The nature of the S-linear algebra: For an S-propagator

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Abstract

This paper is intended to analyse an $\mathcal{S}$-linear algebra's application so as to build an $\mathcal{S}$-propagator's concept. In particular we shall study a *semi*-deterministic propagator via superposition (it is intended the Carfí's notion of superposition).

1 Introduction

We know that the superposition's concept is a generalization of the notion of linear combination, thus it is straightforward its link to an idea of propagator. It can be thought of as [CfS]

Theorem 1.1. *Let A be a continuous curve of $\mathcal{S}$-diagonalizable operators with a same $\mathcal{S}$-basis $\in S_n'$. Let $(t_0, x_0) \in \mathbb{R} \times S_n'$ be an initial condition and x be a curve in S_n' defined by*

$$x : t \mapsto e^{(\int_{t_0}^{t} A d\lambda)(x_0)},$$

for every real time t. Then x is $\sigma(S_n')$-differentiable and it is such that

$$x'(t) = A(t)(x(t))$$

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for every real time t and $x(t_0) = x_0$. Moreover for every pair of times s and t we have

$$x(t) = S(s,t)x(s)$$

where $S : \mathbb{R}^2 \to \mathcal{L}(S'_n)$ is the propagator[1] *defined by*

$$S(s,t) = \exp\bigg(\int_s^t A d\lambda\bigg).$$

Furthermore, in [CfS], we shall find the concept of $^{\mathcal{S}}$Green's family.

Definition 1.2. Let L be a linear endomorphism on the space of tempered distribution S'_n. Any family $G = (G_s)_{s\in\mathbb{R}} \in S'_n$ is said a Green's family of the linear operator L if it satisfies the equality

$$L(G_s) = \delta_s,$$

for any point $s \in \mathbb{R}^n$, where δ_s is the Dirac distribution centered at s. In other words, a family $G = (G_s)_{s\in\mathbb{R}} \in S'_n$ is said a Green's family of the linear operator L iff the distribution G_s is a fundamental solution for the linear operator L at s, $\forall$ index p of the family.

The superposition of a family G is a continuous right inverse of $L \in \mathcal{L}(S'_n)$:

$$L \circ {}^t\hat{G} = (\cdot)_{S'_n}$$

or equivalently

$$L \circ \int_{\mathbb{R}^n} (\cdot, G) = (\cdot)_{S'_n}.$$

Thus, for every $a \in S'_n$ we have

$$\begin{aligned} L \circ \int_{\mathbb{R}^n} (\cdot, G)(a) &= L\bigg(\int_{\mathbb{R}^n} aG\bigg) \\ &= \int_{\mathbb{R}^n} aL(G) \\ &= \int_{\mathbb{R}^n} a\delta \\ &= a \end{aligned}$$

Thus we solve an equation so as to solve an infinity of them. This kind of function is a very important one and a family made of them is a generalization that could be quite a useful tool. Let us see something of utmost importance [CfS]

[1]the term $\mathcal{L}(S'_n)$. deontes the space of $\mathcal{S}$-endomorphism on the space S'_n

Definition 1.3. If we have a function

$$f(y) = \int_{\mathbb{R}^n} f(x)\delta(x-y)dx, \tag{1.1}$$

we consider the distribution $\delta(x-y)$ as the base ket $|x>$ $\forall x \in \mathbb{R}$. But, following [CfS], we have to consider that δ_y is not defined on the real line $\mathbb{R}$ but on the space of test functions $\mathcal{D}(\mathbb{R}, \mathbb{C})$. Thus the (1.1) is not a vector $|x>$ labeled by x. If $|x> = \delta(x-y)$, then y becomes an abuse of notation, rather we may say that we cannot consider both x and y as real numbers simultaneously [CfS]. Thus, given a tempered distribution $u \in S_1'$, we cannot justify the desired expansion

$$u(y) = \int_{\mathbb{R}} u(x)\delta(x-y)dx.$$

Now all we need is the concept of Dirac's family $\delta = (\delta_y)_{y\in\mathbb{R}}$. Hence, given $f \in C^0(\mathbb{R}, \mathbb{C})$

$$f(y) = \int_{\mathbb{R}} f\delta_y.$$

We denote by u the functional

$$u \mapsto \int_{\mathbb{R}} u$$

w.r.t. the Lebesgue measure, and we can define it as the integral on the space of distribution with compact support $\epsilon_1' = \epsilon'(\mathbb{R}, \mathbb{C})$ [CfS] i.e. the functional

$$\int_{\mathbb{R}} (\cdot)_{\epsilon_1'} : \epsilon_1' \to \mathbb{C} : u \mapsto u(1_{\mathbb{R}}).$$

Thus it acts on the constant functional

$$1_{\mathbb{R}} : \mathbb{R} \to \mathbb{C} : x \mapsto 1,$$

that $1_{\mathbb{R}} \in \epsilon_1 = C^\infty(\mathbb{R}, \mathbb{C})$. Hence we can write

$$f = \int_{\mathbb{R}} f(\delta_y)_{y\in\mathbb{R}} \equiv \left(\int_{\mathbb{R}} f\delta_y\right)_{y\in\mathbb{R}}.$$

But, we need an operator

$$\int_{\mathbb{R}} : S_1' \times S^1 \to S_1'$$

such that, following [CfS]:

i S^1 is some set of families in S_1' indexed by the real line $\mathbb{R}$;

ii S^1 contains exactly the summable families;

iii the Dirac's family $(\delta_x)_{x\in\mathbb{R}}$ or $(|x>)_{x\in\mathbb{R}}$ belongs to the family space S^1;

iv every tempered distribution $u \in S'_1$ can be expanded as

$$\int_{\mathbb{R}} u(|x>)_{x\in\mathbb{R}} = u,$$

Thus, it is needed an operator that to any $\mathcal{S}'$-system of coefficients $a \in S'_1$ and any family of distribution $u = (u_k)_{k\in\mathbb{R}} \in S^1$, associates a distribution so as to define a superposition of u w.r.t. a.

We already know that an ${}^{\mathcal{S}}$Green's family is defined by $L(G) = \delta$ (where L is an ${}^{\mathcal{S}}$linear operator). Hence δ is of the type we have seen before ([DEF1.3]), that is to say that an ${}^{\mathcal{S}}$Green's family ought to be equal to a Dirac's family. Moreover, given $L : S'_n \to S'_n$ it is surjective because every Dirac basis must have an anti-image. We say that the Dirac bases are dense in S'_n. Now the space of S'_n is a Fréchet space, it is continuous, bounded and, given the Dieudonné-Schwartz theorem, the image ${}^t u(S'_n)$ is closed in the dual S'_n for the weak-$\star$topology $\sigma(S'_n, S_n)$. Hence we say that L admits an ${}^{\mathcal{S}}$Green's family iff L is surjective.
Given the ([DEF1.3]) of i.e. δ_s we cannot treat that s as a simple parameter as well as we cannot do that in G_s where $L(G_s) = \delta_s$. The meaning of the ([DEF1.3]) is extended to the definition of Green's families.

2 The $\mathcal{S}$-Propagator

We have studied a general definition of Green's functions, we have learned that this kind of function is important so as to find a solution of a differential equation, so in doing this, we define a *propagator*. The concept of propagator is that of foresee a path which begins in the present and moves forward to the future. Now, let us build our propagator.
In [FrS] I introduced a particular kind of propagator which boils down to the Feynman's one. In this paper I want to bolster the Feynman-Strati propagator [FrS]. First of all we define

$$u(t) = \int_{\mathbb{R}} u(t_0)P(t_0, t)$$

the superposition by which the propagator $P(t_o, t)$ and the $u(t_0)$ grow given the initial condition (u_0, t_0). Now in [FrS] there was a particular object between t_0 and t that I called the probabilistic box $\mathfrak{L}$. The $\mathfrak{L}$ is a probabilistic object by which one can use $P(t_o, t)$ in a powerful fashion. Given some information bundles [FrS] $\mathbb{I}^\alpha$ we could define an eigen-expansion given a benchmark propagator $P(t_0, t)$, thus we could define every path given $\mathbb{I}^\alpha$. Besides, we defined a particular drift operator H in which we could find all the information bundles by which the eigen-expansions drift away from the benchmark path. In this paper we want to define the Strati $\mathcal{S}$-propagator, which it is different from that of [FrS]. If we write this new concept in Dirac's notation (bra $< \cdot|$, ket $|\cdot >$), we shall define this situation

$$< x|e^{-(t-t_0)S}|x_0 > \tag{2.1}$$

or in the Carfí's notation

$$x(t) = \int_{\mathbb{R}} x(t_0)P(x, t; x_0). \tag{2.2}$$

It is obvious that

$$P(x, t; x_0) = e^{-(t-t_0)S}.$$

The $\mathfrak{L}$ box can be thought of as the S term of the exponential. Hence, let us discover what this term S is. In few words, we want to consider a differential object by which it is possible to define a probabilisic movement without considering any path integrals. In order to do that we could consider that S is in the space of $S'_n(\mathbb{R}, \mathbb{C})$. One has to remember that a Fourier transform is a well-behaved transformation in the space of tempered distribution. The S is the heat equation

$$\frac{\partial}{\partial t}u(x, t) = k\Delta_x u(x, t). \tag{2.3}$$

In (2.3) we have the x-laplacian Δ_x and a constant k. In order to find a solution of (2.3) we have to use the Fourier Transofrm, and thus we obtain

$$\begin{aligned} &= \frac{\partial}{\partial t}u(x, t) = k\Delta_x u(x, t) \\ &= \frac{d}{dt}\mathcal{F}_x u(\epsilon, t) = -k|\epsilon|^2 \mathcal{F}_x u(\epsilon, t) \\ &= \mathcal{F}_x u(\epsilon, t) = c(\epsilon)e^{-kt|\epsilon|^2}. \end{aligned} \tag{2.4}$$

But so as to find a solution we have to use the inverse transformation of (2.4). Hence, given $c(\epsilon) = \hat{f}(\epsilon)$,

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}(e^{-kt|\epsilon|^2}\hat{f}(\epsilon)) \\ &= \mathcal{F}^{-1}(^{-kt|\epsilon|^2}) \star f \\ &= \frac{1}{(4\pi kt)^{\frac{n}{2}}} \int e^{-\frac{|x-y|^2}{4kt}} f(y) \end{aligned} \tag{2.5}$$

where $\star$ is a convolution. Now, we have to say that the more t increses the smoother $u(x,t)$ is. Moreover, the Fourier transform decreases at infinity more rapidly as t increases. Given a time $\mathtt{t} = (t - t_0)$ we obtain the probabilistic amplitude by $e^{-\mathtt{t}S}|x_0>$. We have found a Green's function $u(x,t)$ given by the inverse of our Fourier transform. Now, let us to take into account the same equation of (2.4) but in another form

$$\left(\frac{\partial}{\partial t} - k\Delta_x\right)G(x,t) = \delta(t)\delta(x), \tag{2.6}$$

thus

$$\left(\frac{\partial}{\partial t} - k|\epsilon|^2\right)\hat{G}(\epsilon,t) = \delta(t). \tag{2.7}$$

Hence, so as to find the Green's function of (2.6)

$$\begin{aligned} G(t,x) &= \mathcal{F}_x^{-1}\left(\hat{G}(\epsilon,t)\right) \\ &= \frac{1}{(2\pi)^n}\mathcal{F}_x\left(\hat{G}(-k,t)\right) \\ &= \frac{\Theta(t)}{(2\pi)^n}\int_{\mathbb{R}} e^{i\epsilon x - kt|\epsilon|^2}d\epsilon \end{aligned} \tag{2.8}$$

where Θ is the Heaviside's function. Hence we obtain the general solution

$$u(x,t) = \frac{\Theta(t)}{(8\pi kt)^{\frac{n}{2}}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}}f(y). \tag{2.9}$$

We have found (2.8) a Green's function and we can extend our idea in order to obtain a Green's family. But the pourpose of this paper is to give an introduction of this kind of study. Now, given

$$x(t) = \int_{\mathbb{R}} x(t_0)e^{-\mathtt{t}S} \tag{2.10}$$

we could say that this propagator is of ^{S}family. This statement is correct. Rather, we have a Fourier transform $\mathcal{F}_x : S'_n \to S'_n$ given that $x(t) \in S'_n$ and $e^{-\mathtt{t}S} \in S_n$. The $\mathcal{F}_x$ is a homeomorphism in the weak-$\star$topology $\sigma(S'_n)$. It is an F-space that maps an F-space that is continuous, bounded and closed [FrT]. Of course $x(t)$ belongs to the S'_n space because of the nature of the superposition operators. The $\mathcal{F}_x$ is (at least) of the space S_n is a linear transformation in S *per se*, and thus it is continuous because of the convergence in S. Rather, for every $\hat{g} \in S$ we have a unique $g \in S$ such that $\mathcal{F}[g] = \hat{g}$.

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