Finite Difference Approximation for Linear Stochastic Partial Differential Equations with Method of Lines

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Abstract

A stochastic partial differential equation, or SPDE, describes the dynamics of a stochastic process defined on a space-time continuum. This paper provides a new method for solving SPDEs based on the method of lines (MOL). MOL is a technique that has largely been used for numerically solving deterministic partial differential equations (PDEs). MOL works by transforming the PDE into a system of ordinary differential equations (ODEs) by discretizing the spatial dimension of the PDE. The resulting system of ODEs is then solved by application of either a finite difference or a finite element method. This paper provides a proof that the MOL can be used to provide a finite difference approximation of the boundary value solutions for two broad classes of linear SPDEs, the linear elliptic and parabolic SPDEs.

Key words: Finite difference approximation, linear stochastic partial differential equations (SPDEs), the method of lines (MOL).

1 Introduction

A stochastic partial differential equation, or SPDE, describes the dynamics of a stochastic process defined on a space-time continuum. This paper provides a new method for solving SPDEs based on the method of lines (MOL). The MOL is a two step numerical procedure that is used for solving a deterministic partial differential equation (PDE). The technique was developed in Liskovets [7]. The first step in the MOL involves discretizing the spatial dimension of the PDE. This transforms the PDE into

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a system of ordinary differential equations (ODEs). The second stage of the algorithm uses one of the many numerical methods available for solving ODEs to provide a numerical solution for the transformed PDE. The main advantage of the MOL is that it preserves the characteristics of the problem being solved. For example, if the original problem is a boundary value problem, then the resulting system of ODEs also forms a boundary value problem.

This paper provides a proof that the MOL can be used to provide a finite difference approximation of the boundary value solutions for two broad classes of linear SPDEs, the linear elliptic and parabolic SPDEs. As with deterministic PDEs, the approximation method works by transforming the SPDE to a system of stochastic differential equations (SDEs). A finite difference method is then used to approximate the solution of the system of SDEs. The numerical technique constructed in this paper is applied to the integral form solution of these two boundary value problems. The technique could also be applied on the weak form solutions of these two boundary value problems, leading to a finite element approximation of the two solutions.

To motivate and provide an introduction for this procedure, section two of this paper provides a discussion of the linear elliptic SPDE and the linear parabolic SPDE, while section three discusses the integral and weak form solutions of the boundary value problems for these two equations. Section four develops a smooth approximation of the noise processes forcing the two SPDEs, leading to smoother versions of the two boundary value solutions. The MOL is applied to these smoother solutions. Section five gives the numerical technique and provides the error analysis of this technique.

2 A Review of Stochastic Partial Differential Equations

This paper will focus on the following two boundary value problems. The first boundary value problem is associated with the solution of the linear stochastic elliptic equation,

$$\begin{array}{l} \Delta u\left(x\right)+bu\left(x\right)=g\left(x\right)+\dot{W}\left(x\right), \quad 0\leq x\leq 1, \\ u\left(0\right)=u\left(1\right)=0, \end{array} \tag{1}$$

where u is a real valued function of $x\in\mathbb{R}_+^d$, $\Delta=\sum_{i=1}^d \partial^2/\partial x_i^2$ is the Laplace operator, b is a constant and $\dot{W}\left(x\right)$ denotes the Gaussian white noise process. The second boundary value problem to be examined is associated with the solution of the linear stochastic parabolic equation,

$$\begin{array}{l} \frac{\partial u}{\partial t}\left(t,x\right)+\frac{\partial^2 u}{\partial x^2}\left(t,x\right)+bu\left(t,x\right)=g\left(t,x\right)+\dot{W}\left(t,x\right), \quad 0\leq t<\infty, \\ u\left(0,x\right)=u_0\left(x\right), \quad 0\leq x\leq 1, \\ u\left(t,0\right)=u\left(t,1\right)=0, \end{array} \tag{2}$$

where u is a real valued function of $t\in\mathbb{R}_+$ and $x\in\mathbb{R}_+^d$, with initial value $u_0\left(x\right)\in C_0\left([0,1]\right)$, b is a constant and $\dot{W}\left(t,x\right)$ denotes the space-time white noise process.

To place this review of SPDEs in the context of an application, the following boundary value problem for the deterministic linear parabolic equation is introduced,

$$\begin{aligned}
&\tfrac{\partial u}{\partial t}(t,x)+\tfrac{\partial^2 u}{\partial x^2}(t,x)+bu(t,x)=F(t,x)\,, \quad 0\le t<\infty,\\
&u(0,x)=u_0(x)\,, \quad 0\le x\le 1,\\
&u(t,0)=u(t,1)=0.
\end{aligned} \tag{3}$$

The above boundary value problem is often used in engineering and the physical sciences to model the diffusion of an electrical current along a cylindrical cable. In this model $u(x,t)$ describe the electrical potential at time t and at the point x. The PDE indicates that $u(x,t)$ is a function of its partial growth rate u_t, its rate of diffusion u_{xx}, and a forcing term $F(x,t)$, which describes the arrival of a current at (x,t). The noise term is introduced into this equation through the forcing term $F(x,t)$,

$$F(x,t)=g(x,t)+\Pi(x,t)\,, \quad 0\le t<\infty,\ 0\le x\le 1, \tag{4}$$

where $g(\cdot,\cdot)$ and $\Pi(\cdot,\cdot)$ respectively describe deterministic and noisy signals.

The noisy component $\Pi(\cdot,\cdot)$ in the forcing equation models the random signal. It is assumed that the arrival of these random signals will be Poisson distributed. There is also no constraint on the amplitudes of these noisy signals, and therefore they can be either excitatory or inhibitory. $\Pi(\cdot,\cdot)$ is therefore a compound Poisson process, centered so that it has a mean of zero. As these noise terms will be small in size and large in number, it is common to model the $\Pi(\cdot,\cdot)$ as a two parameter singular white noise process $\dot{W}(x,t)$, where $\cdot$ indicates the singularity of the process rather than a time derivative. This gives the following linear stochastic parabolic equation

$$\frac{\partial u}{\partial t}(t,x)+\frac{\partial^2 u}{\partial x^2}(t,x)+bu(t,x)=g(x,t)+\dot{W}(x,t)\,, \quad 0\le t<\infty,\ 0\le x\le 1, \tag{5}$$

which in turn gives the boundary value problem (2). The reader will note that the nomenclature of SPDEs is analogous to that of deterministic PDEs.

The white noise process which is defined in this SPDE, is related to the two parameter Brownian motion or Brownian sheet $W(t,x)$ by the following differential equation:

$$\dot{W}(x,t)=\frac{\partial^2 W}{\partial x\partial t}(x,t)\,, \quad 0\le t<\infty,\ 0\le x\le 1,$$

where $\frac{\partial^2 W}{\partial x\partial t}(x,t)$ denotes the mixed derivative of Brownian sheet. It should be noted that this is not a derivative in the ordinary sense, as the Brownian sheet is nowhere differentiable. Instead, this is a distributional derivative in the sense of a Schwartz distribution.

There are three important properties of the standard Brownian sheet that should be discussed before going any further. Firstly, if χ_S is the characteristic function of the rectangle S, then for $S\subset(0,T)\times(a,b)$

$$\int_0^T\int_a^b \chi_S dW(t,x)=W(S)\,, \tag{6}$$

where $W(S)$ is a Gaussian random variable with zero mean and variance $|S|$, where $|S|$ denotes the area of S. This implies, for the rectangle $S = \{(t,x)\,; a \le t < b,\ c \le x < d\}$, that

$$\int_c^d \int_a^b \chi_S dW(t,x) = W(b,d) - W(a,d) - W(b,c) - W(a,c). \qquad (7)$$

Secondly, if $E\left(\int_0^T \int_a^b f^2(t,x)\,dxdt\right) < \infty$, then

$$E\left(\int_0^T \int_a^b f(t,x)\,dW(t,x)\right)^2 = E\left(\int_0^T \int_a^b f^2(t,x)\,dxdt\right). \qquad (8)$$

3 Solutions of Stochastic Partial Differential Equations

There are two ways of giving a precise meaning to the two boundary value problems that were discussed in section two. These are the weak form and integral form solutions of the above two boundary value problems. Walsh [16] has shown for the above two boundary value problems that the integral and weak form solutions are equivalent. Of the two solution forms, the integral form solution can be regarded as being more important. This is because as indicated in Pardoux [14], the integral form solution is the one which is used when proving the existence and uniqueness of a solution.

This section provides descriptions of the weak form and integral form solutions for both boundary value problems; however the solution method discussed in section four will only focus on the integral form solution. For the linear stochastic elliptic equation the weak solution for the boundary value problem (1) is given by

$$-\int_0^1 u(x)\,\Delta\phi(x)\,dx \int_0^1 bu(x)\,\phi(x)\,dx = \int_0^1 g(x)\,\phi(x)\,dxs + \int_0^1 \phi(x)\,dW(x), \quad (9)$$

where the test function $\phi \in C^2[0,1] \cap C^0[0,1]$. The integral solution is given as

$$u(x) + \int_0^1 k(x,y)\,bu(y)\,dy = \int_0^1 k(x,y)\,g(y)\,dy + \int_0^1 k(x,y)\,dW(y), \qquad (10)$$

where

$$k(x,y) = x \wedge y - xy \qquad (11)$$

is the Green's function associated with the elliptic equation $-\Delta v(x) = \phi(x)$, $v(0) = v(1) = 0$ so that $v(x) = \int_0^1 k(x,y)\,\phi(y)\,dy$. It will also be assumed that b is sufficiently small so that for equation (1)

$$\lambda^2 = b^2 \int_0^1 \int_0^1 k(x,y)\,dxdy < 1. \qquad (12)$$

For the linear stochastic parabolic equation that the weak solution for the boundary value problem (2) is defined as follows:

$$
\begin{aligned}
& \int_0^1 u(t,x)\,\phi(x)\,dx + \int_0^t \int_0^1 u(s,x)\,\frac{\partial^2 \phi}{\partial x^2}(x)\,dxds \\
& + \int_0^t \int_0^1 bu(s,x)\,\phi(x)\,dxds \\
= & \int_0^1 u_0(x)\,\phi(x)\,dx + \int_0^t \int_0^1 \phi(x)\,dW(s,x) \\
& + \int_0^t \int_0^1 g(s,x)\,\phi(x)\,dxds,
\end{aligned} \tag{13}
$$

where the test function $\phi \in C^2[0,1] \cap C^0[0,1]$. The integral solution for (2) is given by

$$
\begin{aligned}
& u(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)\,bu(s,y)\,dsdy \\
= & \int_0^1 G_t(x,y)\,u_0(y)\,dy + \int_0^t \int_0^1 G_{t-s}(x,y)\,dW(s,y) \\
& + \int_0^t \int_0^1 G_{t-s}(x,y)\,g(s,y)\,dsdy,
\end{aligned} \tag{14}
$$

where the Green's function

$$
G_t(x,y) = 2\sum_{n=1}^{\infty} e^{-(n\pi)^2 t} \sin n\pi x \sin n\pi y \tag{15}
$$

solves the following deterministic boundary value problem

$$
v_t(t,x) - v_{xx}(t,x) = 0, v(0,x) = \phi(x), v(t,0) = v(t,1) = 0, \tag{16}
$$

so that $v(t,x) = \int_0^1 G_t(x,y)\,\phi(y)\,dy$. In addition it will be assumed that b is sufficiently small so that for equation (2) $\tilde{\lambda} < 1$, where

$$
\tilde{\lambda}^2 = b^2 \int_0^T \int_0^1 \int_0^t \int_0^1 G_{t-s}(x,y)\,dydsdxdt. \tag{17}
$$

4 Approximate White Noise and Regularity

This section follows the approach in Allen et al. [1] who have suggested using the following smoother approximation for the white noise process when computing the approximate solutions of stochastic partial differential equations. They have suggested the following approximation for the one-dimensional white noise process $\dot{W}(x)$,

$0 \leq x \leq 1$. The partition $0 = x_1 < x_2 < ... < x_{N+1} = 1$ is defined on the interval $[0, 1]$, where $x_i = (i-1)\Delta x$ and $\Delta x = 1/N$. Then the following approximation is defined for the white noise process $\dot{W}(x)$ on this partition,

$$\frac{d\hat{W}}{dx}(x) = \frac{1}{\Delta x}\sum_{i=1}^{N} \eta_i \sqrt{\Delta x}\chi_i(x) \tag{18}$$

where

$$\eta_i = \frac{1}{\sqrt{\Delta x}}\int_{xi}^{x_{i+1}} dW(t, x) \quad , i = 1, ..., N, \tag{19}$$

i.e. $\eta_i \sim N(0, 1)$, and

$$\chi_i(x) = \begin{cases} 1 \text{ if } x_i \leq x < x_{i+1} \\ 0 \text{ otherwise.} \end{cases} \tag{20}$$

Note that this is of a similar form to the discrete time approximation of continuous time white noise employed when numerically simulating the solutions of stochastic differential equations (see for example Kloeden and Platen [6]).

Now $d\hat{W}(x)$ can be substituted for $dW(x)$ to obtain the following smoothed version of the linear stochastic elliptic equation (1),

$$\hat{u}(x) + \int_0^1 k(x, y)\, b\hat{u}(y)\, dy = \int_0^1 k(x, y)\, g(y)\, dy + \int_0^1 k(x, y)\, d\hat{W}(y). \tag{21}$$

The reason for the interest in this equation is that its solution $\hat{u}(x)$ is smoother than $u(x)$ and therefore standard numerical procedures can be applied to compute its approximate solution. It is therefore necessary to show that the solution of (21), $\hat{u}(x)$, is a good approximation of $u(x)$, the solution of equation (1). To show this the following lemma is required, which provides the *a priori* estimate of the error for the approximate noise process.

Lemma 1 *For the non-random function* $f(x)$, *let* f *be Holder continuous of order* $0 < \alpha \leq 1$ *on the interval* $[0, 1]$, *i.e. for any* $x, y \in [0, 1]$ *there exists a constant* $\gamma > 0$ *such that*

$$|f(x) - f(y)| \leq \gamma |x - y|^{\alpha}. \tag{22}$$

Then

$$E\left[\int_0^1 f(x)\, dW(x) - \int_0^1 f(y)\, d\hat{W}(y)\right]^2 \leq \gamma^2 (\Delta x)^2. \tag{23}$$

Proof. To see why this is so:

$$E\left[\int_0^1 f(x)\, dW(x) - \int_0^1 f(y)\, d\hat{W}(y)\right]^2 \tag{24}$$

$$\begin{aligned}
&= E\left[\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(f(x)-\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}}f(y)\,dy\right)dW(x)\right]^2 \\
&= \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(f(x)-\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}}f(y)\,dy\right)^2 dx \\
&= \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}}(f(x)-f(y))\,dy\right)^2 dx \\
&\leq \frac{\gamma^2}{(\Delta x)^2}\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(\int_{x_i}^{x_{i+1}}|x-y|^{\alpha}\,dy\right)^2 dx \leq \gamma^2(\Delta x)^{2\alpha},
\end{aligned}$$

since $|x-y|\leq \Delta x$ for $x,y\in[x_i,x_{i+1}]$ and $\sum_{i=1}^{N}\Delta x=1$. ■

This lemma will now be used within the following theorem, which provides an approximation of the mean squared error for the difference between the true solution of the boundary value problem (1), u, and the solution of the smoothed boundary value problem (21), $\hat{u}$. This theorem shows that $\hat{u}(x)\to u(x)$ as$(\Delta x)\to 0$, which implies that the smoothed solution $\hat{u}$ will be a good approximation of u as long as the mesh used to construct the smoothed noise process $\hat{W}$ is sufficiently fine. This theorem and its proof are now given below.

Theorem 2 *Let $\hat{u}$ be the solution of the boundary value problem (21) and u be the solution of the boundary value problem (1). Then*

$$E\left[\int_0^1(u(x)-\hat{u}(x))^2\,dx\right]\leq\frac{2(\Delta x)^2}{(1-\lambda)^2}, \tag{25}$$

where $\lambda^2=b^2\int_0^1\int_0^1 k(x,y)\,dydx<1$ and $k(x,y)=x\wedge y-xy$.

Proof. To show that $\hat{u}(x)$ is a good approximation of $u(x)$, let

$$e(x)=u(x)-\hat{u}(x) \tag{26}$$

and

$$F(x)=\int_0^1 k(x,y)\,dW(y)-\int_0^1 k(x,y)\,d\hat{W}(y). \tag{27}$$

This leads to the following inequality

$$\begin{aligned}
\int_0^1 e^2(x)\,dx \;\leq\; & \lambda^2 E\left[\int_0^1 e^2(x)\,dx\right]+E\int_0^1 F^2(x)\,dx+ \\
& 2\lambda\left[E\left[\int_0^1 e^2(x)\,dx\right]\right]^{1/2}\left[E\left[\int_0^1 F^2(x)\,dx\right]\right]^{1/2},
\end{aligned} \tag{28}$$

where $\lambda^2 = b^2 \int_0^1 \int_0^1 k(x,y)\, dydx$ and it is assumed that $\lambda < 1$. Taking the expectation of both sides of this inequality leads to

$$\begin{aligned} E\left[\int_0^1 e^2(x)\, dx\right] \leq\ & \lambda^2 E\left[\int_0^1 e^2(x)\, dx\right] + E\int_0^1 F^2(x)\, dx + \\ & 2\lambda\left[E\left[\int_0^1 e^2(x)\, dx\right]\right]^{1/2}\left[E\left[\int_0^1 F^2(x)\, dx\right]\right]^{1/2}. \end{aligned} \tag{29}$$

Setting $\hat{e} = E\int_0^1 e^2(x)\, dx$ and $\hat{G} = E\int_0^1 F^2(x)\, dx$ and rearranging the inequality (29) gives

$$\left(1-\lambda^2\right)\hat{e} - 2\lambda\hat{e}^{1/2}\hat{G}^{1/2} - \hat{G} \leq 0, \tag{30}$$

which implies that

$$(\hat{e})^{1/2} \leq \frac{\left(\hat{G}\right)^{1/2}}{1-\lambda}. \tag{31}$$

Expanding this out and applying Lemma 1:

$$E\left[\int_0^1 (u(x) - \hat{u}(x))^2\, dx\right] \leq \frac{2(\Delta x)^2}{(1-\lambda)^2}, \tag{32}$$

where $\lambda^2 = b^2 \int_0^1 \int_0^1 k(x,y)\, dydx < 1$ and $k(x,y) = x \wedge y - xy$. ■

Similarly, an approximate noise process is now constructed to the generalized zero mean Gaussian process. Following the approach of Allen et al. [1], the space $[0,T] \times [0,1]$ is partitioned by rectangles $[t_i, t_{i+1}] \times [x_j, x_{j+1}]$, where $t_i = (i-1)\Delta t$, $x_j = (j-1)\Delta x$ for $i = 1, ..., M$ and $j = 1, ..., N$. The following approximation for the mixed derivative of the generalized Gaussian white noise process can then be made with respect to the partition,

$$\frac{\partial^2 \hat{W}}{\partial t \partial x}(t,x) = \frac{1}{\Delta t \Delta x}\sum_{j=1}^{n}\sum_{i=1}^{m} \eta_{ij}\sqrt{\Delta t \Delta x}\chi_i(t)\chi_j(t), \tag{33}$$

where $\eta_{ij} \sim N(0,1)$, $\Delta t = T/M$ and $\Delta x = 1/N$,

$$\chi_i(x) = \begin{cases} 1 \text{ if } x_i \leq x < x_{i+1} \\ 0 \text{ otherwise} \end{cases} \tag{34}$$

defines the characteristic function for x with $\chi_j(t)$ defined similarly for t, and

$$\eta_{ij} = \frac{1}{\sqrt{\Delta t \Delta x}}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}} dW(t,x). \tag{35}$$

The following lemma is now required, which constructs the *a priori* estimate for the noise approximation.

Lemma 3 *Let $f(t,x)$ be a non-random function defined on $[0,T]\times[0,1]$, which is Holder continuous on $[0,T]\times[0,1]$, i.e. for any $(t,x),(u,y)\in[0,T]\times[0,1]$,*

$$|f(t,x)-f(u,y)|\leq\gamma\left(|t-u|^{\beta}+|x-y|^{\alpha}\right),\quad 0\leq\alpha,\beta\leq 1. \tag{36}$$

Then for constants $0\leq\alpha,\beta\leq 1$ and $\gamma\geq 0$,

$$\begin{aligned}&E\left[\int_0^T\int_0^1 f(t,x)\,dW(t,x)-\int_0^T\int_0^1 f(t,x)\,d\hat{W}(t,x)\right]^2 \\ \leq\;& 2T\gamma^2\left((\Delta t)^{2\beta}+(\Delta x)^{2\alpha}\right).\end{aligned} \tag{37}$$

Proof. Upon applying the Holder inequality (36), the inequality relationship expressed in equation (37) can be derived as follows:

$$\begin{aligned}&E\left[\int_0^T\int_0^1 f(t,x)\,dW(t,x)-\int_0^T\int_0^1 f(t,x)\,d\hat{W}(t,x)\right]^2 \\ =\;& \sum_{i=1}^{M}\sum_{j=1}^{N}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\left(\left(f(t,x)-f(u,v)\right)dudv\right)^2dtdx \\ \leq\;& \frac{\gamma^2}{(\Delta t)^2(\Delta x)^2}\sum_{i=1}^{M}\sum_{j=1}^{N}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}}\left(\left(f(t,x)-f(u,v)\right)dudv\right)^2dtdx \\ \leq\;& 2T\gamma^2\left((\Delta t)^{2\beta}+(\Delta x)^{2\alpha}\right),\quad 0\leq\alpha,\beta\leq 1,\gamma\geq 0.\end{aligned} \tag{38}$$

■

Now the smoothed Brownian sheet $\hat{W}(t,x)$ can be substituted for $W(t,x)$ in the boundary value problem (2) to obtain the following equation

$$\begin{aligned}&\hat{u}(t,x)+\int_0^t\int_0^1 G_{t-s}(x,y)\,b\hat{u}(s,y)\,dsdy \\ =\;& \int_0^1 G_t(x,y)\,u_0(y)\,dy+\int_0^t\int_0^1 G_{t-s}(x,y)\,d\hat{W}(s,y) \\ &+\int_0^t\int_0^1 G_{t-s}(x,y)\,g(s,y)\,dsdy,\end{aligned} \tag{39}$$

where $G_t(x,y)$ is as provided in equation (15) and

$$G_{t-s}(x,y)=2\sum_{n=1}^{\infty}e^{-(n\pi)^2(t-s)}\sin n\pi x\sin n\pi y. \tag{40}$$

The *a priori* estimate provided by Lemma 3 can now be used to show when $\hat{u}$ will be a reasonable approximation of u, the solution of the original boundary value problem (2). This theorem shows that $\hat{u}(t,x)\to u(t,x)$ as $(\Delta x)^2/(\Delta t)^{1/2}\to 0$. The theorem and its proof are now provided below.

Theorem 4 *Let u be the solution of the original boundary value problem (2) and $\hat{u}$ be the solution of smoothed boundary value problem (39). Assuming that $\tilde{\lambda} < 1$, then*

$$E\int_0^T\int_0^1 (u(t,x) - \hat{u}(t,x))\,dxdt \le \frac{1}{\left(1-\tilde{\lambda}\right)^2}\left(c_1(\Delta t)^{1/2} + c_2\frac{(\Delta x)^2}{(\Delta t)^{1/2}}\right), \tag{41}$$

where c_1 and c_2 are constants independent of Δt and Δx.

Proof. To consider the error produced by this approximation, let

$$e(t,x) = u(t,x) - \hat{u}(t,x) \tag{42}$$

and

$$F(t,x) = \int_0^T\int_0^1 G_{t-s}(x,y)\,dW(x,y) - \int_0^T\int_0^1 G_{t-s}(x,y)\,d\hat{W}(x,y), \tag{43}$$

where $G_{t-s}(x,y)$ is given as (40). This then leads to the following inequality

$$\begin{aligned}\int_0^T\int_0^1 e^2(t,x)\,dxdt \;&\le\; \tilde{\lambda}^2\int_0^T\int_0^1 e^2(t,x)\,dxdt + E\int_0^T\int_0^1 F^2(x)\,dx \\ &\quad +2\tilde{\lambda}E\left[\int_0^T\int_0^1 e^2(t,x)\,dx\right]^{1/2}\left[E\int_0^T\int_0^1 F^2(t,x)\,dx\right]^{1/2}\end{aligned} \tag{44}$$

where

$$\tilde{\lambda}^2 = b^2\int_0^T\int_0^1\int_0^t\int_0^1 G^2_{t-s}(x_j,y)\,dydsdxdt \le \frac{b^2T}{24} \tag{45}$$

and it is assumed that $\tilde{\lambda} < 1$. Taking the expectation of both sides of this inequality and setting $\hat{e} = E\int_0^T\int_0^1 e^2(t,x)\,dxdt$ and $\hat{F} = E\int_0^T\int_0^1 F^2(x,t)\,dxdt$ gives

$$\left(1-\tilde{\lambda}^2\right)\hat{e} - 2\tilde{\lambda}\hat{e}^{1/2}\hat{F}^{1/2} - \hat{F} \le 0, \tag{46}$$

implying that

$$\hat{e}^{1/2} \le \frac{\hat{F}^{1/2}}{1-\tilde{\lambda}}. \tag{47}$$

Using the inequalities

$$\sum_{n=1}^{\infty} e^{-n^2\Delta t} \le (\Delta t)^{-1/2} \text{ and } \sum_{n=1}^{\infty}\frac{1-e^{-n^2\Delta t}}{n^2} \le 2(\Delta t)^{1/2}, \tag{48}$$

the following bound on $\hat{F}$ is obtained by applying Lemma 3 and noting that $G_{t-s}(x,y) = 2\sum_{n=1}^{\infty} e^{-(n\pi)^2(t-s)}\sin n\pi x\sin n\pi y$,

$$\hat{F} \le c_1(\Delta t)^{1/2} + c_2\frac{(\Delta x)^2}{(\Delta t)^{1/2}} \tag{49}$$

where c_1 and c_2 are constants independent of Δt and Δx. This implies that under the assumption that $\tilde{\lambda} < 1$

$$E\int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 \, dxdt \leq \frac{1}{\left(1-\tilde{\lambda}\right)^2} \left(c_1 (\Delta t)^{1/2} + c_2 \frac{(\Delta x)^2}{(\Delta t)^{1/2}} \right). \quad (50)$$

This implies that $\tilde{u}(t,x) \to u(t,x)$ provided that $(\Delta x)^2 / (\Delta t)^2 \to 0$ as the mesh is refined. If this is the case then standard numerical procedures can be applied to approximate $\hat{u}(t,x)$. ■

5 Solving SPDEs by Method of Lines: Numerical Method and Error Analysis

To illustrate how MOL operates, we use the following example of a deterministic non-linear PDE, which has been borrowed from Ames [2, 33–35],

$$u_t = u_{xx} + (u_x)^2, \quad 0 < x < 1, 0 < t, \quad (51)$$

with boundary values $u(x,0) = x(1-x)$ and $u(0,t) = 0$, $u(1,t) = \sin t$. In this case, we shall use a discretization of the state variable x; the time variable being left alone at this stage. The domain of x, $[0,1]$ is replaced by a discrete set of points $x_i = i\Delta x$, $i = 1, ..., n$. We now use a Taylor series for $u(x + \Delta x, y)$ about (x,y) to give

$$\begin{aligned} u(x+\Delta x, y) &= u(x,y) + \Delta x \frac{\partial u}{\partial x}(x,y) \quad (52) \\ &\quad + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3}(x,y) + o\left((\Delta x)^4\right), \end{aligned}$$

which upon division by Δx gives the forward difference

$$\frac{\partial u}{\partial x} = \frac{(u(x+\Delta x, y) - u(x,y))}{\Delta x} + o(\Delta x). \quad (53)$$

In terms of the discrete approximation space we have

$$\left.\frac{\partial u}{\partial x}\right|_{ij} = \frac{1}{\Delta x} (u(i+\Delta x, j) - u(i,j)) + o(\Delta x) \quad i,j = 1, ..., 5, \quad (54)$$

where $o(h)$ is the truncation error of the approximation. The forward difference approximation for the second order partial derivative is derived as follows

$$\frac{1}{(\Delta x)^2} [u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y)] = \frac{\partial^2 u}{\partial x^2} + o\left((\Delta x)^2\right). \quad (55)$$

In index form this becomes

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} = \frac{u\left(i+\Delta x, j\right) - 2u\left(i, j\right) + u\left(i-h, j\right)}{\Delta x^2} + o\left(\left(\Delta x\right)^2\right), \quad i, j = 1, ..., 5. \quad (56)$$

Hence neglecting the discretization error $o\left(h^2\right)$, we arrive at the following coupled system of ODEs

$$\begin{aligned} u'\left(i, j\right) &= \frac{1}{0.4}\left(u\left(i+\Delta x, j\right) - 2u\left(i, j\right) + u\left(i-\Delta x, j\right)\right) \\ &\quad + \frac{1}{0.16}\left(u\left(i+\Delta x, j\right) - u_i\left(i, j\right)\right)^2, \quad j = 1, ..., 5, \end{aligned} \quad (57)$$

where $u\left(1, 0\right) = 0.16$, $u\left(2, 0\right) = 0.24$, $u\left(3, 0\right) = 0.24$ and $u\left(4, 0\right) = 0.16$ and the boundary conditions are accounted for by $u\left(0, t\right) = 0$ and $u(5, t) = \sin t$.

Consider the one dimensional smoothed elliptic boundary value problem expressed in equation (21). MOL provides the following finite difference approximation of $\hat{u}\left(x\right)$,

$$\begin{cases} u_j + b\sum_{i=1}^{N} u_i K_{ij} = \sum_{i=1}^{N} r_{ij} + \sum_{i=1}^{N} K_{ij}\frac{\eta_i}{\sqrt{\Delta x}}, \quad j = 2, 3, ..., N \\ u_1 = u_{N+1} = 0, \end{cases} \quad (58)$$

where

$$K_{ij} = \int_{x_i}^{x_{i+1}} k\left(x_j, y\right) dy, \; r_{ij} = \int_{x_i}^{x_{i+1}} k\left(x_j, y\right) g\left(y\right) dy \quad (59)$$

and $\eta_i \sim N\left(0, 1\right)$ for $i = 1, 2, ..., N$. The error of the discrete approximation of $\hat{u}$ is now derived in the following theorem.

Theorem 5 *Let $\lambda^2 < 1$. Given that $\varepsilon > 0$, there is a constant $N_0 > 0$ such that $\lambda_N^2 \leq \lambda^2 + \varepsilon < 1$ for $N > N_0$ and the error in the finite difference approximation (58) satisfies*

$$\left[E\left[\frac{1}{N}\sum_{i=1}^{N}\left(\hat{u}\left(x_j\right) - \hat{u}_j\right)^2 dx\right]\right]^{1/2} \leq \frac{C}{1-\lambda_N^2}\Delta x. \quad (60)$$

Proof. In the finite difference equation (58), u_j approximates $\hat{u}\left(x_j\right)$. It can be seen that $\hat{u}\left(x_j\right)$ solves the following SPDE

$$\begin{aligned} &\hat{u}\left(x_j\right) + b\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}} k\left(x_j, y\right)\hat{u}\left(y\right) dy \\ &= \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}} k\left(x_j, y\right) g\left(y\right) dy + \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}} k\left(x_j, y\right) dy \frac{\eta_i}{\sqrt{\Delta x}}. \end{aligned} \quad (61)$$

Now let $\varepsilon_j = \hat{u}\left(x_j\right) - u_j$ denote the approximation error, where u_j solves equation (58). Then subtracting equation (58) from (61) gives the inequality

$$\left(1-\lambda_N^2\right)\bar{\varepsilon} - 2\lambda_N\bar{\varepsilon}^{1/2}\hat{F}^{1/2} - \hat{F} \leq 0, \quad (62)$$

where

$$\bar{\varepsilon}^2 = E\left[\frac{1}{N}\sum_{j=1}^{N}\varepsilon_j^2\right], \hat{F} = E\left[\frac{1}{N}\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(\hat{u}\left(x\right)-\hat{u}\left(x_i\right)\right)^2 dx\right] \tag{63}$$

and

$$\lambda_N^2 = b^2\int_0^1\sum_{j=1}^{N} k\left(x_j, y\right)\Delta x dy. \tag{64}$$

It can seen that λ_N^2 is a numerical quadrature approximation of λ, since for $\lambda^2 < 1$ and $\varepsilon > 0$ such that $\lambda^2 + \varepsilon < 1$, for N sufficiently large $\lambda_N^2 \leq \lambda^2 + \varepsilon < 1$. This implies for sufficiently large N,

$$\bar{\varepsilon} \leq \frac{\hat{F}\left(\lambda_N^2 + \lambda_N\right)}{1-\lambda_N^2}. \tag{65}$$

All that remains is to construct the error bound on $\hat{F}^2$. Using equation (21) the following inequality can be constructed,

$$\begin{aligned} E\left(\hat{u}\left(x+\Delta x\right)-\hat{u}\left(x\right)\right)^2 \quad &\leq \quad 3\int_0^1\left(k\left(x+\Delta x, y\right)-k\left(x,y\right)\right)^2 dy \\ &\quad \left[b^2 E\int_0^1 \hat{u}^2\left(y\right)dy + \int_0^1 g^2\left(y\right)dy + 1\right]. \end{aligned} \tag{66}$$

Since $k\left(x,y\right) = x\wedge y - xy$, the first term on the right hand side is $O\left(\Delta x^2\right)$. Since $g \in L^2\left(0,1\right)$ is fixed and $E\int_0^1 \hat{u}^2\left(y\right)dy$ is bounded, then

$$E\left(\hat{u}\left(x+\Delta x\right)-\hat{u}\left(x\right)\right)^2 \leq C\left(\Delta x\right)^2, \tag{67}$$

and it follows that

$$\hat{F}^2 = E\left[\frac{1}{N}\sum_{i=1}^{N}\int_{x_i}^{x_{i+1}}\left(\hat{u}\left(x\right)-\hat{u}\left(x_i\right)\right)^2 dx\right] \leq C\left(\Delta x\right)^2. \tag{68}$$

This implies that the error in the discrete approximation has the following form for N sufficiently large,

$$\left[E\left[\frac{1}{N}\sum_{i=1}^{N}\left(\hat{u}\left(x_j\right)-\hat{u}_j\right)^2 dx\right]\right]^{1/2} \leq \frac{C}{1-\lambda_N^2}\Delta x. \tag{69}$$

■

Now consider the smoothed parabolic boundary value problem (39). MOL gives the following finite difference approximation of $\hat{u}\left(t,x\right)$,

$$u_{i+1,j} \quad = \quad -b\sum_{l=1}^{i}\sum_{k=1}^{N} u_{lk}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{l+1-s}\left(x_j, y\right)dyds \tag{70}$$

$$+\sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,u_0(y)\,dyds$$
$$+\sum_{l=1}^{i}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,g(s,y)\,dyds$$
$$+\sum_{l=1}^{i}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,\frac{\eta_{lk}}{\sqrt{\Delta t\Delta x}}dyds$$

for $i = 1,2,...,M,\ j = 1,2,...,N$ with $u_{i1} = u_{iN+1} = 0$ for each i and where $\eta_{ij} \sim N(0,1)$. The error for the finite difference approximation given by equation (70) will now be examined in the following theorem.

Theorem 6 *Let $\tilde{\lambda}^2 < 1$. Given $\varepsilon > 0$ there exists an $N_0, M_0 > 0$ such that $\tilde{\lambda}^2_{N,M} \leq \tilde{\lambda}^2 + \varepsilon < 1$ for $N > N_0$ and $M > M_0$ and the error in the finite difference approximation (70) satisfies*

$$E\left[\frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\hat{u}(t_i,x_j)-u_{ij}\right)^2\right] \leq \frac{\left(2\tilde{c}_2\Delta x + 2\tilde{c}_1(\Delta t)^{1/2}\right)^{1/2}}{1-\tilde{\lambda}^2_{NM}} \tag{71}$$

for constants c_1 and c_2 independent of Δx, Δt and $\hat{u}(t,x)$.

Proof. Since $u_{ij} \approx \hat{u}(t_i,x_j)$, the finite difference approximation of $\hat{u}(t,x)$ can now be expressed as follows,

$$\begin{aligned}\hat{u}(t_{i+1},x_j) &= -b\sum_{l=1}^{i-1}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,\hat{u}(s,y)\,dsdy \\ &\quad+\sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} G_{t_{l+1}}(x_j,y)\,\hat{u}_0(y)\,dy \\ &\quad+\sum_{l=1}^{i-1}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,g(s,y)\,dsdy \\ &\quad+\sum_{l=1}^{i-1}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,\frac{\eta_{lk}}{\sqrt{\Delta t\Delta x}}dsdy\end{aligned} \tag{72}$$

for $i = 1,2,...,M$ and $j = 1,2,...,N$. Let $\varepsilon_{ij} = \hat{u}(t_i,x_j) - u_{ij}$, where u_{ij} solves equation (70). Subtracting (70) from (72) gives the following equation specifying the error of the approximation

$$\begin{aligned}\varepsilon_{i+1,j} &= -b\sum_{l=1}^{i-1}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\,\varepsilon_{lk}dsdy \\ &= -b\sum_{l=1}^{i-1}\sum_{k=1}^{N}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} G_{t_{l+1}-s}(x_j,y)\left(\hat{u}(s,y)-u(t_l,x_k)\right)dsdy\end{aligned} \tag{73}$$

for $i = 1, 2, ..., M$ and $j = 1, 2, ..., N$. Let

$$\tilde{\varepsilon}^2 = E\left[\frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}\varepsilon_{ij}^2\right] \tag{74}$$

and

$$\tilde{\lambda}_{NM}^2 = b^2\sum_{i=1}^{M}\int_{t_i}^{t_{i+1}}\sum_{j=1}^{N}\int_{x_j}^{x_{j+1}}\sum_{l=1}^{i-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^{N}\int_{xk}^{x_{k+1}} G_{t_{l+1}-s}\left(x_j, y\right) dydsdxdt. \tag{75}$$

Then the error of the approximation in equation (73) gives

$$\tilde{\varepsilon} \leq \tilde{\lambda}_{NM}^2\left(\tilde{\varepsilon}^2 + 2\tilde{\lambda}\tilde{\varepsilon}\tilde{F} + \tilde{F}^2\right), \tag{76}$$

where

$$\tilde{F}^2 = \sum_{l=1}^{M}\int_{t_l}^{t_{l+1}}\sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} E\left(\hat{u}\left(s, y\right) - u\left(t_l, x_k\right)\right)^2 dsdy. \tag{77}$$

Since for values of N and M sufficiently large $\tilde{\lambda}_{NM}^2$ is the quadrature approximation for $\tilde{\lambda}^2 < 1$, this implies that (76) gives

$$\tilde{\varepsilon} \leq \frac{\hat{F}^{1/2}\left(\tilde{\lambda}_{NM}^2 + \tilde{\lambda}_{NM}\right)}{1 - \tilde{\lambda}_{NM}^2}. \tag{78}$$

The error bound on $\tilde{F}^2$ is derived as follows. For $t > 0$, the following two inequalities can be derived from equation (39):

$$\begin{aligned}
& E\left(\hat{u}\left(t + \Delta t, x\right) - \hat{u}\left(t, x\right)\right)^2 \\
\leq\ & 4\left[\int_t^{t+\Delta t}\int_0^1 G_{t+\Delta t-s}^2\left(x, y\right) dyds\right. \\
& \left. + \int_0^t\int_0^1\left(G_{t+\Delta t-s}\left(x, y\right) - G_{t-s}\left(x, y\right)\right) dyds\right] \\
& \cdot\left[b^2\int_0^t\int_0^1 \hat{u}^2\left(s, y\right) dyds + \int_0^t\int_0^1 g^2\left(s, y\right) dyds + 1\right] \\
& + 4\int_0^1\left(G_{t+\Delta t}\left(x, y\right) - G_t\left(x, y\right)\right) dy\int_0^1 u_0^2\left(y\right) dy
\end{aligned} \tag{79}$$

and

$$\begin{aligned}
& E\left(\hat{u}\left(t, x + \Delta x\right) - \hat{u}\left(t, x\right)\right)^2 \\
\leq\ & 4\left[\int_0^t\int_0^1\left(G_{t-s}\left(x + \Delta x, y\right) - G_{t-s}\left(x, y\right)\right) dyds\right].
\end{aligned} \tag{80}$$

$$\left[b^2 \int_0^t \int_0^1 \hat{u}^2 (s,y)\, dyds + \int_0^t \int_0^1 g^2 (s,y)\, dyds + 1\right]$$
$$+4 \int_0^1 \left(G_t (x+\Delta x, y) - G_t (x,y)\right) dy \int_0^1 u_0^2 (y)\, dy.$$

Note that the bracketed terms in (79) and (80) contain the Green's function $G_t (x,y) = 2\sum_{n=1}^{\infty} e^{-(n\pi)^2 t} \sin n\pi x \sin n\pi y$. For these bracketed terms in (79) and (80) the following two inequalities can be derived:

$$\begin{aligned} & \int_0^t \int_0^1 \left(G_{t+\Delta t-s} (x,y) - G_{t-s} (x,y)\right) dyds \\ = & \int_0^t 2\sum_{n=1}^{\infty} \sin^2 n\pi x \left(e^{-(n\pi)^2 (t+\Delta t-s)} - e^{-(n\pi)^2 (t-s)}\right)^2 ds \\ \leq & \; 2\sum_{n=1}^{\infty} \frac{1-e^{-(n\pi)^2 \Delta t}}{2(n\pi)^2} \leq \tilde{c}_1 (\Delta t)^{1/2} \end{aligned} \tag{81}$$

and

$$\begin{aligned} & \int_0^t \int_0^1 \left(G_{t-s} (x+\Delta x, y) - G_{t-s} (x,y)\right) dyds \\ = & \int_0^t 2\sum_{n=1}^{\infty} \left(\sin n\pi (x+\Delta x) - \sin n\pi x\right)^2 e^{-(n\pi)^2 (t-s)} ds \\ \leq & \; 2\sum_{n=1}^{\infty} \frac{\left(\sin n\pi (x+\Delta x) - \sin n\pi x\right)^2}{2(n\pi)^2} \leq \tilde{c}_2 (\Delta x)\,. \end{aligned} \tag{82}$$

Since it can be shown that $E\int_0^t \int_0^1 \hat{u}^2 (s,y)\, dyds$ is bounded, this implies that

$$\tilde{F}^2 \leq \tilde{c}_2 \Delta x + \tilde{c}_1 (\Delta t)^{1/2}, \tag{83}$$

which can be substituted into (78) to get

$$\tilde{\varepsilon} \leq \frac{\left(2\tilde{c}_2 \Delta x + 2\tilde{c}_1 (\Delta t)^{1/2}\right)^{1/2}}{1-\tilde{\lambda}_{NM}^2}, \tag{84}$$

where c_1 and c_2 are constants independent of Δx, Δt and $\hat{u}(t,x)$. ■

6 Conclusion

This paper has provided a new method of constructing a finite difference approximation for linear SPDEs using MOL. The approximate solutions are constructed for the boundary value problems associated with a linear elliptic SPDE and a linear parabolic SPDE. The error analysis of the finite difference approximations in section five shows that there is a linear error for the linear elliptic SPDE, while for the linear parabolic SPDE the squared error of the approximation is quasi-linear.

The key to the approximation is the use of smoothed noise in place of the original noisy forcing term. This then leads to a smoother solution, which is easier to approximate. It is shown in section four of this paper that the smoother solution is a reasonable approximation of the original solution. Because of this, the finite difference approximation of the smoothed solution will also be a good approximation of the original solution.

In terms of the numeric procedure developed in this paper, within finance there are two applications that have potential for development. The first application is to the forward rate equation formulated under the Musiela parametrization (see Musiela [12], Brace and Musiela [3], Brace, Gatarak and Musiela [4] and Musiela and Sondermann [13]). The bond pricing problem when expressed in terms of this forward rate equation, leads to a boundary value problem for a first order linear stochastic hyperbolic equation. This would be a natural application for the finite difference solution method that was constructed in this paper.

The second application would be to provide a numerical scheme for approximating the solution of a forward backward stochastic differential equation (FBSDE) with random coefficients. When the coefficients are deterministic the FBSDE (i.e. Markovian case) can be solved by using the Four Step Scheme of Ma, Protter and Yong [8], which requires that a deterministic PDE be solved. When the coefficients of the FBSDE are random, it has been shown in Ma and Yong [9, 10, 11] that the Four Step Scheme requires the solution of parabolic and elliptic SPDEs for the finite and infinite cases, respectively.

In addition to the many immeadiate applications in non-linear filtering theory, the numerical procedure outlined in this paper, could be used to provide a numerical procedure for approximating the solutions of this class of FBSDE. Two potential applications would be to provide numerical solutions for the non-Markovian extension of the consol bond model developed in Duffie, Ma and Yong [5] and the stochastic Black-Scholes formula provided in Ma and Yong [10, 11].

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