From the Bochner integral to the superposition integral

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The nature of the $\mathcal{S}$-Linear Algebra: From the Bochner integral to the superposition integral

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Abstract

The purpose of this paper is to give another definition of the superposition integral by using another kind of reasoning. In particular we define it as a generalization of the Bochner integral. We shall see it in section 2 and the superposition integral in section 3.

1 Introduction

The convex analysis is one of the most important branches of mathematics, it is the fundamental tool in optimization theory and it is a very fascinating field of study. In this paper we shall see an elegant generalization of this subject. In particular we want to clarify the relation between the Bochner integral and the Carfí's superposition theory. In order to have a first glance at the theory as a whole you should be somewhat intrested to read [FrT] and [CfS]. This paper is by no means a complete treatment of convex puzzles, thus if one has no idea whatsoever on this subject it would be better off studying the convex analysis' framework on a textbook which deals with.

2 The Bochner integral

We denote by X a Banach space and by $(\mathbb{P}, \varepsilon, \mu)$ a measure space. Moreover we shall consider functions of the type $f : \mathbb{P} \to X$ and we denote by $|| \cdot ||$ the norm of X. A tool of utmost importance is the notion of *simple functions*. We denote by $\zeta(\mathbb{P}, X)$ the space of simple functions, those functions such that, for any $\varphi : \mathbb{P} \to X$, φ takes a finite value $x_1, \dots x_k \in X$ and $\forall$

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$i = 1, \dots k$ the sets $P_i = \{u \in \mathbb{P} : \varphi(u) = x_i\}$ belong to the σ-algebra ε. It is important to notice that $\mu(P_i) < \infty$.

Definition 2.1. It is called *simple* a function $\varphi : \mathbb{P} \to X$ such that $\exists$ a finite sequence $P_i \subset \mathbb{P}$ of measurable sets such that $P_i \cap P_m = \emptyset$ for $k \neq m$ and

$$\mathbb{P} = \bigcup_{i=1}^{k} P_i \qquad \text{where} \qquad \varphi(u) = w_i \in X \ \forall u \in P_i.$$

Lemma 2.2. *[BaC] A function, whose restrictions are measurable for each set of countable family of disjoint sets is measurable.*

Proof. Given a family of countable disjoit sets $(F_i)_{i=1,2,\dots}$ and a sequence $i(f_{i,k})_{k=1,2,\dots}$ of locally constant functions which converges toward f in $P_i - N_i$, where $\mu(\cup_{i=1}^{\infty} N_i) = 0$. Then

$$f(u)_i = \begin{cases} f(u)_{i,n} & \text{for} \ \ u \in P_i - N_i; \quad (i \leq n) \\ 0 & \text{for} \ \ u \notin P_i - N_i \end{cases}$$

is defined

$$g(u) = \lim_{n \to \infty} f(u)_n \ \ \text{then} \ \ g(u) = f(u) \ \ \text{for} \ \ u \in P_i - N_i,$$

thus g is measurable. $\square$

A very important theorem is the *Ergoff's theorem*

Theorem 2.3. *We define by E a measurable set of finite misure and we denote by $\{f_n\}$ a sequence of a.e. finite valued measurable function which converges a.e. on E to a finite valued measurable function f. For every $\epsilon > 0$ $\exists$ a measurable $G \subset E$ such that $\mu(G) < \epsilon$ and such that the sequence $\{f_n\}$ converges uniformly on $E - G$.*

Proof. [Hal] If we omit a set of measure zero, we can assume that $\{f_n\} \to f$ everywhere on E. If

$$E_n^m = \bigcap_{i=n}^{\infty} \left\{ x : |f_i(u) - f(u)| < \frac{1}{m} \right\}$$

then

$$E_1^m \subset E_2^m \subset \cdots$$

and since $\{f_n\} \to f$ on E

$$E \subset \lim_n E_n^m$$

for every $m = 1, 2, \cdots$. Hence $\lim_n \mu(E - E_n^m) = 0$ [Hal]. Thus $\exists$ a $k > 0 \in \mathbb{N}^+$ where $k = k(m)$ such that

$$\mu(E - E_{k(m)}^m) < \frac{\epsilon}{2^m}.$$

Following [Hal], if

$$G = \cup_{m=1}^{\infty} (E - E_{k(m)}^m),$$

then G is a measurable set, $G \subset E$, and

$$\mu(G) = \mu\Bigg(\bigcup_{m=1}^{\infty}(E - E^m_{k(m)})\Bigg) \leq \sum_{m=1}^{\infty}\mu((E - E^m_{k(m)})).$$

It is important to notice that $E - G = E \cap_{m=1}^{\infty} E^m_{k(m)}$ and that $n \geq k(m)$. Thus, for $u \in E - G$, we have $u \in E^m_n$, therefore $|f_n(u) - f(u)| < \frac{1}{m}$. The uniform convergence on $E - G$ is proved. $\square$

If we proceed to compute the theorem 2.3 for every restriction of $f(u)$ on $G, G_1, G_2 \dots$, we shall obtain the measurability of the restriciton of $f(u)$ to a sequence of disjoint sets which cover $S \setminus 0$.

Definition 2.4. A function $\varphi : \mathbb{P} \to X$ is called *strong* measurable if $\exists \{\varphi_n\}_{n\in\mathbb{N}} \subseteq \zeta(\mathbb{P}, X)$ such that

$$\varphi_n(u) \to f(u) \quad \forall u \in \mathbb{P}.$$

Definition 2.5. A function f is called *weakly* measurable if for every $A \in X'$, the function $u \to Af(u)$ is measurable on $\mathbb{P}$.

It follows that $u \to ||f(u)||$ is measurable on $\mathbb{P}$. It is important to notice that the strong measurability implies that, given an open $C \subseteq X$, the counter-image $F^{-1}(C) \in \varepsilon$.

Definition 2.6. Given a function $f : \mathbb{P} \to X$ that is strong measurable, then its integral exists and it is finite

$$\int_{\mathbb{P}} ||f(\cdot)||d\mu < +\infty$$

that is to say it is *summable*.

Thus given a function f (definition 2.6) $\exists$ a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \zeta(\mathbb{P}, X)$ such that

$$\lim_{n\to\infty}\int_{\mathbb{P}} ||f(\cdot) - \psi_n(\cdot)||d\mu = 0,$$

hence the sequence $\{\int_{\mathbb{P}} \psi_n d\mu\}_{n\in\mathbb{N}}$ is of Cauchy in X, rather, following [AcP]

$$\begin{aligned} \left\|\int_{\mathbb{P}} \psi_n d\mu - \psi_m d\mu\right\| &\leq \int_{\mathbb{P}} ||\psi_n(\cdot) - \psi_m(\cdot)||d\mu \leq \\ \leq \int_{\mathbb{P}} ||\psi_n(\cdot) - f(\cdot)||d\mu &+ \int_{\mathbb{P}} ||f(\cdot) - \psi_m(\cdot)||d\mu \to 0 \quad \text{for} \quad n, m \to \infty. \end{aligned} \tag{2.1}$$

The importance of the completeness (2.1) of X stems from the property that, because of it, the sequence converges on X and thus its limit can be thought of as the integral of f on $\mathbb{P}$.

Definition 2.7. [AcP] Given a measurable space $(\mathbb{P}, \varepsilon, \mu)$ and a summable function $f : \mathbb{P} \to X$, where X is a Banach space, the Bochner integral of f on $\mathbb{P}$ is the X's element defined by

$$\int_{\mathbb{P}} f d\mu = \lim_{n\to\infty}\int_{\mathbb{P}} \psi_n d\mu. \tag{2.2}$$

Now we have to recall an important definition

Definition 2.8. Given a subset $E \subseteq X$, we call the function χ_E, *characteristic function* of the set E if it is defined by

$$\chi_E(u) = \begin{cases} 1 & \text{for} \quad u \in E \\ 0 & \text{for} \quad u \notin E \end{cases}$$

The correspondece between sets and their characteristic function is one to one, and all properties of sets and set operations may be expressed by means of characteristic functionns [Hal]. If $\mathbb{P} \in \varepsilon$, then

$$\int_{\mathbb{P}} f d\mu = \int_{\mathbb{P}} f \chi_{\mathbb{P}} d\mu. \tag{2.3}$$

The (2.3) is a straightforward link to the superposition concept (section 3). And it is the core notion of this paper, rather I am not interested in studying the Bochner integral's properties, but its link to the Carfí's superposition.

3 The superposition integral

In section 2 we have seen a generalization of the Lebesgue integral, in particular the Bochner integral can be thought of as an integral of a function with values in a Banach space w.r.t. a scalar measure. Thus, we take the values as a limit of integrals of simple functions. Now, I have to give a definition of integral which generalize that of Bochner one: the *superposition* integral. We need some definitions (one might peruse [FrT] and [CfS]).

Definition 3.1. We denote by $\mathcal{S}_n \equiv \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ the Schwartz space, that is to say the set made of smooth functions of class C^∞ from $\mathbb{R}^n$ to $\mathbb{C}$ whose functions and derivative are rapidly decreasing at infinity, that is, they tend to zero at $\pm\infty$ faster than the reciprocal of any polynomial

$$\mathcal{S}_n \equiv \mathcal{S}(\mathbb{R}^n, \mathbb{C}) = \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) : \forall \alpha, \beta \in \mathbb{N}_0^n \lim_{|x| \to \infty} |x^\beta D^\alpha f(x)| = 0\} \tag{3.1}$$

Definition 3.2. [FrT] The natural topology of $\mathcal{S}$-spaces, $\mathcal{S}_\tau$, is given by a sequence of seminorms given by

$$||f||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty. \tag{3.2}$$

Thus the $\mathcal{S}$-space is a Fréchet space, hence it is metrizable, rather it has in 0 a countable neighborhood basis and thus, it is generated by some countable families of seminorms [FrT].

Definition 3.3. We denote by $\mathcal{S}'_n \equiv \mathcal{S}'(\mathbb{R}^n \mathbb{C})$ the space of *tempered distribution* from $\mathbb{R}^n$ to $\mathbb{C}$. It is the topological dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_\tau)$. From the [FrT]: a locally integrable function can be a tempered distribution if, for some constants a and C

$$\int_{|x| \leq G} |f(x)| dx \leq a G^C, \quad G \to \infty,$$

thus

$$\int_{\mathbb{R}^n} |f(x)\varphi(x)|dx < \infty \quad \forall \varphi \in S' \tag{3.3}$$

and we can say that $\int_{\mathbb{R}^n} f(x)\varphi(x)dx$ is a tempered distribution.

In (3.3) there are two functions $f(x)$ and $\varphi(x)$. The former is a locally integrable function and the latter is called *test function.* It is obvious that ordinary summable functions are included in these spaces, and thus we can compute the integral of the product of such functions with test functions. This kind of function can be intuitively thought of as a sufficently *good* function by which it is possible to integrate singular and ordinary functions. We denote by $\mathcal{K}$ the set of all real functions $\varphi(x)$ with bounded support and continuous derivatives of all orders vanishing outside of some bounded regions. Thus $\mathcal{K}$ is the space of test functions. Now, following [Gel] we can say that f is a continuous linear functional on $\mathcal{K}$ if $\exists$ some rule according to which one can associate with every $\varphi(x) \in \mathcal{K}$ a real number (f, φ) such that

i f is linear;

ii if a sequence $\{\varphi_n(x)\} \to 0$ then $\{(f_n, \varphi_n)\} \to 0$. (continuity of f).

We have defined, albeit in general, some very important and fundamental definitions. Now we have to define the superposition integral.

Definition 3.4. Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}$ (see [CfS] and [FrT]). The operator generated by the family v is

$$\hat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi).$$

It is linear and map

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \hat{v}.$$

Where $\mathrm{Hom}(\mathcal{S}_n, \mathcal{S}_m)$ denotes the set of all linear operators from $\mathcal{S}_n$ to $\mathcal{S}_m$ [FrT].

Theorem 3.5. *[CfQ] Let $a \in \mathcal{S}'_m$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-family. Then, the composition $u = a \circ \hat{v}$, the function $u : \mathcal{S}_n \to \mathbb{K} : \psi \mapsto a(\hat{v}(\psi))$, is a tempered distribution.*

We have seen (theorem 3.5) that v is *summable* w.r.t. a linear functional $a \in \mathcal{S}'_m$. It happens that a Schwartz family is summable with respect to any tempered distribution on its index Euclidean space [CfF].

Proof. Let $a \in \mathcal{S}'_m$ and let δ be the Dirac family $\in \mathcal{S}'_m$. Since the linear hull span(δ) of the Dirac family is $\sigma(\mathcal{S}'_m)$-sequentially dense in the space $\mathcal{S}'_m$, there is a sequence of distributions $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, in the linear hull span(δ) of Dirac family, converging to the distribution a w.r.t. the weak-$\star$topology $\sigma(\mathcal{S}'_m)$; that is we have

$$^{\sigma(\mathcal{S}'_m)}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since for any natural k, the distribution α_k belongs to the linear hull $\mathrm{span}(\delta)$, there exists a finite family $(y_i)_{i=1}^h$ of points in $\mathbb{R}^m$ and there is a finite family of scalars $(\lambda_i)_{i=1}^h$ in the field $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^h \lambda_i \delta_{yi}.$$

Consequently, by obvious calculations, we have

$$\begin{aligned}\alpha_k \circ \hat{v} &= \sum_{i=1}^h \lambda_i (\delta_{yi} \circ \hat{v}) = \\ &= \sum_{i=1}^h \lambda_i v_{yi}.\end{aligned}$$

Hence for every index $k \in \mathbb{N}$, the linear functional $\alpha_k \circ \hat{v}$ belongs to $\mathcal{S}'_n$. Now, let s be the topology of pointwise convergence on the algebraic dual $(\mathcal{S}_n)^\star$, we claim that

$$^s \lim_{k\to+\infty} (\alpha_k \circ \hat{v}) = a \circ \hat{v}.$$

Rather, for every test functions $\phi \in \mathcal{S}_n$, we obtain

$$\begin{aligned}\lim_{k\to+\infty} (\alpha_k \circ (\hat{v})(\phi)) &= \lim_{k\to+\infty} \alpha_k (\hat{v}(\psi)) = \\ &= a(\hat{v}(\psi)).\end{aligned}$$

□

We have proved the pointwise convergence to the linear functional $a \circ \hat{v}$. Hence by the Banach-Steinhaus theorem it is continuous too, i.e. $a \circ \hat{v}$ should be a tempered distribution in $\mathcal{S}'_n$. Thus summbability of the family v holds true [CfF].

Definition 3.6. Given $a \in \mathcal{S}'_m$ and a summable $v \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$, the superposition integral of v on $\mathcal{S}'_m(\mathbb{R}^m)$ is the element of $\mathcal{S}'_n$ defined by

$$\lim_{k\to+\infty} (\alpha_k \circ \hat{v}) = \int_{\mathbb{R}^m} av = a \circ \hat{v}. \tag{3.4}$$

Is is straightforward the link between definition 3.6 and definition 2.6. In the case of the Bochner integral we have defined the notion of a.e. (almost everywhere) convergence and the fundamental concept of summability. In the superposition integral we have seen the pointwise convergence and the summability too. But in superposition we are going to use several functionals, that is to say, we are handle with distributions. We use the notion of local integration rather than measurability. This is so because we are studying locally Hausdorff convex spaces and Fréchet spaces. Thus we are not interested in norms but in studying topologies which generate seminorms (i.e. (3.2)). For example, in $\mathcal{S}'_n(\mathbb{R}^n, \mathbb{C})$ (the space of tempered distribution) we are interested in using the weak-$\star$topology of this space [FrT]. This kind of theory is useful, rather we have continuous functionals and a map $(\cdot) : \mathcal{S}'_m \to \mathcal{S}'_n$ defines projections among Fréchet spaces. Thus, they are bounded and continuous [FrT]. These properties ensure well-behaved functionals which generalize the concept of measure, in particular by the Dirac's distribution. It is important to notice that one can use it like the $\chi_{\mathbb{P}}$ in section 2. Thus

$$\delta_x = \begin{cases} 1 & \text{for} \quad x \in E \\ 0 & \text{for} \quad x \notin E. \end{cases} \tag{3.5}$$

We have to clarify that the (3.5) is not a function mapped to a point, but we are talking about *distributions*, thus it would be an abuse of notation in using $\delta(x-y)$ w.r.t. $y \in \mathbb{R}^n$ and $x \in \mathcal{D}(\mathbb{R}, \mathbb{C})$, the space of test function (so as to a deeper reasoning about the δ_x see [CfS] and [FrP]). We may see how this particular kind of *measure* works.

Definition 3.7. Let δ the Dirac's family in $\mathcal{S}'_n$. Then, for each tempered distribution $u \in \mathcal{S}'_n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} u\delta &= u \circ \hat{\delta} = \\ &= u \circ 1_{\mathcal{S}'_n} = \\ &= u. \end{aligned}$$

Thus any tempered distribution is an ${}^{\mathcal{S}}$linear superposition of the Dirac's family and the coefficients system of this superposition is the distribution u itself: this is a typical property of the canonical basis of the Euclidean spaces $\mathbb{R}^n$ [CfS]. It is clear that one can build a space by means of the Dirac's distribution family as much it happens by using $\chi_{\mathbb{P}}$, but in a more general fashion due to our definition of the δ_x.

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