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ABSTRACT

Three estimation policies for the optimal order quantity of the classical newsvendor model under exponential demand are evaluated in the current paper. According to the principle of the first estimation policy, the corresponding estimator is obtained replacing in the theoretical formula which gives the optimal order quantity the parameter of exponential distribution with its maximum likelihood estimator. The estimator of the second estimation policy is derived in such a way as to ensure that the requested critical fractile is attained. For the third estimation policy, the corresponding estimator is obtained maximizing the a-priori expected profit with respect to a constant which has been included into the form of the estimator. Three statistical measures have been chosen to perform the evaluation. The actual critical fractile attained by each estimator, the mean square error, and the range of deviation of estimates from the optimal order quantity, when the probability to take such a range is the same for the three estimation policies. The behavior of the three statistical measures is explored under different combinations of sample sizes and critical fractiles. With small sample sizes, no estimation policy predominates over the others. The estimator which attains the closest actual critical fractile to the requested one, this estimator has the largest mean square and the largest range of deviation of estimates from the optimal order quantity. On the contrary, with samples over 40 observations, the choice is restricted among the estimators of the first and third estimation policy. To facilitate this choice, at different sample sizes, we offer the required values of the critical fractile which determine which estimation policy eventually should be applied.

Keywords: Classical newsvendor model; Exponential distribution; Demand estimation; Actual critical fractile; Mean square error of estimators.

JEL Codes: C13: Estimation; C44: Operations Research; D24: Production & cost; M11: Production Management.

1. Introduction

The classical newsvendor model offers one-period optimal ordering policies for products whose demand life cycle lasts so long as the duration of the period. The optimal order quantity that minimizes the expected cost of the model is determined by equating the probability not to observe a stock-out during the period to a critical fractile which depends upon the overage and underage costs of the inventory system. The same optimal order quantity is obtained if instead of the expected cost we consider the expected profit. In this case the critical fractile is a function of the revenue parameters (price and salvage value) and the cost parameters (purchase cost and shortage cost).

Two conditions are required for the application of the classical newsvendor model: (a) at the start of any period the inventory system starts with stocking level equal to the optimal order quantity, and (b) no fixed costs are incurred with the delivery of the order quantity. The knowledge of demand distribution is also necessary in order to proceed to specifications of optimal ordering policies. But, in real life conditions at the stage of forming inventory policies, neither the true process of generating demand data nor the values of its parameters are known. To solve this problem, alternative estimation processes have been developed so far in the literature, and their choice mainly depends upon the type and length of historical data regarding demand per period.

The first classification of available estimation processes concerns the capability of keeping track of demand in cases where stock-outs occur. For example, when sales are conducted in an impersonal environment, it is impossible to measure that part of demand which is not met. So, in such cases the available sample constitutes of sales data, and at periods in the sample where stock-outs might have occurred, sales have underestimated the real demand. Under such circumstances, the available estimation processes take into account the unobserved lost part of demand, by modeling demand through censored or truncated distributions. References on this area include the works of Nahmias (1994), Lau and Lau (1996), Ernst and Kamrad (2006), and Halkos and Kevork (2011).

When records of real demand are available, no matter if stock-outs have or have not occurred in the periods included in the sample, two general approaches can be followed. The first approach is the Bayesian and the second is the Frequentist. The Bayesian approach leads to a posterior distribution of parameters of demand distribution, which is continuously updated as the sample is enriched with new data for demand. The posterior distribution is generated from a prior distribution, with the latter to be chosen either using collateral data or subjective judgment. The posterior distribution is then used to estimate the optimal order quantity and the optimum value of the objective function. Early works on this area constitute the papers of Scarf (1959), Iglehart (1964) and Azoury (1985).

In the context of the Frequentist approach, three estimation policies have been suggested in the literature, providing that the demand distribution has been identified using appropriate statistical tests. According to the first estimation policy, the parameters of demand distribution are replaced by their estimates in the formula which determines the optimal order quantity. Replacing, however, demand parameters by their estimates, the optimal order quantity is incorrectly computed and the requested probability of not observing a stock-out is not attained.

This argument leads to a second estimation policy, where adjustments are made to the estimator of the first estimation policy in order the requested probability to be achieved. The third estimation policy, which is the oldest one, is based on the Expected Total Operating Cost (ETOC) function introduced by Hayes (1969). The ETOC is the expected value of the expected cost of the newsvendor model, after having replaced the optimal order quantity by an appropriate estimator, whose form depends upon the type of demand distribution. This form is then specified by minimizing ETOC with respect to a certain constant which has already been included into the form of the estimator.

The last category of estimation processes refers to the case where records of demand per period are available, but the form of demand distribution cannot be identified. Then, researchers have two alternatives to estimate the optimal order quantity. The first alternative is to follow a non-parametric approach which includes the sampling-based policy or the use of order statistics and bootstrapping techniques. In the sampling-based policy, demand is modeled by the empirical distribution function of historical demand data. The second alternative is followed when partial information about the demand (in terms of moments) is available. In this case the optimal order quantity is determined by maximizing the worst case expected profit considering all distributions with the same values of the available moments. The relevant literature review of the last two alternatives can be found in Liyanage and Shanthikumar (2005), Janssen et al. (2009), and Akcay et al. (2011).

In the current paper, which is classified to the area of the frequentist inferential approach, we evaluate for the first time the three aforementioned estimation policies when demand follows the exponential distribution. To perform the evaluation, three statistical measures are considered and their analytic forms are derived. The first statistical measure is the actual critical fractile that each policy attains, that is, the effective probability the estimated order quantity to meet the total demand occurred during the period. The second statistical measure is the Mean Square Error (MSE) of the estimator of each policy. The need for using MSE is the biasedness that arises to those estimators which either ensure the requested probability of not observing a stock-out during the period or minimize the ETOC function. Finally, the last statistical measure refers to the range of deviations of estimates from the optimal order quantity, when the probability to take such a range is the same for the three estimation policies.

The choice of exponential distribution has been made for three reasons. The first reason is that we take tractable results for the ETOC function, and so comparisons among the three estimation policies can be made at an analytic base. The second reason is existent evidences about the demand distribution from real-life inventory problems. Lau (1997) points out that demand for some seasonal fashion items is characterized by high uncertainties, and in such cases it is more appropriate the demand to be modeled by the exponential distribution. The third reason refers to a theoretic finding which Halkos and Kevork (2012) resulted in. Replacing the parameter of exponential distribution with its maximum likelihood estimator, they showed that validity and precision of the asymptotic confidence interval for the optimal order quantity do not depend upon the values assigned to the revenue and cost parameters of the newsvendor model. This theoretic result leads us in the current work to present the three statistical measures as functions of the critical fractile without making specific assumptions about the values that revenue and cost parameters take on.

The aforementioned discussion leads the rest of the paper to be structured as follows. The next section presents literature review concerning the use of the three estimation policies and existent results from comparative studies among alternative estimation processes. The specifications of estimators for the optimal order quantity according to the three estimation policies are obtained in section 3. In sections 4 and 5 we derive the analytic forms of the three statistical measures and we study their performance for different combinations of sample size and requested critical fractile. Finally, section 6 summarizes the most important findings of the current work.

2. Theoretical background – Literature review

Let $f(\mathrm{x};\boldsymbol{\theta})$ and $F(\mathrm{x};\boldsymbol{\theta})$ be respectively the probability density function and the distribution function of demand X for any inventory cycle (or period), with the vector $\boldsymbol{\theta}$ to include the parameters of demand distribution. For example, if demand is exponentially distributed, the vector $\boldsymbol{\theta}$ will include the parameter of exponential distribution, while with normal demand the vector $\boldsymbol{\theta}$ includes the mean and the variance. In the context of the classical newsvendor model and denoting by,

Q : the order quantity,

p : selling price per unit,

c : purchasing cost per unit,

v : salvage value,

s : shortage penalty cost per unit,

$\mathrm{C_u}$: per unit underage cost, where $\mathrm{C_u = p - c + s}$, and

$\mathrm{C_o}$: per unit overage cost, with $\mathrm{C_o = c - v}$,

the optimization is performed either to the expected cost of the model (Lau, 1997),

$$\zeta(\mathrm{Q},\boldsymbol{\theta}) = \mathrm{C_u}[\mathrm{E(X)} - \mathrm{Q}] + (\mathrm{C_o} + \mathrm{C_u})\mathrm{Q}F(\mathrm{Q};\boldsymbol{\theta}) - (\mathrm{C_o} + \mathrm{C_u})\int_0^{\mathrm{Q}} \mathrm{x}f(\mathrm{x};\boldsymbol{\theta})\mathrm{dx}, \tag{1}$$

or to the expected profit (Khouja, 1999),

$$\xi(\mathrm{Q},\boldsymbol{\theta}) = (\mathrm{p - c + s})\mathrm{Q} - (\mathrm{p - v + s})\mathrm{Q}F(\mathrm{Q};\boldsymbol{\theta}) + (\mathrm{p - v + s})\int_0^{\mathrm{Q}} \mathrm{x}f(\mathrm{x};\boldsymbol{\theta})\mathrm{dx} - \mathrm{sE(X)}. \tag{2}$$

Taking first and second order derivatives using Leibniz's rule, the optimal order quantity, Q^*, is the same either minimizing ζ or maximizing ξ, and satisfies the sufficient condition

$$F(\mathrm{x};\boldsymbol{\theta}) = \frac{\mathrm{C_u}}{\mathrm{C_u + C_o}} = \frac{\mathrm{p - c + s}}{\mathrm{p - v + s}} = \mathrm{R},$$

where R is the critical fractile representing the probability the optimal stocking level at the start of any period to be sufficient to meet the total demand which will occur during the period. So, at the beginning of each period, the inventory system will start with the optimal stock

$$Q^* = F^{-1}(R;\boldsymbol{\theta}). \tag{3}$$

Following Schweitzer and Cachon (2000), R can be either less or greater than 0.5. When, the specified values for p, c, v, s give $R < 0.5$ (or $R > 0.5$), then the product is classified as a low-profit (or high-profit) product.

Suppose that data on demand are available for the most recent n periods. Then, according to the first estimation policy of the frequentist inferential approach, $\boldsymbol{\theta}$ is replaced in (3) with its estimator $\hat{\boldsymbol{\theta}}$, leading to the estimator $\hat{Q}_{n+1} = F^{-1}(R;\hat{\boldsymbol{\theta}})$ of the optimal order quantity for period $n+1$. For the analysis which follows, we name this policy as "*Direct Estimation Policy (DEP)*". The following works have used the principle of DEP to derive confidence intervals or to develop test of hypotheses concerning the optimal order quantity and the maximum expected profit.

Assuming normal demand, and replacing $\boldsymbol{\theta}$ in (3) with its MLE, Kevork (2010) developed an appropriate estimator for the maximum expected profit and explored its statistical properties for both small and large samples. With normal demand, and using the sample mean and the unbiased estimator of the variance, Su and Pearn (2011) developed a statistical hypothesis testing methodology to select among two newsboy-type products the one which has a higher probability of achieving a target profit under the optimal ordering policy. When demand follows the Rayleigh or the exponential distribution, Halkos and Kevork (2012) replaced in (3) the parameter of each distribution by its MLE, and derived the distributions of the estimators for the optimal order quantity and the maximum expected profit.

Replacing, however, $\boldsymbol{\theta}$ with $\hat{\boldsymbol{\theta}}$, the optimal order quantity is incorrectly computed and the requested critical fractile, namely, the probability not to experience a stock-out during the period, is not attained. Ritchken and Sankar (1984) were the first who raised this problem and suggested appropriate adjustments to be made to the DEP estimator $\hat{Q}_{n+1}$. Assuming normal demand with unknown mean and unknown variance, Ritchken and Sankar made the necessary adjustment to that part of the DEP estimator which refers to the safety stock. The same problem of not attaining the requested critical fractile was studied by Katircioglou (1996), who resulted in the same modified estimator with that one of Ritchken and Sankar. Janssen et al. (2009) also handled the same problem and obtained an estimator attaining the requested critical fractile when demand follows the normal distribution with unknown mean but known variance.

To the extent of our knowledge, in the current paper we derive for the first time the adjusted estimator which ensures under exponential demand the requested critical fractile using the Dirichlet and Beta distributions. For the modified estimators of these studies the term "unbiasdness" is used from a different perspective than the traditional one which is related to the sampling distribution of the estimator. Treating the optimal order quantity as the R^{th} percentile of demand distribution, the modified estimators of the aforementioned studies are unbiased from the sense that they ensure the requested critical fractile. This is the reason why in the remaining analysis we name this policy as "*Unbiased Percentile Estimation Policy (UPEP)*".

Hayes (1969) introduced the concept of the Expected Total Operating Cost (ETOC) to investigate the inaccuracy in the estimation of inventory targets such as the optimal order quantity in the newsvendor model. ETOC is the expected value of (1) after having replaced Q with a function of $\hat{\boldsymbol{\theta}}$, $g(\hat{\boldsymbol{\theta}})$, whose form depends upon the type of demand distribution. Specifications for $g(\hat{\boldsymbol{\theta}})$ are obtained minimizing ETOC with respect to constant(s) which are included into $g(\hat{\boldsymbol{\theta}})$. Hayes derived the specifications of $g(\hat{\boldsymbol{\theta}})$ under an exponential and normal demand, when the latter one has unknown mean and known or unknown variance. Katircioglou (1996) also used the ETOC concept to study the effect of biasing in estimation when demand follows the normal and gamma distributions. Modeling demand by the Johnson Translation System (JTS), Akcay et al. (2011) seek the specification of $g(\hat{\boldsymbol{\theta}})$ which minimizes ETOC within a class of estimators implied by the JTS.

An equivalent to the ETOC concept is the a-priori Expected Profit (apEP) introduced by Liyanage and Shanthikumar (2005). The apEP is the expected value of (2) after having replaced again Q with $g(\hat{\boldsymbol{\theta}})$. Under exponential demand, Liyanage and Shanthikumar derived the specification of $g(\hat{\boldsymbol{\theta}})$ setting salvage value and shortage cost equal to zero. In the current paper with exponential demand, we also use the apEP function of Liyanage and Shanthikumar, but we present the constant involved in $g(\hat{\boldsymbol{\theta}})$ as a function of the critical fractile. For the rest of the analysis, we refer to the estimation policy based on the ETOC or apEP concept as *"Hayes Estimation Policy (HEP)"*.

A number of comparative studies between alternative estimation processes exist in the literature. So, we are closing this section by presenting some indicative ones. Conrad (1976) showed that using sales data to estimate the parameter of Poisson distributed demand results in order quantities different than the optimal ones. Hill (1997) compared the Bayesian against the Frequentist approach with exponential, Poisson, and Binomial demand, and showed that Bayesian approach can result in lower expected total cost when a meaningful prior is available. Under a general continuous distribution, Ding et al. (2002) showed that in the presence of lost sales optimal order quantities are higher compared to the case where demand would be fully observed.

Similarly, modeling demand by the JTS, Akcay et al. (2011) quantified the inaccuracy in the DEP estimator as a function of the length of the historical data, the critical fractile, and the shape parameters of the demand distribution and suggested the use of the HEP instead of the DEP for setting order quantities in the presence of this inaccuracy. But, according to the statistical measures which we have chosen to compare the three estimation policies under consideration, when demand follows the exponential distribution, our work concludes that no estimation policy predominates over the others, and the choice is left in inventory managers. If managers can accept a reasonable reduction to the requested probability of not having a stock-out during the period, the choice between the DEP and the UPEP depends on the value of the critical fractile.

3. Estimators for the optimal order quantity

Given that a sample $X_1, X_2, ..., X_n$ is available, and denoting by $\hat{\boldsymbol{\theta}}$ the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$, the order quantity that will be used for period $n+1$ is determined from $\hat{Q}_{n+1} = F^{-1}(R; \hat{\boldsymbol{\theta}})$. To estimate Q^* from $\hat{Q}_{n+1}$, we shall assume that in every period in the sample, the salvage value had been set up at a level which ensured that if any excess inventory remained at the end of period, this was disposed of through either consignment stocks or buyback arrangements.

Under an exponential demand we have $f(x;\theta) = \theta^{-1} e^{-x/\theta}$ and $F(x;\theta) = 1 - e^{-x/\theta}$. So, from (2), the expected profit becomes

$$\xi(Q,\theta) = \theta(p - v + s)\left(1 - e^{-Q/\theta}\right) - (c - v)Q - s\theta, \tag{4}$$

the optimal order quantity is determined from (3) as

$$Q^* = \theta \ln\left(\frac{(p - v + s)}{c - v}\right) = \theta \ln(1 - R)^{-1}, \tag{5}$$

and the MLE of θ is the sample average $\hat{\theta} = \sum_{t=1}^{n} X_t \Big/ n$.

When demand follows the exponential distribution, the ordering policy is expressed by the class of estimators of the form $\hat{Q}^{(j)}_{n+1} = \kappa_j \hat{\theta}$, where κ_j is differentiated according to each estimation policy. So, for each policy we take the following:

Direct Estimation Policy (DEP)

Replacing in (5) θ with $\hat{\theta}$, the estimator of Q^* becomes $\hat{Q}^{(1)}_{n+1} = \kappa_1 \hat{\theta}$, with $\kappa_1 = \ln(1-R)^{-1}$. Since $\hat{\theta}$ is unbiased, $\hat{Q}^{(1)}_{n+1}$ is also unbiased for Q^*.

Percentile Unbiased Estimation Policy (PUEP)

The estimator $\hat{Q}^{(2)}_{n+1} = \kappa_2 \hat{\theta}$ is specified by finding κ_2 such that the following probability statement is true:

$$\Pr\left(X_{n+1} < \kappa_2 \hat{\theta}\right) = \Pr\left(\frac{\sum_{t=1}^{n} X_t}{\sum_{t=1}^{n+1} X_t} < \frac{n}{n+\kappa_2} \right) = R .$$

Since demand is formed independently in successive periods and follows the exponential distribution, X_j's for $j = 1,2,\ldots,n,n+1$ are independent $\text{gamma}(1,\theta)$ random variables. Defining $Y_j = X_j \Big/ \sum_{j=1}^{n+1} X_j$ and following theorem 4.1 of Devroye (1986, pp. 594), the vector of random variables $\begin{bmatrix} Y_1 & Y_2 & \ldots & Y_n \end{bmatrix}$ follows the Dirichlet (Dir) distribution with $\alpha_j = 1$ for $j = 1,2,\ldots,n,n+1$. Then, applying the aggregation property of the Dirichlet distribution (e.g. see Frigyik et al., 2010), the sum $Y_1 + Y_2 + \ldots + Y_n$ follows $\text{Dir}(n,1)$. But $\text{Dir}(n,1)$ is the Beta distribution with parameters $\alpha = n$ and $\beta = 1$. Hence

$$\frac{\sum_{t=1}^{n} X_t}{\sum_{t=1}^{n+1} X_t} = \frac{X_1 + X_2 + \ldots + X_n}{X_1 + X_2 + \ldots + X_n + X_{n+1}} \sim B(n,1), \tag{6}$$

and $n/(n+\kappa_2)$ must be $B_{1-R}(n,1)$, where $B_{1-R}(n,1)$ is the $(1-R)^{th}$ percentile of the Beta distribution.

When α, β are integers, the regularized incomplete Beta function is derived from 8.17.5 of Paris (2010), and has the form

$$I_y(\alpha,\beta) = \sum_{j=\alpha}^{\alpha+\beta-1} \binom{\alpha+\beta-1}{j} y^j (1-y)^{\alpha+\beta-1-j}.$$

Then the cumulative distribution function of $B(n,1)$ is

$$\Pr(B(n,1) \le y) = I_y(n,1) = y^n, \tag{7}$$

from which we take $B_{1-R}(n,1) = (1-R)^{1/n}$, and finally

$$\kappa_2 = \frac{n\left(1-(1-R)^{1/n}\right)}{(1-R)^{1/n}}.$$

Hayes Estimation Policy (HEP)

Defining the estimator $\hat{Q}_{n+1}^{(3)} = \kappa_3 \hat{\theta}$, and using (4), the a-priori expected profit becomes

$$apEP = E\left[\xi\left(\hat{Q}_{n+1}^{(3)}, \theta\right)\right] = \theta(p-v+s)\left\{1-E\left(\exp\left(-\hat{Q}_{n+1}^{(3)}/\theta\right)\right)\right\} - (c-v)E\left(\hat{Q}_{n+1}^{(3)}\right) - s\theta.$$

Because $\sum_{t=1}^{n} X_t \sim \text{gamma}(n,\theta)$, by using the scaling property of the gamma distribution, we take the distributional result

$$U = \omega \sum_{t=1}^{n} X_t \sim \text{gamma}(n, \omega\theta), \tag{8}$$

where in this case $\omega = \kappa_3/(n\theta)$. Then the expected value of $\exp(-U)$ is derived as

$$E(\exp(-U)) = \int_0^\infty \exp(-u) \frac{(\omega\theta)^{-n} u^{n-1} \exp(-u/(\omega\theta))}{\Gamma(n)} du =$$

$$=\frac{(\omega\theta)^{-n}}{\Gamma(n)}\int_0^\infty\left(\frac{\omega\theta}{1+\omega\theta}t\right)^n \exp(-t)dt=(1+\omega\theta)^{-n}, \tag{9}$$

where $\Gamma(n)$ is the gamma function evaluated at n.

Result (9) is also stated in Liyanage and Shanthikumar (2005) but without proof. Using (9), apEP takes the form

$$\text{apEP}=E\left[\xi\left(\hat{Q}_{n+1}^{(3)},\theta\right)\right]=\theta(p-v+s)\left[1-\left(\frac{n}{n+\kappa_3}\right)^n\right]-(c-v)\theta\kappa_3-s\theta,$$

and maximizing it with respect to κ_3, we finally obtain

$$\kappa_3=n\left[(1-R)^{-\frac{1}{n+1}}-1\right].$$

4. Actual Critical Fractile

To start with evaluating the three estimation policies under exponential demand, in the current section, for each estimator, we derive the analytic form of the actual critical fractile, that is, the effective probability to cover the total demand occurred during the period, when the order quantity is determined by each estimation policy. As a benchmark, we take the PUEP estimator as its use ensures that the actual critical fractile equals to the requested one. For the analysis which follows, we remind the reader that $j=1$ stands for the DEP estimator, $j=2$ for the PUEP estimator, and $j=3$ for the HEP estimator.

For the j^{th} estimation policy, the actual critical fractile is defined as the probability $R_{act}^{(j)}=\Pr\left(X_{n+1}\le Q_{n+1}^{(j)}\right)$. Using (6) and (7), its analytical form is

$$R_{act}^{(j)}=\Pr\left(X_{n+1}\le\kappa_j\hat{\theta}\right)=\Pr\left(\frac{\sum_{t=1}^{n}X_t}{\sum_{t=1}^{n+1}X_t}\ge\frac{n}{n+\kappa_j}\right)=1-\Pr\left(B(n,1)\le\frac{n}{n+\kappa_j}\right)=1-\left(\frac{n}{n+\kappa_j}\right)^n \tag{10}$$

To carry on the analysis, we consider the following pairwise differences, which are obtained through (10):

$$\psi_1(n,R) = R_{act}^{(2)} - R_{act}^{(1)} = \left[\frac{n}{n-\ln(1-R)}\right]^n - (1-R), \tag{11a}$$

$$\psi_2(n,R) = R_{act}^{(2)} - R_{act}^{(3)} = (1-R)\left((1-R)^{-1/n} - 1\right) > 0, \tag{11b}$$

$$\psi_3(n,R) = R_{act}^{(3)} - R_{act}^{(1)} = \left[\frac{n}{n-\ln(1-R)}\right]^n - (1-R)^{n/(n+1)}. \tag{11c}$$

Asymptotically the DEP and HEP estimators give the requested critical fractile since

$$\lim_{n\to\infty}\left[\frac{n}{n-\ln(1-R)}\right]^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{\ln(1-R)^{-1}}{n}\right)^n} = \frac{1}{\exp\left(\ln(1-R)^{-1}\right)} = (1-R)$$

and hence $\lim_{n\to\infty}\psi_j(n,R) = 0$ for $j = 1,2,3$.

The sign of differences $R_{act}^{(2)} - R_{act}^{(1)}$ and $R_{act}^{(3)} - R_{act}^{(1)}$ depends upon the behavior of functions $\psi_1(n,R)$ and $\psi_3(n,R)$. Due to their complexity, we explored their mathematical properties numerically. At first, it is easily deduced that $\lim_{R\to 0}\psi_j(n,R) = \lim_{R\to 1}\psi_j(n,R) = 0$. In figures (1a) and (1b) we present for different sample sizes the plots of $\psi_1(n,R)$ and $\psi_3(n,R)$ against R. From figure (1a), we find out that $\psi_1(n,R)$ is positive, while from figure (1b) we observe that there is a value R_o for which $\psi_3(n,R) < 0$ when $R < R_o$, and $\psi_3(n,R) > 0$ when $R > R_o$. Therefore, combining the findings of figures (1a), (1b) with (11b), we deduce that, for any $R < R_o$, it holds that $R_{act}^{(3)} < R_{act}^{(1)} < R_{act}^{(2)} = R$, while for $R > R_o$, we take $R_{act}^{(1)} < R_{act}^{(3)} < R_{act}^{(2)} = R$.

Figure 1a: Plot of $\psi_1(n,R)$ against R

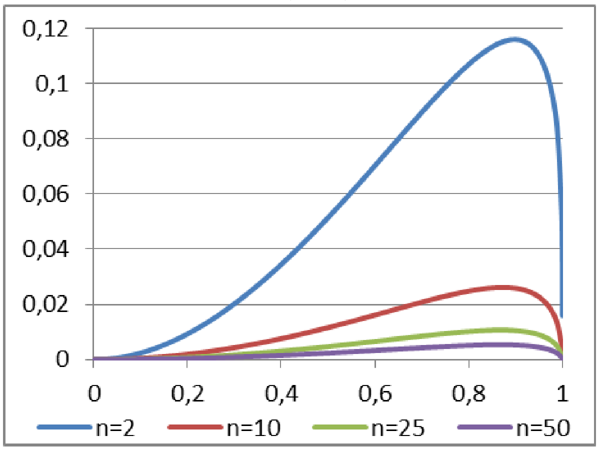
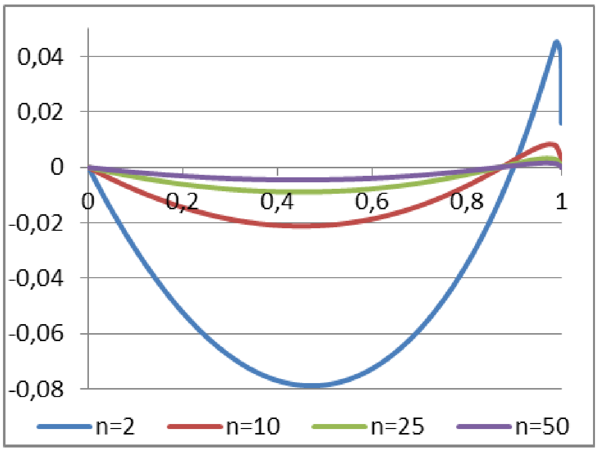


Figure 1b: Plot of $\psi_3(n,R)$ against R



The next two propositions offer further insights for the behavior of $R_{act}^{(j)}$.

Proposition 1: For a finite $n \geq 2$, let $R_m^{(j)}$ be the value of R where the deviations $R - R_{act}^{(1)}$ become maximum. If $R_m^{(1)} > (e-1)/e$, then $R_m^{(1)}$ equals to R_o and satisfies the equation

$$\frac{n}{n - \ln(1 - R_o)} - (1 - R_o)^{1/(n+1)} = 0$$

Proof: See in the Appendix.

Proposition 2: For $n \geq 2$ and finite, the deviations $R - R_{act}^{(3)}$ are maximized at

$$R_m^{(3)} = 1 - \left(\frac{n}{n+1}\right)^{n+1},$$

with $\frac{19}{27} \leq R_m^{(j)} < \frac{e-1}{e}$

Proof: See in the Appendix.

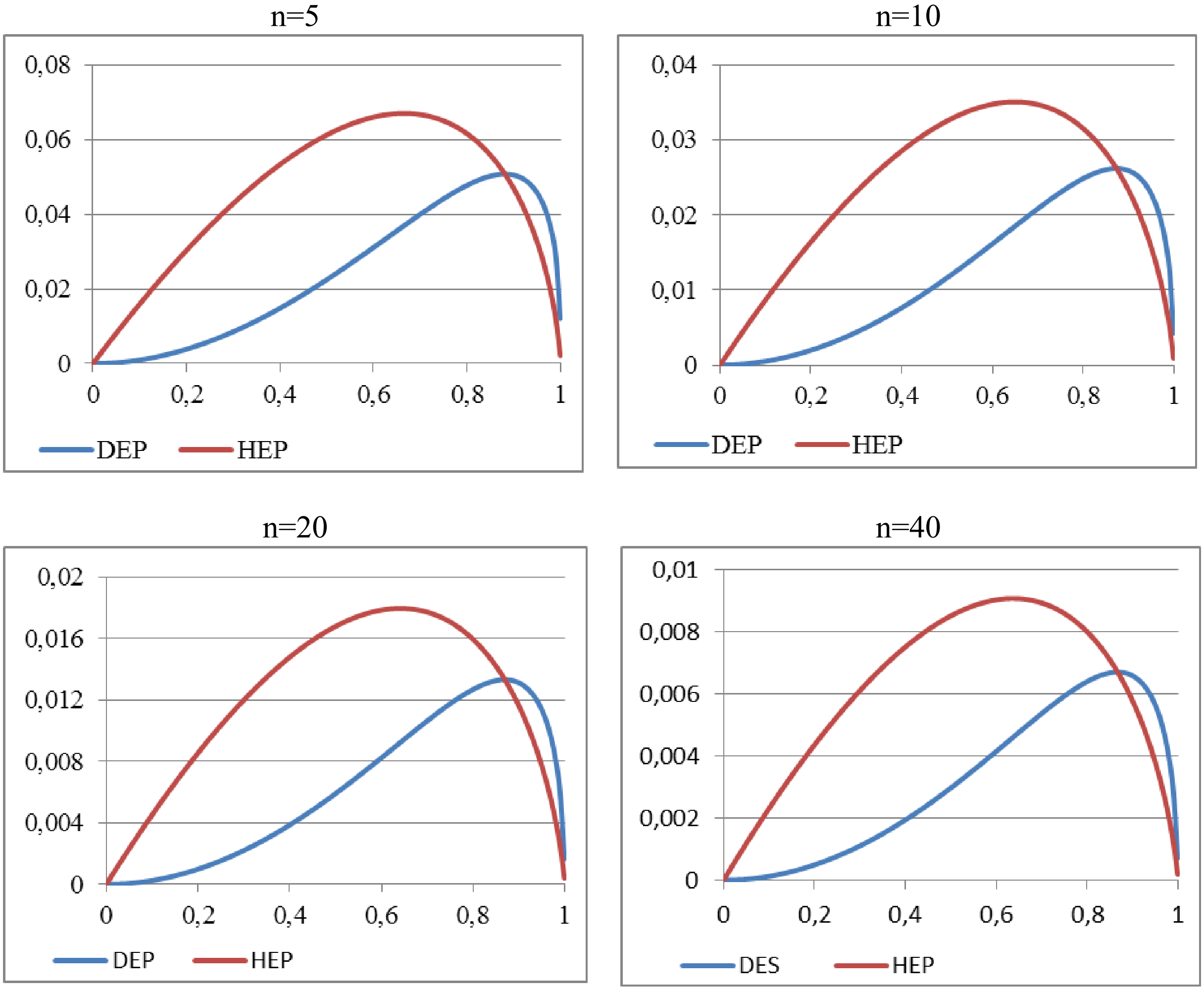
The values of R_o and $R_m^{(3)}$ are displayed in table 1. Particularly, the values of R_o have been obtained numerically solving the equation $\psi_3(n,R)=0$, and keeping for each sample size the difference of the two terms of $\psi_3(n,R)$ approximately at the same size. Besides, the findings of propositions 1 and 2 are illustrated in figure 2, where for different sample sizes we plot the deviations $R-R_{act}^{(j)}$ against R.

From table 1 and figures 2a and 2b, important conclusions are drawn. At the value of R_o the deviations from R of the actual critical fractile attained by the DEP estimator becomes maximum. The maximum deviation from R of the actual critical fractile attained by the HEP estimator is met at a value $R_m^{(3)}$ which is smaller than R_o. Finally, as the sample size is getting larger, R_o is declining with a relatively slow rate , and $R_m^{(3)}$ is always smaller than the inflection point, $(e-1)/e$, of the graph of deviations $R-R_{act}^{(1)}$ against R.

Table 1: Values of R_o and $R_m^{(3)}$ for different sample sizes

n	R_o	$\psi_3(n,R)$	$R_m^{(3)}$
2	0.8984	5.90×10^{-5}	0.7037
3	0.8891	5.58×10^{-5}	0.6836
4	0.8838	4.74×10^{-5}	0.6723
5	0.8803	5.44×10^{-5}	0.6651
10	0.8726	5.07×10^{-5}	0.6495
15	0.8696	5.37×10^{-5}	0.6439
20	0.8680	5.07×10^{-5}	0.6411
25	0.8669	5.05×10^{-5}	0.6393
30	0.8660	5.30×10^{-5}	0.6381
40	0.8647	5.41×10^{-5}	0.6367
50	0.8638	5.25×10^{-5}	0.6358
100	0.8603	5.24×10^{-5}	0.6339
300	0.8492	5.24×10^{-5}	0.6327

Figure 2: Plot of $R - R_{act}^{(j)}$ against R for the DEP and HEP estimators



But the most important remark after studying the deviations $R - R_{act}^{(j)}$ at different R's is traced by looking at the graph for n=40 of figure 2. With samples of at least 40 observations, the issue of using estimators not giving the requested critical fractile ceases to exist as an estimation problem, because the deviations $R - R_{act}^{(j)}$ are considered negligible from the inventory managerial practice point of view, as they are ranging below 1%.

5. Mean Square Error and deviations of estimates from the optimal order quantity

The analytic form of the Mean Square Error of each estimator is derived in this section and its relation to the actual critical fractile is investigated. As we mention in section 3, the DEP estimator is unbiased for Q^*, while the other two estimators are not. Relevant to the MSE is the range of deviations of estimates from the optimal order quantity. The lower limit and the upper limit of the range are derived, when the probability of such a range to occur is the same for the three estimation policies. Finally, the relation between the MSE and the range of deviations of estimates from the optimal order quantity is investigated with regard to the patterns of behavior of the actual critical fractile which have been discussed in the previous section.

To start with, the analytic form of MSE is derived through (8). By setting $\omega = \kappa_j / n$, the sampling distribution of $\hat{Q}^j_{n+1}$ is the $\mathrm{gamma}(n, \kappa_j\theta/n)$. Hence

$$\mathrm{MSE}\left(\hat{Q}^j_{n+1}\right) = \mathrm{Var}\left(\hat{Q}^j_{n+1}\right) + \left[\mathrm{Bias}\left(\hat{Q}^j_{n+1}\right)\right]^2 = \left\{\frac{\kappa_j^2}{n} + \left(\kappa_j - \kappa_1\right)^2\right\}\theta^2, \tag{12}$$

as $\mathrm{Var}\left(\hat{Q}^j_{n+1}\right) = n\left(\kappa_j\theta/n\right)^2$ and

$$\mathrm{Bias}\left(\hat{Q}^j_{n+1}\right) = E\left(\hat{Q}^j_{n+1}\right) - Q^* = \kappa_j\theta - \theta\ln(1-R)^{-1} = \left(\kappa_j - \kappa_1\right)\theta.$$

By setting $\psi_1(n, R) > 0$ in (11a), we take the inequality

$$\frac{n}{n - \ln(1-R)} - (1-R)^{1/n} > 0, \tag{13}$$

which holds for $2 \le n < \infty$, and R between zero and one. Setting also $\psi_3(n,R) < 0$ in (11c) we obtain the inequality

$$\frac{n}{n - \ln(1-R)} - (1-R)^{1/(n+1)} < 0, \tag{14}$$

which holds for finite $n \geq 2$ and $R < R_o$. The inequality in (14) becomes positive when $R > R_o$.

Having available (13) and (14), we reach the next proposition which ranks the sizes of $MSE\left(\hat{Q}^{j}_{n+1}\right)$ for the three estimation policies.

Proposition 3: Given the sample size, n, and the corresponding value of R_o, for $R < R_o$

$$MSE\left(\hat{Q}^{(3)}_{n+1}\right) < MSE\left(\hat{Q}^{(1)}_{n+1}\right) < MSE\left(\hat{Q}^{(2)}_{n+1}\right),$$

while when $R > R_o$

$$MSE\left(\hat{Q}^{(1)}_{n+1}\right) < MSE\left(\hat{Q}^{(3)}_{n+1}\right) < MSE\left(\hat{Q}^{(2)}_{n+1}\right).$$

Proof: See in the Appendix.

To relate the range of deviations of estimates from the optimal order quantity to the actual critical fractile and the MSE of the three estimators, we consider the next probability:

$$\Pr\left(L^{(j)}_{\ell} \leq \hat{Q}^{(j)}_{n+1} - Q^{*} \leq L^{(j)}_{u}\right) = 1 - \alpha .$$

Since $\hat{Q}^{j}_{n+1} \sim \text{gamma}\left(n, \kappa_j \theta / n\right)$, we take the following lower and upper limits for the deviation $\hat{Q}^{(j)}_{n+1} - Q^{*}$ after applying the scaling property of the Gamma distribution:

$$L^{(j)}_{\ell} = \Gamma_{\alpha/2}\left(n, \kappa_j \theta / n\right) - Q^{*} = \frac{\theta \kappa_1}{n} \left\{ \frac{\kappa_j}{\kappa_1} \Gamma_{\alpha/2}(n,1) - n \right\} < 0,$$

and

$$L^{(j)}_{u} = \Gamma_{1-\frac{\alpha}{2}}\left(n, \kappa_j \theta / n\right) - Q^{*} = \frac{\theta \kappa_1}{n} \left\{ \frac{\kappa_j}{\kappa_1} \Gamma_{1-\frac{\alpha}{2}}(n,1) - n \right\} > 0,$$

where $\Gamma_v(n,1)$ is the v^{th} percentile of the $gamma(n,1)$. As $gamma(n,1)$ is a right-skewed distribution, it is easily deduced that for each estimation policy $\left|L_u^{(j)}\right| > \left|L_\ell^{(j)}\right|$.

Proposition 4: Given the sample size, n, for any $R < R_o$ we have

(i) $\left|L_\ell^{(2)}\right| < \left|L_\ell^{(1)}\right| < \left|L_\ell^{(3)}\right|$,

(ii) $L_u^{(3)} < L_u^{(1)} < L_u^{(2)}$,

(iii) $L_u^{(3)} - L_\ell^{(3)} < L_u^{(1)} - L_\ell^{(1)} < L_u^{(2)} - L_\ell^{(2)}$,

while when $R > R_o$

(i) $\left|L_\ell^{(2)}\right| < \left|L_\ell^{(3)}\right| < \left|L_\ell^{(1)}\right|$,

(ii) $L_u^{(1)} < L_u^{(3)} < L_u^{(2)}$,

(iii) $L_u^{(1)} - L_\ell^{(1)} < L_u^{(3)} - L_\ell^{(3)} < L_u^{(2)} - L_\ell^{(2)}$.

Proof: See in the Appendix.

The findings of propositions 1 up to 4 are accompanied with results displayed in Tables 2 and 3. The general conclusion is that no estimation policy predominates over the others, and the choice is left in inventory managers. At the stage of selecting an estimation policy, inventory managers should balance the loss in the requested critical fractile against increases which they will face in mean square errors and maximum deviations of estimates from the optimal order quantity with a pre-specified probability. In the current work, under an exponential demand, we offer the necessary formulae for constructing at any requested critical fractile tables similar to table 2 (or table 3), in order practitioners to have the appropriate information to come up with a decision.

Table 2: Sizes of the statistical criteria when the requested critical fractile takes on those values where the differences between them and the actual critical fractiles attained by the HEP estimator become maximum

n	$R_m^{(3)}$	$R_{act}^{(3)}$	$\frac{MSE\left(Q_{n+1}^{(2)}\right)}{MSE\left(Q_{n+1}^{(3)}\right)}$	$\frac{L_u^{(2)} - L_\ell^{(2)}}{L_u^{(3)} - L_\ell^{(3)}}$	$\frac{L_u^{(2)}}{L_u^{(3)}}$
2	0,7037	0,5556	2,946	1,674	2,197
5	0,6651	0,5981	1,512	1,223	1,478
10	0,6495	0,6145	1,225	1,105	1,273
20	0,6411	0,6231	1,106	1,051	1,166
30	0,6381	0,6261	1,069	1,034	1,127
40	0,6367	0,6276	1,051	1,025	1,105
50	0,6358	0,6285	1,041	1,020	1,092
100	0,6339	0,6303	1,020	1,010	1,060
300	0,6327	0,6315	1,007	1,003	1,033

Table 3: Sizes of the statistical criteria when the requested critical fractile takes on those values where the differences between them and the actual critical fractiles attained by the DEP estimator become maximum

n	R_o	$R_{act}^{(1)}$	$\frac{MSE\left(Q_{n+1}^{(2)}\right)}{Var\left(Q_{n+1}^{(1)}\right)}$	$\frac{L_u^{(2)} - L_\ell^{(2)}}{L_u^{(1)} - L_\ell^{(1)}}$	$\frac{L_u^{(2)}}{L_u^{(1)}}$
2	0,8984	0,7823	5,006	1,869	2,356
5	0,8803	0,8295	1,854	1,246	1,480
10	0,8726	0,8464	1,355	1,110	1,266
20	0,8680	0,8547	1,162	1,052	1,161
30	0,8660	0,8571	1,105	1,034	1,122
40	0,8647	0,8580	1,077	1,025	1,102
50	0,8638	0,8584	1,061	1,020	1,089
100	0,8603	0,8576	1,030	1,010	1,058
300	0,8492	0,8483	1,009	1,003	1,030

For example, suppose that demand is available for a sample of 10 periods, and the values of parameters p, c, v, s result in a requested critical fractile equal to $R_m^{(3)}$, where the deviation $R - R_{act}^{(3)}$ becomes maximum. In such a case, from the data of table 2, the manager can evaluate whether by selecting the HEP estimator a reduction of 3.5% from the requested critical fractile could compensate for a more than 22% increase of the MSE and the maximum deviations of estimates from the optimal order quantity (with probability 95%) which he will face by using the UPEP estimator. In the same hypothetical example, if

$R = R_o$, the inventory manager would face a similar dilemma, that is, the choice among the DEP and UPEP estimators. With $n = 10$, the use of the DEP estimator would lead to a reduction of 2,6% from the requested critical fractile. But on the other hand, the selection of the UPEP estimator would give a MSE increased by 35.5%, and a maximum deviation of estimates from the optimal order quantity increased by 26.6 % (with probability 95%).

We are concluding therefore, that the use of the UPEP estimator can offer the requested critical fractile, but, unfortunately, its use has as consequences larger MSE's and larger range of deviation of estimates from the optimal order quantity. Of course such problems do not exist in large samples as the sizes of all the statistical criteria under consideration are approximately the same.

6. Conclusions

In this per we consider the classical newsvendor model with exponential demand and for the first time we evaluate three estimation policies for the optimal order quantity. We name them as Direct Estimation Policy (DEP), Unbiased Percentile Estimation Policy (UPEP) and Hayes Estimation Policy (HEP). According to the principle of the DEP, the estimator for the optimal order quantity was obtained after we had replaced the parameter of the exponential distribution with its maximum likelihood estimator in the formula which determines the optimal order quantity.

In the context of the UPEP, we modified the DEP estimator in order the new adjusted estimator to attain the requested critical fractile, namely, the probability of not having a stock-out during the period. The HEP is based on the concept of the Expected Total Operating Cost (ETOC) or the equivalent concept of the a-priori Expected Profit (apEP). In the current work, we obtained the apEP by taking the expected value of the expected profit after we had replaced the order quantity with a linear function of the maximum likelihood estimator of the parameter of exponential distribution. The linear function was then specified by maximizing the apEP.

To perform the evaluation we considered three statistical measures for which we derived analytic forms. The first statistical measure is the actual critical fractile which each policy attains. The actual critical fractile is the effective probability the estimated order quantity to meet the total demand which will occur during the period. By definition, only the PUEP estimator attains the requested critical fractile. The second statistical measure is the Mean Square Error (MSE) of the estimator of each policy, and only the DEP estimator is unbiased for the optimal order quantity. The final measure refers to the range of deviations of estimates from the optimal order quantity, when the probability to take such a range is the same for the three estimation policies.

The behavior of the aforementioned three statistical measures was studied analytically for different combination of sample sizes and requested critical fractiles. The general conclusion is that no estimation policy predominates over the others. The closer the actual critical fractile lies to the requested one, the larger the MSE of the estimator and the larger the range of deviations of estimates form the optimal order quantity we take. So, although the UPEP estimator gives the requested critical fractile, it has the largest mean square error, the largest range of deviations, and the largest deviation of estimates from the optimal order quantity. Among the DEP and HEP estimators, which one attains the closest actual critical fractile to the requested one depends upon the value of the requested critical fractile. When this value is quite close to 1, the HEP estimator gives a better actual critical fractile, but it has larger MSE and larger range of deviations of estimates from the optimal order quantity compared to the DEP estimator.

Consequently, the choice among the UPEP estimator on the one side and the DEP or the HEP estimator on the other side can be made on a subjective base accordingly to the preferences of inventory managers. So, if a manager requires a high degree of confidence that the requested critical fractile will eventually occur, he should use the UPEP estimator, knowing however the consequences of such a choice regarding the relatively greater size of the MSE and the size of deviations of estimates from the optimal order quantity. But, when demand follows the exponential distribution, this problem of choice is in existence only for the case of small samples.

We showed that with samples over 40 observations, the deviation of the actual critical fractile attained by the DEP and the HEP estimator from the corresponding requested critical fractile is negligible. Thus, with samples over 40 observations, the choice is restricted among the DEP and the HEP estimator. In this paper, for different sample sizes, we give the values of the requested critical fractile which eventually enable us to make the choice between the latter two estimation policies.

APPENDIX

Proof of Proposition 1

Taking n as a fixed quantity, first and second order conditions to maximize $R - R_{act}^{(1)}$ are obtained from the following derivatives:

$$\frac{d\left(R - R_{act}^{(1)}\right)}{dR} = \frac{d}{dR}\left\{-(1-R) + n^n\left[n - \ln(1-R)\right]^{-n}\right\} = 1 - \left[\frac{n}{n - \ln(1-R)}\right]^{(n+1)} \frac{1}{(1-R)} = 0$$

and

$$\frac{d^2\left(R - R_{act}^{(1)}\right)}{dR^2} = \frac{n^{n+1}}{(1-R)^2}\left\{\frac{n+1}{\left[n - \ln(1-R)\right]^{n+2}} - \frac{1}{\left[n - \ln(1-R)\right]^{n+1}}\right\} =$$

$$= \frac{n^{n+1}}{(1-R)^2}\left[n - \ln(1-R)\right]^{-(n+2)}\left\{1 + \ln(1-R)\right\}.$$

Hence, $R_m^{(1)}$ maximizes the deviations $R - R_{act}^{(1)}$ when

$$\frac{n}{n - \ln\left(1 - R_m^{(1)}\right)} - \left(1 - R_m^{(1)}\right)^{1/(n+1)} = 0, \tag{A1}$$

and $n + \ln\left(1 - R_m^{(1)}\right) < 0$ or $R_m^{(1)} > \dfrac{e-1}{e}$.

At R_o the difference $R_{act}^{(1)} - R_{act}^{(3)}$ is eliminated. But from (8c), $R_{act}^{(1)} = R_{act}^{(3)}$ when $\psi_3(n, R_o) = 0$ or

$$\frac{n}{n - \ln(1 - R_o)} - (1 - R_o)^{1/(n+1)} = 0. \tag{A2}$$

From (A1) and (A2) it is concluded that $R_m^{(1)} = R_o$, which completes the proof.

Proof of Proposition 2

Taking n as a fixed quantity, we take the following derivatives:

$$\frac{d\left(R - R_{act}^{(3)}\right)}{dR} = \frac{d}{dR}\left\{-(1-R) + (1-R)^{n/(n+1)}\right\} = 1 - \frac{n}{n+1}(1-R)^{-1/(n+1)} = 0 \tag{A3}$$

and

$$\frac{d^2\left(R - R_{act}^{(3)}\right)}{dR^2} == -\frac{n}{(n+1)^2}(1-R)^{-(n+2)/(n+1)} < 0. \tag{A4}$$

Hence, from (A3) and (A4), the deviations $R - R_{act}^{(3)}$ are maximized at

$$R_m^{(3)} = 1 - \left(\frac{n}{n+1}\right)^{n+1}.$$

For $n = 2$

$$R_m^{(3)} = \left(1 - \frac{2}{3}\right)\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2\right) = \frac{19}{27} \approx 0.7037$$

and

$$\lim_{n\to\infty} R_m^{(3)} = 1 - \frac{1}{\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n} \lim_{n\to\infty} \frac{n}{n+1} = 1 - \frac{1}{e} = \frac{e-1}{e}$$

which complete the proof.

Proof of Proposition 3

(i) $\mathrm{MSE}\left(\hat{Q}_{n+1}^{(2)}\right) - \mathrm{MSE}\left(\hat{Q}_{n+1}^{(1)}\right) = \left\{\frac{1}{n}(\kappa_2 - \kappa_1)(\kappa_2 + \kappa_1) + (\kappa_2 - \kappa_1)^2\right\}\theta^2 > 0$,

since

$$\kappa_2 - \kappa_1 = \frac{n\left(1 - (1-R)^{1/n}\right)}{(1-R)^{1/n}} - \ln(1-R)^{-1} = \frac{n - \ln(1-R)}{(1-R)^{1/n}}\left\{\frac{n}{n - \ln(1-R)} - (1-R)^{1/n}\right\} > 0, \quad \text{(A5)}$$

which follows from (10).

(ii) $\mathrm{MSE}\left(\hat{Q}_{n+1}^{(2)}\right) - \mathrm{MSE}\left(\hat{Q}_{n+1}^{(3)}\right) = \left\{(\kappa_2 - \kappa_3)\left[\left(\frac{n+1}{n}\kappa_2 - \kappa_1\right)\right] + \left(\frac{n+1}{n}\kappa_3 - \kappa_1\right)\right\}\theta^2 > 0$,

as

$$\kappa_2 - \kappa_3 = \frac{n\left(1 - (1-R)^{1/n}\right)}{(1-R)^{1/n}} - n\left[(1-R)^{-1/(n+1)} - 1\right] = n\left\{(1-R)^{-1/n} - (1-R)^{-1/(n+1)}\right\} > 0, \quad \text{(A6)}$$

and

$$\frac{n+1}{n}\kappa_3 - \kappa_1 = (n+1)(1-R)^{-1/(n+1)} - (n+1) + \ln(1-R) =$$

$$= \frac{n - \ln(1-R)}{(1-R)^{1/r}}\left\{\frac{r}{r - \ln(1-R)} - (1-R)^{1/r}\right\} > 0, \quad \text{(A7)}$$

which follows from (A1), by setting $r = n+1$.

(iii) $\mathrm{MSE}\left(\hat{Q}_{n+1}^{(3)}\right) - \mathrm{MSE}\left(\hat{Q}_{n+1}^{(1)}\right) = \left(\kappa_3 - \kappa_1\right)\left(\frac{n+1}{n}\kappa_3 - \frac{n-1}{n}\kappa_1\right)\theta^2$.

But

$$\kappa_3 - \kappa_1 = n\left[(1-R)^{-1/(n+1)} - 1\right] - \ln(1-R)^{-1} = \frac{n - \ln(1-R)}{(1-R)^{-1/(n+1)}}\left\{\frac{n}{n - \ln(1-R)} - (1-R)^{1/(n+1)}\right\},$$

and from (A3) $\frac{n-1}{n}\kappa_1 < \kappa_3 < \frac{n+1}{n}\kappa_3$. Hence from (11) and (A4):

For $R < R_o$, $\kappa_3 - \kappa_1 < 0$ and $\mathrm{MSE}\left(\hat{Q}_{n+1}^{(3)}\right) - \mathrm{MSE}\left(\hat{Q}_{n+1}^{(1)}\right) < 0$. (A8.1)

For $R > R_o$, $\kappa_3 - \kappa_1 > 0$ and $\mathrm{MSE}\left(\hat{Q}_{n+1}^{(3)}\right) - \mathrm{MSE}\left(\hat{Q}_{n+1}^{(1)}\right) > 0$. (A8.2)

The proof is completed, by using (9) and combining (A1), (A2), and (A4) to get:

(i) From (A1), $\frac{\kappa_1}{\kappa_2} < 1$, and $\beta_n^{(2)} - \beta_n^{(1)} < 0$

(ii) From (A2), $\frac{\kappa_1}{\kappa_2} < \frac{\kappa_1}{\kappa_3}$, and $\beta_n^{(2)} - \beta_n^{(3)} < 0$

(ii) From (A4) and for $R < R_o$, $\frac{\kappa_1}{\kappa_3} > 1$, and $\beta_n^{(1)} - \beta_n^{(3)} < 0$, while

for $R < R_o$, $\frac{\kappa_1}{\kappa_3} < 1$, and $\beta_n^{(1)} - \beta_n^{(3)} > 0$.

Proof of Proposition 4

(i) Since $L_\ell^{(j)}$'s are negative:

$$L_\ell^{(2)} - L_\ell^{(1)} = \frac{\theta\kappa_1}{n}\Gamma_{\alpha/2}(n,1)\left(\frac{\kappa_2}{\kappa_1} - 1\right)$$

and $\left|L_\ell^{(2)}\right| < \left|L_\ell^{(1)}\right|$ which follows from (A5) ,

$$L_\ell^{(2)} - L_\ell^{(3)} = \frac{\theta\kappa_1}{n}\Gamma_{\alpha/2}(n,1)\left(\frac{\kappa_2 - \kappa_3}{\kappa_1}\right)$$

and $\left|L_\ell^{(2)}\right| < \left|L_\ell^{(3)}\right|$ which follows from (A6),

$$L_\ell^{(3)} - L_\ell^{(1)} = \frac{\theta\kappa_1}{n}\Gamma_{\alpha/2}(n,1)\left(\frac{\kappa_3}{\kappa_1} - 1\right)$$

and $\left|L_\ell^{(1)}\right| < \left|L_\ell^{(3)}\right|$ for $R < R_o$ which follows from (A8.1),

while $\left|L_\ell^{(1)}\right| > \left|L_\ell^{(3)}\right|$ for $R > R_o$ which follows from (A8.2).

(ii) Since $L_u^{(j)}$'s are positive, from (A5), (A6), and (A8):

$$L_u^{(2)} - L_u^{(1)} = \frac{\theta\kappa_1}{n}\Gamma_{1-\frac{\alpha}{2}}(n,1)\left(\frac{\kappa_2}{\kappa_1} - 1\right) > 0,$$

$$L_u^{(2)} - L_u^{(3)} = \frac{\theta\kappa_1}{n}\Gamma_{1-\frac{\alpha}{2}}(n,1)\left(\frac{\kappa_2 - \kappa_3}{\kappa_1}\right) > 0,$$

and

$$L_u^{(3)} - L_u^{(1)} = \frac{\theta\kappa_1}{n}\Gamma_{1-\frac{\alpha}{2}}(n,1)\left(\frac{\kappa_3}{\kappa_1} - 1\right)$$

which is negative for $R < R_o$ and positive for $R > R_o$.

(iii) The proof is completed by using again (A5), (A6), and (A8) in the following differences:

$$\left(L_u^{(2)} - L_\ell^{(2)}\right) - \left(L_u^{(1)} - L_\ell^{(1)}\right) = \frac{\theta\kappa_1}{n}\left(\Gamma_{1-\frac{\alpha}{2}}(n,1) - \Gamma_{\alpha/2}(n,1)\right)\left(\frac{\kappa_2}{\kappa_1} - 1\right) > 0,$$

$$\left(L_u^{(2)} - L_\ell^{(2)}\right) - \left(L_u^{(3)} - L_\ell^{(3)}\right) = \frac{\theta\kappa_1}{n}\left(\Gamma_{1-\frac{\alpha}{2}}(n,1) - \Gamma_{\alpha/2}(n,1)\right)\left(\frac{\kappa_2 - \kappa_3}{\kappa_1}\right) > 0,$$

and

$$\left(L_u^{(3)} - L_\ell^{(3)}\right) - \left(L_u^{(1)} - L_\ell^{(1)}\right) = \frac{\theta\kappa_1}{n}\left(\Gamma_{1-\frac{\alpha}{2}}(n,1) - \Gamma_{\alpha/2}(n,1)\right)\left(\frac{\kappa_3}{\kappa_1} - 1\right)$$

which is negative for $R < R_o$ and positive for $R > R_o$.

REFERENCES

Akcay, A., Biller, B., Tayur, S. (2011). Improved inventory targets in the presence of limited historical data. *Manufacturing & Service Operations Management*, DOI 10.1287/ msom. 1100.0320.

Azoury, K.S. (1985). Bayes solution to dynamic inventory models under unknown demand distribution. *Management Science* **31**: 1150-1160.

Conrad, S.A. (1976). Sales Data and the Estimation of Demand. *Operational Research Quarterly*, **27:** 123–127.

Devroye, L. (1986). Non uniform random variate generation. Springer-Verlag, New York.

Ding, X., Ruterman, M.L., Bisi, A. (2002). The censored newsvendor and the optimal acquisition of information. *Operations Research* **50**: 517-527.

Ernst, R., Kamrad, B. (2006). Estimating demand by using sales information: inaccuracies encountered. *European Journal of Operational Research* **174**: 675-688.

Frigyik, B., Kapila, A., Gupta, M.R. (2010). Introduction to the Dirichlet distribution and related processes. Technical Report, UWEETR-2010-0006, Department of Electrical Engineering, University of Washinghton.

Halkos, G.E., Kevork, I.S. (2011). Non-negative demand in newsvendor models: The case of singly truncated normal samples. MPRA Paper No. 31842, Online at http://mpra.ub.uni-muenchen.de/31842/.

Halkos, G.E., Kevork, I.S. (2012). Validity and precision of estimates in the classical newsvendor model with exponential and Rayleigh demand. MPRA Paper No. 36460, Online at http://mpra.ub.uni-muenchen.de/36460/.

Hayes, R.H. (1969). Statistical estimation problems in inventory control. *Management Science* **15**: 686-701.

Hill RM. (1997). Applying Bayesian Methodology with a Uniform Prior to the Single Period Inventory Model. *European Journal of Operational Research* **98:** 555–562.

Iglehart, D.L. (1964). The dynamic inventory problem with unknown demand distribution. *Management Science* **10**: 429-440.

Janssen E, Strijbosch L, Brekelmans R. (2009). Assessing the effects of using demand parameters estimates in inventory control and improving the performance using a correction function. *International Journal of Production Economics* **118**: 34-42.

Katircioglou, K. (1996). Essays in inventory control. Ph.D thesis, University of British Columbia, Vancouver, BC, Canada.

Kevork IS. (2010). Estimating the optimal order quantity and the maximum expected profit for single-period inventory decisions. *Omega* **38**: 218–27.

Khouja M. (1999). The single-period (news-vendor) problem: literature review and suggestions for future research. *Omega* **27**:537–53.

Lau H. (1997). Simple Formulas for the Expected Costs in the Newsboy Problem: an educational note. *European Journal of Operational Research* **100:** 557–61.

Lau, H., Lau, A.H. (1996). Estimating the demand distributions of single-period items having frequent stock-outs. *European Journal of Operational Research* **92**: 254-265.

Liyanage, L.H., Shanthikumar, J.G. (2005). A practical inventory control policy using operational statistics. *Operations Research Letters* **33**: 341-348.

Nahmias S. (1994). Demand Estimation in Lost Sales Inventory Systems. *Naval Research Logistics* **41:** 739–757.

Paris, R.B. (2010). Incomplete gamma and related functions, in NIST Handbook of Mathematical Functions, Cambridge University Press, Edited by Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.

Ritchken, P.H., Sankar, R. (1984). The effect of estimation risk in establishing safety stock levels in an inventory model. *The Journal of the Operational Research Society* **35**: 1091-1099.

Scarf, H.E. (1959). Bayes solutions of the statistical inventory problem. *Annals of Mathematical Statistics* **30**: 490-508.

Schweitzer ME, Cachon GP. (2000). Decision Bias in the Newsvendor Problem with a Known Demand Distribution: Experimental Evidence. *Management Science*, **46:**. 404–420.

Su, R.H., Pearn, W.L. (2011). Product selection for newsvendor-type products with normal demands and unequal costs. *International Journal of Production Economics*, doi: 10.1016/j.ijpe.2011.04.010.