Game theory model for European government bonds market stabilization: a saving-State proposal

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Game Theory Model for European Government Bonds Market Stabilization: a Saving-State Proposal

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Abstract

The aim of this paper is to present a proposal regarding the possible stabilization of the rapid variations on the value of government bonds issued by the States, using the "Game Theory". In particular, we focus our attention on three players: a large bank that has immediate access to the market of government bonds (hereinafter called Speculator, our first player), the European Central Bank (ECB, the second player) and the State in economic difficulty (our third player). We propose on financial transactions the introduction of a tax (cashed directly by the State in economic difficulty), which hits only the speculative profits. We show that the above tax would probably be able to avert the speculation, and, even in case of speculation on its government bonds, the State manages to pull itself out of the crisis. Finally, we also propose a cooperative solution that enables all economic actors involved (the Speculator, the ECB and the State) to obtain a profit.

JEL: C7,E4,G1,G2.

1 Introduction

Lately, the global economic crisis is increased, affecting also States considered very important in the economic field (as for example Italy). One of the causes of the crisis is the exponential growth in government bonds yields, which has increased the public debt of the States.

In the Fig.1 we can see that up to May 2011 the Italian 10-years and 3-years government bonds offered a yield of approximately 4.80% and 3.15%, while in December 2011 both rose above the 7.50% (see [14]).

Figure 1: Trend of Italian 10-years and 3-years government bonds

In the figure 2 we can see the trend upwards of Irish, Portuguese and Spanish 10-years government bonds from January 2010 to July 2011 (see the [15]).

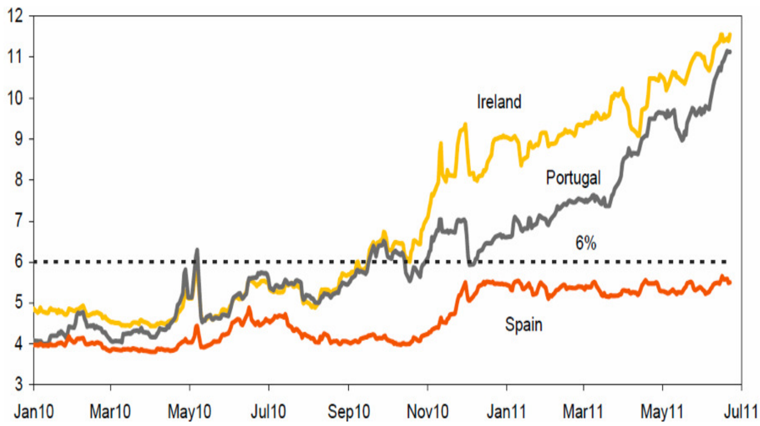

Figure 2: Trend of Irish, Portuguese and Spanish 10-years government bonds

In the figure 3 we can see the trend of main European States 10-years government bonds from June 2011 to January 2012 (see [16]).

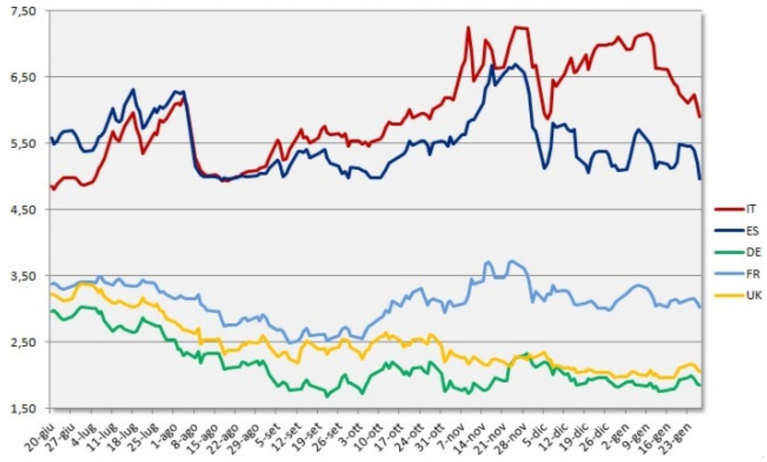

Figure 3: Trend of main European 10-years government bonds

In this regard, with our paper, we intend to propose (using Game Theory [for a complete study of a game see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]]) a possible method to stabilize the government bonds market of the States in economic difficulty, without any losses of collective gain. In this way, with the introduction of a simple but effective tax, the market would be able by itself to reduce yields on government bonds, without further economic measures at global level: thus the States in financial difficulty could finally begin (hopefully) a slow but steady economic recovery.

2 Methodologies

The normal-form game G, that we propose to model our financial interaction, requires a construction which takes place on 3 times, which we say time 0, time 1/2 and time 1.

• At time 0 the Speculator (the first player) can decide

1) to sell short government bonds, in order to obtain greater profit betting on a greater future yield of the bonds;

2) not to intervene in the government bonds market.

• At time 1/2 the ECB may decide to intervene in the bonds market in order to limit the growth of the bonds yield. In this way, even in case of lack of demand of government bonds, the issuer States finds the funds necessary to the national financial requirement.

• At time 1 the Speculator must eventually close its position (opened at time 0) by buying government bonds.

Remark. During the game, we will refer to an interest i which determines the yield on government bonds. When we pass from one time period to another one, we should actualize or capitalize the values that must be "transferred". But because the interest i_u (used in the capitalization factor and discount factor) is much lower than that one we use to get the yield of government bonds, we assume i_u equal to 0. Therefore, in this model, the values referred to different time period are not capitalized or discounted.

3 Financial preliminaries

Here we recall the financial concepts that we shall use in the present article.

1. M represents the quantity expressed in money of issued bonds (for example Italy has to issue a quantity equal to M of government bonds in order to face its financial commitments).
2. Short selling of bonds is a financial transaction involving the sale of bonds without having their property, hoping to buy them later at a lower price. So the short seller would realize a profit. In the event that we examine, talking about government bonds, the hope of short sellers consists in a increase of the yield on government bonds.
3. The government bonds are not normal goods with a purchase price and a sale price. The concept that characterizes them is the yield, which depends upon the interest to which they are sold. The yield on a government bond is given by the interest that remunerates the capital "loaned" to the State.

4 The description of the game

Our first player, the Speculator, may choose to sell short government bonds, in order to cause an increase in the their yield and so to obtain a profit. In fact, at time 1 the Speculator must close the position opened in the government bonds market with a purchase transaction. Otherwise the Speculator can decide not to intervene in the government bonds market.

Thus, the Speculator has the possibility to choose among the strategies $x \in [0, 1]$ which represents the percentage of the quantity of government bonds M that the Speculator decides to buy, depending it intends:

1. not to make any financial transaction ($x = 0$);
2. to sell short government bonds (xM is the quantity of short sold bonds) ($0 < x \leq 1$).

On the other hand, the European Central Bank, that is our second player, operates in the bonds market in consequence of the operation of the first player. It may choose a strategy $y \in [0, 1]$, which represents the percentage of the quantity of government bonds M that the ECB purchases, depending it intends:

1. to buy government bonds of the State in economic difficulty ($y > 0$);
2. not to intervene in the government bonds market ($y = 0$).

In Fig.4 we illustrate graphically the bi-strategy space $E \times F$ of our game G.

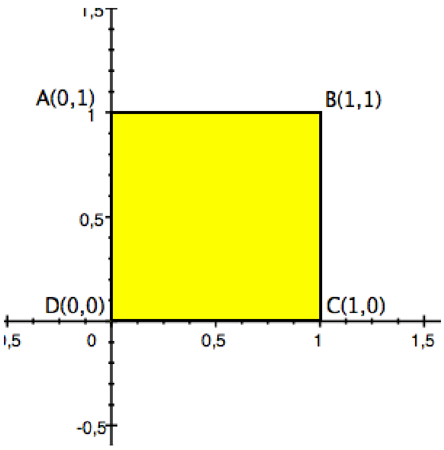

Figure 4: The bi-strategy space of the game

5 The no tax game

5.1 The payoff function of the Speculator in the no tax game

The payoff function of the Speculator, which is the function that represents the gain of the first player, is given by the quantity expressed in money of purchased bonds xM, multiplied by the difference $R_1(x, y) - R_0$ between the value at time 1 of the yield to be cashed (at time 1 the Speculator buys the same amount of securities that it has sold short at time 0) and the value at time 0 of the yield to be paid (at time 0 the Speculator sells short a certain amount of government bonds).

The payoff function of the Speculator is given by:

$$f_1(x, y) = xM(R_1(x, y) - R_0), \tag{1}$$

where:

1. xM is the amount of government bonds that the Speculator short sells at time 0
2. $R_1(x, y)$ is the value of the government bonds yield at time 1. We suppose it is given by
$$R_1(x, y) = i + mx - ny,$$
where
   - i is the interest that remunerates the capital "loaned" to the State;
   - m is a marginal coefficient which indicates the incidence of x on $R_1(x, y)$;

- n is a marginal coefficient which indicates the incidence of y on $R_1(x, y)$.

$R_1(x, y)$ depends on x because if the Speculator intervenes in the government bonds market with a strategy $x \neq 0$, the yield $R_1(x, y)$ is modified because a decline in demand has a positive effect on the interest charged on the government bond. $R_1(x, y)$ depends on y because if the ECB intervenes in the government bonds market with a strategy $y \neq 0$, the value $R_1(x, y)$ is modified because an increase in demand has a negative effect on the government bonds yield (the interest that remunerates the bond goes down). We are assuming as a hypothesis both for x and y a linear dependence.

3. R_0 is the value of the yield at time 0. It is given by $R_0 = i$, where i is the interest that remunerates the capital that is "loaned" to the State. R_0 is a constant because on it does not have impact our strategies x and y.

The payoff function of the Speculator. Therefore, recalling the function R_1, the definition of R_0 and the function f_1, we have

$$f_1(x, y) = xM(mx - ny). \tag{2}$$

5.2 The payoff function of the ECB in the no tax game

The payoff function of the ECB, that is the function representing the algebraic gain of the ECB, is given by the multiplication of the quantity expressed in money of government bonds yM (that the ECB buys at time $1/2$) by the bonds yield at time $1/2$, that is

$$R_{1/2}(x) = i + mx.$$

So the payoff function of the ECB is given by:

$$f_2(x, y) = yM(R_{1/2}(x)), \tag{3}$$

where

1. yM is the quantity of bonds expressed in money that the ECB buys at time $1/2$;

2. $R_{1/2}(x)$ is the bonds yield at time $1/2$. It is given by:

$$R_{1/2}(x) = i + mx.$$

   On it has impact the strategy x because at time 0 the Speculator has already operated in the market, changing the bonds yield.

The payoff function of the ECB. Recalling functions $R_{1/2}$ and f_2, we have

$$f_2(x, y) = yM(i + mx). \tag{4}$$

5.3 The payoff functional relation of the State

In addition to the payoff functions of the Speculator and the ECB must also be considered the payoff functional relation of the State. It is given by the quantity M of issued government bonds, multiplied by the difference between the yield R_0 (which the State would pay without the intervention on the market of the Speculator and of the ECB) and the yield R_1 (which actually pays in consequence of the strategies x of the Speculator and y of the ECB).

Payoff functional relation of the State. It is given by:

$$f_3 = M(R_0 - R_1(x, y)). \tag{5}$$

Recalling the definition of R_0, the function R_1, and the functional relation f_3, we have

$$f_3(x, y) = M(-mx + ny). \tag{6}$$

The payoff function of the no tax game is so given, for every $(x, y) \in E \times F$, by:

$$f(x, y) = (xM(mx - ny), yM(i + mx), M(-mx + ny)) \tag{7}$$

6 Study of the no tax game

6.1 Critical space of the no tax game

Since we are dealing with a non-linear game, it is necessary to study in the bi-win space also the points of the critical zone that belong to the bi-strategy space. In order to find the critical area of the game, we consider the Jacobian matrix and we put its determinant equal 0.

About the gradients of f_1 and f_2, we have

$$\nabla f_1(x, y) = (M(2mx - ny), -nxM)$$

$$\nabla f_2(x, y) = (Mmy, M(i + mx)).$$

The determinant of the Jacobian matrix is:

$$\det J_f(x, y) = M^2(2mx - ny)(i + mx) + M^2 mxny.$$

Therefore, the critical space of the game is:

$$Z_f = \{(x, y) : M^2(2mx - ny)(i + mx) + M^2 mxny = 0\}.$$

Dividing by $M^2 m$, which are all positive numbers (strictly greater than 0), after calculations finally we have: $Z_f = \{(x, y) : y = 2mx(i - mx)(ni)^{-1}\}$.

Assuming that $m = 1/2$, $n = 1/2$ and $i = 1/4$, we obtain

$$Z_f = \{(x, y) : y = (1/2)x^2 + (1/4)x\}.$$

The critical area of our bi-strategy space is represented in the Fig.5 by the segment $[D, H]$.

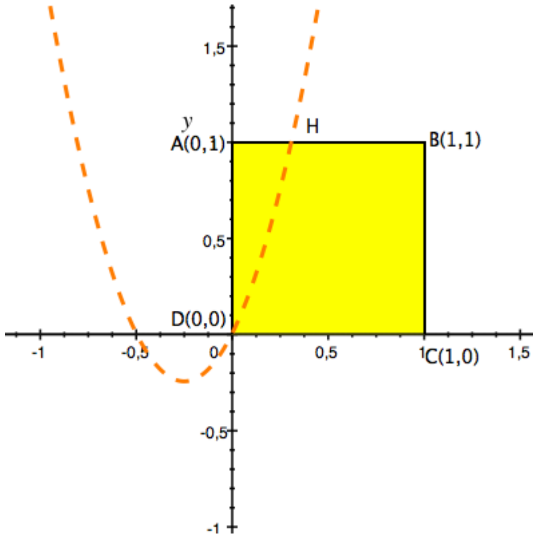

Figure 5: The critical zone of the no tax game

6.2 Payoff space of the no tax game

In order to represent graphically the payoff space $f(E \times F)$, we transform, by the function f, all the sides of bi-strategy square $E \times F$ and the critical space Z of the game G.

The segment $[B, C]$ is the set of all the bi-strategies (x, y) such that $x = 1$ and $y \in [0, 1]$.

Calculating the image of the generic point $(1, y)$, we have: $f(y, 1) = (M(m - ny), yM(i + m))$. Therefore, setting

$$X = M(m - ny) \wedge Y = yM(i + m),$$

and assuming $M = 1$, $i = 1/4$ and $n = m = 1/2$, we have

$$X = 1/2 - (1/2)y \text{ and } Y = (3/4)y.$$

Replacing Y instead of y in the first equation, we obtain the image of the segment $[B, C]$, defined as the set of the bi-wins (X, Y) such that

$$X = 1/2 - (1/2)(4/3)Y = 1/2 - (2/3)Y \ \wedge \ Y \in [0, 3/4].$$

It is a line segment with extremes $B' = f(B)$ and $C' = f(C)$.

Following the procedure described above for the other sides of the bi-strategy square and for the critical space, that are the segments $[A, B]$, $[C, D]$, $[D, A]$ and $[D, H]$, we obtain the Fig.6 on the payoff space $f(E \times F)$ of our game G.

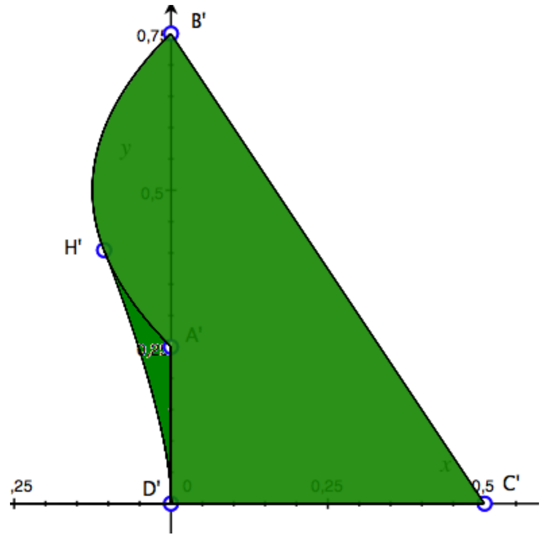

Figure 6: The payoff space of the no tax game

We note that we get a sail-formed figure, but the results must now be interpreted according to the payoff functional relation of the issuer State. Recalling the functional relation f_3, and that $M = 1$ and $n = m = 1/2$, we note that:

- if the two players arrive on the points D' and B' the yield (that the issuer State must pay for its government bonds) remains unchanged because it is balanced by two equal opposing forces. This solution is undesirable because it does not solve the problems of the State and not gives breath to its economy;
- if the two players arrive on the side $]B', C']$, the yield paid by the State for its government bonds increases inexorably, bringing it closer to default;
- if the two players arrive in $[A', D'[$, the yield paid by the State for its government bonds decreases, and thus the State could emerge from the crisis.

According to these considerations, is morally, ethically and economically desirable that the Speculator and the ECB arrive to the point $A = (0, 1)$, so that the paid yield goes down as more is possible and the State comes out of the crisis.

Remark. The point A' and the point B' have the same collective gain about the three subjects of our game. In fact, if we arrive to point A' the State in economic difficulty has a profit equal to $1/2$, the Speculator wins 0 and the ECB wins $1/4$. Instead if we arrive to point B' the State has a profit equal to 0, the Speculator wins 0 and the ECB wins $3/4$. In both points, the total gain of the game is $3/4$.

7 Equilibria of the no tax game

7.1 Nash equilibria of the no tax game

If the two players decide to adopt a selfish behavior, they choose their own strategy maximizing their partial gain. In this case, we should consider the classic Nash best reply correspondences. The best reply correspondence of the Speculator is the correspondence $B_1 : F \to E$ given by $y \mapsto \max_{f_1(\cdot,y)} E$, where $\max_{f_1(\cdot,y)} E$ is the set of all strategies in E which maximize the section $f_1(\cdot, y)$.

Symmetrically, the best reply correspondence $B_2 : E \to F$ of the ECB is given by $x \mapsto \max_{f_2(x,\cdot)} F$.

Choosing $M = 1$, $n = 1/2$ and $m = 1/2$, which are always positive numbers (strictly greater than 0), and recalling that $f_1(x, y) = xM(mx - ny)$, we have $\partial_1 f_1(x, y) = 2mxM - nyM$. So we have:

$$B_1(y) = \begin{cases} \{1\} & \text{if} \quad y < 1 \\ \{0,1\} & \text{if} \quad y = 1 \end{cases} .$$

Recalling also that $f_2(x, y) = yM(i + mx)$, we have $\partial_2 f_2(x, y) = M(i + mx)$ and so:

$$B_2(x) = \{1\} \ \forall\, x \in E.$$

In the Fig.7 we have in red the inverse graph of B_1, and in blue that one of B_2.

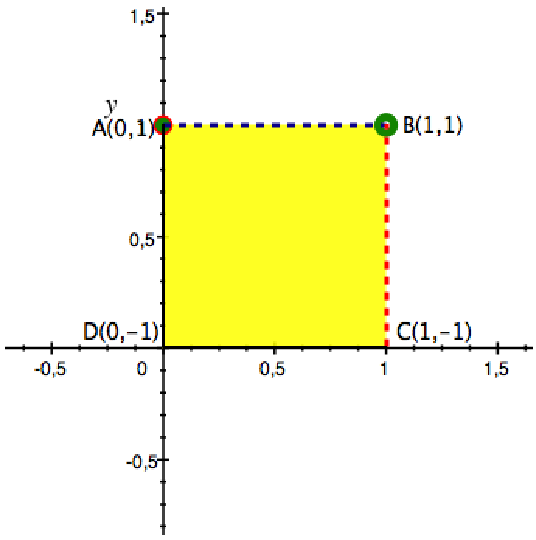

Figure 7: The Nash equilibria of the no tax game

The set of Nash equilibria, that is the intersection of the two best reply graphs (graph of B_2 and the symmetric of B_1), is $\mathrm{Eq(B_1, B_2)} = \{(1,1),(0,1)\}$.

Analysis of Nash equilibria. The Nash equilibrium $B = (1, 1)$ can be considered very good for the two players, because they are on the proper maximal Pareto boundary. The selfishness, in this case, pays well.

But the Nash equilibrium $B = (1, 1)$ does not solve the problems of the State that issues the government bonds, because it should pay a yield that the strategy $x = 1$ increases, and that is returned to its original level by the strategy $y = 1$. In a word, the State continues to fund its public spending with a government bonds yield too high for its possibilities. In the long term the State will end on the brink of the abyss.

The Nash equilibrium $A = (0, 1)$, instead, is good for the State, because the yield to pay on government bonds goes downward. But the point A' isn't on the maximal Pareto boundary.

Moreover, most likely, the Speculator will choose the strategy $x = 1$, because the strategy $x = 0$ precludes the opportunity of profit for the Speculator, which is stuck on the ordinate axis. With $x = 1$, instead, the Speculator tries to win something depending on the strategy of the ECB, and still manages to not lose. Basically, the most likely Nash equilibrium is the point $B = (1, 1)$: almost certainly the achievement of a Nash equilibrium would leave the issuer State in trouble and at risk default.

Remark. At this point, the ECB could consider splitting the win 3/4 obtained in the most likely Nash equilibrium B with the issuer State, in order to cancel the effects of the increase of the yield on government bonds. Thus, the ECB would give 1/2 (value that the State loses because of a strategy $x = 1$) to the State, taking for itself the sum of 1/4. But this seemingly simple solution is not feasible for several reasons:

1. the ECB has a policy that usually does not interfere with that one of the European States, therefore this kind of action is difficult to accomplish;
2. the "payback" to the State by the ECB could have very long timescales, and therefore the State could sink even deeper into economic crisis.
3. the amount cashed by the State cancels its loss (suffered because of the strategy of Speculator), but it would simply postpone the problem over time without dealing with it. In fact, if in the future other financial institutions buy government bonds, the State should pay them a yield which is remained at unsustainable levels, ending in bankruptcy (it is impossible to think that the ECB intervenes each time to save the State, giving him the major part of its profit)

For these reasons, it is important to find a method that allows the State to prevent speculation and not to be constantly "cured". Anyway, it is obvious that a vaccine made only once, is better than a medicine taken continuously, medicine which in future will lose its effectiveness.

Note. We can note that there are three possible cases:

1. If $m = n$ we have the case that we are studying.
2. If $m > n$ we have: $B_1(y) = \{1\}\ \forall\ y \in F$.

3. If $m < n$ we have: $B_1(y) = \begin{cases} \{1\} & \text{if} \quad y < m/n \\ \{0,1\} & \text{if} \quad y = m/n \\ \{0\} & \text{if} \quad m/n < y \le 1 \end{cases}$.

In order that our model achieves the aims which will be explained further on, is necessary that the marginal incidence m of the strategy x on R_1 isn't lower than the marginal incidence n of the strategy y on R_1 (in fact in this case the point $A = (0,1)$ is already the only Nash equilibrium).

But very likely, the value n is lower than the value m because the purchase of government bonds by the ECB could be less accepted by the market players. In fact, the action of the ECB could be seen as a behavior dictated (also, or even only) by political motivations, and not by economic reasons (like for example low risk and high profit about government bonds).

7.2 Defensive phase of the no tax game

We suppose that the two players are aware of the will of the other one to destroy it economically, or are by their nature cautious, fearful, paranoid, pessimistic or risk averse, and then they choose the strategy that allows them to minimize their loss. In this case, we talk about defensive strategies.

Conservative value and meetings. *Conservative value of a player.* It is defined as the maximization of its function of worst win.

Therefore, the conservative value of the Speculator is $v_1^\sharp = \sup_{x\in E} f_1^\sharp(x)$, where $f_1^\sharp$ is the function of worst win of the Speculator, and it is given by $f_1^\sharp(x) = \inf_{y\in F} f_1(x,y)$, for every x in E.

Choosing $M = 1$, $n = 0.5$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), and recalling that $f_1(x,y) = xM(mx - ny)$, we have: $f_1^\sharp(x) = \inf_{y\in F} xM(mx - ny)$.

Since the offensive strategies of the ECB are

$$O_2(x) = \begin{cases} \{1\} & \text{if} \quad x > 0 \\ \{F\} & \text{if} \quad x = 0 \end{cases},$$

we obtain:

$$f_1^\sharp(x) = \begin{cases} \{xM(mx - n)\} & \text{if} \quad x > 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}.$$

In Fig.8 appears $f_1^\sharp$ graphically.

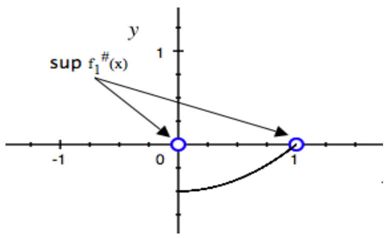

Figure 8: The function of worst win of the Speculator in the no tax game

So the defense (or conservative) strategy set of the Speculator is $E^\sharp = \{0,1\}$ and the conservative value of the Speculator is

$$v_1^\sharp = \sup_{x \in E} \inf_{y \in F} xM(mx - ny) = 0. \tag{8}$$

On the other hand, the conservative value of the ECB is given by $v_2^\sharp = \sup_{y \in F} f_2^\sharp$, where $f_2^\sharp$ is the function of the worst win of the ECB. It is given by $f_2^\sharp(y) = \inf_{x \in E} f_2(x, y)$, for every $y \in F$.

Choosing $M = 1$, $i = 0.25$ and $m = 0.5$, which are always positive numbers (strictly greater than 0), and recalling that $f_2(x, y) = yM(i + mx)$, we have: $f_2^\sharp(y) = \inf_{x \in E} yM(i + mx)$.

Since the offensive strategies of the Speculator are

$$O_1(y) = \begin{cases} \{0\} & \text{if} \quad y > 0 \\ \{E\} & \text{if} \quad y = 0 \end{cases},$$

we obtain:

$$f_2^\sharp(y) = \begin{cases} \{yMi\} & \text{if} \quad y > 0 \\ \{0\} & \text{if} \quad y = 0 \end{cases}.$$

In Fig.9 appears $f_2^\sharp$ graphically.

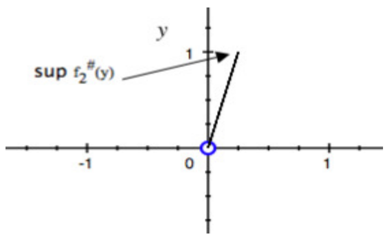

Figure 9: The function of worst win of the ECB in the no tax game

So the defense (or conservative) strategy set of the ECB is given by $F^\sharp = \{1\}$ and the conservative value of the ECB is

$$v_2^\sharp = \sup_{y \in F} \inf_{x \in E} yM(i + mx) = Mi. \tag{9}$$

Therefore, recalling that $M = 1$ and $i = 0.25$, the conservative bi-value is

$$v_f^\sharp = (v_1^\sharp, v_2^\sharp) = (0, 1/4).$$

Conservative meetings. They are represented by the bi-strategies $(x_\sharp, y_\sharp)$, that are represented by the values $B = (1, 1)$ and $A = (0, 1)$.

The conservative meeting $B = (1, 1)$ can be considered good for the Speculator and the ECB, because it is on the maximal Pareto boundary, but it is mediocre for the State. In fact, recalling that $f_3(x, y) = M(-ny + mx)$, the yield to pay on government bonds goes down by 1/2, in accordance with the strategy $y = 1$ of the ECB, but it re-increases by the same amount because of the strategy $x = 1$ of the Speculator.

The conservative meeting $A = (0, 1)$, instead, is good for the State, because the yield to pay on government bonds goes downward. But the point A' isn't on the maximal Pareto boundary.

Moreover, most likely, the Speculator will choose the strategy $x = 1$, because the strategy $x = 0$ precludes the opportunity for profit for the Speculator, which is stuck on the ordinate axis. With $x = 1$, instead, the Speculator tries to win something depending on the strategy of the ECB, and still manages to not lose. Basically, the most likely conservative meeting is $B = (1, 1)$: almost certainly the achievement of a conservative meeting would leave the issuer State in trouble and at risk default.

Note. Recalling that $f_1^\sharp(x) = \begin{cases} \{xM(mx-n)\} & \text{if} \quad x > 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}$, we can note that there are three possible cases:

1. If $m = n$ we have the case that we are studying.
2. If $m > n$ we have: $x_\sharp = 1$.
3. If $m < n$ we have: $x_\sharp = 0$.

In order that our model achieves the aims which will be explained further on, is necessary that the marginal incidence m of the strategy x on R_1 isn't lower than the marginal incidence n of the strategy y on R_1 (in fact in this case the point $A = (0, 1)$ is already the only defensive equilibrium). But very likely, the value n is lower than the value m because the purchase of government bonds by the ECB could be less accepted by the market players. In fact, the action of the ECB could be seen as a behavior dictated (also, or even only) by political motivations, and not by economic reasons (like for example low risk and high profit about government bonds).

7.3 Cooperative solutions of the no tax game

The best way for the two players to get both a win without causing the default of the State in economic difficulty, is to find a cooperative solution.

Cooperative solution. The Speculator and the ECB play the strategies $x = 0$ and $y = 1$ in order to arrive at the payoff A' (which allows the State to reduce the yield on its government bonds) and then they split the bi-win A' by means of a contract.

The Speculator benefits by cooperating with the ECB because following the Nash strategy it does not win anything (while in this way wins $1/10$); the ECB is able to save the State in difficulty, but gives up a significant part of its win than the Nash equilibrium.

Practically: the Speculator does not act with any speculative movement on the securities market, and the ECB, that manages to save the State in economic crisis, shares with the Speculator its winning $W = 1/4$, obtained arriving to A'.

For a possible quantitative division of this win $W = 1/4$, between the ECB and the Speculator, we use the transferable utility solution, applying to the transferable utility Pareto boundary of the payoff space a non-standard Kalai-Smorodinsky solution.

Remark. We consider the infimum and the supremum of the maximal Pareto boundary for a better view of the game in its entirety.

We proceed finding the supremum of our maximal Pareto boundary, which is $\sup \partial^* f(E \times F) =: \alpha = (1/2, 3/4)$; then we join it with the infimum of our maximal Pareto boundary, which is given by $\inf \partial^* f(E \times F) = (0, 0)$.

The coordinates of the intersection of the point P, between the straight line of collective win (i.e. $X + Y = 1/4$) and the straight line joining the supremum of the maximal Pareto boundary with the infimum (i.e. the line $Y = (3/2)X$), give us the desirable division of the collective win $W = 1/4$ between the two players.

In order to find the coordinates of the point P is enough to put in a system of equations $X + Y = 1/4$ and $Y = (3/2)X$. Substituting the Y in the first equation we have $X + (3/2)X = 1/4$ and therefore $X = 1/10$.

Substituting now the X in the second equation, we have $Y = 3/20$. Thus $P = (1/10, 3/20)$ suggests as solution that the Speculator receives $1/10$ by contract by the ECB, while at the ECB remains the win $3/20$.

We can see the Fig.10 in order to make us more aware of the situation.

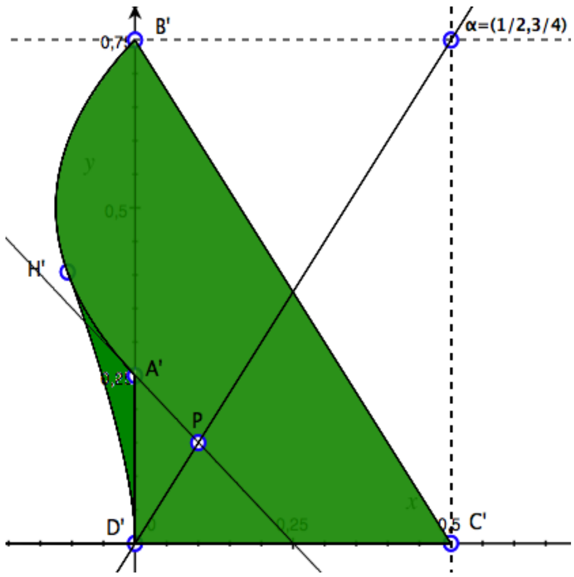

Figure 10: A possible cooperative solution of the no tax game

Remark. But the cooperative solution leaves us dissatisfied. In fact the cooperative solution is difficult to implement because the ECB should achieve an agreement with the Speculator before that the Speculator plays a strategy $x > 0$, and is almost impossible to know in advance the intentions of all the potential speculators in the bonds market. For this reason, it is necessary a preventive economic measure.

8 A new anti-speculative proposal

In the paper [11] we propose, in order to avoid speculations of the first player about the current and future yield of the government bonds, to introduce by regulatory authorities a tax that affects the gain obtained through speculative trades involving the government bonds . We hypothesized that the tax increased the reserves of the ECB.

We obtained the payoff space in figure 11.

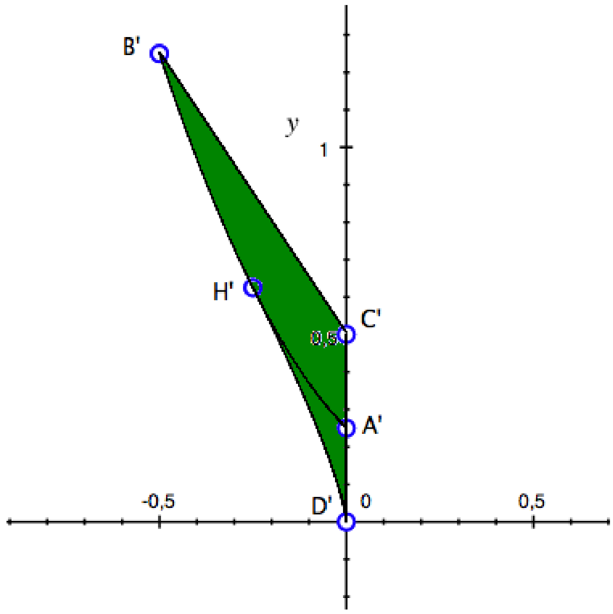

Figure 11: The payoff space of the game with tax cashed by the ECB

Moreover, we noted the movement from point B to point A of the Nash and defensive equilibria. The point A is an optimal point for the State in economic difficulty, because the yield on its government bonds decreases, and at the same time is a quite good point for the Speculator and the ECB because they are on the weak maximal Pareto boundary.

The new anti-speculative proposal. We propose that the tax is not cashed by ECB, but directly by the State in economic difficulty. In this way, even if the Speculator intervenue in government bonds market with a strategy $x \neq 0$, the State cancels whole or in part the effect upwards of the strategy x on government bonds yield.

8.1 The new payoff of the Speculator

We assume that the tax eliminates completely the possibility of speculative profits created by the Speculator itself: the tax is equal to the incidence mx of the strategy x on the yield R_1.

With the introduction of the tax, recalling the Eq. (2), that is $f_1(x, y) = xM(R_1(x, y) - R_0)$, the payoff function of the Speculator becomes:

$$f_1(x, y) = xM(R_1(x, y) - T(x) - R_0).$$

We assume that $T(x) = mx$. After the calculations, we obtain:

$$f_1(x, y) = xM(-ny) \tag{10}$$

8.2 The new payoff function of the ECB

We assume that the introduction of the tax has no effect on the payoff function of the ECB. So its payoff function is equal to Eq. (4), that is $f_2(x, y) = yM(i+mx)$.

8.3 The new payoff function of the State

IIn the case of adoption of the tax, the payoff function of the State without tax is added to the total tax paid by the Speculator. The total tax paid by the Speculator is given by the amount of government bonds xM purchased by the Speculator, multiplied by the tax $T(x)$ applied.

In mathematical language, the payoff function of the State with the tax is given by

$$f_3 = M(R_0 - R_1) + xMT(x). \tag{11}$$

Recalling that $R_0 = i$, that $R_1(x, y) = i + mx - ny$, assuming that $T(x) = mx$, and replacing them in the Eq. (11), that is $f_3(x, y) = M(R_0 - R_1) + xMT(x)$, we have

$$f_3(x, y) = M(i - (i + mx - ny)) + xMmx.$$

After the calculations, we have

$$f_3(x, y) = M(-mx + ny) + xMmx. \tag{12}$$

The payoff function of the game with tax cashed by the State is

$$f(x, y) = (xM(-ny), yM(i + mx), M(-mx + ny) + xMmx)$$

9 Study of the game with tax cashed by the State

9.1 Critical space of the new game

Since we are dealing with a non-linear game, it is necessary to study in the bi-win space also the points of the critical zone that belong to the bi-strategy space. In order to find the critical area of the game, we consider the Jacobian matrix and we put its determinant equal 0.

About the gradients of f_1 and f_2, we have

$$grad \ \ f_1 = (-Mny), -nxM)$$

$$grad \ \ f_2 = (Mmy, M(i + mx).$$

The determinant of the Jacobian matrix is

$$\det J_{f(x,y)} = -M^2ny(i + mx) + M^2nxmy.$$

Therefore, the critical space of the game is

$$Z_f = \{(x, y) : -M^2ny(i + mx) + M^2nxmy = 0\}.$$

Dividing by M^2n, which are all positive numbers (strictly greater than 0), we have:

$$Z_f = \{(x, y) : xmy - y(i + mx) = 0\}.$$

Finally, after the calculations, we have $Z_f = \{(x, y) : y = 0\}$.

The critical area of our bi-strategy space is represented in the figure 12 by the segment $[D, C]$.

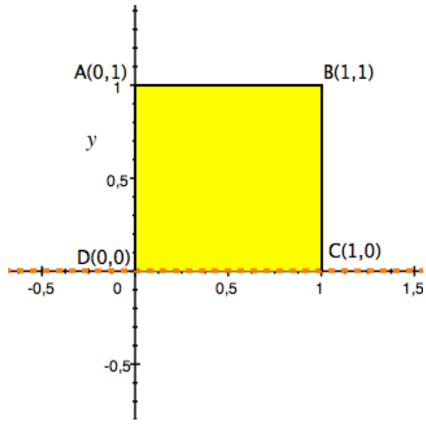

Figure 12: The critical zone of the game with tax cashed by the State

9.2 New payoff space of the game

In order to represent graphically the payoff space $f(E \times F)$, we transform, by the function f, all the sides of bi-strategy square $E \times F$ and the critical space Z of the game G.

We obtain, on the payoff space $f(E \times F)$ of our game G, the figure 13:

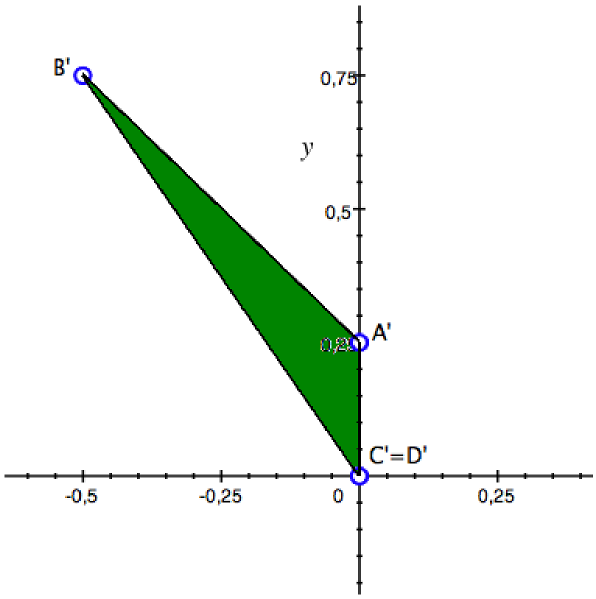

Figure 13: The payoff space of the game with tax cashed by the State

Remark. Now the results must be interpreted according to the payoff function of the issuer State. Recalling the Eq. (12), that is

$$f_3(x, y) = M(-mx + ny)) + xMmx,$$

and that $M = 1$ and $n = m = 1/2$ we note that

- if the two players arrive to the points D' the yield that the issuer State has to pay for its securities remains unchanged. This solution is undesirable because it does not solve the problems of the State and does not gives breath to its economy.
- if the two players arrive to the points C' the yield (that the issuer State has to pay on its government bonds) increases, but the tax cashed balances the increase of the yield. This solution is undesirable because it does not solve the problems of the State, leaving the yield unchanged.
- if the two players arrive to the point A' the yield paid by the State for its government bonds decreases, and thus the State emerges from the crisis.
- if the two players arrive to the point B' the yield paid by the State for its government bonds remains unchanged, but the State cashes the Tax on the government bonds speculation by the Speculator, and thus the State emerges from the crisis.
- if the two players arrive to the side $]B'A'[$ the yield paid by the State for its government bonds decreases in a lower extent than the point A', and the tax cashed cancels only partially the effect of the strategy x of the Speculator.

According to these considerations, is morally, ethically and economically desirable that the Speculator and the ECB arrive to the points $A = (0, 1)$ or $B = (1, 1)$.

Remark. Comparing the payoff space of the no tax game (see the figure 6) and that one of the game with the tax cashed by the State (see the figure 13), we note that the latter seems smaller. At first glance, it seems that the tax has caused a loss of global wealth. But is not so: in fact the collective profit of the three players remains unchanged. The big difference is that the Speculator is effectively unable to make a profit, and the “lost” due to the tax increases the win of the State.

Remark. The point A' and the point B' have the same collective gain about the three subjects of our game. In fact, if we arrive to point A' the State in economic difficulty has a profit equal to 1/2, the Speculator wins 0 and the ECB wins 1/4. Instead if we arrive to point B' the State has a profit equal to 1/2, the Speculator loses 1/2 and the ECB wins 3/4. In both points, the total gain of the game is 3/4 (which is the same of the point A' and B' of the no tax game).

10 Equilibria of the game with tax cashed by the State

10.1 New Nash equilibria of the game

If the two players decide to adopt a selfish behavior, they choose their own strategy maximizing their partial gain. In this case, we should consider the classic Nash best reply correspondences. The best reply correspondence of the

Speculator is the correspondence $B_1 : F \to E$ given by $y \mapsto \max_{f_1(\cdot,y)} E$, where $\max_{f_1(\cdot,y)} E$ is the set of all strategies in E which maximize the section $f_1(\cdot, y)$.

Symmetrically, the best reply correspondence $B_2 : E \to F$ of the ECB is given by $x \mapsto \max_{f_2(x,\cdot)} F$.

Choosing $M = 1$, $n = 1/2$ and $m = 1/2$, which are always positive numbers (strictly greater than 0), and recalling the Eq. (10), that is $f_1(x, y) = xM(-ny)$, we have

$$\partial_1 f_1(x, y) = -Mny.$$

This derivative is positive if $-nyM > 0$, and so:

$$B_1(y) = \begin{cases} \{0\} & \text{if} \quad y > 0 \\ \{E\} & \text{if} \quad y = 0 \end{cases}.$$

Recalling also the Eq. (4), that is $f_2(x, y) = yM(i + mx)$, we have

$$\partial_2 f_2(x, y) = M(i + mx).$$

This derivative is positive if $M(i + mx) > 0$, and so:

$$B_2(x) = \{1\} \quad \forall \quad x \in E$$

In Fig.14 we have in red the inverse graph of B_1, and in blue that one of B_2.

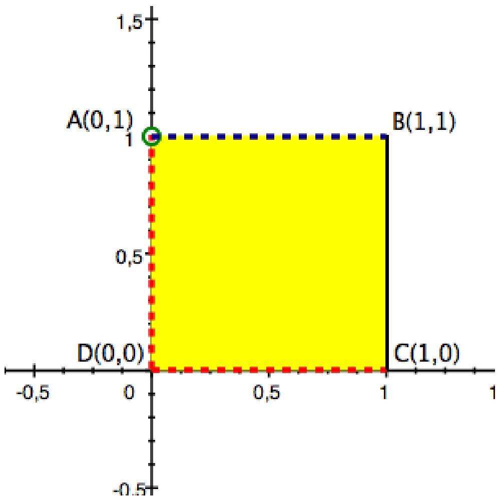

Figure 14: The Nash equilibria of the game with tax cashed by the State

The set of Nash equilibria, that is the intersection of the two best reply graphs, is

$$\mathrm{Eq(B_1, B_2) = (0, 1)}.$$

Analysis of Nash equilibria. The Nash equilibria can be considered optimal for the two players, because they are on the proper maximal Pareto boundary. The selfishness, in this case, pays very well.

The Nash equilibria of the game with tax, moreover, is great for the State that issues the government bonds, because the yield (that the State has to pay) goes downward because of the strategy $y = 1$, while the strategy $x = 0$ does not affect upward. In a word, the State finances its public spending with a lower government bonds yield, and this allows it to overcome the economic crisis that has invested it.

10.2 New defensive phase of the game

We suppose that the two players are aware of the will of the other one to destroy it economically, or are by their nature cautious, fearful, paranoid, pessimistic or risk averse, and then they choose the strategy that allows them to minimize their loss. In this case, we talk about defensive strategies.

10.2.1 Conservative value and meetings.

Conservative value and meetings. *Conservative value of a player.* It is defined as the maximization of its function of worst win. Therefore, the conservative value of the Speculator is $v_1^\sharp = \sup_{(x\in E)} f_1^\sharp$, where $f_1^\sharp$ is the function of worst win of the Speculator, and it is given by $f_1^\sharp(x) = \inf_{(y\in F)} f_1(x, y)$.

Recalling the Eq. (10), that is $f_1(x, y) = xM(-ny)$, and choosing $M = 1$ and $n = 0.5$, which are always positive numbers strictly greater than 0, we have:

$$f_1^\sharp = \inf_{y\in F} xM(-ny).$$

Since the offensive strategies of the ECB are $O_2(x) = \begin{cases} \{1\} & \text{if} \quad x > 0 \\ \{F\} & \text{if} \quad x = 0 \end{cases}$, we obtain:

$$f_1^\sharp(x) = \begin{cases} \{xM(-n)\} & \text{if} \quad x > 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}.$$

Graphically $f_1^\sharp$ appears as:

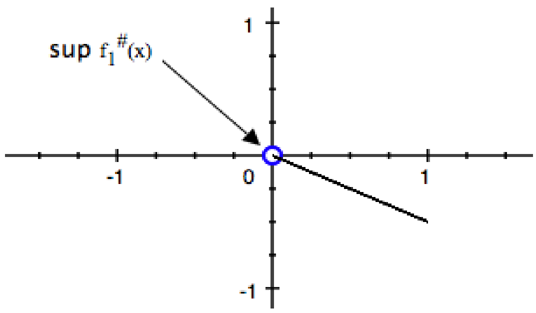

Figure 15: The function of worst win of the Speculator in the game with tax cashed by the State

So the defense (or conservative) strategy of the Speculator is given by $x_\sharp = 0$, and the conservative value of the Speculator is

$$v_1^\sharp = \sup_{x\in E} \inf_{y\in F} xM(-ny) = 0. \tag{13}$$

On the other hand, the conservative value of the ECB is given by $v_2^\sharp = \sup_{y\in F} f_2^\sharp$, where $f_2^\sharp$ is the function of the worst win of the ECB. It is given by $f_2^\sharp(y) = \inf_{x\in E} f_2(x,y)$.

Recalling the Eq. (4), that is $f_2(x,y) = yM(i+mx)$, and choosing $M=1$, $i=0.25$ and $m=0.5$, which are always positive numbers (strictly greater than 0), we have:

$$f_2^\sharp = \inf_{x\in E} yM(i+mx).$$

Since the offensive strategies of the Speculator are $O_1(y) = \begin{cases} \{0\} & \text{if} \quad y>0 \\ \{E\} & \text{if} \quad y=0 \end{cases}$, we obtain:

$$f_2^\sharp(y) = \begin{cases} \{yMi\} & \text{if} \quad y>0 \\ \{0\} & \text{if} \quad y=0 \end{cases}.$$

Graphically $f_2^\sharp(y)$ appears as:

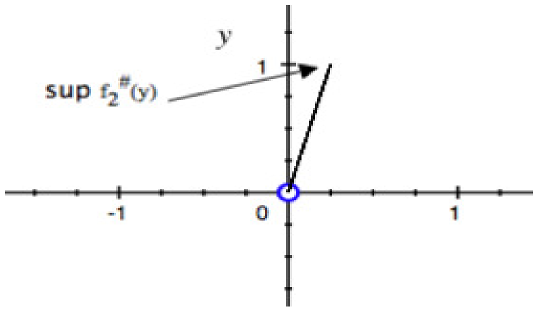

Figure 16: The function of worst win of the ECB in the game with tax cashed by the State

So the defense (or conservative) strategy of the ECB is given by $y_\sharp = 1$, and the conservative value of the ECB is

$$v_2^\sharp = \sup_{y\in F} \inf_{x\in E} yM(i+mx) + xMmx = Mi. \tag{14}$$

Therefore, choosing $M=1$ and $i=0.25$, the conservative bi-value is

$$v_f^\sharp = (v_1^\sharp, v_2^\sharp) = (0, 1/4).$$

Conservative meetings. They are represented by the bi-strategies $(x_\sharp, y_\sharp)$, that are represented by the value $A=(0,1)$.

Analysis of conservative meeting. The conservative meeting can be considered good because it is located on the proper maximal Pareto boundary, and it is also great for the State. In fact, recalling the Eq. (11), that is

$$f_3(x,y) = M(ny-mx) + xMmx,$$

and that $M=1$ and $n=m=0.5$ are always positive numbers (strictly greater than 0), the yield paid on government bonds falls by 1/2, because of the strategy $x=0$ of the Speculator and the strategy $y=1$ of the ECB.

11 Cooperative solution of the new game

We assumed that on the bonds yield the incidence n of y is equal to the incidence m of x. But if we assume that $n < m$, our payoff space changes. **Remark.** Very likely, the value n is lower than the value m because the purchase of government bonds by the ECB could be less accepted by the market players. In fact, the action of the ECB could be seen as a behavior dictated (also, or even only) by political motivations, and not by economic reasons (like for example low risk and high profit about government bonds).

So, for the cooperative solution we assume that $n = 1/3$ and $m = 1/2$. In the figure 17 we can see the new payoff space.

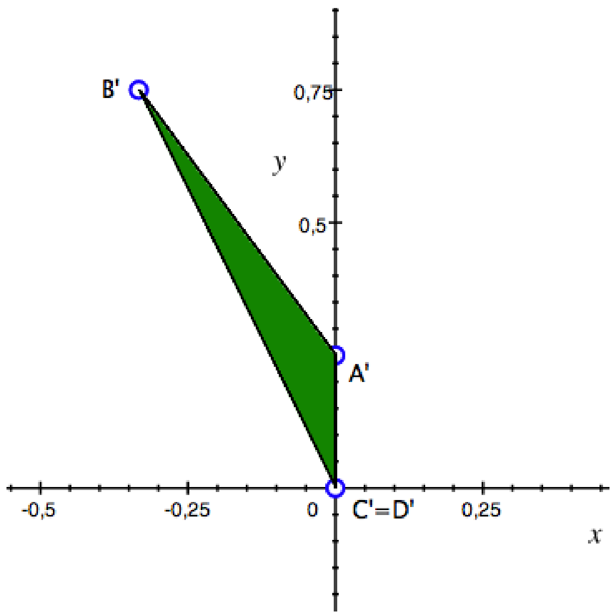

Figure 17: The payoff space of the game with $n < m$

We note that the point B' moves upward.

Cooperative solution. The Speculator and the ECB play the strategies $x = 1$ and $y = 1$ in order to arrive at the point B', which allows the State to win the value $1/3$ (in fact, the increase of the bond yield is totally balanced by the tax cashed). After, the ECB divides its win $5/12$ with the Speculator by contract.

Financial point of view. The Speculator plays the strategy $x = 1$, and the ECB shares with the Speculator its winning $W = 5/12$, obtained arriving to B'. At the same time, the State in economic difficulty is saved.

For a possible quantitative division of this win $W = 5/12$, between the ECB and the Speculator, we use the transferable utility solution, applying to the transferable utility Pareto boundary of the payoff space a non-standard Kalai-Smorodinsky solution.

We proceed finding the infimum of our maximal Pareto boundary, which is $\inf \partial_* f(E \times F) =: \beta = (-1/3, 0)$; then we join it with the Nash equilibrium of the game, which is given by $A' = (0, 1/4)$.

We can see the figure 18 in order to make us more aware of the situation.

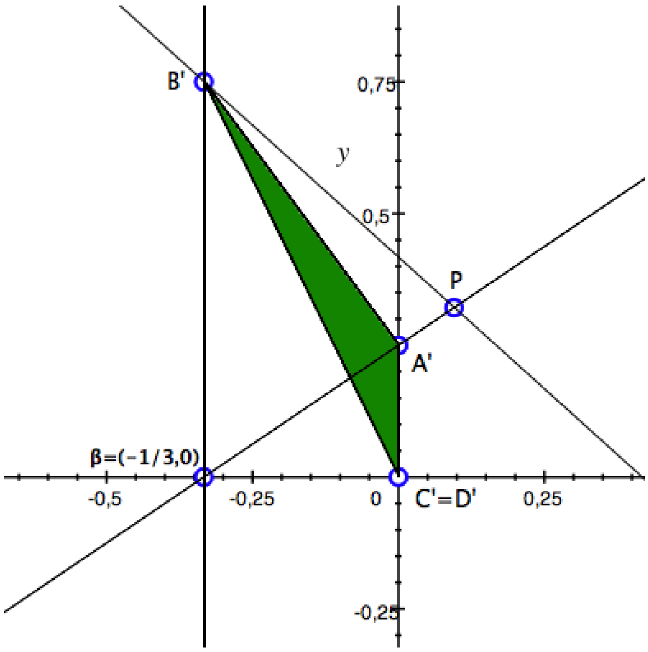

Figure 18: The cooperative solution of the game with $n < m$

The coordinates of the intersection of the point P, between the straight line of collective win (i.e. $X + Y = 5/12$) and the straight line joininig the infimum of the maximal Pareto boundary with the Nash equilibrium (i.e. the line $Y = (3/4)X + 1/4$), give us the desirable division of the collective win $W = 5/12$ between the two players.

In order to find the coordinates of the point P is enough to put in a system of equations $X + Y = 5/12$ and $Y = (3/4)X + 1/4$. Substituting the Y in the first equation we have $X + (3/4)X = 1/6$ and therefore $X = 2/21$. Substituting now the X in the second equation, we have $Y = 9/28$.

Thus $P = (2/21, 9/28)$ suggests as solution that the Speculator receives $2/21$ by contract by the ECB, while at the ECB remains the win $9/28$.

12 Conclusions

We just studied two games with the same agents: the first game is a simplified representation of the reality; in the second one we suggest a possible regulatory model that provides the stabilization of the government bonds market through the introduction of a tax on government bonds transactions.

No tax game. Without the introduction of the tax, the defensive and Nash equilibria lead most likely in the point B. But the point B is not a good point of arrival for the State in economic difficulty, because the yield on its bonds remains at high levels and unchanged. In this regard, the only possible satisfactory solution is a cooperative solution between the two players: the Speculator and the ECB play the strategies $x = 0$ and $y = 1$ arriving to point A' and dividing the collective win by contract (at the same time, the yield on government bonds of the State decreases, and so the total gain of the three subjects of our game is the same than that one in the point B'). But the cooperative solution leaves us dissatisfied. In fact the cooperative solution is difficult to implement because the ECB should achieve an agreement with the Speculator before that the Speculator plays a strategy $x > 0$, and is almost impossible to know in advance the intentions of all the potential speculators in the bonds market. For this reason, it is necessary a preventive economic measure.

Game with the tax cashed by ECB. In the regulatory model that we proposed in [11] with the introduction of the tax cashed bye the ECB, we note that the Nash and defensive equilibria of the game move to point A. The point A is an optimal point for the State in economic difficulty, because the yield on its bonds is reduced, allowing the State to move the first step towards economic recovery. But in this case, the point A' is also a quite good point for the Speculator and the ECB, because they are on the weak maximal Pareto boundary: we transformed the politically more desirable solution in a solution convenient for all parties involved. In this way, the collective gain is not subject to losses than the no tax game, and once and for all we solve the problem of too high yield on government bonds. Moreover, the introduction of the tax is a preventive deterrent for the presence of the speculators in the bonds market.

Game with the tax cashed by State. In this case, the results achieved with the introduction of the tax cashed by the ECB remain valid. Moreover, we make a further step forward. The point A' becomes a point of the proper maximal Pareto boundary (in the game between the Speculator and the ECB), and not only a point of the weak maximal Pareto boundary. Moreover, also the point B' becomes an optimal point for the State (the profit of the State is the same of that one in the point A': in fact, even if the Speculator for any reason decides to play the strategy $x = 1$, the tax cashed directly by the State balances the increase in the yield on its government bonds and so the State wins $1/2$ equally.

Thanks to this result (and assuming a higher incidence on government bonds yield of actions by the Speculator than these one by ECB), we can propose also a cooperative solution between the Speculator and the ECB. They divide the win $W = 5/12$ of the point B' by contract (the Speculator wins $2/21$, the BCE wins $9/28$ and at the same time the State in economic difficulty wins $1/3$). In this way, all three economic subjects of our game win something: the Speculator and the ECB win more than Nash and defensive equilibria, and the State goes out of the economic crisis.

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