Risk Measures for Skew Normal Mixtures

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Abstract

Finite mixtures of Skew distributions have become increasingly popular in the last few years as a flexible tool for handling data displaying several different characteristics such as multimodality, asymmetry and fat-tails. Examples of such data can be found in financial and actuarial applications as well as biological and epidemiological analysis. In this paper we will show that a convex linear combination of multivariate Skew Normal mixtures can be represented as finite mixtures of univariate Skew Normal distributions. This result can be useful in modeling portfolio returns where the evaluation of extremal events is of great interest. We provide analytical formula for different risk measures like the Value-at-Risk and the Expected Shortfall probability.

Keywords: Finite mixtures, Skew Normal distributions, Value-at-Risk, Expected Shortfall probability.

1 Introduction

The implementation of finite mixture has become increasingly popular in many disciplines, such as biological sequences analysis, econometrics, machine learning, actuarial science, finance and epidemiology. Often data display strong asymmetry, fat tails and multimodality features that are usually shared by different subpopulations. In this context mixtures of asymmetric distributions have been adopted in different areas, see for example Frühwirth-Schnatter and Pyne [6], Bernardi et al. [4] and Haas and Mittnik [8]. Among the skewed distrubutions, the Skew Normal of Azzalini [1] and the Skew Student-t of Azzalini and Capitanio [3] have become widely employed because of the attractive properties they share with their symmetric counterparts, and the greater shape flexibility introduced by the additional asymmetry parameter.

In what follows we will consider finite mixture of multivariate Skew Normal distributions and linear combinations of them. We will show that a linear combination of mixtures of multivariate Skew Normals admits a closed form representation as a finite mixtures of univariate Skew Normals. This result can be very useful for example when we model the distribution of the portfolio return, defined as a convex linear combination of the portfolio assets assumed to be a mixture of multivariate Skew Normals. In fact, it is well known in the financial literature that financial returns strongly deviates from the Gaussian assumption.

Moreover, since investors holding short or long positions on portfolios of risky assets are mainly interested in evaluating the risk associated to their portfolios, it is important to calculate some risk measures like the Value-at-Risk, (VaR), and the Expected Shortfall probability, (ES). Usually those risk measure are estimated by historical or Monte Carlo simulations. In this paper, under the assumption of a joint multivariate Skew Normal mixture for the assets, we will provide an analytical closed form expression for the VaR, the ES and related measures, as functions of the model parameters.

The paper is organized as follows: in Section 2 we prove that the multivariate Skew Normal mixtures are closed with respect to linear combinations. In Section 3 we provide analytical formulas for the Value-at-Risk, the Expected Shortfall probability, and related risk measures both for Skew Normals and mixtures of them. Few remarks and possible extensions are discussed in Section 4.

2 Linear combinations of multivariate Skew Normal mixtures

Finite mixture of multivariate Skew Normal distributions (see Azzalini and Dalla Valle [2]) for the d-dimensional observation data $\mathbf{y} = (\mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_d)$ can be defined as

$$\pi\left(\mathbf{y}|\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha}, \boldsymbol{\eta}\right) = \sum_{l=1}^{L} \eta_l f_{\mathsf{MSN}}\left(\mathbf{y}|\boldsymbol{\xi}_l, \boldsymbol{\Omega}_l, \boldsymbol{\alpha}_l\right), \tag{2.1}$$

where $f_{\mathsf{MSN}}\left(\mathbf{y}|\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha}\right)$ is the density of the multivariate Skew Normal distribution defined in Azzalini and Dalla Valle [2]:

$$f_{\mathsf{MSN}}\left(\mathbf{y}|\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha}\right) = 2\Phi\left(\frac{\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right)}{\sqrt{1-\boldsymbol{\gamma}^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\gamma}}}\right)\frac{1}{(2\pi)^{\frac{d}{2}}\left|\boldsymbol{\Omega}\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\mathbf{y}-\boldsymbol{\xi}\right)^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right)\right\}, \tag{2.2}$$

where $\boldsymbol{\xi} \in \mathbb{R}^d$ is a $d-$dimensional vector of location parameters, $\boldsymbol{\Omega}$ is a positive definite square matrix of dimension d, and $\boldsymbol{\gamma}$ is defined as a reparameterization of the $d-$dimensional vector of skewness parameters $\boldsymbol{\alpha}$, in the following way

$$\boldsymbol{\gamma} = \frac{\boldsymbol{\Omega}^{\frac{1}{2}}\boldsymbol{\alpha}}{\sqrt{1+\boldsymbol{\alpha}^{\mathsf{T}}\boldsymbol{\alpha}}} = \boldsymbol{\Omega}^{\frac{1}{2}}\boldsymbol{\delta}. \tag{2.3}$$

The parameters $\eta_l, l = 1, 2, \ldots, L$ are the component weights satisfying $0 \le \eta_l \le 1, \forall l = 1, 2, \ldots, L$, and $\sum_{l=1}^{L} \eta_l = 1$. We denote with $\phi(\mathsf{x})$ and $\Phi(\mathsf{x})$ the density function and the cumulative density function of a scalar Gaussian distribution, and with $\delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$, the univariate counterpart of the $\boldsymbol{\delta}$ parameter defined in the previous equation (2.3). The univariate Skew Normal distribution can be obtained from the previous definition by setting the dimension d of the vector of observation $\mathbf{y}$ equal to 1.

In what follows we prove our main result contained in Theorem 2.1 showing that the distribution of linear combinations of multivariate Skew Normal mixtures is a mixture of univariate Skew Normals. Before stating the theorem it is useful to recall the following known results providing the Moment Generating Function (MGF) of univariate and multivariate Skew Normal distributions and their extension to mixtures of them.

Corollary 2.1. *Let* $\mathsf{X} \sim \mathcal{SN}(\xi, \omega, \alpha)$ *be a univariate Skew Normal distribution, and* $\mathbf{X} \sim \mathcal{MSN}(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$ *be a d-dimensional Skew Normal distribution, then the MGF of* X *and* $\mathbf{X}$ *are respectively*

$$M_{\mathsf{X}}(t) = 2\exp\left\{\xi t + \frac{t^2\omega^2}{2}\right\}\Phi(\delta\omega t) \tag{2.4}$$

$$M_{\mathbf{X}}(\mathbf{t}) = 2\exp\left\{\mathbf{t}^{\top}\boldsymbol{\xi} + \frac{\mathbf{t}^{\top}\boldsymbol{\Omega}\mathbf{t}}{2}\right\}\Phi\left(\boldsymbol{\delta}^{\top}\boldsymbol{\Omega}^{\frac{1}{2}}\mathbf{t}\right). \tag{2.5}$$

Proof. *See Genton [7], page 7 and 17.* □□

Corollary 2.2. *Let* $\mathsf{Y} \sim \sum_{l=1}^{L}\eta_l f_{\mathsf{SN}}(\xi_l, \omega_l, \alpha_l)$*, be a univariate Skew Normal mixture, and* $\mathbf{Y} \sim \sum_{l=1}^{L}\eta_l f_{\mathsf{MSN}}(\mathbf{y}|\boldsymbol{\xi}_l, \boldsymbol{\Omega}_l, \boldsymbol{\alpha}_l)$ *be a d-dimensional Skew Normal mixture, then the MGF of* Y *and* $\mathbf{Y}$ *are respectively*

$$M_{\mathsf{Y}}(t) = \sum_{l=1}^{L}\eta_l\left(2\exp\left\{\xi_l t + \frac{t^2\omega_l^2}{2}\right\}\Phi(\delta_l\omega_l t)\right) \tag{2.6}$$

$$M_{\mathbf{Y}}(\mathbf{t}) = \sum_{l=1}^{L}\eta_l\left(2\exp\left\{\mathbf{t}^{\top}\boldsymbol{\xi}_l + \frac{\mathbf{t}^{\top}\boldsymbol{\Omega}_l\mathbf{t}}{2}\right\}\Phi\left(\boldsymbol{\delta}_l^{\top}\boldsymbol{\Omega}_l^{\frac{1}{2}}\mathbf{t}\right)\right). \tag{2.7}$$

Proof. *The proof is straightforward.* □□

Theorem 2.1 (Linear combinations of multivariate Skew Normal mixtures). *Let* $\mathbf{Y} \sim \sum_{l=1}^{L}\eta_l f_{\mathsf{MSN}}(\mathbf{y}|\boldsymbol{\xi}_l, \boldsymbol{\Omega}_l, \boldsymbol{\alpha}_l)$*, assume* $\mathbf{w} \in \mathbb{R}^d$ *is d-dimensional vector of real coefficients, such that* $\sum_{j=1}^{d} w_j = 1$*, then*

$$\mathsf{Z} = \mathbf{w}^{\top}\mathbf{Y} \tag{2.8}$$

has density function

$$f_{\mathsf{Z}}(\mathsf{z}) = \sum_{l=1}^{L}\eta_l f_{\mathsf{SN}}\left(\mathsf{z}|\tilde{\xi}_l, \tilde{\omega}_l, \tilde{\alpha}_l\right),$$

where $\tilde{\xi}_k = \mathbf{w}^{\top}\boldsymbol{\xi}_l$*,* $\tilde{\omega}_l = \left(\mathbf{w}^{\top}\boldsymbol{\Omega}_l\mathbf{w}\right)^{\frac{1}{2}}$*,* $\tilde{\delta}_l = \mathbf{w}^{\top}\boldsymbol{\delta}_l$*, for* $l = 1, 2, \ldots, L$*. The shape parameters* $\tilde{\alpha}_l$ *can be recovered by inverting the relation* $\tilde{\delta}_l = \frac{\tilde{\alpha}_l}{\sqrt{1+\tilde{\alpha}_l^2}}$*, getting* $\tilde{\alpha}_l = \frac{\tilde{\delta}_l}{\sqrt{1-\tilde{\delta}_l^2}}, \forall l = 1, 2, \ldots, L$*.*

Proof. *The MGF of the (scalar) random variable* $\mathsf{Z} = \mathbf{w}^{\top}\mathbf{Y}$ *is equal to*

$$\begin{aligned} M_{\mathsf{Z}}(t) &= \mathbb{E}\left[e^{t\mathsf{Z}}\right] = \mathbb{E}\left[e^{t\left(\mathbf{w}^{\top}\mathbf{Y}\right)}\right] \\ &= \int_{\mathbb{R}^d}\exp\left\{t\left(\mathbf{w}^{\top}\mathbf{y}\right)\right\}\times \\ &\qquad \sum_{l=1}^{L}\eta_l 2\Phi\left(\frac{\boldsymbol{\gamma}_l^{\top}\boldsymbol{\Omega}_l^{-1}(\mathbf{x}-\boldsymbol{\xi}_l)}{\sqrt{1-\boldsymbol{\gamma}_l^{\top}\boldsymbol{\Omega}_l^{-1}\boldsymbol{\gamma}_l}}\right)\frac{1}{(2\pi)^{\frac{d}{2}}|\boldsymbol{\Omega}_l|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\xi}_l)^{\top}\boldsymbol{\Omega}_l^{-1}(\mathbf{y}-\boldsymbol{\xi}_l)\right\}d\mathbf{y} \end{aligned}$$

After some calculations the MGF can be written as

$$M_{\mathsf{Z}}(t) = \sum_{l=1}^{L} \eta_l 2 \exp\left\{t\left(\mathbf{w}^\top \boldsymbol{\xi}_l\right) + t^2 \frac{\mathbf{w}^\top \boldsymbol{\Omega}_l \mathbf{w}}{2}\right\} \int_{\mathbb{R}^d} \Phi\left(\frac{\boldsymbol{\gamma}_l^\top \boldsymbol{\Omega}_l^{-1}(\mathbf{x}-\boldsymbol{\xi}_l)}{\sqrt{1-\boldsymbol{\gamma}_l^\top \boldsymbol{\Omega}_l^{-1} \boldsymbol{\gamma}_l}}\right) \frac{1}{(2\pi)^{\frac{d}{2}} |\boldsymbol{\Omega}_l|^{\frac{1}{2}}} \times$$
$$\exp\left\{-\frac{1}{2}\left(\mathbf{y}-\boldsymbol{\xi}_l-\boldsymbol{\Omega}_l^{\frac{1}{2}}\mathbf{w}t\right)^\top \boldsymbol{\Omega}_l^{-1}\left(\mathbf{y}-\boldsymbol{\xi}_l-\boldsymbol{\Omega}_l^{\frac{1}{2}}\mathbf{w}t\right)\right\} d\mathbf{y}.$$

Now, considering the transformation $\mathbf{z} = \boldsymbol{\Omega}_l^{-\frac{1}{2}}\left(\mathbf{y}-\boldsymbol{\xi}_l-\boldsymbol{\Omega}_l^{\frac{1}{2}}\mathbf{w}t\right)$*, we get*

$$M_{\mathsf{Y}}(t) = \sum_{l=1}^{L} \eta_l 2 \exp\left\{t\left(\mathbf{w}^\top \boldsymbol{\xi}_l\right) + t^2 \frac{\mathbf{w}^\top \boldsymbol{\Omega}_l \mathbf{w}}{2}\right\} \int_{\mathbb{R}^d} \Phi\left(\frac{\boldsymbol{\gamma}_l^\top \boldsymbol{\Omega}_l^{-\frac{1}{2}}\mathbf{z} + \boldsymbol{\gamma}_l^\top \boldsymbol{\Omega}^{-\frac{1}{2}}\mathbf{w}t}{\sqrt{1-\boldsymbol{\gamma}_l^\top \boldsymbol{\Omega}_l^{-1} \boldsymbol{\gamma}_l}}\right) \frac{\exp\left\{-\frac{1}{2}\mathbf{z}^\top \mathbf{z}\right\}}{(2\pi)^{\frac{d}{2}}} d\mathbf{z}.$$

By applying the result presented in Ellison [5], detailed in appendix, we obtain

$$M_{\mathsf{Y}}(t) = \sum_{l=1}^{L} \eta_l 2 \exp\left\{t\left(\mathbf{w}^\top \boldsymbol{\xi}_l\right) + t^2 \frac{\mathbf{w}^\top \boldsymbol{\Omega}_l \mathbf{w}}{2}\right\} \Phi\left(\boldsymbol{\delta}_l^\top \mathbf{w}t\right),$$

which corresponds to the MGF of a univariate mixture of Skew Normal distribution having the following parameters: $\left\{\eta_l, \tilde{\xi}_l = \mathbf{w}^\top \boldsymbol{\xi}_l, \tilde{\omega}_l = \left(\mathbf{w}^\top \boldsymbol{\Omega}_k \mathbf{w}\right)^{\frac{1}{2}}, \tilde{\delta}_l = \boldsymbol{\delta}_l^\top \mathbf{w}\right\}_{l=1}^{L}$. □□

As a byproduct of this result we have that setting $\mathbf{w} = (0, \ldots, 1, \ldots, 0)$ we obtain the marginal distribution of the i-th component of the vector $\mathbf{Y}$.

3 Risk Measures

Investors holding short or long positions on portfolios of risky assets are mainly interested in evaluating the exposure to risk associated to their global positions. Assuming that the distribution of the portfolio returns follow a multivariate mixture of Skew Normals we calculate different risk measures for the resulting linear combination of them. The results stated in previous Section 2 provide the theoretical framework to calculate such risk measures. As risk measures, we consider, the probability of shortfall (PS), i.e. the probability that a portfolio falls short some target level, the probability of outperformance (PO), i.e the probability of outperforming the target level, and the more familiar measures known as target shortfall (TS) and tail conditional expectation (TCE). All these measures can be interpreted as a partial moments of a given order of the corresponding random variable. They are evaluated for univariate Skew Normal distributions in Section 3.1 and subsequently extended to mixtures of Skew Normals in Section 3.2. As a latter result we show how to evaluate the Value-at-Risk and the Expected Shortfall probability using the provided measures.

3.1 Risk measures for univariate Skew Normal distributions

In this Section we provide explicit formulas to evaluate risk measures of a univariate Skew Normal random variable.

Theorem 3.1 (**PS and PO for univariate Skew Normal**). *Let* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, the probability of shortfall PS, i.e. the proability that the random variable* Y *falls short of the target* y_p *is*

$$\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) = F_{\mathsf{X}}\left(\frac{\mathsf{y}_p - \xi}{\omega}, \alpha\right), \tag{3.1}$$

while the probability that the random variable Y *outperforms the target* y_p*, i.e. the probability of outperformance PO, is*

$$\begin{aligned}\mathbb{PO}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= 1 - F_{\mathsf{X}}\left(\frac{\mathsf{y}_p - \xi}{\omega}, \alpha\right) \\ &= 1 - \mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha), \end{aligned} \tag{3.2}$$

where $F_{\mathsf{X}}(\mathsf{x}, \alpha)$ *is the cdf of a standardized Skew Normal distribution, i.e.* $\mathsf{X} \sim \mathcal{SN}(\alpha)$ *evaluated at* x*.*

Proof. *The shortall probability, PS, is the probability that a univariate random variable,* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, falls short the threshold* y_p*, and can be evaluated as the zero-th order lower partial moment (LPM) of the random variable* Y *with respect to the threshold* $\mathsf{y}_p \in \mathbb{R}$

$$\begin{aligned}\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= \mathbb{LPM}_{\mathsf{Y}}(\mathsf{y}_p, 0) \\ &= \mathbb{E}\left\{[\mathsf{y}_p - \mathsf{Y}]_+^0\right\} \\ &= \int_{-\infty}^{\frac{\mathsf{y}_p - \xi}{\omega}} 2\Phi(\alpha \mathsf{x})\, \phi(\mathsf{x})\, d\mathsf{x} \\ &= F_{\mathsf{X}}\left(\frac{\mathsf{y}_p - \xi}{\omega}, \alpha\right).\end{aligned}$$

The probability of outperformance, PO, is the probability that a univariate random variable, $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, outperforms the threshold* y_p*, and can be evaluated as the zero-th order upper partial moment (UPM) of the random variable* Y *with respect to the threshold* $\mathsf{y}_p \in \mathbb{R}$

$$\begin{aligned}\mathbb{PO}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= \mathbb{UPM}_{\mathsf{Y}}(\mathsf{y}_p, 0) \\ &= \mathbb{E}\left\{[\mathsf{Y} - \mathsf{y}_p]_+^0\right\} \\ &= \int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\Phi(\alpha \mathsf{x})\, \phi(\mathsf{x})\, d\mathsf{x} \\ &= 1 - \mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha).\end{aligned}$$

□□

Theorem 3.2 (**TS for univariate Skew Normal**). *Let* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, then the target shortfall TS, of* Y *is*

$$\mathbb{TS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) = (\xi - \mathsf{y}_p)\, \mathbb{PO}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) + \omega\left[2\Phi(\alpha \mathsf{x}_p)\, \phi(\mathsf{x}_p) + \delta b\left(1 - \Phi(\mathsf{z}_p)\right)\right], \tag{3.3}$$

where $b = \sqrt{\frac{2}{\pi}}$*,* $z_p = \sqrt{1+\alpha^2}\mathsf{x}_p$ *and* $\mathsf{x}_p = \frac{\mathsf{y}_p - \xi}{\omega}$*.*

Proof. *The target shortfall, TS, can be evaluated as the first order upper partial moment UPM of the random variable* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, with respect to the threshold* $\mathsf{y}_p \in \mathbb{R}$

$$\begin{aligned}
\mathbb{TS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= \mathbb{UPM}_{\mathsf{Y}}(\mathsf{y}_p, 1) \\
&\equiv \mathbb{E}\left\{[\mathsf{Y} - \mathsf{y}_p]_+^1\right\} \\
&= \int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\Phi(\alpha \mathsf{x})\,\phi(\mathsf{x})\, d\mathsf{x}.
\end{aligned}$$

After some algebra we get:

$$\begin{aligned}
\mathbb{TS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= (\xi - \mathsf{y}_p) \int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\Phi(\alpha \mathsf{x})\,\phi(\mathsf{x})\, d\mathsf{x} + \omega \int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\mathsf{x}\Phi(\alpha \mathsf{x})\,\phi(\mathsf{x})\, d\mathsf{x} \\
&= (\xi - \mathsf{y}_p)\, \mathbb{PO}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) + \omega \int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\mathsf{x}\Phi(\alpha \mathsf{x})\,\phi(\mathsf{x})\, d\mathsf{x}.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
\int_{\frac{\mathsf{y}_p - \xi}{\omega}}^{+\infty} 2\mathsf{x}\Phi(\alpha \mathsf{x})\,\phi(\mathsf{x})\, d\mathsf{x} &= \Big[-2\phi(\mathsf{x})\,\Phi(\alpha \mathsf{x})\Big]_{\mathsf{x}_p}^{+\infty} + \alpha b \int_{\mathsf{x}_p}^{+\infty} \exp\left\{-\frac{\mathsf{x}^2}{2}\right\} \phi(\alpha \mathsf{x})\, d\mathsf{x} \\
&= 2\phi(\mathsf{x}_p)\,\Phi(\alpha \mathsf{x}_p) + \alpha b \int_{\mathsf{x}_p}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1+\alpha^2)\,\mathsf{x}^2}{2}\right\} d\mathsf{x} \\
&= 2\phi(\mathsf{x}_p)\,\Phi(\alpha \mathsf{x}_p) + \frac{\alpha b}{\sqrt{1+\alpha^2}} \int_{\mathsf{z}_p}^{+\infty} \phi(\mathsf{z})\, d\mathsf{z} \\
&= 2\phi(\mathsf{x}_p)\,\Phi(\alpha \mathsf{x}_p) + \delta b\left[1 - \Phi(\mathsf{z}_p)\right]
\end{aligned}$$

where b*,* x_p*,* z_p *and* δ *are defined as before.* □□

Finally, in the following theorem we use previous results to calculate the tail conditional expecation for a univariate Skew Normal distributions.

Theorem 3.3 (TCE for univariate Skew Normal). *Let* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, then the tail conditional expectation of* Y*, i.e. the mean of* Y *truncated below the threshold* y_p*, is*

$$\mathbb{TCE}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) = \xi + \frac{\omega b}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha)} \left[\delta \Phi(\mathsf{z}_p) - \sqrt{2\pi}\phi(\mathsf{x}_p)\,\Phi(\alpha \mathsf{x}_p)\right]. \tag{3.4}$$

where $b = \sqrt{\frac{2}{\pi}}$ *and* $\mathbb{PS}_{\mathsf{Y}}(\mathsf{x}, \xi, \omega, \alpha)$ *denotes the probability of shortfall of a Skew Normal r.v. evaluated at* x*, defined in equation (3.1).*

Proof. *Let* $\mathsf{X} \sim \mathcal{SN}(\alpha)$ *be a standardized Skew Normal density, consider* $\mathsf{Y} = \xi + \omega \mathsf{X}$*, then* Y *has a Skew Normal distribution* $\mathsf{Y} \sim \mathcal{SN}(\xi, \omega, \alpha)$*, the TCE of* Y *is*

$$\begin{aligned}
\mathbb{TCE}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) &= \mathbb{E}(\mathsf{Y} | \mathsf{Y} \leq \mathsf{y}_p) \\
&= \xi + \omega \mathbb{E}\left(\mathsf{X} | \mathsf{X} \leq \frac{\mathsf{y}_p - \xi}{\omega}\right) \\
&= \xi + \omega \mathbb{E}(\mathsf{X} | \mathsf{X} \leq \mathsf{x}_p) \\
&= \xi + \omega \mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha), \qquad (3.5)
\end{aligned}$$

where $\mathsf{x}_p = \frac{\mathsf{y}_p - \xi}{\omega}$, *and* $\mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha)$ *is the TCE of a standardized Skew Normal distribution,* $\mathsf{X} \sim \mathcal{SN}(\alpha)$ *and can be evaluated as follows*

$$\begin{aligned}\mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha) &= \mathbb{E}(\mathsf{X}|\mathsf{X} \leq \mathsf{x}_p) \\ &= \frac{1}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \int_{-\infty}^{\mathsf{x}_p} \mathsf{x} f_{\mathsf{X}}(\mathsf{x}, \alpha)\, d\mathsf{x} \end{aligned} \tag{3.6}$$

where $f_{\mathsf{SN}}(\mathsf{x}, \alpha) = 2\phi(\mathsf{x})\Phi(\alpha\mathsf{x})$, *is the probability density function of the standardized Skew Normal distribution and* $F_{\mathsf{X}}(\mathsf{x}_p, \alpha)$ *is the corresponding cdf. By substituting this last expression for the density in the previous defintion of tail conditional expectation (3.6), it becomes*

$$\mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha) = \frac{b}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \int_{-\infty}^{\mathsf{x}_p} \mathsf{x} \exp\left\{-\frac{\mathsf{x}^2}{2}\right\} \Phi(\alpha\mathsf{x})\, d\mathsf{x},$$

where $b = \sqrt{\frac{2}{\pi}}$. *Integrating by parts, we have*

$$\begin{aligned}\mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha) &= \frac{b}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \left[\left[-\exp\left\{-\frac{\mathsf{x}^2}{2}\right\}\Phi(\alpha\mathsf{x})\right]_{-\infty}^{\mathsf{x}_p} + \alpha \int_{-\infty}^{\mathsf{x}_p} \exp\left\{-\frac{\mathsf{x}^2}{2}\right\}\phi(\alpha\mathsf{x})\, d\mathsf{x}\right] \\ &= \frac{b}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \left[-\exp\left\{-\frac{\mathsf{x}_p^2}{2}\right\}\Phi(\alpha\mathsf{x}_p) + \alpha \int_{-\infty}^{\mathsf{x}_p} \exp\left\{-\frac{\mathsf{x}^2}{2}\right\}\phi(\alpha\mathsf{x})\, d\mathsf{x}\right], \end{aligned} \tag{3.7}$$

where the last integral in the previous formula (3.7) can be evaluated as follows

$$\begin{aligned}\int_{-\infty}^{\mathsf{x}_p} \exp\left\{-\frac{\mathsf{x}^2}{2}\right\}\phi(\alpha\mathsf{x})\, d\mathsf{x} &= \int_{-\infty}^{\mathsf{x}_p} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1+\alpha^2)\mathsf{x}^2}{2}\right\} d\mathsf{x} \\ &= \frac{1}{\sqrt{1+\alpha^2}} \int_{-\infty}^{\mathsf{z}_p} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\mathsf{z}^2}{2}\right\} d\mathsf{z} \\ &= \frac{\Phi(\mathsf{z}_p)}{\sqrt{1+\alpha^2}},\end{aligned}$$

where $\mathsf{z}_p = \sqrt{1+\alpha^2}\mathsf{x}_p$. *By substituting this last expression into the previous equation (3.7) we obtain the final form for the TCE*

$$\begin{aligned}\mathbb{TCE}_{\mathsf{X}}(\mathsf{x}_p, \alpha) &= \frac{b}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \left[\frac{\alpha}{\sqrt{1+\alpha^2}}\Phi(\mathsf{z}_p) - \sqrt{2\pi}\phi(\mathsf{x}_p)\Phi(\alpha\mathsf{x}_p)\right] \\ &= \frac{b}{F_{\mathsf{X}}(\mathsf{x}_p, \alpha)} \left[\delta\Phi(\mathsf{z}_p) - \sqrt{2\pi}\phi(\mathsf{x}_p)\Phi(\alpha\mathsf{x}_p)\right]\end{aligned}$$

and, by exploiting equation (3.5), we obtain

$$\mathbb{TCE}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha) = \xi + \frac{\omega b}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, \xi, \omega, \alpha)} \left[\delta\Phi(\mathsf{z}_p) - \sqrt{2\pi}\phi(\mathsf{x}_p)\Phi(\alpha\mathsf{x}_p)\right],$$

where $\mathsf{z}_p = \sqrt{1+\alpha^2}\mathsf{x}_p$, *and* $\mathsf{x}_p = \frac{\mathsf{y}_p - \xi}{\omega}$. □□

3.2 Risk measures for univariate Skew Normal mixtures

In this Section we extend the risk measures presented in the previous section to the mixtures of univariate Skew Normal distributions.

Theorem 3.4 (PS, PO and TS for mixtures of Skew Normals). *Let* $\mathsf{Y} \sim \sum_{l=1}^{L} \eta_l f_{\mathsf{SN}}(\mathsf{y}|\xi_l, \omega_l, \alpha_l)$*, then the shortfall probability PS, the probability of outperformance, PO, and the target shortfall TS of* Y *are convex linear combinations of the PS, PO and TS, of the component densities evaluated as in equations (3.1), (3.2) and (3.3):*

$$\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L) = \sum_{l=1}^{L} \eta_l \mathbb{PS}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l) \tag{3.8}$$

$$\mathbb{PO}_{\mathsf{Y}}(\mathsf{y}_p, L) = \sum_{l=1}^{L} \eta_l \mathbb{PO}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l) \tag{3.9}$$

$$\mathbb{TS}_{\mathsf{Y}}(\mathsf{y}_p, L) = \sum_{l=1}^{L} \eta_l \mathbb{TS}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l). \tag{3.10}$$

Proof. *The proof is straightforward.* □□

Theorem 3.5 (TCE for Skew Normal mixtures). *Let* $\mathsf{Y} \sim \sum_{l=1}^{L} \eta_l f_{\mathsf{SN}}(\mathsf{y}|\xi_l, \omega_l, \alpha_l)$*, then the tail conditional expectation of* Y *is a convex linear combination of the tail conditional expectations of the components:*

$$\mathbb{TCE}_{\mathsf{Y}}(\mathsf{y}_p, L) = \sum_{l=1}^{L} \pi_l \mathbb{TCE}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l) \tag{3.11}$$

where the weights are $\pi_l = \eta_l \frac{\mathbb{PS}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l)}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L)}$, $l = 1, 2, \dots, L$, *with* $\sum_{l=1}^{L} \pi_l = 1$.

Proof.

$$\begin{aligned}
\mathbb{TCE}_{\mathsf{Y}}(\mathsf{y}_p, L) &= \mathbb{E}(\mathsf{Y}|\mathsf{Y} \le \mathsf{y}_p) \\
&= \frac{1}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L)} \int_{-\infty}^{\mathsf{y}_p} \mathsf{y} \left[\sum_{l=1}^{L} \eta_l f(\mathsf{y}|\xi_l, \omega_l, \alpha_l) \right] d\mathsf{y} \\
&= \frac{1}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L)} \sum_{l=1}^{L} \eta_l \int_{-\infty}^{\mathsf{y}_p} \mathsf{y} f(\mathsf{y}|\xi_l, \omega_l, \alpha_l)\, d\mathsf{y} \\
&= \sum_{l=1}^{L} \frac{\eta_l \mathbb{PS}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l)}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L)} \mathbb{TCE}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l) \\
&= \sum_{l=1}^{L} \pi_l \mathbb{TCE}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l),
\end{aligned}$$

where $\pi_l = \frac{\eta_l \mathbb{PS}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l)}{\mathbb{PS}_{\mathsf{Y}}(\mathsf{y}_p, L)}$, *with* $\sum_{l=1}^{L} \pi_l = 1$, *and* $\mathbb{TCE}_l(\mathsf{y}_p, \xi_l, \omega_l, \alpha_l)$, $\forall l = 1, 2, \dots, L$, *can be evaluated using equation (3.4).* □□

We now briefly discuss the evaluation of the well known measures of risk: the Value-at-Risk (VaR), and the Expected Shortfall Probability (ES).

The VaR of portfolio return $\mathsf{Z} = \mathbf{w}^\mathsf{T}\mathbf{Y}$ at confidence level λ is defined as the smallest number z_0 such that the probability that Z falls short z_0 is not larger than $1-\lambda$

$$\begin{aligned}
\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z}) &= \inf\{\mathsf{z}_0 \ni \mathbb{P}(\mathsf{Z} < \mathsf{z}_0) \leq 1-\lambda\} \\
&= \inf\{\mathsf{z}_0 \ni F(\mathsf{z}_0) \leq 1-\lambda\} \\
&= F_\mathsf{Z}^{-1}(1-\lambda)
\end{aligned}$$

where $F_\mathsf{Z}()$ is the cdf of Z, $F_\mathsf{Z}^{-1}()$ is the inverse function of $F_\mathsf{Z}()$ provided one exists, and the last inequality holds for continuous distributions. Under the assumption that the risky assets $\mathbf{Y}$ have a mixture of multivariate Skew Normal distributions we have shown in Theorem 2.1 that the portfolio return $\mathsf{Z} = \mathbf{w}^\mathsf{T}\mathbf{Y}$ is distributed as a univariate mixture of Skew Normals. Hence, the VaR of the portfolio return $\mathsf{Z} = \mathbf{w}^\mathsf{T}\mathbf{Y}$ at fixed λ confidence level is evaluated as the unique solution with respect to z of the following equation:

$$\mathbb{P}\mathbb{S}_\mathsf{Z}(\mathsf{z}, L) - (1-\lambda) = 0, \tag{3.12}$$

where $\mathbb{P}\mathbb{S}_\mathsf{Z}(\mathsf{z}, L)$ is the probability of shortfall of the distribution of Z.

The Expected Shortfall probability of the portfolio return Z is defined as

$$\begin{aligned}
\mathbb{E}\mathbb{S}_\lambda(\mathsf{Z}) &= \mathbb{E}\big[\mathsf{Z}|\mathsf{Z} \leq \mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z})\big] \\
&= \frac{1}{1-\lambda}\int_{\mathsf{Z}\leq\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z})} \mathsf{z} f(\mathsf{z})\, d\mathsf{z}
\end{aligned} \tag{3.13}$$

that is, the λ level expected shortfall is the average loss smaller or equal than the λ level quantile loss $\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z})$; this value is analytically tractable for mixture of Skew Normals, and coincides with the TCE evaluated at the $\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z})$, i.e.

$$\mathbb{T}\mathbb{C}\mathbb{E}_\mathsf{Z}(\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z}), L) = \sum_{l=1}^{L} \pi_l \mathbb{T}\mathbb{C}\mathbb{E}_l(\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z}), \xi_l, \omega_l, \alpha_l), \tag{3.14}$$

where $\mathbb{T}\mathbb{C}\mathbb{E}_\mathsf{Z}(\mathbb{V}\mathrm{aR}_\lambda(\mathsf{Z}), L)$ is evaluated as in equation (3.11).

4 Conclusion

In this paper we provide a new result for linear combinations of multivariate Skew Normal distributions. This result can be useful when we deal with portfolio returns as convex linear combinations of different risky assets, modeled by mixtures of multivariate Skew Normal distributions. Since investors are interested in evaluating the risk associated to their global portfolios, we calculate risk measures as Value-at-Risk and Expected Shortfall Probability. The provided formulas can be useful in applications for portfolio optimization.

Appendix

In this appendix we state the main result contained in Ellison [5] which is useful in deriving the moment generating function of the Skew Normal distribution and for proving Theorem 2.1.

Theorem 4.1. *Let* X *a Gaussian random variable with mean* μ *and variance* σ^2*, i.e.* $\mathsf{X} \sim \mathcal{N}\left(\mu, \sigma^2\right)$*, then*

$$\mathbb{E}\left(\Phi\left(\mathsf{X}\right)\right) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right). \tag{4.1}$$

For the proof of the theorem we refer the reader to the work of Ellison [5], and in particular to the first corollary to the Theorem 2.

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