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Approval consensus measures

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Abstract

In many realistic group decision making problems where a "representative" collective output must be produced, it is relevant to measure how much consensus this solution conveys to the group. Many aspects influence the final decision in group decision making problems. Two key issues are the experts' individual opinions and the methodology followed to compute such a final decision (aggregation operators, voting systems, etc.). In this paper we consider situations where each member of a population decides upon approving or not approving each of a set of options. The experts express their opinions in a dichotomous way, e.g., because they intend to use approval voting. In order to measure the consensus or cohesiveness that the expression of the individual preferences conveys we propose the concept of approval consensus measure (ACM), which does not refer to any priors of the agents like preferences or other decision-making processes. Then we give axiomatic characterizations of two generic classes of ACMs.

Key words: Approval voting, Consensus measures, Tanimoto similarity index

JEL classification: D70.

1. Introduction

Social Choice and Decision Making Theories try to give answers to many daily real situations. The study, analysis, testing, ... of the way that individual preferences are aggregated in order to obtain a "representative" collective choice is an important research area. In particular, the measurement of the degree of agreement in a group has attracted growing attention although its study is often complex and controversial (because it involves the treatment of individual opinions, aggregation procedures or voting rules, and it

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is conditional on the context, etc) a handicap that is common to virtually all branches of the Social Sciences. Both aspects have been the subject of a joint treatment in [1] and [2].

In this paper we focus on the study of the degree of cohesiveness in a group decision making context when agents express their opinions in a dichotomous way, e.g. because they intend to apply *approval voting* (AV) in order to reach a collective decision. We have become interested in this context because it adapts to many real-world situations. Since the publication of Brams and Fishburn [3, 4], many organizations and scientific societies use AV, to wit, the Mathematical Association of America (MMA), the American Mathematical Society (AMS), the Institute for Operational Research and Management Sciences (INFORMS), the American Statistical Association (ASA), the Institute of Electrical and Electronic Engineers (IEEE), and other smaller societies such as the Society for Judgment and Decision Making, the Society for Social Choice and Welfare, etc. (see [5]). Moreover, many successful theoretical and empirical works on AV have studied this voting system from various points of view (see [6], [7], [8], [9], [10], [11], [12], [13], among others). The conclusion that 'the Approval Voting method is more likely to lead to a consensus vote than polarizing the electorate' (Alós-Ferrer and Granić [13, p. 173]) makes dichotomous assessments worth investigating in relation with the measurement of cohesiveness in a society.

Based on the theoretical and practical importance of this case, in our approach to the measurement of the cohesiveness experts have dichotomous opinions over the set of alternatives: they have to classify them as "acceptable" or "non-acceptable". This is in line with an earlier contribution by Erdamar et al. [14], the main difference being that here we do not make reference to any priors of the agents (like preferences or other decision-making processes). Following the approach initiated in Bosch [15], we propose and characterize some classes of measures of the consensus or cohesiveness that such expression of the individual preferences conveys. We generically refer to them as *approval consensus measures* (ACMs).

We first give an axiomatic characterization of s-Simple ACMs (that measure the probability that a randomly chosen contraction of the set of candidates to a subset with s elements produces unanimity). We then reproduce the analysis for a modification of that class of ACMs, that intends to lessen the influence of irrelevant alternatives (i.e., those whom nobody approves of). To that propose we draw inspiration from the Tanimoto similarity index (see [16] and [17]) in order to define and characterize s-Simple Tanimoto ACMs.

This similarity index and others have been applied in different fields, especially in Biogenetics (see [18], [19] and [20], for instance). As a result, this variation of the s-Simple ACMs verifies an independence of irrelevant alternatives property, which supposes a distinctive feature of s-Simple Tanimoto ACMs.

The paper is structured as follows. Section 1 is devoted to introduce basic terminology, as well as our proposal of measurement of consensus, the approval consensus measure. In Section 3 and Section 4 we set forth the characterization and properties of two families of approval consensus measures, s-Simple and s-Simple Tanimoto ACMs, respectively. Next, in Section 5 we present an illustrative example. Finally, in Section 6 we give some concluding remarks.

2. Notation and definitions

Let $X = \{x_1, ..., x_k\}$ be the finite set of k options, alternatives or candidates. It is assumed that X contains al least two alternatives, i.e., that the cardinality of X is greater or equal than 2, $|X| \geqslant 2$. Abusing notation, on occasions we refer to option x_s as option s for convenience. A population of agents or experts is a finite subset $\mathbf{N} = \{1, 2, ..., N\}$ of natural numbers.

We consider that each expert can vote for or approve of as many options, alternatives or candidates as he/she wishes, thus showing extreme and dichotomous opinions. In order to formalize these assessments we can take three alternative positions.

1. Let $\mathcal{P}(X)$ be the set of all subsets of X. For any expert $i \in \mathbf{N}$, let $B_i \in \mathcal{P}(X)$ be the set of alternatives that he/she approves of. We write $\mathbb{P} = \mathcal{P}(X)^N$ for the set of all the assessments on X, i.e., an element $B_1 \times \ldots \times B_N \in \mathbb{P}$ captures the sets of alternatives that the respective agents approve of.

2. We can also capture the dichotomous opinions of expert $i \in \mathbf{N}$ on X by means of $A_i \in \{0, 1\}^k$, i.e., component j of A_i is 1 if and only if expert i approves of alternative j. We write $\mathbb{V} = \{0, 1\}^k \times \overset{N}{\ldots\ldots} \times \{0, 1\}^k$ for the set of all dichotomous experts' opinions on X, thus $A_1 \times \ldots \times A_N \in \mathbb{V}$ captures the sets of alternatives that the respective agents approve of.

The elements of $\mathbb{P}$ can be identified with elements of $\mathbb{V}$ in a trivial manner.

3. An *approval profile* is an $N \times k$ matrix

$$M = \begin{pmatrix} M_{11} & \dots & M_{1k} \\ \vdots & \ddots & \vdots \\ M_{N1} & \dots & M_{Nk} \end{pmatrix}_{N\times k}$$

where M_{ij} is the opinion of the expert i over the alternative x_j, in the sense

$$M_{ij} = \begin{cases} 1 & \text{if expert } i \text{ approves the alternative } x_j, \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathbb{M}_{N\times k}$ for the set of all $N \times k$ matrices. Again, the elements of $\mathbb{P}$, resp. of $\mathbb{V}$, can be identified with elements of $\mathbb{M}_{N\times k}$ in a trivial manner.

For one thing, and relating to past notation, row i of the profile M can be identified with $A_i \in \{0,1\}^k$ thus it describes the dichotomous assessment of expert i over the alternatives. For another, column j of the profile M captures the experts' assessments on the alternative j. We denote it by M^j.

Any permutation σ of the experts $\{1,2,...,N\}$ determines a profile M^σ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile M. Similarly, any permutation π of the alternatives $\{1,2,...,k\}$ determines a profile ${}^{\pi}M$ by permutation of the columns of M: column i of the profile ${}^{\pi}M$ is column $\pi(i)$ of the profile M.

For each approval profile M, its restriction to a *subprofile* on the alternatives in $I \subseteq X$, denoted M^I, arises from exactly selecting the columns of M that are associated with the respective alternatives in I (in the same order). In particular, and dropping brackets for simplicity, $M^{\{j\}} = M^j$ is column j of M, and $M^{i,j}$ is the two-column submatrix of M that consists of columns i and j (in the same order). An s-restricted profile of M is the restriction of M to a subprofile on s alternatives.

Any partition $\{I_1,...,I_t\}$ of $\{1,2,...,k\}$, that we identify with a partition of X, generates a *decomposition of* M into subprofiles M^{I_1}, ... , M^{I_t}.[1]

For each approval profile M on k alternatives, by n_0 we denote the number of alternatives that all agents disapprove of, and by n_1 we denote the number

[1]A partition of a set S is a collection of pairwise disjoint non-empty subsets of S whose union is S.

of alternatives that all experts approve of. Similarly, $n = n_0 + n_1$ denotes the number of alternatives the agents are unanimous on, i.e., the number of columns of M that are constant. [2]

For convenience, $(1)_{N\times k}$ denotes the $N \times k$ matrix whose cells are universally equal to 1.

The following example illustrates the use of the previous notation.

Example 1. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives, thus $k = 4$ (we also use $X = \{1, 2, 3, 4\}$ for simplicity). We suppose a population of three agents or experts, $\mathbf{N} = \{1, 2, 3\}$, and the following approval profile M:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

This information can also be expressed by stating $B_1 = \{x_1\}$, $B_2 = \{x_1, x_2, x_3\}$ and $B_3 = X$. Or alternatively, that $A_1 = (1, 0, 0, 0)$, $A_2 = (1, 1, 1, 0)$ and $A_3 = (1, 1, 1, 1)$.

Suppose a permutation σ of the experts $\mathbf{N} = \{1, 2, 3\}$ given by

$$\begin{array}{cccc} \sigma: & \mathbf{N} & \to & \mathbf{N} \\ & 1 & & 3 \\ & 2 & & 2 \\ & 3 & & 1 \end{array}$$

, then row 1 of M^σ is row $\sigma(1) = 3$ of M, and so forth,

therefore $M^\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Suppose a permutation π of the alternatives $X = \{1, 2, 3, 4\}$ given by

$$\begin{array}{cccc} \pi: & \mathbf{X} & \to & \mathbf{X} \\ & 1 & & 2 \\ & 2 & & 3 \\ & 3 & & 4 \\ & 4 & & 1 \end{array}$$

, then column 1 of ${}^\pi M$ is column $\pi(1) = 2$ of M, and so forth,

[2]Strictly speaking, the notation $n(M)$, $n_0(M)$, $n_1(M)$ should be used in order to emphasize the dependence of these amounts on the approval profile. We believe that dropping the reference to M does not cause any confusion thus we omit it.

therefore ${}^{\pi}M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Let $I_1 = \{1,4\}$ and $I_2 = \{2,3\}$, then M is decomposed into the following two subprofiles:

$$M^{I_1} = M^{1,4} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } M^{I_2} = M^{2,3} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

An approval profile M is *unanimous* if the set of approved alternatives is the same across experts. In matrix terms, if the columns of $M \in \mathbb{M}_{N\times k}$ are constant. Equivalently: if $n = k$, or if $A_1 = ... = A_N$.

An *extension* of an approval profile M on $X = \{x_1, ..., x_k\}$ is an approval profile $\bar{M}$ on $\bar{X} = \{x_1, ..., x_k, , x_{k+1}, ..., , x_{k'}\}$ such that the restriction of $\bar{M}$ to the first k alternatives of $\bar{X}$ coincides with M.

Definition 1. An *approval consensus measure* (also ACM for simplicity) is a mapping $\mu : \mathbb{M}_{N\times k} \to [0,1]$ that assigns a number $\mu(M) \in [0,1]$ to each approval profile M, with the following properties:

i) $\mu(M) = 1$ if and only if M is unanimous.
ii) *Anonymity*: $\mu(M^{\sigma}) = \mu(M)$ for each permutation σ of the agents and $M \in \mathbb{M}_{N\times k}$
iii) *Neutrality*: $\mu({}^{\pi}M) = \mu(M)$ for each permutation π of the alternatives and $M \in \mathbb{M}_{N\times k}$

3. The class of s-Simple ACMs. Characterization and properties

In this section we analyse a class of approval consensus measures that depend on a parameter $k \geqslant s > 0$. Intuitively, they measure the probability that a randomly chosen s-restricted profile of a given profile is unanimous. Formally:

Definition 2. The s-Simple approval consensus measure is the mapping $\mathcal{C}_s : \mathbb{M}_{N\times k} \to [0,1]$ given by

$$\mathcal{C}_s(M) = \frac{n(n-1)....(n-(s-1))}{k(k-1)....(k-(s-1))} = \frac{\binom{n}{s}}{\binom{k}{s}} = \frac{C_n^s}{C_k^s} \tag{1}$$

for each approval profile M on k alternatives, where n denotes the cardinality of the set of alternatives the agents are unanimous on.

It is not difficult to check that these expressions produce approval consensus measures. The key part is that their values lie in $[0,1]$, and particularly that $\mathcal{C}_s(M) \geqslant 0$ throughout. This holds true because when $n \geqslant s$ the numerator is a product of positive numbers, and when $n < s$ one of the factors in the numerator is 0.

We proceed to explore some general properties of the class of s-Simple approval consensus measures. Afterwards we provide an axiomatic characterization in our main result in this Section, namely Theorem 1. Because particular instances of this class of approval consensus measures have specific interpretations we also perform additional *ad-hoc* analyses of the most relevant cases.

Measuring the approval consensus by means of an s-Simple ACM verifies the following properties. Let M denote an approval profile.

1. $\mathcal{C}_1(M) \geqslant \mathcal{C}_2(M) \geqslant .... \geqslant \mathcal{C}_k(M)$ because $\mathcal{C}_s(M) = 0$ when $s > n$, and for each index s such that $s < k$ and $s \leqslant n$ one has

$$\frac{\mathcal{C}_{s+1}(M)}{\mathcal{C}_s(M)} = \frac{\frac{\binom{n}{s+1}}{\binom{k}{s+1}}}{\frac{\binom{n}{s}}{\binom{k}{s}}} = \frac{n-s}{k-s} \leqslant 1 \text{ since } n \leqslant k$$

2. Reversal invariance: If we define the complementary approval profile M^c of M according to $A_i^c = X - A_i$ (in matrix terms: $M^c = (1)_{N\times k} - M$) then $\mathcal{C}_s(M) = \mathcal{C}_s(M^c)$ because the set of alternatives the agents are unanimous on does not change.
3. The consensus measure does not change if agents are replicated in any number.
4. Suppose that an agent $N+1$ is added to the society $\mathbf{N}$, and that this agent approves, resp. disapproves, of all the alternatives in X. Then the consensus measure does not rise.
   We argue for the case where agent $N+1$ disapproves of all the alternatives in X, the other case being symmetrical. The approval profile $\bar{M}$ that thus arises has n_0 unanimously disapproved alternatives and 0 unanimously approved alternatives. Therefore a comparison must be

made between the following two numbers:

$$\mathcal{C}_s(M) = \frac{C^s_{n_0+n_1}}{C^s_k}, \mathcal{C}_s(\bar{M}) = \frac{C^s_{n_0+0}}{C^s_k}$$

thus $\mathcal{C}_s(M) \geqslant \mathcal{C}_s(\bar{M})$ if and only if $C^s_{n_0+n_1} \geqslant C^s_{n_0}$.

5. Suppose that an alternative $k+1$ is added to the set of alternatives X, and that this alternative is unanimously approved, resp. disapproved, by all agents. If M is not unanimous, and the introduction of the new alternative does not affect the agents' assessments of the original alternatives, then the consensus measurement of the enlarged sets is strictly higher than the original one.
   The argument is as follows. Let $\tilde{M}$ be the profile after enlarging the set of alternatives, we want to show that $\mathcal{C}_s(M) < 1$ implies $\mathcal{C}_s(M) < \mathcal{C}_s(\tilde{M})$. Equivalently,

   $$\mathcal{C}_s(M) = \frac{C^s_n}{C^s_k} < \mathcal{C}_s(\tilde{M}) = \frac{C^s_{n+1}}{C^s_{k+1}}$$

   which after some algebra is equivalent to $k > n$. This is true because M is not unanimous.

6. Convergence to full unanimity can be established if we repeatedly introduce alternatives with the property that all agents agree on their acceptability. Formally: Suppose that alternatives $k+1, ..., k+t$ are added to the set of alternatives X, and that each alternative is either unanimously approved or unanimously disapproved by all agents. If the introduction of new alternatives does not affect the agents' assessments of past sets of alternatives, then the consensus measurement of the extended approval profiles $\tilde{M}^{(t)}$ approaches one when t tends to infinity.
   The argument is as follows. If $n = k$ the claim is trivial, so we assume $n < k$. Since $\tilde{M}^{(t)}$ is the profile after enlarging the set of alternatives with the $k+1, ..., k+t$ new alternatives, we want to show that $\lim_{t\to\infty}(\mathcal{C}_s(\tilde{M}^{(t)})) = \lim_{t\to\infty}(\frac{C^s_{n+t}}{C^s_{k+t}}) = 1$ for each $s \leqslant k, n < k$. Because

   $$\lim_{t\to\infty}(\frac{C^s_{n+t}}{C^s_{k+t}}) = \lim_{t\to\infty}\frac{\frac{(n+t)!}{(n+t-s)!}}{\frac{(k+t)!}{(k+t-s)!}} = \lim_{t\to\infty}(\frac{(k+t-s)}{(k+t)}\cdots\frac{(n+t-s+1)}{(n+t+1)})$$

and this is a finite product of constantly $k-n$ sequences that converge to 1 (when t tends to infinity), the thesis ensues.

7. The computation of the s-Simple ACM of a profile on k alternatives reduces to the average of the corresponding measures of its reductions to $(k-1)$-restricted profiles:

Proposition 1. Let $X = \{x_1, ..., x_k\}$ be a set with k alternatives, and let M be an approval profile on X. Then

$$\mathcal{C}_s(M) = \frac{1}{k}\sum_{j=1}^{k}\mathcal{C}_s(M^{-j})$$

Proof. If the agents are unanimous on exactly n alternatives of X, the neutrality property of approval consensus measures permits to assume that these alternatives are $\{x_1, ..., x_n\}$. Then

$$\begin{aligned}
\frac{1}{k}\sum_{j=1}^{k}\mathcal{C}_s(M^{-j}) &= \frac{1}{k}\left(\sum_{j=1}^{n}\mathcal{C}_s(M^{-j}) + \sum_{j=n+1}^{k}\mathcal{C}_s(M^{-j})\right) = \\
&= \frac{1}{k}\left(n\frac{C_{k-1}^{s-1}}{C_{k-1}^{s}} + (k-n)\frac{C_s^n}{C_{k-1}^{s}}\right) = \\
&= \frac{1}{k}\frac{n\frac{(n-1)!}{s!(n-1-s)!} + (k-n)\frac{n!}{s!(n-s)!}}{\frac{(k-1)!}{s!(k-1-s)!}} = \\
&= \frac{\frac{n!(n-s)}{s!(n-s)!} + \frac{n!(k-n)}{s!(n-s)!}}{\frac{k!(k-s)}{s!(k-s)!}} = \frac{\frac{n!}{s!(n-s)!}}{\frac{(k!}{s!(k-s)!}} = \frac{C_n^s}{C_k^s} = \mathcal{C}_s(M)
\end{aligned}$$

□

3.1. Necessary and sufficient conditions for the s-Simple ACM

We proceed to characterize the s-Simple ACM in terms of the following two properties:

Definition 3. An approval consensus measure μ verifies:

i) s-triviality if and only if for each approval profile M on X and each $I \subseteq X$ with cardinality s,

$$\mu(M^I) = \begin{cases} 1 & \text{if the agents are unanimous on every alternative in } I, \\ 0 & \text{otherwise.} \end{cases}$$

ii) s-reducibility if and only if for each approval profile M on X,

$$\mu(M) = \frac{1}{C_k^s} \sum_{\substack{I \subseteq X \\ |I|=s}} \mu(M^I)$$

Their respective interpretations are as follows. For a given s, s-triviality means that the application of μ to any s-restricted profile behaves in a trivial manner: the profiles have an approval consensus measure of 1 exactly when they are unanimous, the alternative being approval consensus measure of 0. As to s-reducibility, it means that the approval consensus measure of a profile is the average of the approval consensus measures of all its s-restricted profiles.

Theorem 1. Let μ be an approval consensus measure on X. Then $\mu = \mathcal{C}_s$ if and only if μ verifies s-triviality and s-reducibility.

Proof. Let $X_u \subseteq X$ denote the set of alternatives for which the agents have an unanimous opinion, thus $n = |X_u|$.

Let us first prove that $\mathcal{C}_s$ verifies s-triviality and s-reducibility for each $k \geqslant s > 0$. For each approval profile M on X and each $I \subseteq X$ with cardinality s, let $n_I \leqslant s$ denote the number of alternatives of I for which the agents have an unanimous opinion. Then $\mathcal{C}_s(M^I) = \frac{\binom{n_I}{s}}{\binom{s}{s}}$ equals 1 iff $n_I = s$, i.e., iff M^I is unanimous; and it equals 0 otherwise.

To check for s-reducibility we just need to observe that $\sum_{\substack{I \subseteq X \\ |I|=s}} \mathcal{C}_s(M^I) = C_n^s$ because s-triviality implies that this sum is precisely the number of subsets of X with cardinality s, such that the agents have an unanimous opinion on their alternatives. In other words: It is the number of combinations of n distinct elements taken s at a time. Formally:

$$\sum_{\substack{I \subseteq X \\ |I|=s}} \mathcal{C}_s(M^I) = \sum_{\substack{I \subseteq X_u \\ |I|=s}} \mathcal{C}_s(M^I) + \sum_{\substack{I \not\subseteq X_u \\ |I|=s}} \mathcal{C}_s(M^I) = C_n^s + 0$$

Conversely, let μ be an approval consensus measure that verifies s-triviality and s-reducibility. Due to s-triviality,

$$\sum_{\substack{I \subseteq X \\ |I|=s}} \mu(M^I) = |\{I \subseteq X_u : |I| = s\}| = C_n^s$$

and now s-reducibility yields

$$\mu(M) = \frac{C_n^s}{C_k^s} = \mathcal{C}_s(M)$$

□

For focal instances of s we obtain special cases of s-simple approval consensus measures that we proceed to investigate further. The case $s = 1$ will be called the *simple ACM*: it measures the probability that the agents unanimously agree on a randomly chosen alternatives (be it approved or not). The case $s = k$ will be called the *trivial ACM*: it is equal to 1 when the profile is unanimous, and 0 otherwise. In between we have the case $s = 2$, that we call the *Pareto ACM.*

3.2. The simple ACM. Futher properties

The particularization of the s-Simple ACM when $s = 1$ yields $\mathcal{C}_1(M) = \frac{n}{k}$ for each approval profile M on a set with k alternatives. It measures the probability that all agents unanimously approve/disapprove of a randomly selected alternative. This has been called the simple approval consensus measure. We proceed to give an alternative characterization of it in terms of the following property:

Definition 4. An approval consensus measure μ verifies *convexity* if and only if for each approval profile M on X, and each decomposition of M into two subprofiles M_1 and M_2,

$$\mu(M) = \frac{k_1\mu(M_1) + k_2\mu(M_2)}{k}$$

A routine checking shows that $\mathcal{C}_1$ verifies convexity. This property means that the measure of a profile is a weighted average of the measures of any decomposition into subprofiles, the weights being given by the respective relative sizes of the subprofiles.

We are in a position to establish the following necessary and sufficient conditions for the simple approval consensus measure:

Theorem 2. Let μ be an approval consensus measure on X. The following statements are equivalent:

1. μ is the simple approval consensus measure.
2. μ verifies 1-triviality and 1-reducibility.
3. μ verifies 1-triviality and convexity.

Proof. Due to Theorem 1, we only need to check that conditions 1 and 3 are equivalent. We already know that 1 implies 3. Let us assume that μ verifies 1-triviality and convexity. We proceed by induction on k to prove $\mu(M) = \frac{n}{k}$ for each approval profile M on a set with k alternatives. The case $k = 1$ holds by 1-triviality. Assume that the statement is true for sets with k alternatives or lesser. Let $X = \{x_1, ..., x_k, x_{k+1}\}$ be a set with $k + 1$ alternatives, and let M be an approval profile on X. Denote by M_1 the restriction of the profile M to $\{x_1, ..., x_k\}$. Convexity assures

$$\mu(M) = \frac{k\,\mu(M_1) + \mu(M^{k+1})}{k+1}$$

and the induction hypothesis yields

$$\mu(M) = \frac{k\,\mathcal{C}_1(M_1) + \mathcal{C}_1(M^{k+1})}{k+1}$$

Because $\mathcal{C}_1$ verifies convexity, the latter expression boils down to $\mu(M) = \mathcal{C}_1(M)$. □

3.3. The Pareto ACM. Futher properties

The particularization of the s-Simple ACM when $s = 2$ yields $\mathcal{C}_2(M) = \frac{n(n-1)}{k(k-1)}$ for each approval profile M on a set with k alternatives. It measures the probability that a shrink of the set of alternatives to two randomly chosen alternatives produces full consensus among the agents. This is related to a possible adapted variation of Bosch's Pareto measure [15, pp. 81-82], which "is based on the number of pairs on which the voters agree": here we interpret that the voters agree on x_i and x_j when they are unanimous on x_i and also on x_j, i.e., when the voters have exactly the same opinion about which of the two alternatives must be approved.

A routine checking proves the following relationships between $\mathcal{C}_1$ and $\mathcal{C}_2$ for each approval profile M on a set with k alternatives,

$$\mathcal{C}_2(M) = \mathcal{C}_1(M)\frac{k\,\mathcal{C}_1(M) - 1}{k-1}$$

$$\mathcal{C}_1(M) = \frac{1 - \sqrt{4\,\mathcal{C}_2(M)\,k\,(k-1)+1}}{2k}$$

Convexity explains the behavior of the simple approval consensus measure when a new alternative is added without affecting the agent's opinion on the original alternatives. In order to state the corresponding exact relation for the Pareto ACM we prove the following result:

Proposition 2. Let $X = \{x_1, ..., x_k\}$ be a set with k alternatives, and let M be an approval profile on X such that the agents are unanimous on n alternatives. Denote by $\bar{M}$ an extension of the profile M to $\bar{X} = \{x_1, ..., x_k, x_{k+1}\}$. Then

$$\mathcal{C}_2(\bar{M}) = \frac{k(k-1)\mathcal{C}_2(M) + 2n\mathcal{C}_1(\bar{M}^{k+1})}{k(k+1)} = \frac{C_k^2\,\mathcal{C}_2(M) + n\,\mathcal{C}_1(\bar{M}^{k+1})}{C_{k+1}^2}$$

Proof. The second equality is trivial. In order to check the first equality we recall $\mathcal{C}_2(M) = \frac{n(n-1)}{k(k-1)}$. If the agents are unanimous on x_{k+1} one has $\mathcal{C}_2(\bar{M}) = \frac{n(n+1)}{k(k+1)}$ and our claim becomes

$$\mathcal{C}_2(\bar{M}) = \frac{k(k-1)\mathcal{C}_2(M) + 2n}{k(k+1)}$$

which can be checked easily. Otherwise our claim becomes

$$\mathcal{C}_2(\bar{M}) = \frac{k(k-1)\mathcal{C}_2(M) + 0}{k(k+1)}$$

which holds true too because now $\mathcal{C}_2(\bar{M}) = \frac{n(n-1)}{k(k+1)}$. □

4. The class of *s*-Simple Tanimoto ACMs. Characterization and properties

In this section we explore a variant of s-Simple ACMs that satisfies an independence of irrelevant alternatives criterion. It is inspired in the Tanimoto similarity index. At this point we have to introduce some additional notations. An alternative x_j is called *irrelevant* on profile M if all agents disapprove it of, i.e $M_{ij} = 0$ for all $i \in \mathbf{N}$, otherwise it is relevant. An approval profile M is *irreducible* if it lacks for irrelevant alternatives. That means that each alternative is approved of by at least one agent. Given a non-unanimous approval profile M, we denote by M_R its unique irreducible subprofile, i.e., M_R arises from M after removing irrelevant alternatives.

We now introduce the class of Simple Tanimoto approval consensus measures that depend on a parameter s. Intuitively, for each non-unanimous profile they measure the probability that a randomly chosen s-set of relevant alternatives are approved by all agents. Formally:

Definition 5. The s-Simple Tanimoto ACM is the mapping $\mathcal{T}_s : \mathbb{M}_{N\times k} \to [0,1]$ given by

$$\mathcal{T}_s(M) = \begin{cases} 1 & \text{if } M \text{ is unanimous} \\ 0 & \text{if } M \text{ is not unanimous and } k - n_0 < s \\ \frac{C^s_{n_1}}{C^s_{k-n_0}} & \text{otherwise} \end{cases} \tag{2}$$

for each approval profile M on k alternatives.

Since $k - n_0 \geqslant n_1$ it is immediate to check that the above expresion provides an approval consensus measure.

Now we proceed as in the previous section: we proceed to enumerate some general properties of the class of s-Simple Tanimoto ACMs and then we give their axiomatic characterization. Therefore, let M denote an approval profile.

1. $\mathcal{T}_1(M) \geqslant \mathcal{T}_2(M) \geqslant \ldots \geqslant \mathcal{T}_k(M)$. If M is unanimous, $k - n_0 < s$ or $n_1 < s$, then the chain of inequalities is trivial. Otherwise $s \leqslant n_1 < k - n_0$ yields
$$\frac{\mathcal{T}_s(M)}{\mathcal{T}_{s-1}(M)} = \frac{n_1 - s + 1}{k - n_0 - s + 1} \leqslant 1.$$
2. $\mathcal{C}_s(M) \geqslant \mathcal{T}_s(M)$. We only discuss non-trivial cases, that is, $s \leqslant n_1 < k - n_0$. Exploiting the inequality $\frac{x}{y} > \frac{x+1}{y+1}$ with $x, y > 0$ and $x < y$, we obtain
$$\frac{n-i}{k-i} > \frac{n-n_0-i}{k-n_0-i} = \frac{n_1-i}{k-n_0-i} \qquad \text{for } 0 \leqslant i \leqslant s-1.$$
From this, the assertion is straightforward.
3. The consensus measure does not change if agents are replicated in any number.

4. Suppose that M is not unanimous, and that an agent $N+1$ is added to the society $\mathbf{N}$. If this agent disapproves of all the alternatives in X, then the consensus measure is zero. It suffices to observe that the number of approved alternatives by unanimity is zero.
   On the other hand, if the new agent approves of all the aternatives in X the consensus measure does not rise. Let $\hat{M}$ be the profile after adding the new agent, then note that $\hat{n}_0 = 0 \leqslant n_0$, $\hat{n}_1 = n_1$ and $\hat{k} = k$. We have either $k - n_0 < s$ or $k - n_0 \geqslant s$. In the first case, since M is not unanimous, we get $\hat{n}_1 = n_1 < k - n_0 < s$ and then $\mathcal{T}_s(\hat{M}) = 0 = \mathcal{T}_s(M)$. In the second case, a simple computation arrives at:

$$\begin{aligned}\mathcal{T}_s(\hat{M}) &= \frac{n_1(n_1-1)\dots(n_1-s+1)}{k(k-1)\dots(k-s+1)} \\ &\leqslant \frac{n_1(n_1-1)\dots(n_1-s+1)}{(k-n_0)(k-n_0-1)\dots(k-n_0-s+1)} = \mathcal{T}_s(M).\end{aligned}$$

5. Suppose that M is not unanimous, that an alternative $k+1$ is added to set of alternatives X, and that this alternative is unanimously approved of by all agents. If the introduction of the new alternative does not affect the agents' assessments of the original alternatives, then the consensus measurement of the enlarged sets does not decrease.
   Let $\hat{M}$ be the elarged profile, we then have $\hat{n}_0 = n_0$, $\hat{n}_1 = n_1 + 1 < s$ and $\hat{k} = k + 1$. We distinguish two cases:
   - Case $0 < k - n_0 < s$ or $n_1 < s$. This implies $\mathcal{T}_s(M) = 0$ and then $\mathcal{T}_s(\hat{M}) \geqslant \mathcal{T}_s(M)$.
   - Case $k - n_0 \geqslant s$ and $n_1 \geqslant s$. This implies $\hat{k} - \hat{n}_0 > s$ and $\hat{n}_1 > s$. A simple computation gives

$$\begin{aligned}\mathcal{T}_s(\hat{M}) =& \frac{C^s_{\hat{n}_1}}{C^s_{\hat{k}-\hat{n}_0}} = \frac{C^s_{n_1+1}}{C^s_{k-n_0+1}} \\ =& \frac{n_1+1}{k-n_0+1}\frac{k-n_0+1-s}{n_1+1-s}\mathcal{T}_s(M).\end{aligned} \tag{3}$$

   Now, using the fact that $\frac{x}{y} > \frac{x+1}{y+1}$ when $x, y > 0$ and $x > y$, we infer

$$\frac{k-n_0+1-s}{n_1+1-s} > \frac{(k-n_0+1-s)+1}{(n_1+1-s)+1} > \dots > \frac{k-n_0+1}{n_1+1} \tag{4}$$

Combining (3) and (4) we obtain the desired assertion $\mathcal{T}_s(\hat{M}) > \mathcal{T}_s(M)$.

6. The s-Simple Tanimoto ACM satisfies an independence of irrelevant alternatives criterion. Suppose that M is not unanimous, that an alternative $k+1$ is added to set of alternatives X, and that this alternative is unanimously disapproved of by all agents. Then the consensus measurement does not change. This is trivial because $\hat{n}_1 = n_1$ and $\hat{k} - \hat{n}_0 = k - n_0$.
7. Convergence to full unanimity can be established if we repeatedly introduce alternatives that are unanimously approved. Formally: Suppose that alternatives $k+1, ..., k+t$ are added to the set of alternatives X, and that each alternative is unanimously approved by all agents. If the introduction of new alternatives does not affect the agents' assessments of past sets of alternatives, then the consensus measurement of the extended approval profiles $\tilde{M}^{(t)}$ approaches one when t tends to infinity. The argument is analogous to that of section 3 and so omitted.

We now characterize the s-Simple Tanimoto ACMs. We first introduce the folowing definitions.

Definition 6. An approval consensus measure μ verifies:

i) *Independence of irrelevant alternatives* if and only if $\mu(M) = \mu(M_R)$ for any non-unanimous profile.
ii) *s-reducibility on irreducible profiles* if μ is a s-reducible measure on the set of irreducible approval profiles, that is: for each irreducible approval profile M on X
$$\mu(M) = \frac{1}{C_k^s} \sum_{\substack{I \subseteq X \\ |I|=s}} \mu(M^I)$$
iii) *s-nullity* if and only if the consensus measurement of any non unanimous profile that approves of less than s alternatives is zero.

The first property reveals that the unanimously disregarded alternatives do not play any role in the consensus measurement. The second one is a weak version of s-reduciblity at Section 5.1. Combining both properties we infer that the consensus measurement of a profile M only depends on its s-restricted profiles that are irreducible. The last property means that the consensus only can be positive if the agents approve of at least s alternatives.

Theorem 3. Let μ be an approval consensus measure on X. Then $\mu = \mathcal{T}_s$ if and only if μ verifies s-triviality, independence of irrelevant alternatives, s-reducibility on irreducible profiles and s-nullity.

Proof. Given an approval profile, let $X_u(M), X_1(M) \subseteq X(M)$ be the set of alternatives for which the agents have an unanimous opinion and the set of approved alternatives by unanimity, respectively. Thus $n = |X_u(M)|$, and $n_1 = |X_1(M)|$.

Let us first prove that $\mathcal{T}_s$ verifies the four properties above. For each approval profile M and each $I \subseteq X$ with cardinality s, let n_0^I and n_1^I be the number of alternatives of I that all agents disapprove of and approve of, respectively. If M^I is unanimous then it is obvious that $\mathcal{T}_s(M^I) = 1$. If M^I is not unanimous, we have either $n_0^I > 0$ or $n_0^I = 0$. In the first case it must be the case that $|I| - n_0^I < s$ and then $\mathcal{T}_s(M^I) = 0$. In the second case we deduce that $n_1^I < s$, thus $C^s_{n_1^I} = 0$. It again implies $\mathcal{T}_s(M^I) = 0$.

To prove that $\mathcal{T}_s$ satisfies the independence of irrelevant alternatives is enought to note that an approval profile M and its associated irreducible profile M_R have the same number of approved alternatives and the same number of approved alternatives by unanimity.

Note that s-nullity is a simple consequence of the definition of $\mathcal{T}_s$.

We finally check for s-reducibility on irreducible profiles. Let M be an irreducible profile, since $n_0 = 0$ and $X_u(M) = X_1(M)$ we deduce by s-triviality that

$$\mathcal{T}_s(M) = \frac{1}{C_k^s} \sum_{\substack{I \subseteq X(M) \\ |I| = s}} \mathcal{T}_s(M^I).$$

Conversely, let μ be an approval consensus measure that satisfies the four properties above. We can assume that M is not unanimous, because in other case any measure provides consensus one. By independence of irrelevant alternatives we have

$$\mu(M) = \mu(M_R),$$

where the cardinality of the set of evaluated alternatives in M_R is $k - n_0$. If $k - n_0 < s$ by s-nullity we deduce $\mu(M) = 0 = \mathcal{T}_s(M)$.

We now analyse the case $k - n_0 \geqslant s$. Due to s-reducibility on irreducible profile we infer

$$\mu(M_R) = \frac{1}{C^s_{k-n_0}} \sum_{\substack{I \subseteq X(M_R) \\ |I| = s}} \mu(M_R^I).$$

Since $X_u(M_R) = X_1(M_R)$, using s-triviality we obtain

$$\sum_{\substack{I \subseteq X(M_R) \\ |I|=s}} \mu(M_R^I) = C_{n_1}^s,$$

and we then arrive at the desired assertion

$$\mu(M) = \frac{C_{n_1}^s}{C_{k-n_0}^s} = \mathcal{T}_s(M).$$

□

5. Illustrative example

In order to performe a practical exploration of the behavior of our consensus measures proposals, in this section we develop an illustrative example that considers different separate scenarios to illustrate some differences between s-Simple approval consensus measures (s-SACM) and s-Simple Tanimoto consensus measures (s-STACM). Moreover, different values of the parameters are considered in this fictitious exercise.

We suppose, without loss of generality, an imaginary small experts committee of 16 members, $X = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P\}$. This committee has to elect a president among its members. The voting system selected to carry out this choice is approval voting. Thus each member of the committee should pick as many candidates as she/he considers good fits for the position. In addition, throughout the example we assume that J and N are two major candidates.

We recall that n_0 represents the number of candidates that all committee members disapprove, n_1 represents the number of candidates that all committee members approve and that our proposals of approval consensus measures are computed by means of Equations (1) and (2).

We now explore in detail four real possible scenarios.

5.1. First scenario

In the simplest non-trivial scenario, there are only two candidates ($k = 2$), namely, the set of options is $X_1 = \{J, N\}$. The corresponding approval profile is M_1 (see Figure 1). We note $n_0 = 0$ and $n_1 = 0$ in this case. Table 1 gives the recount of the ballots which are represented by the approval profile M_1.

In this first scenario, the two favorites candidates obtain similar results. The degree of agreement among the experts about the candidates is zero for all values of the s parameter and for each ACM under inspection.

Table 1: Approval voter breakdown (1st scenario)

	Candidates	
	J	N
Approval voting results	9	8

5.2. Second scenario

In this case we have the same two favorites candidates, but now a third candidate, the G candidate joins the list. Then, $X_2 = \{J, N, G\}$ is the set of the alternatives. The G candidate does not have any real chance of winning the election as we can observe in the approval profile M_2 (see Figure 1) and Table 2 .

Table 2: Approval voter breakdown (2nd scenario)

	Candidates		
	J	N	G
Approval voting results	9	8	0

The number of candidates approved or disapproved for all committee members are $n_1 = 0$ and $n_0 = 1$, respectively.

We can note in Table 3 that the degree of agreement among the experts for s-SACM and $s = 1$ is equal to $\frac{1}{3}$ but it is zero for s-STACM. It is clear that at least the experts agree on all hands that the candidate G is not a "good" candidate for such a position.

Table 3: Consensus measures for 3 alternatives

	s				
	1	2	3	4	...
s-SACM	$\frac{1}{3}$	0	0	0	0
s-STACM	0	0	0	0	0

5.3. Third scenario

Suppose a third scenario where there are four candidates, the J, N, G and F candidates, thus the set of options is $X_3 = \{J, N, G, F\}$. The F candidate exposes a mix-candidacy between J's and N's ideas. Additionally, F is totally hostile to G's ideas. In this voting setting, M_3 is the updated approval profile (see Figure 1) and Table 4 gathers the approval voter breakdown.

Table 4: Approval voter breakdown (3rd scenario)

	Candidates			
	J	N	G	F
Approval voting results	6	4	0	16

As we can observe in Table 4 there is a candidate that has been accepted by all committee members ($n_1 = 1$), namely the J candidate, and one candidate has been rejected by all committee members ($n_0 = 1$), namely the G candidate.

The relevant alternatives are J, N and F. Table 5 collects the measurement given by the s-Simple and s-Simple Tanimoto ACMs.

Table 5: Consensus measures for 4 alternatives

	s				
	1	2	3	4	...
s-SACM	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0
s-STACM	$\frac{1}{3}$	0	0	0	0

5.4. Fourth scenario

Finally, we are going to suppose that after calling for nominations, nobody is running for president. Then, following statutory provision, all committee members are appointed as candidates. In this case, $k = 16$ and the alternatives set is X. Table 6 gives the corresponding approval voter breakdown.

$$
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \quad
M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad
M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}
$$

Figure 1: Approval profiles for the 1st, 2nd and 3rd scenarios

We can observe that there are six candidates that have been disapproved by all members, $n_0 = 6$ and there is only one candidate approved for all members, $n_1 = 1$.

Table 6: Approval voter breakdown (4th scenario)

	Candidates															
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
Results	1	1	0	0	0	16	0	1	4	12	2	1	6	8	0	0

The numerical results are shown in Table 7.

6. Concluding remarks

We have explored the problem of measuring the degree of cohesiveness in a group decision making setting where experts express their opinions in a dichotomous way. To that purpose, we defined the concept of approval consensus measures (ACMs) and gave the first necessary and sufficient con-

Table 7: Consensus measures for 16 alternatives

	s						
	1	2	3	4	5	6	...
s-SACM	$\frac{7}{16}$	$\frac{21}{120}$	$\frac{35}{560}$	$\frac{35}{1820}$	$\frac{21}{4368}$	$\frac{7}{8008}$	0
s-STACM	$\frac{1}{10}$	0	0	0	0	0	0

ditions that characterize relevant primary classes of ACMs, namely s-Simple ACMs and s-Simple Tanimoto ACMs.

In this initial contribution to the topic we have ellaborated on approval consensus measures that adapt to simple real-world situations. An obvious future development is the study and characterization of other consensus measures that can be used to analyze complex situations more faithfully. Moreover, our approach (or other similar proposals) could be applied in conflict resolution as a component of a decision aid system. But this would be the topic of another paper.

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References

[1] J. C. R. Alcantud, R. de Andrés Calle, J. M. Cascón, Measurement of consensus with a reference, Journal of Microeconomics Forthcoming.

[2] J. C. R. Alcantud, R. de Andrés Calle, J. M. Cascón, A unifying model to measure consensus solutions in a society, Mathematical and Computer Modelling (2011) doi:10.1016/j.mcm.2011.12.020.

[3] S. Brams, P. C. Fishburn, Approval voting, American Political Science Review 72 (1978) 831–847.

[4] S. Brams, P. C. Fishburn, Approval Voting, Birkhauser, Boston, 1983.

[5] J. Laslier, M. R. Sanver (Eds.), Handbook on Approval Voting, Springer-Verlag, Berlin Heidelberg, 2010.

[6] C. Alós-Ferrer, A simple characterization of approval voting, Social Choice and Welfare 27 (2006) 621–625.

[7] W. Cox, Electoral equilibrium under approval voting, American Journal of Political Science 29 (1985) 112–118.

[8] M. Lines, Approval voting and strategy analysis: A Venitian example, Theory and Decision 20 (1986) 155–172.

[9] R. Weber, Approval voting, The Journal of Economic Perspectives 9 (1995) 39–49.

[10] J. Laslier, Analysing a preference and approval profile, Social Choice and Welfare 20 (2003) 229–242.

[11] M. Vorsatz, Approval voting on dichotomous preferences, Social Choice and Welfare 28 (2007) 127–141.

[12] J. Massó, M. Vorsatz, Weighted approval voting, Economic Theory 36 (2008) 129–146.

[13] C. Alós-Ferrer, D.-G. Granić, Two field experiments on approval voting in Germany, Social Choice and Welfare 39 (2012) 171–205.

[14] B. Erdamar, J. L. García-Lapresta, D. Pérez-Román, M. R. Sanver, Measuring consensus in a preference-approval context, Information Fusion (2012) doi:10.1016/j.inffus.2012.02.004.

[15] R. Bosch, Characterizations of voting rules and consensus measures, Ph.D. thesis, Tilburg University (2005).

[16] T. T. Tanimoto, An elementary mathematical theory of classification and prediction, Tech. rep., IBM (1958).

[17] D. Rogers, T. Tanimoto, A computer program for classifying plants, Science 132 (1960) 1115–1118.

[18] M. Fligner, J. Verducci, P. Blower, A modification of the Jaccard-Tanimoto similarity index for diverse selection of chemical compounds using binary strings, Technometrics 44 (2002) 110–119.

[19] L. Touil, F. Guesmi, K. Fares, A. Ferchichi, Genetic diversity of some mediterranean populations of the cultivated alfalfa (*Medicago sativa* L.) using ISSR markers, Biotechnology 7 (2008) 808–812.

[20] D. Rodríguez-Baena, A. Pérez-Pulido, J. S. Aguilar-Ruiz, A biclustering algorithm for extracting bit-patterns from binary datasets, Bioinformatics 27 (2011) 2738–2745.