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Abstract

The purpose of paper is to investigate how the interplay of trade, commuting and communication costs shapes economy at both inter-regional and intra-urban level. Specifically, we study how economic integration affects the internal structure of cities and and how decentralizing the production and consumption of goods in secondary employment centers allows firms located in a large city to maintain their predominance.

JEL Classification: F12, F22, R12, R14

Key-words: city structure, secondary business centers, commuting costs, trade costs, communication costs

Introduction

A weakness in urban economic theory is that it has relied too heavily on the monocentric city model. A single job center runs counter to the evidence that has accumulated in the empirical literature on employment subcenters. But, the main drawback of the monocentric model is that it fails to explain that job location – even in a single center – is not exogenous but depends on other determinants of urban form. A reasonably tractable polycentric model can be based on the assumption that production and residential uses can occur everywhere in an initially featureless space but become interdependent by the commuting decisions of workers and the communication linkages among firms. Producers value access to other producers, to labor, and to facilities that help to run its business. The location of production and, hence, of jobs is endogenous as is the location of residences and, hence, of labor. From this perspective the monocentric city arises as the total clustering of jobs.

Moreover, cities are involved into the process of inter-regional trade. It is, therefore, fundamental to understand (i) how the intensity of trade is influenced by their size and structure and, conversely, (ii) how economic integration affects the internal structure of cities. This is what we undertake in this study by modeling the interplay between trade costs, commuting costs and communication costs. Our approach combines basic ingredients from urban economics and new economic geography.

Creation of subcenters within a city, i.e. the formation of a polycentric city, appears to be a natural way to alleviate the burden of urban costs. It is, therefore, no surprise that Anas et al. (1998) observe that “polycentricity is an increasingly prominent feature of the landscape.” Thus, the escalation of urban costs in large cities seems to prompt a redeployment of activities in a polycentric pattern, while smaller cities retain their monocentric shape. However, for this to happen, firms set up in the secondary centers must maintain a very good access to the main urban center, which requires low communication costs.

Comparison to other approaches

Trying to explain the emergence of cities with various sizes our framework, unlike Helpman (1998), Tabuchi (1998) and others, allows cities to be polycentric. Moreover, in contrast to A. Sullivan (1986), K. Wieand (1987), and (Helsley and Sullivan, 1991), in our treatment, there are no pre-specified locations or numbers of subcenters, and our model is a fully closed general equilibrium spatial economy. As mentioned above, emergence of additional job centers is based on the urge towards decreasing of urban costs, rather than mysterious consumer’s “propensity to big malls”, as suggested (Anas and Kim, 1996). Our approach, that takes into account various types of costs (trade, commuting, and

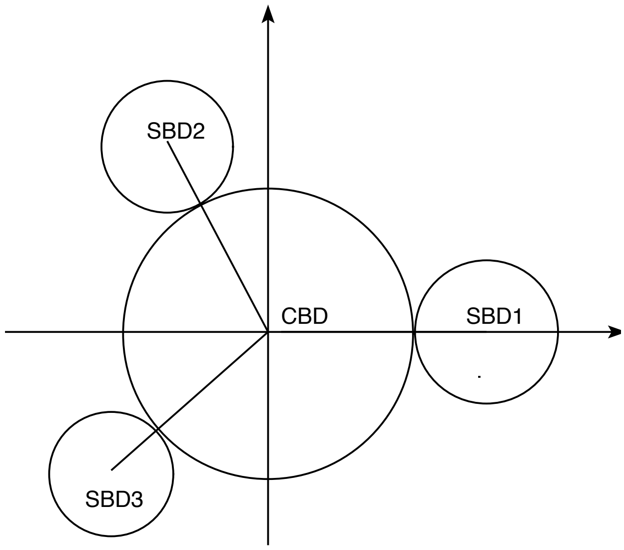


Figure 1: Polycentric city

communication) is similar to J. Cavailhès et al. (2007) with one important exception. We drop very convenient (yet non-realistic) assumption on "long narrow city". Our analysis is extended to the two-dimension because the geographical space in the real world is better approximated by a two-dimensional space.

1 Model overview

1.1 Spatial structure

Consider an economy with $R \geq 1$ regions, one sector and two primary goods, labor and land. Each region can be urbanized by accommodating firms and workers within a city, and is formally described by a two-dimensional space $X = \mathbb{R}^2$ (or by sufficiently large area around origin). Whenever a city exists, it has a central business district (in short CBD) located at the origin $0 \in X$. Residence zone around CBD assumed to be a circle due to geographical homogeneity. One would expect us to explain why this CBD exists as well as why firms leaving the CBD want to be together and form a secondary business districts, in short SBDs.

Firms are free to locate in the CBD or to set up in the suburbs of the metro where they form a SBDs. Both the CBD and SBDs are assumed to be dimensionless, while residence zones around SBDs are also circles. In what follows, the superscript C is used to describe variables related to the CBD, whereas S describes the variables associated with a SBDs. Without loss of generality, we focus on the only one of SBDs, because all SBDs are supposed to be identical. *Locations* are expressed by variable $x \in X$ while *distances* are measured as Euclidean norm $||x||$ for CBD-zone, whereas the SBD in city r, if any, is established at $x_r^S \neq 0$, which is endogenous. Even though firms consume services supplied in each SBD, the higher-order functions (specific local public goods and non-tradeable business-to-business services such as marketing, banking, insurance) are still located in the CBDs.

Hence, for using such services, firms set up in a SBD must incur a communication cost, which is given by

$$\mathcal{K}(x_r^S) = K + k \cdot ||x_r^S|| \tag{1}$$

where K and k are two positive constants. Indeed, communicating requires the acquisition of specific facilities, thus explaining why communication costs have a fixed component. However, relationships between the CBD and a SBD also involves face-to-face communication. Therefore, some workers must go to the CBD, thus making communication costs dependent on the distance $||x_r^S||$ between the CBD and the SBD. Both the CBD and the SBD are surrounded by residential areas occupied by workers. Furthermore, as the distance between the CBD and SBD is small compared to the intercity distance, we disregard the intra-urban transport cost of goods.

Finally, we consider the case where the CBD of urbanized region r is surrounded by $m_r \geq 1$ SBDs. Under those various assumptions, the location, number and size of the SBDs as well as the size of the CBD will be endogenously determined. In other words, apart from the assumed existence of the CBD, the internal structure of each city is endogenous.

1.2 Workers

The economy is endowed with L mobile workers. The welfare of a worker depends on her consumption of the following three goods. The first good is unproduced and homogeneous. It is assumed to be costlessly tradeable and chosen as the numéraire. The second good is produced as a continuum n of varieties of a horizontally differentiated good under monopolistic competition and increasing returns, using labor as the only input. Any variety of this good can be shipped from one city to the other at a unit cost of $\tau > 0$ units of the numéraire. The third good is land; without loss of generality, we set the opportunity cost of land to zero. Each worker living in city $1 \leq r \leq R$ consumes a residential plot of fixed size chosen as the unit of area. The worker also chooses a quantity $q(i)$ of variety $i \in [0, n]$, and a quantity q_0 of the numéraire. She is endowed with one unit of labor and $\bar{q}_{0r} > 0$ units of the numéraire. The source of this endowment will be discussed in proper time. At this moment initial $\bar{q}_{0r}$ are considered as exogenous parameters. Each worker commutes to her employment center – without cross-commuting – and bears a unit commuting cost given by $t > 0$, so that for the worker located at x the commuting cost is either $t||x||$ or $t||x - x_r^S||$ according to the employment center.

The budget constraint of an individual residing at $x \in X$ in city r and working in the corresponding CBD can then be written as follows:

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 + R_r^C(||x||) + t||x|| = w_r^C + \bar{q}_{0r} \tag{2}$$

where $R_r^C(x)$ is the land rent prevailing at location x (in fact, it depends on distance $||x||$ from the CBD only). The budget constraint of an individual working in the SBD, located at specific place x_r^S is

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 + R_r^S(||x - x_r^S||) + t||x\text{-}x_r^S|| = w_r^S + \bar{q}_{0r}. \tag{3}$$

Preferences over the differentiated product and the numéraire are identical across workers and represented by a utility function $U(q_0; q(i), i \in [0, n])$.

The considerable part of our results does not depend on the utility specification. For more detailed study of the wage and welfare aspects we use Ottaviano's quasi-linear utility function

$$U(q_0; q(i), i \in [0, n]) = \alpha \int_0^n q(i)\mathrm{d}i - \frac{\beta}{2} \int_0^n [q(i)]^2 \mathrm{d}i - \frac{\gamma}{2} \left[\int_0^n q(i)\mathrm{d}i \right]^2 + q_0 \tag{4}$$

where $\alpha, \beta, \gamma > 0$. Reason of this choice is that this function was used in comprehensive analysis of *linear city with two SBDs* in Cavailhès et al. (2007). Comparing our results with ones from this paper we reveal some features of two-dimensional model of city, which could not be obtained in framework of linear city.

1.3 Firms

Technology in manufacturing is such that producing $q(i)$ units of variety i requires a given number φ of labor units.[1] There are no scope economies so that, due to increasing returns to scale, there is a one-to-one relationship between firms and varieties. Thus, the total number of firms is given by $n = L/\varphi$. Labor market clearing implies that the number of firms located (or varieties produced) in city r is such that $n_r = \lambda_r n$, where λ_r stands for the share of workers residing in r.

Denote by Π_r^C (respectively Π_r^S) the profit of a firm set up in the CBD of city r (respectively the SBD). Let θ_r be the share of firms located in the CBD and, therefore, by $(1-\theta_r)/m$ the share of firms in each SBD. When the firm producing variety i is located in the CBD, its profit function is given by:

$$\Pi_r^C(i) = I_r(i) - \varphi \cdot w_r^C \tag{5}$$

where

$$I_r(i) = p_{rr}(i) \cdot Q_{rr}(i) + \sum_{s \neq r} (p_{rs}(i) - \tau) \cdot Q_{rs}(i)$$

stands for the firm's revenue earned from local sales $Q_{rr}(i)$ and from exports $Q_{rs}(i)$ from r to s. When the firm sets up in the SBDs of the same city, its profit function becomes:

$$\Pi_r^S(i) = I_r(i) - \mathcal{K}(x_r^S) - \varphi \cdot w_r^S \tag{6}$$

the firm's revenue is the same as in the CBD because shipping varieties within the city is costless so that prices and outputs do not depend on firm's location in the city.

1.4 Two-dimensional features

The short overview of the model allows to point out some differences two-dimensional model from model of "long narrow city" that could be substantial for final outcomes. Urban costs (commuting and communication) mainly depend on *distances* or *geographic size* of the city, which is proportional to *population* or *demographic size* in linear city and *less than proportional* in two-dimensional model (to be more specific, geographic size increases as *square root* of population). Moreover, an additional economy on scale in urban costs comes from possibility to allocate more than two SBDs around central zone. It means that linear model (possibly) *overestimates* dispersion forces (caused by urban costs) in comparison to agglomeration forces (related to monopolistic competition). In other words, two-dimensional model is "more favorable" to formation of larger city agglomeration.

[1]One may assume that producing one unit of variety i requires additionally $c \geq 0$ units of numéraire. This is not substantive generalization, however, because this model is technically equivalent to one with $c = 0$ (see Ottaviano et al., 2002)

2 Urban Costs and Decentralization within a City

A city equilibrium is such that each individual maximizes her utility subject to her budget constraint, each firm maximizes its profits, and markets clear. Individuals choose their workplace (CBD or SBDs) and their residential location with respect to given wages and land rents. In each workplace (CBD or SBDs), the equilibrium wages are determined by a bidding process in which firms compete for workers by offering them higher wages until no firm can profitably enter the market. Given such equilibrium wages and the location of workers, firms choose to locate either in the CBD or in the SBDs. At the city equilibrium, no firm has an incentive to change place within the city, and no worker wants to change her working place and/or her residence. In this section, we analyze such an equilibrium, taking as fixed the number of workers $l_r = \lambda_r L$. To ease the burden of notation, we drop the subscript r.

2.1 Land rents, wages and workplaces

Within each city, a worker chooses her location so as to maximize her utility $U(q_0, q(i); i \in [0, n])$ under the corresponding budget constraint, (2) or (3). Let $\Psi^C(x)$ and $\Psi^S(x)$ be the bid rent at $x \in X$ of an individual working respectively in the CBD and the SBD. Land is allocated to the highest bidder. Because there is only one type of labor, at the city equilibrium it must be that $R(x) = \max\left\{\Psi^C(x), \Psi^S(x), 0\right\}$. We assume that some non-profit collecting agency redistributes *uniformly* all of collected rent among all population of city. Thus *initial endowment* $\bar{q}_0$ is in fact an individual share of numéraire good, financed by the equal share of total raised rent in specific city.

For given CBD's share of firms θ, number of SBDs m and city's population l we obtain the following values of central zone radius r^C and distance $||x^S||$ from CBD to any SBD:

$$r^C = \sqrt{\frac{\theta l}{\pi}}, \quad ||x^S|| = \sqrt{\frac{\theta l}{\pi}} + \sqrt{\frac{(1-\theta)l}{m\pi}}. \tag{7}$$

The equilibrium land rents, equalizing the disposable income for all locations $x \in X$ are given by

$$R^C(x) = t \cdot \left(\sqrt{\frac{\theta l}{\pi}} - ||x||\right), \quad \text{for} \quad ||x|| \leq \sqrt{\frac{\theta l}{\pi}} \tag{8}$$

and

$$R^S(x) = t \cdot \max\left\{0, \sqrt{\frac{(1-\theta)l}{m\pi}} - ||x^S - x||\right\}, \quad \text{for} \quad ||x|| > \sqrt{\frac{\theta l}{\pi}} \tag{9}$$

where $x^S \in X$ is SBD closest to $x \in X$. Initial endowment (i.e., redistributed total rent):

$$\bar{q}_0 = \frac{1}{l}\int_X R(x)\mathrm{d}x = \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}}\left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right] \tag{10}$$

The wage difference between CBD and SBD:

$$w^C - w^S = t\left(\sqrt{\frac{\theta l}{\pi}} - \sqrt{\frac{(1-\theta)l}{m\pi}}\right). \tag{11}$$

Thus, the difference in the wages paid in the CBD and in the SBD compensates exactly the worker for the difference in the corresponding commuting costs.

2.2 Urban Costs

City equilibrium state implies that for each consumer in both centrum and suburbia should be equivalent, i.e.

$$E^C = w^C + \bar{q}_0 - R^C(x) - t||x|| \equiv E^S = w^S + \bar{q}_0 - R^S(x) - t||x - x^S||.$$

Using this equivalence we focus on consumption of CBD workers. By the obvious reason E^C should be at least non-negative. Let's define *urban costs* as a sum of rent and commuting costs minus initial endowment $\bar{q}_0$.[2] Due to (8-10) these urban costs are as follows

$$\begin{aligned} C_u^C = R^C(x) + t||x|| - \bar{q}_0 = t\sqrt{\frac{\theta l}{\pi}} - \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}} \left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right], \\ C_u^S = R^S(x) + t||x - x^S|| - \bar{q}_0 = t\sqrt{\frac{(1-\theta)l}{\pi}} - \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}} \left[\theta^{3/2} + \frac{(1-\theta)^{3/2}}{\sqrt{m}}\right]. \end{aligned} \tag{12}$$

The city equilibrium implies that the following identity holds

$$w^C - C_u^C = w^S - C_u^S.$$

In these terms, the wage wedge identity may be rewritten as difference of urban costs:

$$w^C - w^S = C_u^C - C_u^S.$$

In what follows we focus on CBD dropping taking SBD into account due to equilibrium equivalency of consumption in centrum and suburbia.

2.3 Equilibrium wages and the city structure

Regarding the labor markets, the equilibrium wages of workers are determined by the zero-profit condition. In other words, operating profits are completely absorbed by the wage bill. Hence, the equilibrium wage rates in the CBD and in the SBDs must satisfy the conditions $\Pi^C(w^{C*}) = 0$ and $\Pi^S(w^{S*}) = 0$, respectively. Thus, setting (5) (respectively (6)) equal to zero, solving for w^{C*} (respectively w^{S*}), we get:

$$w^{C*} = \frac{I}{\varphi}, \quad w^{S*} = \frac{I - \mathcal{K}(x^S)}{\varphi} \tag{13}$$

Hence

$$w^{C*} - w^{S*} = \frac{K + k||x^S||}{\varphi},$$

which means that the equilibrium wage wedge is proportional to the level of the communication cost that prevails at the SBD. Substituting of (11) and (7) into previous formula yields:

$$(\varphi t - k)\sqrt{m\theta l} = K\sqrt{m\pi} + (\varphi t + k)\sqrt{(1-\theta)l} > 0 \tag{14}$$

which implies that inequality $\varphi t - k > 0$ is necessary condition for equilibrium existence. More exactly, the opposite inequality $k \geq \varphi t$ means that distance-sensitive communication costs are too large in comparison to commuting ones, so in fact we have rather *communicatively separated* cities,

[2] For technical reasons it is convenient to treat $\bar{q}_0$ as some kind of rent compensation, subtracting it from costs rather adding to wage.

than connected CBD and SBD.

Assuming from now on that $\varphi t - k > 0$ holds, we have to solve this equation with respect to θ. Admissible solution θ^* of equation (14) will be referred as *equilibrium share.* First we consider more simple limit case $K = 0$, i.e. fixed communication costs supposed to be negligible. Then the solution is

$$\theta^* = \frac{(\varphi t + k)^2}{(\varphi t + k)^2 + (\varphi t - k)^2 m} = \frac{1}{1 + m\delta^2}$$

where δ stands for $\dfrac{\varphi t - k}{\varphi t + k} \in (0, 1)$. Note that $\theta^* > \dfrac{1}{1+m}$, because $\delta^2 < 1$.

Remark. Value of δ measures the *relative difference of commuting costs and distance-sensitive communication costs.* The larger is this difference, the more firms tend to transfer their activity to suburbia, thus the lesser is central share of firms θ^*. Obviously δ increases with respect to commuting costs t.

Note that $\varphi t \leq k$ immediately implies that polycentric structure is cost-inefficient, because communication costs are too large, thus we will assume from now on $\varphi t > k$. Then we can define the following city characteristics

$$l^M = \frac{\pi K^2}{(\varphi t - k)^2} = \frac{\pi K^2}{4k^2} \cdot \frac{(1-\delta)^2}{\delta^2}.$$

Proposition 1. *i) Let $l \leq l^M$ then the unique solution of equation (14) is $\theta^* = 1$ with $m = 0$, i.e. city is, in fact, monocentric.*

ii) Let $l \leq l^M$ then for each $m \geq 1$ equation (14) has unique solution $\theta^* \in \left(\dfrac{1}{1+m\delta^2}, 1\right)$, i.e. there exists an equilibrium distribution of firms.

iii) The CBD share of firms $\theta^*(l, m)$ decreases with respect to both population l and number of SBDs m and

$$\lim_{l\to\infty} \theta^*(l, m) = \frac{1}{1+m\delta^2}, \quad \lim_{m\to\infty} \theta^*(l, m) = \frac{l^M}{l} \in (0, 1). \tag{15}$$

Let's substitute function $\theta^*(m, l)$ into the urban cost function

$$C_u^C = t\sqrt{\frac{\theta l}{\pi}} - \frac{t}{3} \cdot \sqrt{\frac{l}{\pi}} \left[\theta^{3/2} + (1-\theta)\sqrt{\frac{1-\theta}{m}}\right]$$

defined in previous subsection. Then

$$C_u^C(l, 0) = \frac{2}{3} t \sqrt{\frac{l}{\pi}} \text{ for all } l \geq 0$$

and

$$C_u^C(l, m) = \frac{2}{3} t \sqrt{\frac{l}{\pi}} \text{ for all } m > 0 \text{ and for all } l \leq \frac{\pi K^2}{4k^2} \cdot \frac{(1-\delta)^2}{\delta^2}.$$

Moreover, equation (14) implies that

$$\sqrt{\frac{1-\theta^*(l,m)}{m}} = \delta\sqrt{\theta^*(l,m)} - \frac{K\sqrt{\pi}}{(\varphi \cdot t + k)\sqrt{l}} = \delta\sqrt{\theta^*(l,m)} - (1-\delta)\frac{K\sqrt{\pi}}{2k\sqrt{l}},$$

thus for $m > 0$ and $l > \dfrac{\pi K^2}{4k^2} \cdot \dfrac{(1-\delta)^2}{\delta^2}$ the urban cost function

$$C_u^C(l, m) = \frac{t}{3}\left[2 + (1-\delta)(1-\theta^*(l,m))\right]\sqrt{\frac{\theta^*(l,m) \cdot l}{\pi}} + \frac{tK \cdot (1-\delta)}{6k}(1-\theta^*(l,m)).$$

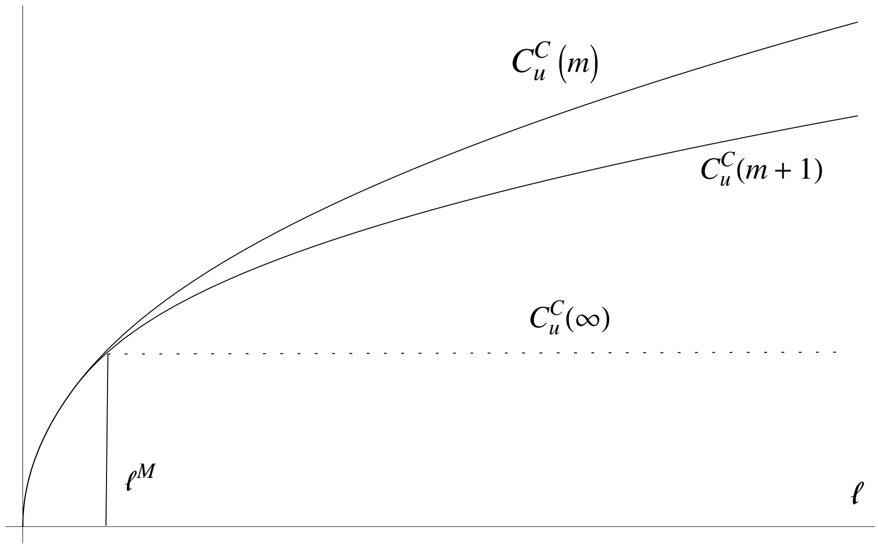


Figure 2: Comparative statics of urban costs

Proposition 2. *Function $C_u^C(m,l)$ is continuous for all $m \geq 0$, $l \geq 0$ and continuously differentiable function for $m > 0$, $l > 0$. Moreover, $C_u^C(l,m)$ strictly increases with respect to l, strictly decreases with respect to m for all $l > \dfrac{\pi K^2}{4k^2} \cdot \dfrac{(1-\delta)^2}{\delta^2}$ and*

$$\lim_{l \to +\infty} C_u^C(l,m) = +\infty, \quad \lim_{m \to +\infty} C_u^C(l,m) = \frac{2t}{3}\sqrt{\frac{l^M}{\pi}} = \frac{(1+\delta)K}{3\varphi\delta}.$$

Figure 2 represents results of Proposition 3 in visual way as simulation in Wolfram's Mathematica 8.0.

3 Short-Run Inter-City Equilibrium

Until now we studied equilibrium decentralization within the city, or Intra-City equilibrium. Let's turn to Inter-City Equilibrium assuming that city populations l_r and numbers of SBD m_r are given (in short run) for each city r. This assumption is quite reasonable in short-run. The long-run equilibrium with endogenous population and number of SBD will be considered in section 4. Equilibrium shares of firms located at CBD θ_r^* may be obtained as solutions of equation (14) independently for each city r. These shares in turn allow to determine the equilibrium values of

- central zone radii r_r^C and distances $||x_r^S||$ from CBD to any SBD due to (7);
- land rents in central zone R_r^C and in suburbia R_r^S due to (8) and (9);
- initial endowment of numéraire $\bar{q}_{0r}$ due to (10);
- wage difference $w_r^C - w_r^S$ due to (11).

Note that these values depend on population l_r, SBD number m_r as well as magnitudes of production costs φ, commuting costs t and communication costs K, k and *don't depend on utility $U(q_0; q(i), i \in [0,n])$ and trade costs τ.*

Equilibrium shares θ_r^* determine cost-efficient distribution of *firms*, while the workers need positive disposable income to survive in "city jungle", e.g., for CBD-employee it means

$$w_r^C + \bar{q}_{0r} - R_r^C(x) - t||x|| > 0.$$

This additional "consumer's cut-off condition" completes the notion of City equilibrium. While urban costs $R_r^C(x) + t||x|| - \bar{q}_{0r}$ don't depend on inter-city trade (and even on *existence* of other cities), the wage

$$w_r^C = \frac{p_{rr}(i) \cdot Q_{rr}(i) + \sum_{s \neq r}(p_{rs}(i) - \tau) \cdot Q_{rs}(i)}{\varphi},$$

on the contrary, substantially depends on trade, as well as on specific form of utility $U(q_0; q(i), i \in [0, n])$.

3.1 Trade, Wages and Utility

Consider now our multi-regional setting. To simplify description, assume that there are two regions, Home and Foreign. Let λ be the share of workers residing in Home city, exogenous in short-run. Then populations of both cities are $l_H = \lambda L$ and $l_F = (1 - \lambda)L$, correspondingly. We shall use Ottaviano's quasi-linear utility function incapsulating quadratic sub-utility

$$U(q_0; q(i), i \in [0, n]) = \alpha \int_0^n q(i)\mathrm{d}i - \frac{\beta}{2} \int_0^n [q(i)]^2 \mathrm{d}i - \frac{\gamma}{2} \left[\int_0^n q(i)\mathrm{d}i \right]^2 + q_0$$

where $\alpha > 0$, $\beta > \gamma > 0$. Comprehensive analysis of *linear city with two SBDs* with this function was carried out in Cavailhès et al. (2007).

For Home region we define demand functions of representative consumer: $q_{HH}(i)$ and $q_{FH}(j)$ as solution of consumer problem

$$\max U(q_0; q(i), i \in [0, n_H + n_F])$$

subject to

$$\int_0^{n_H} p_{HH}(i) q_{HH}(i)\mathrm{d}i + \int_0^{n_F} p_{FH}(j) q_{FH}(j)\mathrm{d}j + q_0 = E_H.$$

Similarly demand functions $q_{FF}(j)$ and $q_{HF}(i)$ of representative consumer of Foreign region may be defined. Facing this demands, firms of first region maximize profit

$$I_H(i) = \lambda L \cdot p_{HH}(i) \cdot q_{HH}(i) + (1 - \lambda)L \cdot [p_{HF}(i) - \tau] \cdot q_{HF}(i)$$

and obtain optimal (equilibrium) prices and quantities. Zero-profit condition (13) determines equilibrium wages. It should be mentioned that trade is profitable only if trade costs τ are sufficiently small.[3] Otherwise, we obtain the Autarchy Inter-City Equilibrium with trade bounded by city walls. In particular, the Home firm's income is

$$I_H(i) = \lambda L \cdot p_{HH}(i) \cdot q_{HH}(i).$$

[3] Sufficient condition for trade with arbitrary inter-city distribution of population is $\tau < \tau_{\text{trade}} = \dfrac{2\alpha\beta\varphi}{2\beta\varphi + \gamma L}$ (see Ottaviano et al. (2002)).

Anyways, specifying of utility function and trade costs allows us to determine wages in both CBD and SBD, which completes the equilibrium description. Further both possible cases will be considered separately: **City Equilibrium under Autarchy** and **City Equilibrium with Trade**.

3.2 Equilibrium under Autarchy

Assume at first that trade costs τ a very large and inter-city trade is non-profitable. It implies that firms in each city forced to trade with local consumers only. Thus equilibrium wage of CBD-employee

$$w^{C*} = \frac{I_r^*}{\varphi} = \frac{l_r \cdot p_{rr}^* \cdot q_{rr}^*}{\varphi},$$

where p_{rr}^* is equilibrium (local) price, q_{rr}^* is equilibrium demand of representative (local) consumer. These wages are determined independently for each city, so we may drop regional subscript r.

This representative consumer maximizes utility

$$U(q_0; q(i), i \in [0,n]) = \alpha \int_0^n q(i)\mathrm{d}i - \frac{\beta}{2}\int_0^n [q(i)]^2 \mathrm{d}i - \frac{\gamma}{2}\left[\int_0^n q(i)\mathrm{d}i\right]^2 + q_0$$

subject to

$$\int_0^n p(i)q(i)\mathrm{d}i + q_0 = w^C + \bar{q}_0 - R^C(x) - t||x|| = w^C - C_u^C,$$

where $n = l/\varphi$ is the local local number of firms. First of all, recall some well-known results concerning consumer's problem with this form of utility.

Proposition 3. *Consumer's demand is linear function*

$$q(i) = \frac{\alpha}{\beta + \gamma n} - \frac{1}{\beta}p(i) + \frac{\gamma}{(\beta+\gamma n)\beta} \cdot P,$$

where

$$P = \int_0^n p(i)\,di$$

is price index. Equilibrium prices are uniform by goods

$$p^*(i) \equiv p^* = \frac{\alpha\beta}{2\beta + \gamma n},$$

while equilibrium demand of representative consumer is

$$q^*(i) \equiv q^* = \frac{\alpha}{2\beta + \gamma n}.$$

Consumer's surplus at equilibrium is equal to

$$CS = \frac{\alpha^2 n(\beta + \gamma n)}{2(2\beta + \gamma n)^2}.$$

See for details (Ottaviano et al., 2002).

Using these facts and taking into account that $n = \dfrac{l}{\varphi}$ we obtain the terms of equilibrium wage at

CBD

$$w^{C*} = \frac{l \cdot p^* \cdot q^*}{\varphi} = \frac{\alpha^2 \beta \varphi l}{(2\beta\varphi + \gamma l)^2}$$

and consumer's surplus

$$CS = \frac{\alpha^2(\beta\varphi + \gamma l) l}{2(2\beta\varphi + \gamma l)^2},$$

which does not depend on consumer residence. Moreover sum of wage and surplus (urban gains, for short) is

$$G_u^C = CS + w^{C*} = \frac{\alpha^2(3\beta\varphi + \gamma l) l}{2(2\beta\varphi + \gamma l)}.$$

Finally, consumer's welfare in CBD is a difference of urban gains and urban costs

$$V^C = CS + w^{C*} - C_u^C.$$

Similar to CBD we may calculate the corresponding SBD's characteristics: wage

$$w^{S*} = \frac{\alpha^2 \beta \varphi l}{(2\beta\varphi + \gamma l)^2} - t \left(\sqrt{\frac{\theta^* l}{\pi}} - \sqrt{\frac{(1-\theta^*) l}{m\pi}} \right),$$

urban gains

$$G_u^S = CS + w^{S*} = \frac{\alpha^2(3\beta\varphi + \gamma l) l}{2(2\beta\varphi + \gamma l)} - t \left(\sqrt{\frac{\theta^* l}{\pi}} - \sqrt{\frac{(1-\theta^*) l}{m\pi}} \right),$$

and welfare

$$V^S = CS + w^{S*} - C_u^S.$$

Recall that in city equilibrium an identity

$$w^C - C_u^C = w^S - C_u^S$$

holds. Thus equilibrium welfares in central zone and in suburbia are equal $V^S = V^C$ and feasibility condition for SBD

$$w^S - C_u^S \geq 0 \iff w^C - C_u^C \geq 0.$$

Once again, we may focus on CBD only. Note that central wage w^{C*} as well as consumer's surplus CS does not depend on number of SBD's m, in contract to urban costs C_u^C.

Proposition 4. *Wage function $w^{C*}(l)$ is strictly concave for $0 \leq l \leq \frac{4\beta\varphi}{\gamma}$, strictly convex for $l > \frac{4\beta\varphi}{\gamma}$, reaches its maximum value $\frac{\alpha^2}{8\gamma}$ at $l^* = \frac{2\beta\varphi}{\gamma}$ and*

$$\lim_{l\to+\infty} w^{C*}(l) = 0, \;\; w^{C*}(0) = 0, \;\; \frac{\partial w^{C*}}{\partial l}(0) = \frac{\alpha^2}{2} < +\infty.$$

Urban gains $G_u^C(l)$ is strictly concave and increasing function for all $l \geq 0$,

$$\lim_{l\to+\infty} G_u^C(l) = \frac{\alpha^2}{2\gamma}, \;\; G_u^C(0) = 0, \;\; \frac{\partial G_u^C}{\partial l}(0) = \frac{3\alpha^2}{4} < +\infty.$$

Proof of this proposition is a routine analytic study of functions $w^{C*}(l)$ and $G_u^C(l)$. Figure 3

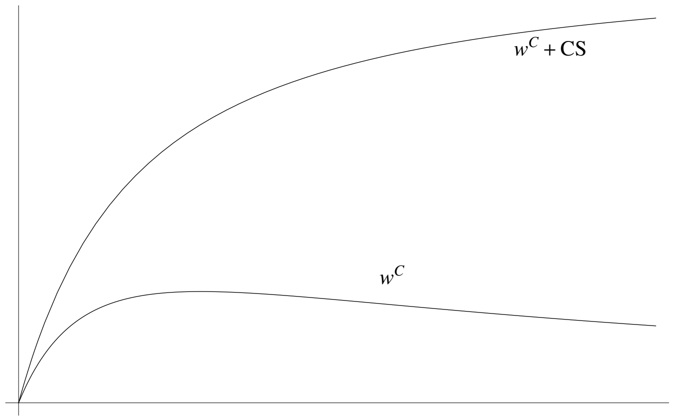


Figure 3: Wage w^C and urban gains $w^C + CS$

represents results of Proposition 4 in visual way as simulation in Wolfram's Mathematica 8.0.

Existence and Comparative Statics

Let's combine results of Propositions 2 and 4. It was mentioned above that equilibrium share θ^* generates city equilibrium only in case of $w^C(l) - C_u^C(l,m)) \geq 0$. Note that wage $w^C(l)$ is bounded function while urban costs increase unrestrictedly with respect to l, hence $w^C(l) - C_u^C(l,m)) < 0$ for all sufficiently large l. Moreover, $w^C(0) = C_u^C(0,m) = 0$, while $\dfrac{\partial w^{C*}}{\partial L}(0) = \dfrac{\alpha^2}{2} < \dfrac{\partial C_u^C}{\partial L}(0,m) = +\infty$, thus $w^C(l) - C_u^C(l,m) < 0$ for all sufficiently small l. If commuting costs t are too large it is possible that city equilibrium does not exist for all l. The simple sufficient condition for equilibrium existence is that *maximum* wage exceeds the urban costs in monocentric city:

$$w^C(l^*) > C_u^C(l^*,0) \iff t < \frac{3\sqrt{\pi}\alpha^2}{16\sqrt{2\beta\gamma\varphi}}.$$

In this case for all $m \geq 0$ and l from neighbourhood of $l^* = \dfrac{2\beta\varphi}{\gamma}$ inequality $w^C(l) \geq C_u^C(l,0) \geq C_u^C(l,m)$ holds. Another sufficient condition is

$$\frac{\alpha^2\beta\varphi l^M}{(2\beta\varphi + \gamma l^M)^2} = w^C(l^M) > C_u^C(l^M,m) \equiv \frac{(1+\delta)K}{3\varphi\delta}, \tag{16}$$

where $l^M = \dfrac{\pi K^2}{4k^2} \cdot \dfrac{(1-\delta)^2}{\delta^2}$ (see Proposition 2).

Let inequality $w^C(l) - C_u^C(l,m)) > 0$ holds for some l and m, then we can define two critical values of city population – maximum and minimum "equilibrium" population:

$$l_{\min}(m) > 0: \ w^C(l_{\min}(m)) = C_u^C(l_{\min}(m),m) \ \text{ and } \forall l < l_{\min}(m): \ w^C(l) < C_u^C(l,m) \ \text{ holds,}$$
$$l_{\max}(m) < +\infty: \ w^C(l_{\max}(m)) = C_u^C(l_{\max}(m),m) \ \text{ and } \forall l > l_{\max}(m): \ w^C(l) < C_u^C(l,m) \ \text{ holds}$$

It is obvious that increasing of m broaden interval $[l_{\min}(m), l_{\max}(m)]$ (more exactly, $l_{\min}(m)$ decreases, while $l_{\max}(m)$ increases with respect to m).[4] Moreover, disposable income $w^C(l) - C_u^C(l,m))$ and welfare $V^C = G_u^C(l) - C_u^C(l,m))$ both increases with respect to m, with exception $l \in [0, l^M]$ when

[4] If $w^C(l^M) > \frac{(1+\delta)K}{3\varphi\delta}$ then $0 < l_{\min}(m) \equiv l_{\min}(0) < l^M < l_{\max}(0) < l_{\max}(m)$ for all $m \geq 0$.

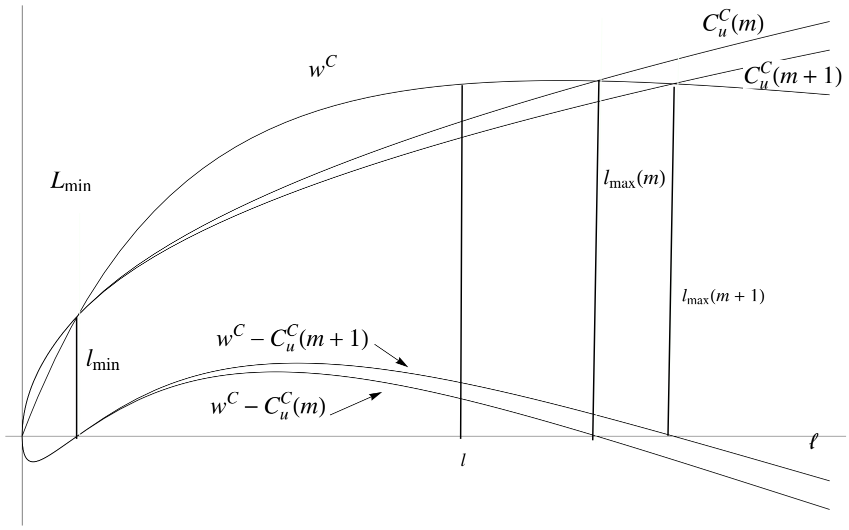


Figure 4: Comparative statics of equilibria

urban costs $C_u^C(l, m)$ do not depend on m. Figure 5 illustrates equilibrium under autarchy and comparative statics with respect to m using simulation in Wolfram's Mathematica 8.0.

Discussion. Previous considerations show that autarchy may be very restrictive to the city sizes: city survives only if its size exceeds the lower threshold $l_{\min} > 0$ and does not exceed the upper one $l_{\max}$. Developing of the city infrastructure (i.e. increasing in m) shifts up the upper bound $l_{\max}$, but cannot help too much with lower one $l_{\min}$.

3.3 Equilibrium with Trade

Assume that trade costs $\tau < \tau_{\text{trade}} = \dfrac{2\alpha\beta\varphi}{2\beta\varphi + \gamma L}$ where $L = l_H + l_F$ is total number of workers. Then trade is profitable for any worker's distribution among cities (see Ottaviano et al. (2002)). Recall that $l_H = \lambda L$, $l_F = (1-\lambda)L$ for $\lambda \in [0,1]$. Case $\lambda = 1$ (resp. $\lambda = 0$) implies that Foreign city (resp. Home city) disappears and all population gathered in Home city (resp. Foreign city). To make it possible, assume $L < \min\{l_{\max}(m_H), l_{\max}(m_F)\}$, in particular, $w_H^{C*}(1) > C_u^C(1)$. Otherwise, some values of $\lambda \in [0,1]$ may be non-feasible.

Proposition 5. *Trade equilibrium prices are uniform by goods*

$$p_{HH}^*(i) \equiv p_{HH}^* = \frac{2\alpha\beta\varphi + \tau\gamma L \cdot (1-\lambda)}{2(2\beta\varphi + \gamma L)}, \; p_{FF}^*(i) \equiv p_{FF}^* = \frac{2\alpha\beta\varphi + \tau\gamma L \cdot \lambda}{2(2\beta\varphi + \gamma L)},$$

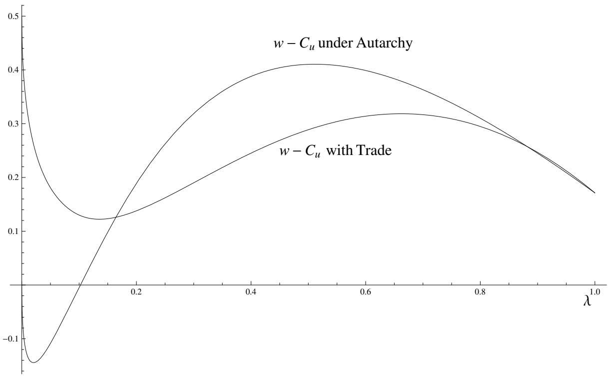
$$p_{HF}^* = p_{FF}^* + \frac{\tau}{2}, \; p_{FH}^* = p_{HH}^* + \frac{\tau}{2}.$$

Home Consumer's Surplus

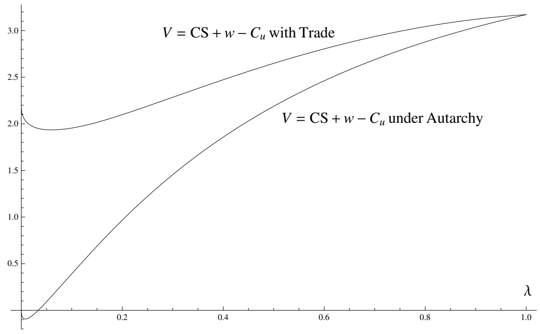
$$CS_H(\lambda) = \frac{\alpha^2 L}{2(\beta\varphi + \gamma L)} - \frac{\alpha L}{\beta\varphi + \gamma L} \cdot [p_{HH}^* \cdot \lambda + p_{FH}^* \cdot (1-\lambda)] +$$

$$+ \frac{L}{2\beta\varphi} \cdot \left[(p_{HH}^*)^2 \cdot \lambda + (p_{FH}^*)^2 \cdot (1-\lambda)\right] - \frac{\gamma L^2}{2\beta\varphi \cdot (\beta\varphi + \gamma L)} \cdot [p_{HH}^* \cdot \lambda + p_{FH}^* \cdot (1-\lambda)]^2 \qquad (17)$$

is increasing and concave function of λ.



a) Disposable Income with Trade and under Autarchy



b) Welfare with Trade and under Autarchy

Figure 5: Autarchy and Trade

Home Equilibrium wage

$$w_H^{C*}(\lambda) = \frac{\beta\varphi L}{(2\beta\varphi + \gamma L)^2}\left[\left(\alpha + \frac{\tau\gamma L}{2\beta\varphi}(1-\lambda)\right)^2 \cdot \lambda + \left((\alpha - \tau) - \frac{\tau\gamma L}{2\beta\varphi}(1-\lambda)\right)^2 \cdot (1-\lambda)\right] \tag{18}$$

is strictly concave function, increasing at $\lambda = 0$.

For analytical proof see Ottaviano et al. (2002) and Cavailhès et al. (2007)

Note that

$$w_H^{C*}(0) = \frac{\alpha^2\beta\varphi L}{(2\beta\varphi + \gamma L)^2}\left(1 - \tau \cdot \frac{2\alpha\beta\varphi + \gamma L}{2\alpha\beta\varphi}\right)^2 > 0 = C_u^C(0)$$

for all $\tau < \tau_{\text{trade}} = \dfrac{2\alpha\beta\varphi}{2\beta\varphi + \gamma L}$. Moreover, for letting $\tau = 0$ in (18) we obtain

$$w_H^{C*}(\lambda) = \frac{\alpha^2\beta\varphi L}{(2\beta\varphi + \gamma L)^2} = w_H^{C*}(1) > C_u^C(1) \geq C_u^C(\lambda)$$

for all $\lambda \in [0,1]$, because $l_{\min} < L < l_{\max}$. Thus, for all sufficiently small τ inequality $w_H^{C*}(\lambda) > C_u^C(\lambda)$ holds for all $\lambda \in [0,1]$. It means that sufficient trade freeness cancels *Problem of lower threshold*, i.e. small cities could survive, trading with the larger ones.

Typical example of simulation is presented at Figure 5.

Summarizing the previous considerations we formulate the following

Proposition 6. *(i) Under autarky there exist threshold values* $0 < l_{\min} < l_{\max} < \infty$*, such that consumer's cut-off condition holds (hence, city equilibrium exists) if and only if city population* $l_r \in (l_{\min}, l_{\max})$.

(ii) Increasing in SBD number m_r *broaden admissibility interval* $(l_{\min}, l_{\max})$ *for specific city, i.e.,* $l_{\min}(m+1) \leq l_{\min}(m)$ *and* $l_{\max}(m+1) \geq l_{\max}(m)$*. Yet if* $l_{\min}(m) \leq l^M$ *then increasing in SBD number does not affect* $l_{\min}$*, i.e.,* $l_{\min}(m') \equiv l_{\min}(m)$ *for all* $m' \geq m$.

(iii) On the other hand, if trade costs τ *are sufficiently small, then* $l_{\min}(m) \equiv 0$ *for all* m.

This Proposition implies that too small ($l < l_{\min}$) or too large ($l > l_{\max}$) cities cannot survive under autarky. Developing of city infrastructure may help to shift up the upper bound $l_{\max}$, the lower threshold, however, cannot shift $l_{\min}$ closer to zero when $l_{\min}$ belongs to "monocentric interval" $(0, l^M)$. Only trade with the larger neighbouring cities helps to survive for the small ones.

3.4 Endogenous SBD number

In what follows we assume that $l > l^M$, which forces firms to move from center to suburbs. Previous considerations show that then appear some number $m \geq 1$ of SBDs. What determines its number? There is no simple and unambiguous answer, because in practice it depends on many factors, that are not always economic ones.

One of the main questions is “Who can afford the building of additional suburb?” If answer is “None”, we find ourself in setting with predefined number of SBDs (like model of Cavailhès et al., 2007). Otherwise, we assume that decision is up to ‘City Developer’, who takes into account the social welfare considerations. This welfare is measures by indirect utility

$$V(\lambda) = CS(\lambda) + w^{C*}(\lambda) - C_u^C(\lambda).$$

Assume that the current number of SBD is $m \geq 0$ and ‘City Developer’ considers prospective developing of city infrastructure. It implies an increasing of SBD number to $m+1$, which requires additional *development costs* $D > 0$ measured by numéraire good. On the other hand, this development increases (*ceteris paribus*) total welfare by magnitude

$$\begin{aligned}\Delta_m = l \cdot (V(l, m+1) - V(l, m)) = l \cdot \big((G_u^C(l) - C_u^C(l, m+1)) - (G_u^C(l) - C_u^C(l, m))\big) = \\ = l \cdot \big(C_u^C(l, m) - C_u^C(l, m+1)\big)\end{aligned}$$

Urban costs $C_u^C(l, m)$ decrease with respect to m, hence $\Delta_m > 0$. On the other hand,

$$\sum_{m=0}^{\infty} \Delta_m = l \cdot (C_u^C(l, 0) - \lim_{m \to \infty} C_u^C(l, m)) = \frac{2t \cdot l}{3 \cdot \sqrt{\pi}} \cdot \left(\sqrt{l} - \sqrt{l^M}\right)$$

where $l^M = \dfrac{\pi K^2}{4k^2} \cdot \dfrac{(1-\delta)^2}{\delta^2} = \dfrac{\pi K^2}{(\varphi t - k)^2}$. Thus $\Delta_m \to 0$ very quickly and $\Delta_m < D$ for all sufficiently large m. Moreover,

$$\sum_{m=0}^{\infty} \Delta_m = \frac{2t \cdot l}{3 \cdot \sqrt{\pi}} \cdot \left(\sqrt{l} - \frac{\pi K}{\varphi t - k}\right)$$

is obviously increasing in both city population l and commuting costs t. The same holds for each Δ_m taken separately, but the proof is cumbersome an requires Implicit Function differentiation.

Consideration of efficiency determine the cost-efficient endogenous number of SBD’s:

$$m^* = \max\{m \mid \Delta_m \geq D\}.$$

In particular, the previous considerations imply that m^* also increases in both l and t. Note that theoretical comparative statics is fully supported by empirical evidences (see MacMillen and Smith, 2003).

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