}\right] \preceq_{cx} c_7\left[(1-\hat{\zeta})a_t^{\hat{\zeta}} + \hat{\zeta}\bar{K}\right]$ since c_7 is independent of $(1-\zeta)a_t^{\zeta} + \zeta\bar{K}$ and $(1-\hat{\zeta})a_t^{\hat{\zeta}} + \hat{\zeta}\bar{K}$. Note that $c_6 l_t$ and $c_7\left[(1-\zeta)a_t^{\zeta} + \zeta\bar{K}\right]$ are independent. And $c_6 l_t$ and $c_7\left[(1-\hat{\zeta})a_t^{\hat{\zeta}} + \hat{\zeta}\bar{K}\right]$ are independent. Thus by part (d) of Theorem 3.A.12 of Shaked and Shanthikumar (2010), we have

$$c_6 l_t + c_7 \left[(1-\zeta) a_t^{\zeta} + \zeta \bar{K} \right] \preceq_{cx} c_6 l_t + c_7 \left[(1-\hat{\zeta}) a_t^{\hat{\zeta}} + \hat{\zeta} \bar{K} \right].$$

Thus we have

$$a_{t+1}^{\zeta} \preceq_{cx} a_{t+1}^{\hat{\zeta}}.$$

By mathematical induction we have

$$a_t^{\zeta} \preceq_{cx} a_t^{\hat{\zeta}}, \ \forall t \geq 1.$$

Since $a_t^{\zeta} \to_{st} a_{\infty}^{\zeta}$ and $a_t^{\hat{\zeta}} \to_{st} a_{\infty}^{\hat{\zeta}}$ thus

$$a_{\infty}^{\zeta} \preceq_{cx} a_{\infty}^{\hat{\zeta}}$$

by part (c) of Theorem 3.A.12 of Shaked and Shanthikumar (2010). By lemma 8 we have

$$a_{\infty}^{\zeta} \succeq_{L} a_{\infty}^{\hat{\zeta}}$$

since $E\left(a_{\infty}^{\zeta}\right) = E\left(a_{\infty}^{\hat{\zeta}}\right) = \bar{K}$. ■