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Abstract

R&D investment has well-known liquidity problems, with potentially important consequences. In this paper, we analyze the effects of monetary policy on economic growth and social welfare in a Schumpeterian model with cash-in-advance (CIA) constraints on consumption, R&D investment, and manufacturing. Our results are as follows. Under the CIA constraints on consumption and R&D (manufacturing), an increase in the nominal interest rate would decrease (increase) R&D and economic growth. So long as the effect of cash requirements in R&D is relatively more important than in manufacturing, the nominal interest rate would have an overall negative effect on R&D and economic growth as documented in recent empirical studies. We also analyze the optimality of Friedman rule and find that Friedman rule can be suboptimal due to a unique feature of the Schumpeterian model. Specifically, we find that the suboptimality or optimality of Friedman rule is closely related to a seemingly unrelated issue that is the overinvestment versus underinvestment of R&D in the market economy, and this result is robust to alternative versions of the Schumpeterian model.

JEL classification: O30, O40, E41

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1 Introduction

In this study, we analyze the effects of monetary policy on economic growth and social welfare in a Schumpeterian growth model featuring cash-in-advance (CIA) requirements. In the well-established tradition of CIA and economic growth, the CIA constraints appear on consumption and on capital investment, as in the seminal study by Stockman (1981), further developed by Abel (1985). In their line of argument, as long as physical capital acquisition has cash requirements, the long-term capital-to-labor ratio is decreased by the nominal interest rate, which acts as a tax on capital. However, existing evidence strongly supports the view that R&D investment is even more severely affected by liquidity requirements[1] than physical capital: for example, Brown and Petersen (2009) find[2] that the investment-cash flow sensitivity largely disappears for physical investment, while it remains comparatively strong for R&D. More recently, Aghion *et al.* (2012) find in their data[3] that R&D is more affected by countercyclical monetary policy than physical investment, due to credit and liquidity constraints.

To address this issue in a neat way, we build a scale-free variant of the quality-ladder model *a la* Grossman and Helpman (1991) and Aghion and Howitt (1992), which incorporates money demand into the quality-ladder model through a CIA constraint on R&D investment, alongside the more conventional CIA constraints on consumption and manufacturing expenditures. Our main results can be summarized as follows. Under the CIA constraints on consumption and R&D, an increase in the nominal interest rate would decrease R&D and economic growth. This could be partially offset by a CIA requirement on manufacturing, whereby an increase in the nominal interest rate may encourage R&D. However, as long as the effect of the CIA constraint on R&D dominates the effect of the CIA constraint on manufacturing, the nominal interest rate would have an overall negative effect on R&D and economic growth, as documented in recent empirical studies, such as Evers *et al.* (2007) and Chu and Lai (2012).

We also analyze the long-run implications on social welfare and compare our results to Friedman's (1969) proposed monetary policy rule, according to which the optimal nominal interest rate should be zero. Since then, a large number of studies has analyzed the optimality of Friedman rule in different economic environments; see for example, Mulligan and Sala-i-Martin (1997) for a discussion on some of the early studies, and Bhattacharya *et al.* (2005), Gahvari (2007, 2012) and Lai and Chin (2010) for recent contributions. Until recently, a close-to-zero nominal interest rate has been little more than a theoretical possibility, rarely occurring in reality. However, since December 2008, the target range for the federal funds rate in the US has been at zero to 0.25%. In June 2012, the Federal Open Market Committee (FOMC) announced that it "currently anticipates that economic conditions [...] are likely to warrant exceptionally low levels for the federal funds rate at least through late 2014."[4] Another example is Japan, where the benchmark interest rate has been between zero and 0.1% also since December 2008. In this paper, we find that a zero-interest-rate

[1] In their empirical survey, Hall and Lerner (2009), state: "The conclusions from this body of empirical work are several: first, there is solid evidence that debt is a disfavored source of finance for R&D investment".

[2] In their 1970-2006 US firm-level data.

[3] For 15 industrial OECD countries in the 1995-2005 period.

[4] Press Release of the FOMC meeting on June 20, 2012.

policy can be suboptimal due to a unique feature of the Schumpeterian model that has been largely ignored in the literature on monetary economics. Specifically, we find that the suboptimality or optimality of Friedman rule is closely related to a seemingly unrelated issue that is the overinvestment versus underinvestment of R&D in the market economy, and this result is robust to alternative versions of the model. Under inelastic labor supply, Friedman rule is suboptimal (optimal) if and only if the equilibrium is characterized by R&D overinvestment (underinvestment). Under elastic labor supply, R&D overinvestment (underinvestment) becomes necessary (sufficient) for Friedman rule to be suboptimal (optimal) due to an interaction between the CIA constraints on consumption and R&D investment.

Our welfare analysis relates to the R&D-based growth literature. In this literature, whether R&D underinvestment or overinvestment emerges in equilibrium is still an open question. Jones and Williams (2000) show that a calibrated R&D-based growth model is likely to feature R&D underinvestment because the positive externalities associated with R&D dominate the negative externalities. A subsequent study by Comin (2004) shows that this result is based on an assumption in the calibration that domestic total factor productivity (TFP) growth is completely driven by domestic R&D. Then, he finds that if domestic R&D only drives a small fraction of domestic TFP growth, there would be R&D overinvestment in the economy, which he argues as the more likely scenario according to his simulation results.

We contribute to this literature by incorporating CIA requirements into a standard R&D-driven growth framework with vertical innovation. In a previous attempt, featuring CIA and horizontal innovation, *a la* Romer (1990), Marquis and Reffett (1994) prove the optimality of the Friedman rule in the presence of CIA in the consumption sector. Their crucial assumption is that the "non-cash good" fraction of consumption requires human capital to process transactions. Therefore, an increase in the interest rate, by discouraging the "cash good" consumption, increases the demand for transaction services, thereby reallocating human capital from manufacturing and research into the payment production. This has a negative level effect and a negative growth effect - by reducing human capital input from R&D. Since in Romer's (1990) structure R&D is always sub-optimal, the Friedman rule would be second-best optimal. Unlike their model[5], which quite unrealistically assumes that liquidity problems are absent in the R&D sector, we here allow for the presence of a CIA constraint in the R&D sector as well, and single out a direct negative effect of a higher nominal interest rate on R&D without the need of any role of human capital in the transaction technology. Moreover, the optimality of a positive nominal interest rate in the present study is driven by the possibility of R&D overinvestment in the Schumpeterian growth model. This property of R&D overinvestment is absent in the neoclassical growth model by construction and also absent in the Romer (1990) model. Finally, Marquis and Reffett (1994) considers a first-generation R&D-based growth model that features scale effects; in contrast, we examine our results in the two main versions of a scale-invariant Schumpeterian growth model.

The rest of this note is organized as follows. Section 2 presents the monetary Schumpeterian growth model. Section 3 analyzes the effects of monetary policy. Section 4 considers alternative versions of the model. The final section concludes.

[5] And a related model by Chu *et al.* (2012).

2 A monetary Schumpeterian growth model

In this section, we present the monetary Schumpeterian growth model. In summary, we modify the quality-ladder model in Grossman and Helpman (1991) by allowing for elastic labor supply and incorporating money demand via CIA constraints on consumption and R&D investment.[6] Furthermore, we allow for population growth and remove scale effects by incorporating a dilution effect on R&D productivity following Laincz and Peretto (2006).[7] Given that the quality-ladder model has been well-studied, the standard features of the model will be briefly described below to conserve space.

2.1 Households

At time t, the population size of each household is N_t, and its law of motion is $\dot{N}_t = nN_t$, where $n \geq 0$ is the exogenous population growth rate. There is a unit continuum of identical households, who have a lifetime utility function given by[8]

$$U = \int_0^\infty e^{-\rho t} \left[\ln c_t + \theta \ln(1 - l_t) \right] dt, \tag{1}$$

where c_t is per capita consumption of final goods and l_t is the supply of labor per person at time t. The parameters $\rho > 0$ and $\theta \geq 0$ determine respectively subjective discounting and leisure preference. Each household maximizes (1) subject to the following asset-accumulation equation:

$$\dot{a}_t + \dot{m}_t = (r_t - n)a_t + w_t l_t + \tau_t - c_t - (\pi_t + n)m_t + i_t b_t. \tag{2}$$

a_t is the real value of assets owned by each member of households, and r_t is the real interest rate. Each member of households supplies labor l_t to earn a real wage rate w_t. Each person also receives a lump-sum transfer τ_t from the government (or pay a lump-sum tax if $\tau_t < 0$). π_t is the inflation rate that determines the cost of holding money, and m_t is the real money balance held by each person partly to facilitate purchases of consumption goods. The CIA constraint is given by $\xi c_t + b_t \leq m_t$, where $\xi \in [0, 1]$.[9] b_t is the amount of money borrowed from each member of households by entrepreneurs to finance R&D investment, and the return on b_t is i_t.

From standard dynamic optimization, we derive a no-arbitrage condition $i_t = r_t + \pi_t$; therefore, i_t is also the nominal interest rate. The optimality condition for consumption is

$$\frac{1}{c_t} = \eta_t (1 + \xi i_t), \tag{3}$$

[6] We consider this version of the model with CIA constraints on consumption and R&D as our benchmark. However, we will also explore the implications of a CIA constraint on manufacturing in an extension of the model; see Section 4.2.

[7] In Section 4.3, we consider a semi-endogenous-growth version of the model. See Jones (1999) and Laincz and Peretto (2006) for a discussion of scale effects in R&D-based growth models.

[8] Here we assume that the utility function is based on per capita utility. Alternatively, one can assume that the utility function is based on aggregate utility in which case the effective discount rate simply becomes $\rho - n$.

[9] The usual CIA constraint on consumption is captured by the special case of $\xi = 1$; see for example, Wang and Yip (1992). Here we follow Dotsey and Ireland (1996) to consider a more general setup, and the parameter ξ determines the relative strength of the CIA constraints on consumption and R&D investment.

where η_t is the Hamiltonian co-state variable on (2). The optimality condition for labor supply is

$$w_t(1-l_t) = \theta c_t(1+\xi i_t), \tag{4}$$

and the familiar intertemporal optimality condition is

$$-\frac{\dot{\eta}_t}{\eta_t} = r_t - \rho - n. \tag{5}$$

2.2 Final goods

Final goods are produced by competitive firms that aggregate intermediate goods using a standard Cobb-Douglas aggregator given by

$$y_t = \exp\left(\int_0^1 \ln x_t(j)dj\right), \tag{6}$$

where $x_t(j)$ denotes intermediate goods $j \in [0,1]$. From profit maximization, the conditional demand function for $x_t(j)$ is

$$x_t(j) = y_t/p_t(j), \tag{7}$$

where $p_t(j)$ is the price of $x_t(j)$.

2.3 Intermediate goods

There is a unit continuum of industries producing differentiated intermediate goods. Each industry is temporarily dominated by an industry leader until the arrival of the next innovation, and the owner of the new innovation becomes the next industry leader.[10] The production function for the leader in industry j is

$$x_t(j) = z^{q_t(j)} L_{x,t}(j). \tag{8}$$

The parameter $z > 1$ is the step size of a productivity improvement, and $q_t(j)$ is the number of productivity improvements that have occurred in industry j as of time t. $L_{x,t}(j)$ is production labor in industry j. Given $z^{q_t(j)}$, the marginal cost of production for the industry leader in industry j is $mc_t(j) = w_t/z^{q_t(j)}$. It is useful to note that we here adopt a cost-reducing view of vertical innovation as in Peretto (1998).

Standard Bertrand price competition leads to a profit-maximizing price $p_t(j)$ determined by a markup $\mu = p_t(j)/mc_t(j)$ over the marginal cost. In the original Grossman-Helpman model, the patentholder is assumed to have complete protection against imitation such that $\mu = z$. Li (2001) considers a more general policy environment with incomplete patent

[10] This is known as the Arrow replacement effect in the literature. See Cozzi (2007) for a discussion of the Arrow effect.

protection against potential imitation such that $\mu \in (1, z)$; see also Goh and Olivier (2002).[11] Here we follow their formulation to model patent breadth μ, which also serves as a simple way to separate the markup from the step size z. The amount of monopolistic profit is

$$\Pi_t(j) = \left(\frac{\mu - 1}{\mu}\right) p_t(j) x_t(j) = \left(\frac{\mu - 1}{\mu}\right) y_t. \tag{9}$$

Finally, production-labor income is

$$w_t L_{x,t}(j) = \left(\frac{1}{\mu}\right) p_t(j) x_t(j) = \left(\frac{1}{\mu}\right) y_t. \tag{10}$$

2.4 R&D

Denote $v_t(j)$ as the value of the monopolistic firm in industry j. Because $\Pi_t(j) = \Pi_t$ for $j \in [0, 1]$ from (9), $v_t(j) = v_t$ in a symmetric equilibrium that features an equal arrival rate of innovation across industries.[12] In this case, the familiar no-arbitrage condition for v_t is

$$r_t = \frac{\Pi_t + \dot{v}_t - \lambda_t v_t}{v_t}. \tag{11}$$

This condition equates the real interest rate r_t to the asset return per unit of asset. The asset return is the sum of (a) monopolistic profit Π_t, (b) potential capital gain $\dot{v}_t$, and (c) expected capital loss $\lambda_t v_t$ due to creative destruction, where λ_t is the arrival rate of the next innovation.

There is a unit continuum of R&D firms indexed by $k \in [0, 1]$. They hire R&D labor $L_{r,t}(k)$ for innovation. The wage payment for R&D labor is $w_t L_{r,t}(k)$; however, to facilitate this wage payment, the entrepreneur needs to borrow money from households subject to the nominal interest rate i_t. Therefore, the CIA constraint on R&D gives the monetary authority an ability to influence the equilibrium allocation of resources across sectors through the nominal interest rate.[13]

The zero-expected-profit condition of firm k is[14]

$$v_t \lambda_t(k) = (1 + i_t) w_t L_{r,t}(k), \tag{12}$$

[11] Their idea is that without sufficient strength of patent protection, the presence of monopolistic profits attracts imitation; therefore, stronger patent protection allows monopolistic producers to charge a higher markup without the threat of imitation.

[12] We follow the standard approach in the literature to focus on the symmetric equilibrium. See Cozzi *et al.* (2007) for a theoretical justification for the symmetric equilibrium to be the unique rational-expectation equilibrium in the Schumpeterian growth model.

[13] Evers *et al.* (2007) provide empirical evidence that the inflation rate and the nominal interest rate have negative effects on total factor productivity growth via R&D.

[14] Here we assume that $b_t(k) \geq \alpha w_t L_{r,t}(k)$, where α can be interpreted as the length of time money must be held to finance R&D; see Feenstra (1985) for a discussion of CIA constraints in continuous time. For now, we normalize α to one for simplicity. In Section 4.2, we consider a more general analysis with $\alpha \in [0, 1]$.

where the *firm-level* arrival rate of innovation is $\lambda_t(k) = \overline{\varphi}_t L_{r,t}(k)$, where $\overline{\varphi}_t = \varphi / N_t$ captures the dilution effect that removes scale effects as in Laincz and Peretto (2006).[15] Finally, the *aggregate* arrival rate of innovation is

$$\lambda_t = \int_0^1 \lambda_t(k) dk = \frac{\varphi L_{r,t}}{N_t} = \varphi l_{r,t}, \tag{13}$$

where we have defined $l_{r,t} \equiv L_{r,t}/N_t$ as R&D labor per capita. Similarly, we will define $l_{x,t} \equiv L_{x,t}/N_t$ as production labor per capita.

2.5 Monetary authority

The nominal money supply is denoted by M_t, and its growth rate is $\dot{M}_t/M_t$. By definition, the aggregate real money balance is $m_t N_t = M_t/P_t$, where P_t denotes the price of final goods. The monetary policy instrument that we consider is i_t because we are interested in analyzing the optimal nominal interest rate. Given an exogenously chosen i_t by the monetary authority, the inflation rate is endogenously determined according to $\pi_t = i_t - r_t$. Then, given π_t, the growth rate of the nominal money supply is endogenously determined according to $\dot{M}_t/M_t = \dot{m}_t/m_t + \pi_t + n$. Finally, the monetary authority returns the seigniorage revenue as a lump transfer $\tau_t N_t = \dot{M}_t/P_t = [\dot{m}_t + (\pi_t + n)m_t]N_t$ to households.

Alternatively, one can consider the growth rate of money supply as the policy instrument directly controlled by the monetary authority. Notice that in our economy, the consolidated public sector, by manipulating the changes in money supply via lump-sum transfers to households, is able to control the money growth rate $\dot{M}_t/M_t$ and hence the nominal interest rate. To see this, by the Fisher equation, $i_t = r_t + \pi_t$, where $\pi_t = \dot{M}_t/M_t - g_t - n$.[16] By the Euler equation, $r_t = \rho + g_t + n$;[17] therefore, the nominal interest rate is

$$i_t = r_t + \pi_t = (\rho + g_t + n) + (\dot{M}_t/M_t - g_t - n) = \rho + \dot{M}_t/M_t,$$

which is determined by the growth rate of money supply.

2.6 Decentralized equilibrium

The equilibrium is a time path of allocations $\{c_t, m_t, l_t, y_t, x_t(j), L_{x,t}(j), L_{r,t}(k)\}$ and a time path of prices $\{p_t(j), w_t, r_t, i_t, v_t\}$. Also, at each instance of time,

- households maximize utility taking $\{i_t, r_t, w_t\}$ as given;
- competitive final-goods firms produce $\{y_t\}$ to maximize profit taking $\{p_t(j)\}$ as given;
- monopolistic intermediate-goods firms produce $\{x_t(j)\}$ and choose $\{L_{x,t}(j), p_t(j)\}$ to maximize profit taking $\{w_t\}$ as given;

[15] In Section 4.3, we consider an alternative specification given by $\overline{\varphi}_t = \varphi / Z_t$ under which the model becomes a semi-endogenous growth model as in Segerstrom (1998).

[16] It can be shown that on the balanced growth path, m_t and c_t grow at the same rate.

[17] It can be shown that on the balanced growth path, $1/\eta_t$ and c_t grow at the same rate.

- R&D firms choose $\{L_{r,t}(k)\}$ to maximize expected profit taking $\{i_t, w_t, v_t\}$ as given;
- the market-clearing condition for labor holds such that $L_{x,t} + L_{r,t} = l_t N_t$;
- the market-clearing condition for final goods holds such that $y_t = c_t N_t$;
- the value of monopolistic firms adds up to the value of households' assets such that $v_t = a_t N_t$; and
- the amount of money borrowed by R&D entrepreneurs is $w_t L_{r,t} = b_t N_t$.

Substituting (8) into (6), we derive the aggregate production function given by

$$y_t = Z_t L_{x,t}, \tag{14}$$

where aggregate technology Z_t is defined as

$$Z_t = \exp\left(\int_0^1 q_t(j)dj \ln z\right) = \exp\left(\int_0^t \lambda_s ds \ln z\right). \tag{15}$$

The second equality of (15) applies the law of large numbers. Differentiating the log of (15) with respect to t yields the growth rate of aggregate technology given by

$$g_t \equiv \dot{Z}_t / Z_t = \lambda_t \ln z = (\varphi \ln z) l_{r,t}. \tag{16}$$

As for the dynamics of the model, Proposition 1 shows that the economy jumps to a unique and saddle-point stable balanced growth path.

Proposition 1 *Given a constant nominal interest rate i, the economy immediately jumps to a unique and saddle-point stable balanced growth path along which each variable grows at a constant (possibly zero) rate.*

Proof. See Appendix A. ■

On the balanced growth path, the equilibrium labor allocation is stationary. Imposing balanced growth on (11) yields $v_t = \Pi_t/(\rho + \lambda)$ because $\dot{\Pi}_t/\Pi_t = g + n$ and $r = g + \rho + n$ from (5). Substituting this condition into (12) yields $\lambda \Pi_t/(\rho + \lambda) = (1 + i) w_t L_{tr}$, where λ is given by (13), Π_t is given by (9) and w_t is given by (10). Using these conditions, we derive

$$(\mu - 1) l_x = (l_r + \rho/\varphi)(1 + i), \tag{17}$$

which is the first equation that solves for $\{l_x, l_r, l\}$. The second equation is simply the per capita version of the labor-market-clearing condition given by

$$l_x + l_r = l. \tag{18}$$

To derive the last equation, we substitute (10) into (4) to obtain

$$l = 1 - \theta(1 + \xi i)\mu l_x. \tag{19}$$

Solving (17)-(19), we obtain the equilibrium labor allocation as follows.

$$l_r = \frac{\mu - 1}{\mu + i + \mu\theta(1+i)(1+\xi i)}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}, \tag{20}$$

$$l_x = \frac{1+i}{\mu + i + \mu\theta(1+i)(1+\xi i)}\left(1 + \frac{\rho}{\varphi}\right), \tag{21}$$

$$l = \frac{\mu + i}{\mu + i + \mu\theta(1+i)(1+\xi i)}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}. \tag{22}$$

Equation (20) shows that R&D labor l_r is decreasing in the nominal interest rate i under both elastic labor supply (i.e., $\theta > 0$) and inelastic labor supply (i.e., $\theta = 0$). Therefore, economic growth $g = (\varphi \ln z)l_r$ is also decreasing in i in both cases. This result is consistent with empirical evidence in Chu and Lai (2012), who document a negative relationship between inflation and R&D. In our model, $\pi = i - r = i - g(i) - \rho - n$; therefore, an increase in i causes an increase in π and l_x, and a decrease in l_r, g and r.

Proposition 2 *R&D and economic growth are both decreasing in the nominal interest rate.*

Proof. Note (20) and then (16). ■

2.7 Socially optimal allocation

In this subsection, we derive the socially optimal allocation of the model. Imposing balanced growth on (1) yields

$$U = \frac{1}{\rho}\left[\ln c_0 + \frac{g}{\rho} + \theta \ln(1-l)\right], \tag{23}$$

where $c_0 = Z_0 l_x$ and $g = \lambda \ln z = (\varphi \ln z)l_r$. We normalize the exogenous Z_0 to unity. Maximizing (23) subject to (18) yields the first-best allocation denoted with a superscript *.

$$l_r^* = 1 - \frac{\rho(1+\theta)}{\varphi \ln z}, \tag{24}$$

$$l_x^* = \frac{\rho}{\varphi \ln z}, \tag{25}$$

$$l^* = 1 - \frac{\rho\theta}{\varphi \ln z}. \tag{26}$$

We restrict the parameter space to ensure that $l_r^* > 0$, which in turn implies that $l^* > 0$.

3 Optimal monetary policy and Friedman rule

In this section, we analyze optimal monetary policy and the optimality of Friedman rule. In Section 3.1, we consider the special case of inelastic labor supply. In Section 3.2, we consider the general case of elastic labor supply. Under elastic labor supply, we consider both cases of the model with and without the CIA constraint on consumption.

3.1 Friedman rule under inelastic labor supply

In this subsection, we consider Friedman rule under inelastic labor supply, which is captured by setting $\theta = 0$. In this case, the equilibrium allocation simplifies to

$$l_r = \frac{\mu - 1}{\mu + i}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}, \tag{27}$$

$$l_x = \frac{1 + i}{\mu + i}\left(1 + \frac{\rho}{\varphi}\right), \tag{28}$$

and $l = 1$. From (27) and (28), it is easy to see that R&D labor l_r is decreasing in the nominal interest rate i, whereas production labor l_x is increasing in i. Furthermore, given the fact that the parameter ξ does not appear in (27) and (28), the CIA constraint on consumption has no effect on l_r and l_x under inelastic labor supply. In this case, the effect of i operates through the CIA constraint on R&D investment under which an increase in the nominal interest rate increases the cost of R&D and leads to a reallocation of labor from R&D to production.

Under inelastic labor supply, the monetary authority may be able to achieve the first-best allocation by choosing the optimal nominal interest rate i^* given by[18]

$$i^*(\underset{+}{\mu}, \underset{+}{\rho}, \underset{-}{\varphi}, \underset{-}{z}) = \max\left[\frac{\mu - (1 + \varphi/\rho)\ln z}{(1 + \varphi/\rho)\ln z - 1}, 0\right]. \tag{29}$$

The inequality $i^* \geq 0$ is imposed to respect the zero lower bound of the nominal interest rate. If $i^* = 0$, then Friedman rule is optimal, but the monetary authority is unable to achieve the first-best allocation (unless $i^* = 0$ holds exactly and is not binding). If $i^* > 0$, then Friedman rule is suboptimal, but the monetary authority is able to achieve the first-best allocation by setting $i = i^*$. It is well known that the quality-ladder model features both positive R&D externalities, such as the intertemporal spillover effect and the consumer-surplus effect, and negative R&D externalities, such as the business-stealing effect.[19] Therefore, the equilibrium with $i = 0$ may feature either overinvestment or underinvestment in R&D. Comparing (27) with (24) under $\theta = 0$, we see that $i^* > 0$ if and only if the equilibrium l_r evaluated at $i = 0$ is greater than the optimal l_r^*. In other words, R&D overinvestment in equilibrium is a necessary and sufficient condition for Friedman rule to be suboptimal. We summarize these results in Proposition 3.

Proposition 3 *Under inelastic labor supply, the optimal nominal interest rate i^* is given by (29). If and only if R&D overinvestment occurs in the zero-nominal-interest-rate equilibrium, then the optimal nominal interest rate would be strictly positive; in this case, Friedman rule is suboptimal.*

[18] It is useful to note that $l_r^* > 0$ is sufficient to ensure that $(1 + \varphi/\rho)\ln z > 1$.

[19] One could also introduce an additional negative externality in the form of an intratemporal duplication effect as in Jones and Williams (2000) by assuming decreasing returns to scale in (13) (i.e., $\lambda_t = \varphi l_{r,t}^{\phi}$, where $0 < \phi < 1$) in order to expand the parameter space for R&D overinvestment. However, this additional feature would complicate our analysis, and the current framework that already features negative R&D externalities is sufficient to illustrate our point.

Proof. Impose $\theta = 0$ on (24) and compare with (27). Then, a few steps of mathematical manipulation show that $l_r|_{i=0} > l_r^* \Leftrightarrow i^* > 0$. ■

Finally, as for the comparative statics of i^*, it is increasing in μ. Intuitively, a larger patent breadth increases R&D, which in turn implies that R&D overinvestment is more likely to occur, so that i^* increases. It is interesting to note that under inelastic labor supply, patent policy and monetary policy are perfectly substitutable in the sense that a lower interest rate has the same effect as a larger patent breadth. Also, i^* is increasing in ρ. When the discount rate is high, R&D overinvestment is more likely to occur, so that i^* increases. Furthermore, i^* is decreasing in φ and z. When R&D productivity φ is high or the step size z of innovation is large, R&D underinvestment is more likely to occur, so that i^* decreases.

3.2 Friedman rule under elastic labor supply

Under elastic labor supply, monetary policy affects the supply of labor. Equation (22) shows that labor supply l is decreasing in i. Given that the nominal interest rate i now has a distortionary effect on the consumption-leisure decision, optimal monetary policy no longer achieves the first-best allocation.

We first consider the case without the CIA constraint on consumption by setting $\xi = 0$. Substituting (20)-(22) into (23) and differentiating U with respect to i, we derive the optimal nominal interest rate i^* for $\xi = 0$ given by

$$i^* = \max\left(\frac{\mu - \Omega}{\Omega - 1}, 0\right), \tag{30}$$

where Ω is a composite parameter defined as follows.

$$\Omega \equiv \frac{1 + \mu\theta}{1 + \theta}\left(1 + \frac{\varphi}{\rho}\right)\ln z - \mu\theta. \tag{31}$$

It can be shown that $l_r^* > 0$ is sufficient for $\Omega > 1$. Therefore, Friedman rule is suboptimal (i.e., $i^* > 0$) if and only if $\mu > \Omega$. It can also be shown that $\mu > \Omega$ is equivalent to R&D overinvestment (i.e., $l_r|_{i=0} > l_r^*$). In other words, R&D overinvestment is necessary and sufficient for Friedman rule to be suboptimal even with elastic labor supply so long as the CIA constraint on consumption is absent (i.e., $\xi = 0$). It is useful to note that when the equilibrium features R&D overinvestment, setting $i = i^*$ yields the first-best allocation of R&D labor (i.e., $l_r|_{i=i^*} = l_r^*$); however, setting $i = i^*$ does not yield the first-best allocation of labor supply. Specifically, we find that $l|_{i=i^*} < l^*$ because the presence of a positive markup $\mu > 1$ reduces the labor share of income and distorts the supply of labor. It can be shown that when $i^* > 0$, the inequality $l|_{i=i^*} < l^*$ simplifies to $\mu > 1$. We summarize these results in Proposition 4.

Proposition 4 *When the CIA constraint on consumption is absent, R&D overinvestment is both necessary and sufficient for Friedman rule to be suboptimal even with elastic labor supply. In this case, when the optimal nominal interest rate is positive, optimal monetary policy achieves the first-best allocation of R&D labor; however, it does not achieve the first-best allocation of labor supply.*

Proof. Proven in text. ■

When the CIA constraint on consumption is present (i.e., $\xi > 0$), there does not exist a closed-form solution for the optimal nominal interest rate i^*. In this case, we analyze whether Friedman rule is optimal. To do so, we substitute (20)-(22) into (23) and differentiate U with respect to i. Then, evaluating $\partial U/\partial i$ at $i = 0$ yields

$$sign\left(\frac{\partial U}{\partial i}|_{i=0}\right) = sign\left[(1+\theta)\left(\frac{1+\mu\theta}{1+\mu\theta(1+\xi)}\right) - \frac{1+\mu\theta}{\mu(1+\theta)}\left(\frac{\varphi+\rho}{\rho}\right)\ln z\right], \tag{32}$$

which can be positive or negative depending on parameter values. Comparing (24) with (20) evaluated at $i = 0$, we find that $l_r|_{i=0} > l_r^*$ is equivalent to the following inequality.

$$l_r|_{i=0} > l_r^* \Leftrightarrow (1+\theta) > \frac{1+\mu\theta}{\mu(1+\theta)}\left(\frac{\varphi+\rho}{\rho}\right)\ln z. \tag{33}$$

From (32) and (33), it is easy to see that when the CIA constraint on consumption is absent (i.e., $\xi = 0$), R&D overinvestment (i.e., $l_r|_{i=0} > l_r^*$) is both necessary and sufficient for $\partial U/\partial i|_{i=0} > 0$, which implies that Friedman rule is suboptimal because social welfare is increasing in i at $i = 0$. However, when the CIA constraint on consumption is present (i.e., $\xi > 0$), R&D overinvestment is no longer sufficient for $\partial U/\partial i|_{i=0} > 0$; on the other hand, R&D underinvestment is sufficient for $\partial U/\partial i|_{i=0} < 0$. In this case, the degree of R&D overinvestment must be substantial enough in order for Friedman rule to be suboptimal. Intuitively, in the presence of the CIA constraint on consumption, the nominal interest rate causes an additional distortionary effect on the consumption-leisure decision. As a result of this additional distortion, R&D overinvestment is necessary but not sufficient to justify a positive nominal interest rate. In other words, the suboptimality of Friedman rule requires that the welfare gain from overcoming R&D overinvestment through the CIA constraint on R&D dominates the welfare loss from distorting leisure through the CIA constraint on consumption. We summarize this result in Proposition 5.

Proposition 5 *When the CIA constraint on consumption is present, R&D overinvestment is necessary but not sufficient for Friedman rule to be suboptimal. However, if the degree of R&D overinvestment is substantial enough, then Friedman rule would be suboptimal.*

Proof. Comparing (32) and (33) shows that $l_r|_{i=0} > l_r^*$ is necessary but not sufficient for $\partial U/\partial i|_{i=0} > 0$. Suppose $l_r|_{i=0} = l_r^* + \chi$, where $\chi > 0$. There exists a threshold value $\overline{\chi}$ such that if and only if $\chi > \overline{\chi}$, then $\partial U/\partial i|_{i=0} > 0$. Furthermore, $\overline{\chi}$ is given by

$$\overline{\chi} \equiv \frac{\xi\mu\theta\,(1+\theta)}{1+\mu\theta(1+\xi)}\frac{\rho}{\varphi\ln z},$$

which is increasing in ξ. ■

4 Friedman rule under alternative cases

In this section, we consider various alternative versions of the model. In Section 4.1, we examine an alternative case of the model in which only the CIA constraint on consumption is present. In Section 4.2, we examine another alternative case in which the model features CIA constraints on R&D and manufacturing. In Section 4.3, we consider a semi-endogenous-growth version of the model.

4.1 Friedman rule under CIA on consumption only

In this subsection, we examine an alternative case in which the model features only the CIA constraint on consumption (but not the CIA constraint on R&D). In this case, (17) becomes

$$(\mu - 1)l_x = l_r + \rho/\varphi. \tag{34}$$

Combining this equation with (18) and (19) yields the equilibrium labor allocation given by

$$l_r = \frac{\mu - 1}{\mu[1 + \theta(1 + \xi i)]}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}, \tag{35}$$

$$l_x = \frac{1}{\mu[1 + \theta(1 + \xi i)]}\left(1 + \frac{\rho}{\varphi}\right), \tag{36}$$

$$l = \frac{1}{1 + \theta(1 + \xi i)}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}. \tag{37}$$

Substituting (35)-(37) into (23) and differentiating U with respect to i yields

$$\frac{\partial U}{\partial i} = -\frac{\theta\xi}{\rho}\left[\frac{\xi i}{1 + \xi i}\left(\frac{1}{1 + \theta(1 + \xi i)}\right) + \frac{\mu - 1}{\mu[1 + \theta(1 + \xi i)]^2}\left(\frac{\varphi + \rho}{\rho}\right)\ln z\right] < 0. \tag{38}$$

Equation (38) shows that welfare is monotonically decreasing in i; therefore, Friedman rule is always optimal when the CIA constraint on R&D investment is absent.

Proposition 6 *When the Schumpeterian growth model features only the CIA constraint on consumption, Friedman rule is always optimal regardless of whether the equilibrium features R&D overinvestment or underinvestment.*

Proof. Note (38). ■

Intuitively, under the CIA constraint on consumption, an increase in i decreases all of $\{l_r, l_x, l\}$. Furthermore, it can be shown that $l_r^* > 0$ implies $l^* > l|_{i=0}$ in (37); therefore, any increase in i that leads to a further reduction in l is socially suboptimal. Also, it is useful to note that the effects of i on l_r and l_x under the two CIA constraints are very different. Recall that under the CIA constraint on R&D investment, an increase in i leads to a reallocation of labor from R&D to production, but this reallocation effect of i is absent under the CIA constraint on consumption. From this analysis, we conclude that the CIA constraint on R&D, which is absent in previous studies, is crucial to the suboptimality of Friedman rule.

4.2 Friedman rule under CIA on manufacturing and R&D

In this subsection, we consider another alternative case in which the model features CIA constraints on manufacturing and R&D. For simplicity, we assume inelastic labor supply. To introduce a CIA constraint on manufacturing, we assume that the financing of wage payment to production workers also requires money borrowed from households. In this case, the total cost of wage payment is $(1+\beta i_t)w_t L_{x,t}(j)$, where $\beta \in [0,1]$. Therefore, the marginal cost of production for the industry leader in industry j is $mc_t(j) = (1+\beta i_t)w_t/z^{q_t(j)}$, and the markup is $\mu = p_t(j)/mc_t(j)$ as before. It can be shown that (9) remains unchanged whereas (10) becomes

$$(1+\beta i_t)w_t L_{x,t}(j) = \left(\frac{1}{\mu}\right) p_t(j)x_t(j) = \left(\frac{1}{\mu}\right) y_t. \tag{39}$$

As for the zero-expected-profit condition for R&D, we consider a more general CIA constraint on R&D such that (12) becomes

$$v_t \lambda_t(k) = (1+\alpha i_t)w_t L_{r,t}(k), \tag{40}$$

where $\alpha \in [0,1]$. The rest of the model is the same as Section 2.

Following similar derivations as in Section 2.6, we find that (17) becomes

$$(\mu - 1)(1+\beta i)l_x = (l_r + \rho/\varphi)(1+\alpha i). \tag{41}$$

Combining this equation with $l_x + l_r = 1$ and performing a few steps of mathematical manipulation yield

$$l_r = \frac{\mu - 1}{\mu - 1 + (1+\alpha i)/(1+\beta i)}\left(1 + \frac{\rho}{\varphi}\right) - \frac{\rho}{\varphi}. \tag{42}$$

Therefore, we find that l_r and $g = (\varphi \ln z) l_r$ are decreasing (increasing) in i if $\alpha > \beta$ ($\alpha < \beta$). Intuitively, an increase in i raises both the cost of production and the cost of R&D; however, the relative strength of the opposing effects of the CIA constraints is determined by α and β. The empirical evidence for a negative effect of inflation and the nominal interest rate on total factor productivity growth documented in Evers *et al.* (2007) implies that $\alpha > \beta$; in other words, R&D requires a higher financing cost than manufacturing.

As for the optimal nominal interest rate i^*, equating (42) and (24) under $\theta = 0$ yields the following condition that characterizes the interior optimal nominal interest rate.

$$\frac{1+\alpha i^*}{1+\beta i^*} = \frac{\mu - 1}{(1+\varphi/\rho)\ln z - 1}. \tag{43}$$

In this case, if $\alpha > \beta$, then we come to the same conclusion that $i^* > 0$ if and only if the equilibrium features R&D overinvestment (i.e., $\mu > (1+\varphi/\rho)\ln z$).[20] However, if $\alpha < \beta$, then we come to the opposite conclusion that $i^* > 0$ if and only if the equilibrium features R&D underinvestment (i.e., $\mu < (1+\varphi/\rho)\ln z$).[21] We summarize these results below.

[20] In order for i^* to achieve the first-best allocation in this case, α needs to be sufficiently larger than β such that $\alpha > \beta(\mu-1)/[(1+\varphi/\rho)\ln z - 1]$.

[21] In order for i^* to achieve the first-best allocation in this case, α needs to be sufficiently smaller than β such that $\alpha < \beta(\mu-1)/[(1+\varphi/\rho)\ln z - 1]$.

Proposition 7 *When there are CIA constraints on both R&D and production, R&D and economic growth are decreasing (increasing) in the nominal interest rate if $\alpha > \beta$ ($\alpha < \beta$). Furthermore, if $\alpha > \beta$ ($\alpha < \beta$), then R&D overinvestment (underinvestment) is necessary and sufficient for Friedman rule to be suboptimal.*

Proof. Note (42) and (43). ■

4.3 Friedman rule in a semi-endogenous growth model

In this subsection, we briefly examine our results in a semi-endogenous growth model with only the CIA constraint on R&D; see Segerstrom (1998) for a semi-endogenous-growth version of the quality-ladder model. For simplicity, we focus on the case of inelastic labor supply by setting $\theta = 0$, so that $l_r + l_x = l = 1$. To introduce semi-endogenous growth, we assume an effect of increasing complexity on innovation such that R&D productivity is decreasing in aggregate technology Z_t. In this case, (13) becomes

$$\lambda_t = \frac{\varphi L_{r,t}}{Z_t}. \tag{44}$$

Under this specification, the steady-state growth rate of Z_t is determined by the exogenous population growth rate such that $g = n > 0$. The rest of the model is the same as Section 2.

Following similar derivations as in Section 2.6, we find that equilibrium R&D labor is characterized by

$$\frac{l_r}{1 - l_r} = \frac{\mu - 1}{1 + i} \left(\frac{\lambda}{\rho + \lambda} \right), \tag{45}$$

where $\lambda = g / \ln z = n / \ln z$ is exogenous on the balanced growth path. Equation (45) shows that equilibrium R&D l_r is decreasing in the nominal interest rate i as before. Using standard dynamic optimization, we maximize (1) subject to (a) $c_t = Z_t l_{x,t}$, (b) $\dot{Z}_t = (\varphi \ln z) l_{r,t} N_t$, and (c) $l_{r,t} + l_{x,t} = 1$. We find that the first-best optimal allocation on the balanced growth path is characterized by

$$\frac{l_r^*}{1 - l_r^*} = \frac{n}{\rho + n}. \tag{46}$$

Equating (45) and (46) yields the optimal nominal interest rate i^* given by

$$i^* = \max \left[(\mu - 1) \left(\frac{\lambda}{\rho + \lambda} \right) \frac{\rho + n}{n} - 1, 0 \right], \tag{47}$$

where $\lambda = n / \ln z$. Therefore, we come to the same conclusion in the monetary semi-endogenous growth model that Friedman rule is suboptimal (i.e., $i^* > 0$) if and only if the equilibrium features R&D overinvestment (i.e., $l_r|_{i=0} > l_r^*$).

Proposition 8 *In a semi-endogenous growth model with a CIA constraint on R&D investment and inelastic labor supply, the optimal nominal interest rate i^* is given by (47). Furthermore, if and only if R&D overinvestment occurs in the zero-nominal-interest-rate equilibrium, then the optimal nominal interest rate would be strictly positive; in this case, Friedman rule is suboptimal.*

Proof. Compare (45) with (46). Then, a few steps of mathematical manipulation show that $l_r|_{i=0} > l_r^* \Leftrightarrow i^* > 0$. ■

5 Conclusion

In this study, we have analyzed the long-run growth and welfare effects of monetary policy in a Schumpeterian growth model with CIA constraints. Although we find that R&D and economic growth are decreasing in the nominal interest rate, a zero interest rate policy does not necessarily maximize social welfare. Specifically, we find that the suboptimality or optimality of Friedman rule is closely related to a seemingly unrelated issue that is the overinvestment versus underinvestment of R&D in the market economy, and this result is robust to both the fully-endogenous-growth and semi-endogenous-growth versions of the Schumpeterian model.

Finally, we conclude with a brief summary of our results and their intuition. Under inelastic labor supply, the CIA constraint on consumption has no distortionary effect on the consumption-leisure decision; therefore, any effect of monetary policy operates through the CIA constraint on R&D investment. If and only if there is too much R&D in equilibrium, then a positive nominal interest rate that increases the cost of R&D would be optimal. Under elastic labor supply, the CIA constraint on consumption distorts the consumption-leisure decision; as a result, a positive nominal interest rate leads to a welfare cost through a reduction in labor supply. In this case, R&D overinvestment is necessary but not sufficient for a positive nominal interest rate to be optimal. In other words, in order for a positive nominal interest rate to be optimal (i.e., Friedman rule being suboptimal), the welfare gain from overcoming R&D overinvestment through the CIA constraint on R&D must dominate the welfare loss from distorting leisure through the CIA constraint on consumption. Furthermore, we consider an alternative version of the model with CIA constraints on R&D and manufacturing. In this case, we find that the optimality of Friedman rule depends on the relative strength of the CIA constraints on R&D and manufacturing. If the effect of the CIA constraint on manufacturing dominates (is dominated by) the effect of the CIA constraint on R&D, then R&D underinvestment (overinvestment) would become a necessary and sufficient condition for Friedman rule to be suboptimal.

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Appendix A: Dynamics of the monetary Schumpeterian growth model

Proof of Proposition 1. We define a transformed variable $\Phi_t \equiv y_t/v_t$, and its law of motion is

$$\frac{\dot{\Phi}_t}{\Phi_t} \equiv \frac{\dot{y}_t}{y_t} - \frac{\dot{v}_t}{v_t}. \tag{A1}$$

From the resource constraint $y_t = c_t N_t$, the law of motion for y_t is

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{c}_t}{c_t} + n = r_t - \rho, \tag{A2}$$

where the second equality comes from (5) and (3) because $i_t = i$ for all t. From (11), the law of motion for v_t is

$$\frac{\dot{v}_t}{v_t} = r_t + \lambda_t - \frac{\Pi_t}{v_t}, \tag{A3}$$

where $\lambda_t = \varphi l_{r,t}$ and $\Pi_t = y_t(\mu - 1)/\mu$. Substituting (A2) and (A3) into (A1) yields

$$\frac{\dot{\Phi}_t}{\Phi_t} \equiv \left(\frac{\mu - 1}{\mu}\right) \Phi_t - \varphi l_{r,t} - \rho. \tag{A4}$$

To derive a relationship between $l_{r,t}$ and Φ_t, we first make use of (10) and (12) to derive

$$l_{x,t} = \left(\frac{1+i}{\mu\varphi}\right) \frac{y_t}{v_t} = \left(\frac{1+i}{\mu\varphi}\right) \Phi_t. \tag{A5}$$

Then, combining (4) and (10) yields

$$l_t = 1 - \mu\theta(1 + \xi i) l_{x,t}. \tag{A6}$$

Finally, combining (A5), (A6) and $l_t = l_{r,t} + l_{x,t}$, we derive

$$l_{r,t} = 1 - [1 + \mu\theta(1 + \xi i)] \left(\frac{1+i}{\mu\varphi}\right) \Phi_t. \tag{A7}$$

Substituting (A7) into (A4) yields an autonomous dynamic system of Φ_t.

$$\frac{\dot{\Phi}_t}{\Phi_t} \equiv \left(\frac{\mu - 1 + (1+i)\left[1 + \mu\theta(1+\xi i)\right]}{\mu}\right) \Phi_t - (\varphi + \rho). \tag{A8}$$

Therefore, the dynamics of Φ_t is characterized by saddle-point stability such that Φ_t jumps immediately to its interior steady state given by

$$\Phi = \frac{\mu(\varphi + \rho)}{\mu - 1 + (1+i)\left[1 + \mu\theta(1+\xi i)\right]}. \tag{A9}$$

Equations (A5) and (A7) imply that if Φ is stationary, then l_x and l_r must be stationary, which in turn implies that l is stationary as well. ■