A dynamic limit order market with fast and slow traders

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Abstract

We study a dynamic limit order market where agents may invest into a trading technology that grants them a speed advantage over others. Being fast is valuable because it allows limit orders to be revised quickly in the light of new information and therefore reduces the risk of being picked off. Even though this can generate more trading, the equilibrium level of investment is excessive and always generates a welfare loss because fast traders exert negative externalities on slow agents and are able to extract any surplus. If the diffusion of trading technology additionally leads to a more efficient trading process, this result may reverse completely. For sufficiently large efficiency gains, fast traders exert positive externalities on slow market participants and their presence leads to an increase in social welfare, albeit the equilibrium level of investment is below the social optimum. Our results imply that the marginal impact of investments related to algorithmic and high-frequency trading on social welfare crucially depends on the pre-investment level of market efficiency.

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1 Introduction

While the proverb "time is money" applies to virtually all economic activities, the accelerated proliferation of electronic trading has taken this wisdom to the extreme. High-frequency trading (HFT), a variant of algorithmic trading, relies on sophisticated computer programs for the implementation of trading strategies that involve a vast amount of orders in very small time intervals. Propietary trading desks, hedge funds and so-called pure-play HFT outlets are investing large sums into human (IT experts, mathematicians, linguists, etc.) and physical (co-location, data feeds and warehouses, etc.) capital in an effort to outpace the competition. Recent estimates suggest that HFTs are now responsible for more than 50% of trading in U.S. equities.[1]

These developments are being accompanied by a heated debate among financial economists, practitioners, and regulators about the implications of an increasing computerization of the trading process. While proponents[2] argue that technology increases market efficiency via improved liquidity and price discovery, critiques[3] claim that HFTs make profits at the expense of other (slow) market participants and have the potential to destabilize markets.

This paper contributes to this debate by presenting a model of trading in a limit order market where agents differ in the speed with which they can react to the arrival of new information, which is thought to capture the difference between (fast) HFTs and (slow) human market participants. We build on the model of Foucault (1999), in which limit orders face the "winner's curse" because they cannot be revised after submission and thus may become stale due to the arrival of new information. In this setting, an increase in the underlying asset's risk induces agents to submit less aggressive limit orders (sometimes referred to as "order shading") such that less gains from trade are realized. We extend Foucault's model by allowing for the possibility that agents invest into a trading technology that endows them with a *relative* speed advantage. Specifically, we assume that fast traders (FTs) face a reduced probability of being "picked off" (Copeland and Galai (1983)) because they have the possibility to revise their limit orders after news releases, albeit only if the next agent is a slow trader (ST).

We analyze the stationary equilibrium of this dynamic limit order market and compare it with the baseline case of identical traders studied by Foucault (1999). Notice that because the choice between limit and market orders in endogenous in this model, agent's profits from either are closely interrelated. In fact, the limit order market is nothing else but a sequential bargaining process, where an agent makes a take-it-or-leave-it offer (limit order) to the following agent, who either accepts this offer (via a market order) or in turn makes another offer to the agent arriving after him.

[1] See Financial Times, "High-frequency boom time hits slowdown", April 12, 2011.

[2] See e.g. Optiver, "High Frequency Trading", Position Paper, 2011, http://fragmentation.fidessa.com/wp-content/uploads/High-Frequency-Trading-Optiver-Position-Paper.pdf

[3] See SEC Chairman Mary Schapiro's speech in front of the Security Traders Association "Remarks Before the Security Traders Association", www.sec.gov/news/speech/2010/spch092210mls.htm.

As a consequence, the expected profit obtained from posting a limit order acts as an endogenous outside option that determines agents' willingness to accept a limit order or not. Now it is crucial to understand that the decision of some agents to become fast does not only have a positive effect on their own outside option (because they avoid being picked off if the next agent is slow) but also a negative effect on the outside option of those agents that remain slow. The latter effect arises because STs now need to adjust their strategy for the fact that some agents have become fast and require better terms of trade in order to be convinced to accept a limit order. While they may choose to i) incur either a reduced execution probability or ii) lower profits conditional on execution (via more aggressive quotes), both reactions lead to strictly lower expected profits.

As the order choice is endogenous, this redistribution in "bargaining power" also affects the distribution of expected profits from market orders (we call the inverse expected trading costs), and FTs generally obtain better prices than STs. While situations may arise where STs face lower expected trading costs than in the absense of FTs, this is entirely due to the increased aggressiveness of other STs quotes, because FTs are able to price discriminate implicitly (if an order can be cancelled, the next agent is slow with probability 1) between trader types due to their speed advantage.

In sum, STs are always worse off in the presence of FTs compared to a market with only STs. This has important implications for social welfare, as the equilibrium level of investment is such that fast and slow agents obtain the same expected utility once the cost of becoming fast is accounted for. Although FTs' ability to avoid the "winner's curse" has the potential to increase the total gains from trade that are realized, investment is excessive in equilibrium due to negative externalities and yields a loss in social welfare.

In an extension we allow for the possibility that investments in trading technology do not only give some agents an edge over others but also improve the overall efficiency of the trading process (e.g. because it facilitates the aggregation of dispersed trading information in a fragmented market). More specifically we assume that slow traders are a source of inefficiency in the sense that there is some positive probability $1-\eta$ with which limit orders are "missed", i.e. the arriving agent submits a market order although the best available limit order is such that he would accept it. FTs never miss orders, such that their presence now also has a positive effect on STs' outside option because it increases the execution probability of limit orders. If the slow market is sufficiently inefficient (i.e. for low η), this effect dominates the negative effect of FTs' higher limit order profits and the equilibrium level of investment may lead to an increase in social welfare albeit falling short of the social optimum (due to positive externalities).

Our results have important implications for policy makers as they suggest that the impact of investments made by trading firms engaging in algorithmic and particularly high-frequency trading on social welfare crucially depends on the efficiency of the marketplace prior to these investments. Hence it is important to develop a broader view that jointly addresses both the benefits as well as the potential concerns related to the ongoing transformation of the market structure. Certainly,

the widespread automatization of the trading process has brought along numerous benefits as it has facilitated liquidity provision via electronic market making and inter-market competition via smart oder routing systems. On the other hand, the HFT community is spending vast sums with the sole aim to outpace the competition, while the resulting gains for overall market efficiency are likely to be marginal. Given that there is scope for intervention from either side (taxes or subsidies), the real-world implementation of adequate policies is likely to face measurement problems.

The literature on algorithmic and high-frequency trading has grown substantially in the past couple of years (see e.g. the surveys by Biais and Woolley (2012) and Focuault (2012)). Most closely related to our paper are the two recent papers by Jovanovic and Menkveld (2011) and Biais et al. (2012). The first paper studies competitive middlemen that intermediate between early limit order traders and late market order traders. As in this study, HFTs' speed advantage may reduce order shading by updating quotes quickly and therefore increase trade. On the other hand, they offer another explanation for why HFT may also reduce trade (unawareness of information by STs when submitting market orders). A calibration exercise reveals a slight increase in welfare. The main difference between this paper and theirs is that they assume FTs to be competitive and act as pure intermediaries, while we allow for some degree of market power and do not consider pure intermediation. Moreover, we additionally consider the social cost of investments into technology when evaluating welfare. In Biais et al. (2011), investing into trading technology helps to reap gains from trade by facilitating the search for trading opportunities, but at the same time increases adverse selection for slow market participants. This negative externality (and the associated overinvestment in equilibrium) is very similar to the one arising in our model, but it is based on reduced-form assumptions instead of arising from speed in connection with the trading protocol. Moreover, because they consider a dealer market, STs cannot benefit from the gains in efficiency brought about by investments into trading technology. Also closely related, Cartea and Penalva (2011) propose a model where their increased speed allows HFT to impose a haircut on liquidity traders, which increases trading volume and price volatility, but lowers the welfare of liquidity traders.

Several studies empirically examine the impact of algorithmic and high-frequency trading on market quality. In summary, this stream of the literature concludes that automated trading strategies improve liquidity (Hendershott et al. (2011), Hasbrouck and Saar (2011)), are highly profitable (e.g. Brogaard (2010) and Menkveld (2011)), and significantly contribute to price discovery (e.g. Hendershott and Riordan (2011a, 2011b)). In line with our findings on the effects of speed in a limit order market, Hendershott and Riordan (2011a) find that algorithmic traders "supply liquidity when it is expensive and consume liquidity when it is cheap". Moallemi and Sağlam (2011) provide some empirical estimates of the "cost of latency" and find a dramatic increase between 1995 and 2005. Chaboud et al. (2009)) study computer- and human-generated order flow in the FX market and conclude that the trading strategies by automated traders are more correlated among each other than those of human market participants (see also Brogaard (2010)). Kirilenko et al. (2011) examine

the recent "flash crash" in U.S. equity markets and find that HFT may have exacerbated volatility during this brief liquidity crisis, although they are not to blame for the crash itself. In contrast, the results in Brogaard (2011) do not confirm the concerns that HFT activity leads to an increase in volatility. While most empirical studies have only analyzed U.S. data (exceptions are Chaboud et al. (2009) and Hendershott and Riordan (2011a)), Boehmer et al. (2012) examine a sample of 39 exchanges around the world. In sum, they confirm the view that algorithmic trading has a positive effect on liqudity and price efficiency, but also find that it increases volatility (this effect is not due to improved price efficiency).

This paper is organized as follows. Section 2 provides an outline of the model, whose solution is presented subsequently in Section 3. An analysis of trading profits and the order flow composition follows in Section 4, while we endogenize the proportion of fast traders and examine social welfare in Section 5. The extended model with efficiency gains due to investments into trading technology is presented in Section 6. Finally, a numerical solution with normally distributed innovations is contained in Section 7, followed by the conclusion. Proofs are relegated to Appendix A, while Appendices B and C contain tables and figures for the closed-form and numerical solution, respectively.

2 The model

2.1 The limit order market

We consider an infinite-horizon[4] version of Foucault's (1999) dynamic limit order market. There is a single risky asset whose fundamental value follows a random walk, i.e.

$$v_t = v_{t-1} + \varepsilon_t$$

where the innovations can take values of $\sigma \geq 0$ and $-\sigma$ with equal probability and are independent over time. Trading takes place sequentially at time points $t = 1, 2, \ldots$ and the order size is fixed at one unit. In this model, trading arises due to differences in private values. Specifically, we assume that at time t', the reservation price of a trader arriving at time $t \leq t'$ is given by

$$R_{t'} = v_{t'} + y_t$$

which is the sum of the asset's fundamental value and the time-invariant private valuation y_t. We assume that this private valuation can take two values $y_h = +L$ and $y_l = -L$ with equal

[4] This assumption is merely for convenience as it simplifies the algebra. Foucault (1999) assumes that the terminal date is stochastic, as the trading process stops after each period with constant probability $1 - \rho > 0$. An infinite horizon may be interpreted as the limiting case where $\rho \to 1$.

probability, where $L > 0$. The y's are independent and identically distributed across traders, and moreover independent from the asset value innovations. All traders are risk-neutral and maximize their expected utility. The utility obtained by an agent purchasing or selling the asset is given by

$$U(y_t) = (v_{t'} + y_t - P_{t'})q_{t'}$$

where $t' \geq t$ denotes the time of the transaction, $P_{t'}$ is the transaction price, and $q_{t'}$ is a trade direction indicator that takes the value of $+1$ for buy transactions and -1 for sell transactions. The utility of an agent that does not trade is assumed to be equal to zero.

Trading is organized as a limit order market. Consider a buyer (i.e. an agent with private valuation y_h). Upon his arrival, he can either a) submit a market buy order or b) submit a buy limit order for one unit of the asset.[5] We assume that he decides to submit a limit order if he is indifferent between both choices. Similarly, sellers choose between market and limit sell orders. All limit orders are valid for one period, i.e. they expire unless being executed by the following agent.

Besides their private valuations agents may also differ in their type θ_t, which is determined by the trading technology available to them. For simplicity, we assume that technology is solely related to speed, such that agents are either fast traders ($\theta_t = FT$) or slow traders ($\theta_t = ST$). Let $\alpha \in [0, 1]$ denote the proportion of fast traders (henceforth FTs, slow traders are abbreviated as STs). We assume that each trader may choose to become fast at a cost c before entering the market and learning his private valuation y_t. As it is customary in the literature (see e.g. Grossman and Stiglitz (1980), we first compute the equilibrium taking α as exogenous (and independent of ε and y) and then pin down its equilibrium value by equating the utilities of fast and slow traders.

Unless limit orders are monitored perfectly (Foucault, Röell and Sandås (2003) and Liu (2009) study the cost of monitoring limit orders), the arrival of new information may render them stale and thereby grant a free option to other market participants (Copeland and Galai (1983)). Clearly, the ability to react faster than others is very valuable as it reduces this risk of being "picked off". In order to model this effect in the most parsimonious way we assume that FTs can revise (or update) their limit orders after the arrival of new information (i.e. the realization of ε_{t+1}) yet before the arrival of the next agent, but only provided he is a ST. If the next trader is a FT as well, the order cannot be revised. STs can never cancel their orders. Notice that for both $\alpha = 0$ and $\alpha = 1$ our model collapses to the model of Foucault (1999), where limit orders can never be revised once they are submitted. Hence speed only matters in relative terms, that is being fast is is only an advantage as long as there is someone else that is slow. This assumption (which will be relaxed at a later stage) is quite natural given that our focus is on the winner's curse problem in a limit order market.

[5] Foucault (1999) assumes that traders always submit a buy and sell limit order, which is without loss of generality as limit prices can always be chosen such that limit orders have a zero execution probability. In fact, in equilibrium, the ask (bid) quotes of buyers (sellers) are never executed, such that we directly assume that buyers (sellers) only submit buy (sell) limit orders.

Let $s_t = (B_t^m, A_t^m)$ denote the best bid and ask quote in the market. If there is no bid (ask) quote posted, we write $B_t^m = -\infty$ ($A_t^m = \infty$). Upon entering the market, a trader learns his type θ_t and private valuation y_t, and observes the state of the limit order book s_t as well as the current fundamental value of the asset v_t. Call $S_t = (s_t, v_t)$ the state of the market.

2.2 Payoffs

Consider a buyer that arrives at time t when the state of the market is S_t. If he chooses to submit a buy market order (which executes at the best available ask price), his payoff is equal to

$$U_{t,k}^{b,MO}(A_t^m) = v_t + L - A_t^m \qquad k \in \{ST, FT\} \tag{1}$$

Instead, he can choose to submit a buy limit order. The expected payoff of a slow buyer submitting a buy limit order with bid price $B_{t,ST}$ is given by

$$E(U_{t,ST}^{b,LO}(B_{t,ST})) = p_t^b(B_{t,ST})(v_t + E_{Ex}[\varepsilon_{t+1}] + L - B_{t,ST}) \tag{2}$$

where $p_t^b(B_{t,ST})$ denotes the execution probability of a buy limit order with bid price $B_{t,ST}$ and $E_{Ex}(\cdot)$ is an expectation conditional on the execution of the respective limit order. Given that a FT may revise his limit order in case the next arriving trader is a ST, a fast buyer that decides to use a limit order chooses three different bid prices, $(B_{t,FT}, B_{t,FT}^{+\sigma}, B_{t,FT}^{-\sigma})$), and his payoff is given by

$$\begin{aligned} E(U_{t,FT}^{b,LO}(B_{t,FT}, B_{t,FT}^{+\sigma}, B_{t,FT}^{-\sigma}) &= \alpha p_t^b(B_{t,FT}|\theta_{t+1} = 1)(v_t + E_{Ex}[\varepsilon_{t+1}|\theta_{t+1} = 1] + L - B_{t,FT}) \\ &+ (1-\alpha)\frac{1}{2}p_t^b(B_{t,FT}^{+}|\theta_{t+1} = 0, \varepsilon_{t+1} = +\sigma)(v_t + \sigma + L - B_{t,FT}^{+\sigma}) \\ &+ (1-\alpha)\frac{1}{2}p_t^b(B_{t,FT}^{-\sigma}|\theta_{t+1} = 0, \varepsilon_{t+1} = -\sigma)(v_t - \sigma + L - B_{t,FT}^{-\sigma}) \end{aligned} \tag{3}$$

where the $p_t^b(\cdot|\theta_{t+1}, \varepsilon_{t+1})$ denote execution probabilities conditional on the realization of next period's trader type and asset value innovation. Similarly, a seller submitting a sell market order obtains

$$U_{t,k}^{s,MO}(B_t^m) = B_t^m - (v_t - L) \qquad k \in \{ST, FT\} \tag{4}$$

while the expected payoffs for STs and FTs from posting sell limit orders with ask prices equal to $A_{t,ST}$ and $(A_{t,FT}, A_{t,FT}^{+\sigma}, A_{t,FT}^{-\sigma})$, respectively, are given by

$$E(U_{t,ST}^{s,LO}(A_{t,ST})) = p_t^s(A_{t,ST})(A_{t,ST} - (v_t + E_{Ex}[\varepsilon_{t+1}] - L)) \tag{5}$$

and

$$
\begin{aligned}
E(U^{s,LO}_{t,FT}(A_{t,FT}, A^{+\sigma}_{t,FT}, A^{-\sigma}_{t,FT}) = {} & \alpha p^s_t(A_{t,FT}|\theta_{t+1}=1)(A_{t,FT} - (v_t + E_{Ex}[\varepsilon_{t+1}|\theta_{t+1}=1] - L)) \quad (6) \\
& + (1-\alpha)\frac{1}{2}p^s_t(A^{+\sigma}_{t,FT}|\theta_{t+1}=0, \varepsilon_{t+1}=+\sigma)(A^{+\sigma}_{t,FT} - (v_t + \sigma - L)) \\
& + (1-\alpha)\frac{1}{2}p^s_t(A^{-\sigma}_{t,FT}|\theta_{t+1}=0, , \varepsilon_{t+1}=-\sigma)(A^{-\sigma}_{t,FT} - (v_t - \sigma - L))
\end{aligned}
$$

2.3 Equilibrium Definition

Let $B^*_{t,ST}$ denote the optimal bid price chosen by a slow buyer that decides to place at limit order at time t. Thus, upon arrival, a slow buyer chooses between a) a buy market order at ask price A^m_t and b) a buy limit order with bid price $B^*_{t,ST}$. We call his choice the slow buyer's order placement strategy $O^b_{ST}(S_t) \in \left\{b^m_t, B^*_{t,ST}\right\}$ where b^m_t denotes a market buy order at time t. Similarly, let $(B^*_{t,FT}, B^{-\sigma*}_{t,FT}, B^{+\sigma*}_{t,FT})$ be the optimal bid prices for a fast buyer that opts for limit orders when arriving at time t. He then chooses between a) a buy market order at ask price A^m_t and b) a buy limit order with bid price $B^*_{t,FT}$, which, unless the next agent is a FT, is revised to $B^{+\sigma*}_{t,AT}$ ($B^{-\sigma*}_{t,AT}$) after the arrival of positive (negative) fundamental information. Hence, $O^b_{FT}(S_t) \in \left\{b^m_t, (B^*_{t,FT}, B^{-\sigma*}_{t,FT}, B^{+\sigma*}_{t,FT})\right\}$. The choices of slow and fast sellers are completely symmetric, i.e. they choose between a) a market sell at B^m_t and b) limit sell orders with ask prices equal to $A^*_{t,ST}$ and $(A^*_{t,FT}, A^{-\sigma*}_{t,FT}, A^{+\sigma*}_{t,FT})$, respectively, such that their order placement strategies are $O^s_{ST}(S_t) \in \left\{s^m_t, A^*_{t,ST}\right\}$ and $O^s_{FT}(S_t) \in \left\{s^m_t, (A^*_{t,FT}, A^{-\sigma*}_{t,FT}, A^{+\sigma*}_{t,FT})\right\}$, where s^m_t denotes a market sell. As in Foucault (1999) and Colliard and Foucault (2012), we focus on stationary Markov-perfect equilibria, which is natural because trader's profits do not depend on the history of the game but only on the state of the market upon their arrival.

Definition 1 *A Markov-perfect equilibrium of the limit order market consists of order placement strategies $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ and $O^{s*}_{FT}(\cdot)$ such that, for each possible state of the market S_t, i) $O^{b*}_{ST}(S_t)$ ($O^{b*}_{FT}(S_t)$) maximizes the expected utility of a slow (fast) buyer arriving in state S_t if all other traders follow the strategies $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ and $O^{s*}_{FT}(\cdot)$ and ii) $O^{s*}_{ST}(S_t)$ ($O^{s*}_{FT}(S_t)$) maximizes the expected utility of a slow (fast) seller arriving in state S_t if all other traders follow the strategies $O^{b*}_{ST}(\cdot)$, $O^{s*}_{ST}(\cdot)$, $O^{b*}_{FT}(\cdot)$ and $O^{s*}_{FT}(\cdot)$.*

Foucault (1999) shows that it is possible to characterize traders' optimal decisions by means of cutoff prices that depend on a trader's private valuation and the current fundamental value of the asset. The buy (sell) cutoff price is the highest (lowest) ask (bid) price at which an arriving buyer (seller) submits a market buy (sell) order instead of a buy (sell) limit order. Let $V^{LO*}_{ST}(y_t)$ and $V^{LO*}_{FT}(y_t)$ denote equilibrium expected profits from posting limit orders for STs and FTs, respectively, that is

$$V_{ST}^{LO*}(y_t) \equiv \begin{cases} E(U_{t,ST}^{b,LO}(B_{t,ST}^*)) & if \quad y_t = y_h \\ E(U_{t,ST}^{s,LO}(A_{t,ST}^*)) & if \quad y_t = y_l \end{cases} \tag{7}$$

$$V_{FT}^{LO*}(y_t) \equiv \begin{cases} E(U_{t,FT}^{b,LO}(B_{t,FT}^*, B_{t,FT}^{-\sigma*}, B_{t,FT}^{+\sigma*})) & if \quad y_t = y_h \\ E(U_{t,FT}^{s,LO}(A_{t,FT}^*, A_{t,FT}^{-\sigma*}, A_{t,FT}^{+\sigma*})) & if \quad y_t = y_l \end{cases} \tag{8}$$

Then, the equilibrium buy and sell cutoff prices are given by

$$C_{ST}^{s*}(v_t, y_t) - (v_t + y_t) = V_{ST}^{LO*}(y_t) \tag{9}$$

$$C_{FT}^{s*}(v_t, y_t) - (v_t + y_t) = V_{FT}^{LO*}(y_t) \tag{10}$$

$$(v_t + y_t) - C_{ST}^{b*}(v_t, y_t) = V_{ST}^{LO*}(y_t) \tag{11}$$

$$(v_t + y_t) - C_{FT}^{b*}(v_t, y_t) = V_{FT}^{LO*}(y_t) \tag{12}$$

Intuitively, the limit order market can be interpreted as a sequential bargaining process, where a limit order trader makes a take-it or leave-it offer to the next arriving agent, who in turn may either accept this offer or decide to post a new offer to the following agent. Therefore, the expected profits from submitting limit orders effectively constitute an endogenous outside option (the profits when not accepting the available offer) that determine a reservation price (the cutoff price). The above system of equations can be solved for the equilibrium cutoff prices, which in turn give rise to traders' equilibrium quotation strategy.

3 Equilibrium

For ease of exposition, we will in the following always assume that the limit order trader is a "buyer" (with private valuation y_h) and the market order trader is a "seller" (with private valuation y_l). This is without loss of generality because the symmetric arguments apply to limit order sellers and market order buyers. It is easy to see that it is never optimal for a limit order buyer to target another buyer, because both agents will value the asset at $v_t + L$ when transacting at time t such that there are no gains from trade to be shared. Moreover, in understanding the construction of the equilibrium, it is crucial to notice that any optimal bid price must be such that it is marginally above some seller's cutoff price for a particular realization of the asset value innovation ε_{t+1}. Clearly, a slight increase in the bid price does not lead to a higher execution probability, while a small decrease in the bid price

leads to a strictly lower execution probability. In the following, we will abuse notation by equating equilibrium quotes to cutoff prices as in Foucault (1999), because they can be made arbitrarily close.

Lemma 1 *In equilibrium, FTs' revised bid quotes are given by*
$B^{-\sigma*}_{t,FT} = C^{s*}_{ST}(v_t - \sigma, -L) \qquad B^{+\sigma*}_{t,FT} = C^{s*}_{ST}(v_t + \sigma, -L)$

Proof. See the Appendix. ■

Due to their speed advantage, FTs may re-price their limit orders after observing the change in the fundamental value in case the next arriving agent is a ST. In essence, this enables FTs to discriminate between FTs and STs, because the former face the initial quotes while the latter face the updated quotes. Hence the updated quotes are effectively set in the knowledge of both of ε_{t+1} and θ_{t+1}, such that the optimally revised bid price is just equal to the cutoff sell price of a slow seller given the fundamental asset value v_{t+1}. Consequently, FTs' decision boils down to choosing between initial quotes (which are only aimed at FTs) with a high fill rate and a low fill rate.[6] One the other hand, STs may employ four different quotation strategies in equilibrium, which reflects that their quotes always face both FTs and STs.

Lemma 2 *Let $\alpha \in (0, 1)$. Then*

i) $C^{s*}_{FT}(v_t, -L) > C^{s*}_{ST}(v_t, -L)$
ii) $C^{s*}_{ST}(v_t, +L) > C^{s*}_{FT}(v_t, -L)$
iii) $C^{s*}_{ST}(v_t + \sigma, -L) > C^{s*}_{FT}(v_t - \sigma, -L)$

Proof. See the Appendix. ■

Unsurprisingly, fast sellers have a higher sell cutoff price because their endogenous outside option of posting limit orders is more valuable due to the ability of revising limit orders. Parts b) and c) of the Lemma state that there this advantage, which we henceforth denote $\delta^* \equiv C^{s*}_{FT}(v_t, -L) - C^{s*}_{ST}(v_t, -L)$, is naturally limited both by the gains from trade and the risk of being picked off. We are now ready to determine the equilibrium quotation strategies.

Proposition 1 *For fixed parameters (α, σ, L), there exists a unique Markov-perfect equilibrium in the limit order market. The type of equilibrium is as follows.*

Type 1: If $\alpha \leq \alpha^*_1$ and $\sigma \geq \sigma^*_1$, then

[6] A fast buyer may either post a bid price equal to $C^{s*}_{FT}(v_t - \sigma, -L)$, which only executes after a decrease in the asset value (low fill rate), or a bid price equal to $C^{s*}_{FT}(v_t + \sigma, -L)$, which may execute both after an increase or a decrease of the asset value (high fill rate).

$B^*_{t,FT} = C^{s*}_{FT}(v_t - \sigma, -L) \qquad B^*_{t,ST} = C^{s*}_{ST}(v_t - \sigma, -L)$

Type 2: If $\alpha^*_1 < \alpha \leq \alpha^*_2$ *and* $\sigma \geq \sigma^*_4$ *or* $\alpha^*_2 < \alpha$ *and* $\sigma \geq \sigma^*_5$*, then*

$B^*_{t,FT} = C^{s*}_{FT}(v_t - \sigma, -L) \qquad B^*_{t,ST} = C^{s*}_{FT}(v_t - \sigma, -L)$

Type 3: If $\alpha \leq \alpha^*_1$ *and* $\sigma^*_3 \leq \sigma < \sigma^*_1$ *or* $\alpha^*_1 < \alpha \leq \alpha^*_2$ *and* $\sigma^*_3 \leq \sigma < \sigma^4_*$*, then*

$B^*_{t,FT} = C^{s*}_{FT}(v_t - \sigma, -L) \qquad B^*_{t,ST} = C^{s*}_{ST}(v_t + \sigma, -L)$

Type 4: If $\alpha \leq \alpha^*_2$ *and* $\sigma^*_2 \leq \sigma < \sigma^*_3$*, then*

$B^*_{t,FT} = C^{s*}_{FT}(v_t + \sigma, -L) \qquad B^*_{t,ST} = C^{s*}_{ST}(v_t + \sigma, -L)$

Type 5: If $\alpha \leq \alpha^*_2$ *and* $\sigma < \sigma^*_2$ *or* $\alpha^*_2 < \alpha$ *and* $\sigma < \sigma^*_5$*, then*

$B^*_{t,FT} = C^{s*}_{FT}(v_t + \sigma, -L) \qquad B^*_{t,ST} = C^{s*}_{FT}(v_t + \sigma, -L)$

The variables α^*_1*,* α^*_2*,* σ^*_1*,* σ^*_2*,* σ^*_3*,* σ^*_4*,* σ^*_5 *are defined in the Appendix.*

Proof. See the Appendix. ■

In the following, we briefly describe the different equilibria, focusing on the type of limit orders chosen by slow and fast limit order sellers. Figure 1 in Appendix B graphically depicts the regions in the (α, σ)-space that correspond to the different equilibria, where we have set $L = 1$ (this is without loss of generality as only the ratio of σ and L is relevant).

Type 1: In the type-1 equilibrium, the limit orders posted by slow buyers are only executed by slow sellers in the case of a decrease in the fundamental asset value. The execution probability of such of such an order is $(1 - \alpha)/4$. Fast buyers follow a low fill rate strategy, i.e. they initially set the bid price slightly above $C^{s*}_{FT}(v_t - \sigma, -L)$, such that the order is executed if the next agent is a fast seller and the asset value has decreased. If the next agent turns out to be a ST, the FT is able to revise the bid price according to the latest realization in the asset value process and posts a bid price that is equal to a slow seller's cutoff price, i.e. either $C^{s*}_{ST}(v_t - \sigma, -L)$ or $C^{s*}_{ST}(v_t + \sigma, -L)$ depending on the realization of ε_{t+1} (see Lemma 1). The execution probability of this quotation strategy is $(2 - \alpha)/4$.

Type 2: In this type of equilibrium, slow buyers that post limit orders choose the bid price such that the order is executed if the next agent is either a slow or a fast seller and the asset value has decreased, such that these orders have an execution probability of 1/4. Fast buyers behave as in the type-1 equilibrium.

Type 3: Here, slow buyers submit buy limit orders with a bid price slightly above the sell cutoff price of a slow seller after an asset value increase, that is $B^*_{t,ST} = C^{s*}_{ST}(v_t + \sigma, -L)$. Such orders are executed if a) the next arriving agent is slow seller independently of the asset value innovation, or b)

the next trader is a fast seller and the asset value has decreased, and their equilibrium probability of execution is equal to $(2-\alpha)/4$. Fast buyers behave as in the type-1 equilibrium.

Type 4: In the type-4 equilibrium, slow buyers behave as in the type-3 equilibrium. Fast buyers employ a high fill rate strategy, i.e. they initially post a bid price slightly above $C^{s*}_{FT}(v_t+\sigma,-L)$, and then revise it according to Lemma 1 if the next agent turns out to be a ST. The execution probability of this quotation strategy is equal to 1/2.

Type 5: Slow buyers that opt for a limit order choose a bid price such that the order is executed if the next agent is a seller, such that their orders have an execution probability of 1/2. Fast buyers behave as in a type-4 equilibrium.

The effect of FTs speed advantage on limit order profits is understood best by looking at the type-3 equilibrium, where the quotation strategies of both types of agents have the same execution probability of $(2-\alpha)/4$. Nevertheless, STs' limit orders always execute at $B^*_{t,ST} = C^{s*}_{ST}(v_t+\sigma,-L)$, while FTs' limit orders may also execute at $C^{s*}_{ST}(v_t-\sigma,-L) < B^*_{t,ST}$ or $C^{s*}_{FT}(v_t-\sigma,-L) < B^*_{t,ST}$, which implies higher expected profits due to the avoidance of picking-off risks and the ability to price discriminate between FTs and STs.

The main goal of our analysis is to examine how trading technology affects equilibrium outcomes such as the composition of order flow, trading profits, and welfare. This basically amounts to comparing the outcome for some level of $\alpha \in (0,1)$ with the case where all traders are slow. Unfortunately, the fact that ε is assumed to be both discrete and bounded (which is necessary to obtain a closed-form solution) complicates this endeavour slightly as it leads to bang-bang solutions where a slight change in parameters can lead to a jump from one extreme (low trade) outcome to another (high trade). Hence we will frequently make the following assumption on σ.

Assumption 1 $\sigma \in \Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1 = [0, \sigma^*_2(\alpha^*_2))$ and $\Sigma_2 = [\sigma^*_1(\alpha^*_1), \infty)$.

Intuitively, this assumption means that that σ is not "too close" to the tipping point $\sigma^*_1(0)$ of the original Foucault (1999) model (at this point, agents are indifferent between following a low fill rate and a high fille rate quotation strategy with execution probabilities of 1/4 and 1/2, respectively), which allows us to reap the benefits of having a closed-form solution while minimizing the potential distortions due to bang-bang type solutions. The range for values of σ that is excluded under Assumption 1 is indicated by the dashed lines in Figure 1 (Appendix B). In Section 7 we provide the results of a numerical solution where ε follows a normal distribution. The obtained results are virtually identical to those derived in closed form under the above assumption.

In the following sections, we will frequently provide plots of equilibrium outcomes as functions of α for the purpose of illustration. Notice that for a fixed level of σ, different values of α may give rise to different equilibria and for $\sigma \in \Sigma_1$ the exact values of α for which we move from one equilibrium

to another additionally depend on σ itself. For $0 \leq \sigma < \sigma_3^*(0)$, we may end up in a type-4 or a type-5 equilibrium, and for $\sigma_3^*(0) \leq \sigma < \sigma_2^*(\alpha_2^*)$ a type-3, type-4 or a type-5 equilibrium may arise. Therefore, we choose $\hat{\sigma}_1 = \sigma_3^*(0)/2$ and $\hat{\sigma}_2 = (\sigma_2^*(\alpha_2^*) - \sigma_3^*(0))/2$ as representative levels of σ for illustration purposes. Because for $\sigma \in \Sigma_2$ the equilibrium outcomes do not depend on σ we simply set $\hat{\sigma}_3 = 1$.

4 Trading profits and the composition of order flow

4.1 Limit order Profits

By definition, the equilibrium cutoff prices are the endogenous reservation prices that represent the outside option of posting limit orders. While Lemma 2 implies that $V_{FT}^{LO*} > V_{ST}^{LO*}$ (suppressing the y_t), we can make a stronger statement by relating each trader type's limit order profits to those obtained by agents in the absence of fast traders, V_0^{LO*}.

Proposition 2 *Let $\alpha \in (0,1)$. Then, $V_{FT}^{LO*} > V_0^{LO*} > V_{ST}^{LO*}$.*

Proof. See the Appendix. ■

In any equilibrium with a positive fraction of FTs, STs' expected profits from posting limit orders are lower than in the absence of FTs. Intuitively, there are two ways in which STs can respond to the arrival of FTs with a higher outside option ($\delta^* > 0$). They can either *i*) increase the aggressiveness of their limit orders or *ii*) incur a decreased order execution probability (i.e. they decide to post quotes that are not (always) executed by FTs). It is immediate that the first reaction always harms STs expected profits from limit orders, as they simply offer better quotes, but the execution probability of these orders is as in the case where $\alpha = 0$. On the other hand, choosing *ii*) allows STs to post less aggressive quotes than in the absence of FTs, as the outside option of other STs has suffered ($V_{ST}^{LO*} < V_0^{LO*}$). Nevertheless, the effect of a reduced execution probability dominates, such that their expected profits are also lower in this case. For illustration, Figure 2 in Appendix B depicts V_{FT}^{LO*},V_{ST}^{LO*} and δ^* as functions of α for $\sigma = \hat{\sigma}_l$, where $l = 1, 2, 3$.

4.2 Order flow composition

We next turn to the analysis of the order flow composition, i.e. the equilibrium mix between limit and market orders. On the equilibrium path, four events (actions) are possible. The arriving agent can be 1) a ST submitting a limit order, 2) a ST submitting a market order, 3) a FT submitting a limit

order, or 4) a FT submitting a market order. Now let $\varphi^i = (\varphi_1^i, \varphi_2^i, \varphi_3^i, \varphi_4^i)$ denote the stationary probability distribution of the above market events in a type-i equilibrium, where $i \in \{1, 2, 3, 4, 5\}$. As in Colliard and Foucault (2012), we call the probability of a randomly arriving investor submitting a market (limit) order the trading (make) rate, which are given by

$$TR^i = \varphi_2^i + \varphi_4^i \tag{13}$$

$$MR^i = \varphi_1^i + \varphi_3^i = 1 - TR^i \tag{14}$$

Similarly, we can calculate the trading and make rates for ST and FT separately as

$$TR_{ST}^i = \varphi_2^i/(\varphi_1^i + \varphi_2^i) \tag{15}$$

$$MR_{ST}^i = \varphi_1^i/(\varphi_1^i + \varphi_2^i) = 1 - TR_{ST}^i \tag{16}$$

$$TR_{FT}^i = \varphi_4^i/(\varphi_3^i + \varphi_4^i) \tag{17}$$

$$MR_{FT}^i = \varphi_4^i/(\varphi_3^i + \varphi_4^i) = 1 - TR_{FT}^i \tag{18}$$

Let TR^*, TR_{ST}^* and TR_{FT}^* denote the equilibrium trading rates (e.g. $TR^* = TR^1$ if $\alpha \leq \alpha_1^*$ and $\sigma \geq \sigma_1^*$), and let TR_0^* be the equilibrium trading rate when all traders are slow.

Proposition 3 *$TR_{ST}^* \geq TR^* \geq TR_{FT}^*$ for all $\alpha \in (0, 1)$. Now assume that $\sigma \in \Sigma$.*

i) If $\sigma \in \Sigma_2$, then $TR_{ST}^* > TR^* > TR_0^* > TR_{FT}^*$ for all $\alpha \in (0, 1)$.

ii) If $\sigma \in \Sigma_1$, then $TR_{ST}^* \geq TR_0^* \geq TR^* \geq TR_{FT}^*$ for all $\alpha \in (0, 1)$.

Proof. See the Appendix. ■

In equilibrium, STs are more likely to use market orders than FTs because they are more likely to encounter a quote they find worth accepting when arriving at the market. This follows directly from FTs' higher endogenous outside option, which may diminish the aggressiveness of STs limit order such that they sometimes only "target" STs for a given realization of the asset value innovation.

To understand the results on the overall trading rate, it is important to understand that the introduction of FTs has two opposing effects. First, FTs' speed advantage reduces their risk of being picked off and thereby the need for order shading, which leads to an increase in trading volume. Second, it may be optimal for STs to post less aggressive limit orders because FTs advantage δ^* is too high, which implies more order shading and therefore implies a decrease in trading volume.

If $\sigma \in \Sigma_2$, the first effect dominates. Absent FTs ($\alpha = 0$), slow buyers submit limit orders that only execute in case the asset value decreases (increases), i.e. trading volume is low because the risk of being picked off is too severe. Now, the introduction of FTs leads to more trade as their ability of updating quotes quickly after the arrival of new information allows transactions to occur after

price increases as well. On the other hand, the first effect is absent for $\sigma \in \Sigma_1$, because there is no order shading when $\alpha = 0$ (because ε is bounded, it is possible to set e.g. a bid price that any seller will find worth executing, independent of the realization ε). Hence the presence of FTs leads to a decrease in trading volume because it is optimal for STs to post less aggressive limit orders for a low level of α. Once the proportion of FTs is sufficiently high, this effect also disappears, and the level of trading volume is as in the slow market.

Our results regarding trading volume are very similar to those found in Jovanovic and Menkveld (2010) in the sense that the introduction of FTs may lead both to an increase or a decrease in trading volume. In fact, both our and their paper suggest that the increased speed of FTs may reduce order shading because their quotes can reflect new fundamental information instantaneously. Nevertheless, the reason for a possibly lower trading activity in their paper (unawareness of hard information by STs when submitting market orders) differs considerably from the mechanism at work in this model (a higher endogenous outside option of FTs induces STs to post less aggressive orders). Other literature rather *assumes* that the presence of FTs increases trading volume, either via intermediation (Cartea and Penalva (2011) or due to more effective search for trading opportunities (Biais et al. (2012)).

For illustration, Figure 3 in Appendix B depicts TR^*_{ST}, TR^*_{FT} and TR^* as functions of α for $\sigma = \hat{\sigma}_l$, where $l = 1, 2, 3$.

4.3 Trading costs

Let V_k^{MO*} denote the equilibrium expected profit from posting a market order for a trader with type $k \in \{ST, FT\}$. We can then write

$$V_k^{MO*} = L - E(\tau_k^*)$$

where $E(\tau_k^*)$ is the expected trading cost in equilibrium, i.e. the average premium paid above the assets true value. For example, a seller's utility derived from submitting a market order is given by equation (4), such that his trading cost is equal to

$$\tau_t = v_t - B_t^m$$

It is important to note that the best available bid crucially depends on the type of both the market order trader and the limit order trader whose quote is executed. This is due to the assumption that FTs may revise their quotes, but only in the case the agent arriving after him is a ST. Additionally, the trading cost also depends on the most recent realization of the fundamental asset value due to the picking off risk faced by the limit order trader. Let $\tau_{t,j,k}$ denote the trading cost for a type-j seller that arrives at time t and submits a sell market order that executes against a buy limit order

posted by a type-k buyer at time $t-1$, where $j, k \in \{ST, FT\}$. In particular, consider a slow seller who arrives at time t and submits a sell market order that executes against the best available bid. If the bid stems from a ST, the trading cost is given by

$$\tau^{+\sigma}_{t,ST,ST} = v_{t-1} + \sigma - B_{t-1,ST} \tag{19}$$

for $\varepsilon_t = +\sigma$ and

$$\tau^{-\sigma}_{t,ST,ST} = v_{t-1} - \sigma - B_{t-1,ST} \tag{20}$$

for $\varepsilon_t = -\sigma$. In case the best available bid was posted by a fast buyer (and therefore was revised before the arrival of the market order trader), the trading cost is given by

$$\tau^{+\sigma}_{t,ST,FT} = v_{t-1} + \sigma - B^{+\sigma}_{t-1,FT} \tag{21}$$

for $\varepsilon_t = +\sigma$ and

$$\tau^{-\sigma}_{t,ST,FT} = v_{t-1} - \sigma - B^{-\sigma}_{t-1,FT} \tag{22}$$

for $\varepsilon_t = -\sigma$. In order to calculate the expected trading costs, we simply have to weight the trading costs for each possible event by its stationary probability. Let $\pi^{+\sigma*}_{j,k}(\pi^{-\sigma*}_{j,k})$ denote the equilibrium probability that the asset value increases (decreases) and subsequently a buy limit order posted by a type-k trader is executed by a sell market order from a type-j trader, where $j, k \in \{ST, FT\}$. Then the equilibrium expected trading cost of the slow seller is given by[7]

$$E(\tau^*_{ST}) = \frac{\varphi^*_1(\pi^{+\sigma*}_{ST,ST}\tau^{+\sigma*}_{ST,ST} + \pi^{-\sigma*}_{ST,ST}\tau^{-\sigma*}_{ST,ST}) + \varphi^*_3(\pi^{+\sigma*}_{ST,FT}\tau^{+\sigma*}_{ST,FT} + \pi^{-\sigma*}_{ST,FT}\tau^{-\sigma*}_{ST,FT})}{\varphi^*_1(\pi^{+\sigma*}_{ST,ST} + \pi^{-\sigma*}_{ST,ST}) + \varphi^*_3(\pi^{+\sigma*}_{ST,FT} + \pi^{-\sigma*}_{ST,FT})} \tag{23}$$

Following exactly the same logic, the expressions for the trading costs of fast sellers are given by

$$\tau^{+\sigma}_{t,FT,ST} = v_{t-1} + \sigma - B_{t-1,ST} \tag{24}$$

$$\tau^{-\sigma}_{t,FT,ST} = v_{t-1} - \sigma - B_{t-1,ST} \tag{25}$$

$$\tau^{+\sigma}_{t,FT,FT} = v_{t-1} + \sigma - B_{t-1,FT} \tag{26}$$

$$\tau^{-\sigma}_{t,FT,FT} = v_{t-1} - \sigma - B_{t-1,FT} \tag{27}$$

and their equilibrium expected trading costs are given by

$$E(\tau^*_{FT}) = \frac{\varphi^*_1(\pi^{+\sigma*}_{FT,ST}\tau^{+\sigma*}_{FT,ST} + \pi^{-\sigma*}_{FT,ST}\tau^{-\sigma*}_{FT,ST}) + \varphi^*_3(\pi^{+\sigma*}_{FT,FT}\tau^{+\sigma*}_{FT,FT} + \pi^{-\sigma*}_{FT,FT}\tau^{-\sigma*}_{FT,FT})}{\varphi^*_1(\pi^{+\sigma*}_{FT,ST} + \pi^{-\sigma*}_{FT,ST}) + \varphi^*_3(\pi^{+\sigma*}_{FT,FT} + \pi^{-\sigma*}_{FT,FT})} \tag{28}$$

Then the *average* expected trading cost in equilibrium is given by

$$E(\tau^*) = \frac{TR^*_{ST}}{TR^*_{ST} + TR^*_{FT}} E(\tau^*_{ST}) + \frac{TR^*_{FT}}{TR^*_{ST} + TR^*_{FT}} E(\tau^*_{FT}) \tag{29}$$

Defining $E(\tau^*_0)$ as the equilibrium average expected trading cost in the slow market, we obtain the following.

[7] Notice that we may omit the time subscripts due to stationarity.

Proposition 4 *$E(\tau^*_{ST}) > E(\tau^*) > E(\tau^*_{FT})$ for all $\alpha \in (0,1)$. Now assume that $\sigma \in \Sigma$. Then*

i) $E(\tau^*_{FT}) < E(\tau^*_0)$ for all $\alpha \in (0,1)$.

ii) For every σ, there exists some α^1_τ such that $E(\tau^*_{ST}) > E(\tau^*) > E(\tau^*_0)$ for all $\alpha < \alpha^1_\tau$.

iii) For every $\sigma \in \Sigma_2$ and every $\sigma < \sigma^*_2(1/4)$, there exists some interval $(\alpha^2_\tau, \alpha^3_\tau)$ such that $E(\tau^*) < E(\tau^*_{ST}) < E(\tau^*_0)$ for all $\alpha \in (\alpha^2_\tau, \alpha^3_\tau)$.

Proof. See the Appendix. ■

The intuition behind the fact that FTs enjoy lower expected trading costs than STs is straightforward. Because they are slow, STs miss profits due to picking off stale limit orders, as some of them stem from fast traders and therefore have been set in the knowledge of the latest asset value innovation. Additionally, this effect is amplified by the fact that FTs quotation strategies discriminate between FTs and STs such that the latter do not profit from more attractive quotes targeted at FTs (who have a higher cutoff sell price).

There is ample empirical evidence that supports the view that being slow leads to trading at less favourable prices. Garvey and Wu (2010) document that geographical distance to the market center is negatively related to execution speed and positively related to transactions costs. Hasbrouck & Saar (2009) show that the "lifetimes" of limit orders have decreased considerably over the last years, suggesting that a large proportion of quotes are in fact not accessible for slower market participants. Hendershott and Riordan (2011a) study algorithmic trading on the German Stock Exchange and find that "algorithmic traders consume liquidity when it is cheap", i.e. they pay lower effective spreads than human traders. Consistent with this, Moallemi and Saĝlam (2011) develop a model of the "cost of latency" and estimate that it has increased threefold in the period 1995-2005. In fact, some exchanges have effectively supported implicit price discrimination via speed by introducing so-called "flash orders" (see Skjeltorp et al. (2011) for details), a practice that has been banned in the meantime.

While the fact that $E(\tau^*_{FT}) < E(\tau^*_0)$ is not very surprising (FTs large outside option implies that attractive quotes are needed to convince them to submit market orders), the relationship between the presence of FTs and $E(\tau^*_{ST})$ (and hence also $E(\tau^*)$) is somewhat more complex. For simplicity, consider the case where $\sigma \in \Sigma_2$, such that only a type-1 or a type-2 equilibrium may arise. If $\alpha = 0$, the equilibrium bid price is set such that a buy limit order is only executed in case the next agent is a seller *and* the asset value has decreased in between trader arrivals. In other words, the risk of being picked off is sufficiently high to induce order shading by limit order traders. Now consider what happens if we introduce a small proportion of FTs ($\alpha \leq \alpha^*_1$). Given that α is small, it is optimal for STs to submit buy limit orders that are only executed by other STs, but not by FTs (δ^* is relatively large for small α as FTs' limit orders can only be picked off by other FTs). Because $V^{LO*}_{ST} < V^{LO*}_0$ (see Proposition 2), STs post lower bid prices as in the absence of FTs and thus $E(\tau^*_{ST}) > E(\tau^*_0)$.

Now consider what happens if we increase α further. Once we have $\alpha > \alpha_1^*$, it becomes optimal for STs to post buy limit orders that are also executed by fast sellers in the case of a price decrease. This happens for two reasons. First, a higher level of α leads to a considerable increase in the execution probability of limit orders if they are also targeted at FTs. Second, FTs exert a negative externality on each others limit order profits by increasing the picking off risk (note that FTs may not cancel their limit orders if the next trader is also a FT), such that it becomes "cheaper" for STs to target them. But because STs now post more aggressive limit orders, the expected trading costs of other STs decrease below $E(\tau_0^*)$, as we have $V_{FT}^{LO*} > V_0^{LO*}$ for all $\alpha \in (0,1)$. As α approaches unity, this effect diminishes as $V_{FT}^{LO*} \to V_0^{LO*}$ and because STs are less and less likely to find a quote submitted by another ST (FTs can discriminate between STs and FTs, offering worse quotes to the former). It is important to notice that FTs do not contribute to an eventual decrease in STs expected trading costs.

In principle, this intuition also applies to the case where $\sigma \in \Sigma_1$. Nevertheless, in this case there is no order shading without FTs (i.e. limit orders have an execution probability of 1/2) and therefore market orders may benefit if the asset value moves against the limit order trader. Because they are slow, STs no longer obtain these additional profits when facing limit orders from FTs because those are already incorporating the latest asset value innovation. Therefore, the more aggressive quoting behaviour by STs for intermediate values of α will only lead to $E(\tau_{ST}^*) < E(\tau_0^*)$ as long as σ is sufficiently low ($\sigma < \sigma_2^*(1/4)$). Otherwise the reduction in picking off profits will dominate and trading costs are higher than absent FTs.

For illustration, Figure 4 in Appendix B depicts $E(\tau_{ST}^*)$, $E(\tau_{FT}^*)$ and $E(\tau^*)$ as functions of α for $\sigma = \hat{\sigma}_l$, where $l = 1, 2, 3$.

While we find that expected trading costs may both increase or decrease in the presence of fast traders, the empirical evidence suggests a rather one-sided picture. Several studies (see e.g. Hendershott et al. (2011), Boehmer et al. (2012), and Hasbrouck and Saar (2012)) find that algorithmic trading is associated with an improvement in market liquidity in the sense of lower quoted and the effective spreads. Some of these studies also suggest that the effect is causal. While it is possible to reconcile these findings with our model, some caution is warranted. Most importantly, our measure of trading costs merely reflects the distribution of bargaining power between limit and market order traders, which is not necessarily in line with the more traditional notion of the "cost of immediacy" payable to an intermediary. In fact, trading costs can be negative in our model (as e.g. in Goettler et al. (2009)) because we may have $B_t^m > v_t$, while most empirical measures assume that the midquote is equal to the true value and hence $B_t^m < v_t$ by assumption.

5 Equilibrium investment in trading technology and social welfare

We next turn to the equilibrium determination of the proportion of fast traders and the analysis of social welfare. As mentioned in Section 2, we assume that all agents are born slow but may choose to become fast prior to the trading game at a cost $c > 0$. Now let $W^*_{ST}(\alpha)$ and $W^*_{FT}(\alpha)$ denote the equilibrium expected trading profits for slow and fast traders respectively (given some proportion α).[8] Then, an interior equilibrium requires that the profits of slow traders are equal to those of fast traders net of the cost for becoming fast, that is

$$\Delta W^*(\alpha^*) \equiv W^*_{FT}(\alpha^*) - W^*_{ST}(\alpha^*) = c \tag{30}$$

On the other hand, corner equilibria may arise either because the cost c is prohibitively high, i.e.

$$\max_{\alpha} \Delta W^*(\alpha) < c \tag{31}$$

in which case all agents choose to remain slow, $\alpha^* = 0$, or because the incremental benefit of being fast is high enough to justify the cost for any level of α (hence $\alpha^* = 1$), which is the case iff

$$\min_{\alpha} \Delta W^*(\alpha) > c \tag{32}$$

Equipped with the results from the previous sections, it is straightforward to calculate the expected equilibrium trading profits for both types of agents as

$$W^*_{ST} = \frac{\varphi^*_1}{\varphi^*_1 + \varphi^*_2} V^{LO*}_{ST} + \frac{\varphi^*_2}{\varphi^*_1 + \varphi^*_2} V^{MO*}_{ST} \tag{33}$$

$$W^*_{FT} = \frac{\varphi^*_3}{\varphi^*_3 + \varphi^*_4} V^{LO*}_{FT} + \frac{\varphi^*_4}{\varphi^*_3 + \varphi^*_4} V^{MO*}_{FT} \tag{34}$$

Then, emphasizing the dependence of all equilibrium outcomes on α, equilibrium social welfare is given by

$$\begin{aligned} W^*(\alpha^*) &= (1-\alpha^*) W^*_{ST}(\alpha^*) + \alpha^* (W^*_{FT}(\alpha^*) - c) \\ &= TR^*(\alpha^*) \times (2L) - \alpha^* c \end{aligned} \tag{35}$$

In Figure 5 (Appendix B) we plot $W^*_{ST}(\alpha)$, $W^*_{FT}(\alpha)$ and $\Delta W^*(\alpha)$ for $\sigma = \hat{\sigma}_l$, where $l = 1, 2, 3$. Note that an equilibrium may fail to exist because $\Delta W^*(\alpha)$ is not continuous in α. Furthermore, an interior equilibrium need not be unique because there may be more than one value of α that satisfies $\Delta W^*(\alpha) = c$ for fixed (σ, L, c). In any case, given our previous results we can conclude the following.

[8] Notice that the results from the previous sections trivially imply that $\Delta W^*(\alpha) > 0$ for all α.

Proposition 5 *Suppose that the parameters* (σ, L, c) *are such that an equilibrium with endogenous investment in trading technology exists and* $\sigma \in \Sigma$*. Then any positive equilibrium level of investment* $\alpha^* > 0$ *exceeds the socially optimal level of investment* α^+ *and moreover yields a social welfare loss, that is we have* $W^*(\alpha^*) < W^*(0)$*.*

Proof. See the Appendix. ■

From a social welfare point of view it is clearly inefficient that all agents become fast because $W^*(1) = W^*(0) - c < W^*(0)$. Moreover, it directly follows from Proposition 3 that an interior equilibrium can never be efficient for $\sigma \in \Sigma_1$ because in this case $TR^* < TR_0^*$. It turns out that although $TR^* > TR_0^*$ for $\sigma \in \Sigma_2$, the potential increase in gains from trade is never sufficient to cover the social cost of investment in technology. This occurs because fast agents do not internalize their negative impact on slow traders, such that given the socially optimal investment level α^+ there are still incentives for slow agents to become fast. While this negative externality is very similar in spirit to that in Biais et al. (2012), it is based on considerably weaker assumptions. Moreover, as we will see in the next section, a simple extension may reverse this result.

6 Technology with efficiency gains

In this section we entertain the possibility that an investment in technology does not only give agents an edge over others but also contributes to a more efficient trading process. To this end we generalize our framework and suppose that slow traders are a potential source of inefficiency. In particular, we assume that if a slow seller arrives and the limit order book contains an order that he would find worthwile executing (i.e. he finds a bid price that exceeds his sell cutoff price), he misses this trading opportunity with probability $(1 - \eta)$, where $\eta \in (0, 1]$ is the efficiency of the "slow" market. Because fast traders never miss a limit order, an increase in α improves the average efficiency of traders $\bar{\eta} = (1 - \alpha)\eta + \alpha$. While our assumption is very stilized, it is quite realistic. For example, automation may help reaping gains from trade by facilitating the consolidation of dispersed trading information in a fragmented market such as the US or European equity markets. Although the algebra is quite cumbersome, it is straightforward to compute the equilibrium of the extended model because it suffices to adjust the execution probabilities accordingly. For example, the expected utilities of slow and fast limit order buyers (equations (2) and (3)) now become

$$E(U_{t,ST}^{b,LO}(B_{t,ST})) = \bar{\eta} p_t^b(B_{t,ST})(v_t + E_{Ex}[\varepsilon_{t+1}] + L - B_{t,ST}) \tag{36}$$

and

$$
\begin{aligned}
E(U^{b,LO}_{t,FT}(B_{t,FT}, B^{+}_{t,FT}) = {} & \alpha p^b_t(B_{t,FT}|\theta_{t+1}=1)(v_t + E_{Ex}[\varepsilon_{t+1}|\theta_{t+1}=1] + L - B_{t,FT}) \\
& + (1-\alpha)\eta\frac{1}{2}p^b_t(B^{+}_{t,FT}|\theta_{t+1}=0, \varepsilon_{t+1}=+\sigma)(v_t + \sigma + L - B^{+\sigma}_{t,FT}) \\
& + (1-\alpha)\eta\frac{1}{2}p^b_t(B^{-\sigma}_{t,FT}|\theta_{t+1}=0, \varepsilon_{t+1}=-\sigma)(v_t - \sigma + L - B^{-\sigma}_{t,FT}) \qquad (37)
\end{aligned}
$$

Proposition 6 *For fixed parameters $(\alpha, \sigma, L, \eta)$, there exists a unique Markov-perfect equilibrium in the limit order market.*

Proof. See the Appendix. ■

It is important to notice that we obtain the same equilibria as before (in terms of quotation strategies), just with different cutoff prices and execution probabilities. Figure 6 in Appendix B plots the partitioning of the (α, σ)-space for different values of η (we set $L = 1$ as before). Notice that as that η decreases, the range that gives rise to the type-1, type-3 and type-4 equilibria shrinks. This is quite intuitive, because it becomes less and less attractive to only target slow traders as they are the source of the inefficiency.

Given the equilibrium cutoff prices and quotation strategies, it is straightforward to compute the equilibrium order flow composition, trading costs, and welfare. It is easy to see that the presence of fast traders may now potentially improve social welfare if the "slow" market is sufficiently inefficient. While slow traders are still negatively afffected by FTs' better outside option, this effect can be overcompensated by an increase in the execution probability of their limit orders (the lower η, the larger this effect), such that we may have $V^{LO*}_{ST} > V^{LO*}_{0}$. Then, increased limit order profits for STs additionally imply an increased "bargaining power" for market orders and therefore also lower trading costs $E(\tau^*_{ST}) < E(\tau^*_0)$. Hence we have the following.

Proposition 7 *Fix (σ, L) and some positive level of equilibrium investment $\alpha^* > 0$. Then, there exist parameters (η, c) with $\eta < 1$ such that $W^*(\alpha^*) > W^*(0)$.*

Proof. See the Appendix. ■

The proposition states that we can always find a sufficiently low level of η such that the equilibrium investment in technology yields an increase in social welfare (naturally we require that c is such that this equilibrium exists). Notice that we cannot make the stronger statement that there is some level $\tilde{\eta}$ such that any equilibrium with endogenous investment will always result in a social welfare gain for all $\eta < \tilde{\eta}$. This is owed to our assumption that ε is bounded. To see this, let $\sigma \in \Sigma_2$, such that there always exists some small α ($< \alpha^*_1$) that gives rise to a type-1 equilibrium. In this type of equilibrium it is optimal for STs to submit limit orders that are never executed by FTs, which

implies that STs do not benefit from the efficiency gains and suffer a welfare loss as $V_{ST}^{LO*} < V_0^{LO*}$ and $E(\tau_{ST}^*) > E(\tau_0^*)$. As η converges to zero, this effect disappears because $\alpha_1^* \to 0$. Figure 7 in Appendix B plots W_{ST}^*, W_{FT}^* and $\Delta W^*(\alpha)$ for different values of η and $\sigma = \hat{\sigma}_l$, where $l = 1, 3$.[9]

Because the presence of fast traders increases the efficiency of the trading process via allowing more gains from trade to be realized, investment in technology may now in fact exhibit positive externalities. This implies that the result from the previous section may reverse and we can obtain *underinvestment* in equilibrium. To see this, consider the following numerical example. Assume that $\sigma = L = 1$ and $c = 0.1$. Then the following table provides the (unique) equilibrium level of investment α^* and the optimal level of investment α^+ together with the associated levels of social welfare. For comparison, we also provide social welfare for $\alpha = 0$. Notice that under the chosen parameter configuration there always exists some positive level of α that leads to an increase in social welfare because we have $TR^* > TR_0^*$.

η	α^*	α^+	$W^*(\alpha^*)$	$W^*(\alpha^+)$	$W^*(0)$
1.00	0.759	0.347	0.383	0.446	0.400
0.75	0.691	0.507	0.358	0.407	0.316
0.50	0.549	0.616	0.311	0.391	0.222
0.25	0.125	0.683	0.157	0.381	0.118

It is easy to see that we move from excessive investment ($\alpha^* > \alpha^+$) to underinvestment ($\alpha^* < \alpha^+$) as η decreases. Although Biais et al. (2012) also assume that investment in trading technology may bring efficiency gains, their setting does not allow for positive externalities because they consider a dealer market and assume that fast traders are more likely to "find" a dealer (i.e. trading opportunity). This difference suggests that the trading protocol may have an important impact on the way technology affects welfare. Our results are consistent with the observation that algorithmic trading is particularly prevalent in those markets where the limit order market is the dominant trading protocol.

In conclusion, the overall effect of trading technology on welfare crucially depends on whether its sole purpose is to outpace other market participants or instead also contributes to an improvement in the effectiveness of the marketplace. In general, the equilibrium level of investment will differ from the social optimum, such there is room for policy intervention such as taxes (for overinvestment) or subsidies (for underinvestment) on expenditures for trading technology.

[9] The reason why we do not plot the graph for $\sigma = \hat{\sigma}_2$ is that this parameter value does not give rise to the same equilibrium combinations for all η (which is the case for $\hat{\sigma}_1$ and $\hat{\sigma}_3$, see Figure 6). Recall from Section 3 that the $\hat{\sigma}_l$ were chosen under Assumption 1, whose purpose is to avoid a "jump" from a low-trade to a high-trade equilibrium. Notice that one would have to adjust Assumption 1 for $\eta < 1$ because the parameter regions that give rise to a specific equilibrium now also depend on η.

7 Robustness: Normally distributed innovations

While assuming that $\varepsilon \in \{\sigma, -\sigma\}$ has allowed us to solve for the equilibrium cutoff prices and quotation strategies in closed form, the resulting bang-bang type solutions complicate the analysis of equilibrium outcomes as small variations in α may induce a jump from on extreme strategy to another. Also, the fact that ε is bounded means that in some occasions limit orders cannot be stale as they already condition on a "worst case scenario". In order to minimize the complications arising from discontinuities we have frequently evoked Assumption 1 and effectively excluded a specific parameter range for σ. In order to verify that this strategy has achieved the desired effect without leading us to draw wrong or incomplete conclusions, we here alternatively assume that ε follows a normal distribution with zero mean and standard deviation σ. Let $F_\varepsilon(\cdot)$ denote the associated distribution function.

Now suppose the equilibrium quotation strategy of a slow buyer arriving at time t is to post a bid price of $B^*_{t,ST}$. Given the definition of a cutoff sell price, a slow seller that arrives one period later will opt for a market order iff $B^*_{t,ST} \geq C^{s*}_{ST}(v_{t+1}, -L)$. Given our distributional assumption regarding ε_{t+1}, there exists some ε^*_{SS} such that $B^*_{t,ST} = C^{s*}_{ST}(v_t + \varepsilon^*_{SS}, -L)$, which implies that $p^b_t(B^*_{t,ST}|\theta_{t+1} = 0) = F_\varepsilon(\varepsilon^*_{SS})/2$. Similarly, there exists some ε^*_{SF} such that $B^*_{t,ST} = C^{s*}_{FT}(v_t + \varepsilon^*_{SF}, -L)$ and hence $p^b_t(B^*_{t,ST}|\theta_{t+1} = 1) = F_\varepsilon(\varepsilon^*_{SF})/2$. It is easy to see that we then must have $\delta^* = \varepsilon^*_{SS} - \varepsilon^*_{SF} > 0$ (notice that parts i) and ii) of Lemma 2 are still valid). Symmetry between buyers and sellers implies that a slow trader's expected limit order profits in equilibrium satisfy

$$\begin{aligned} C^{s*}_{ST}(v_t, -L) - (v - L) = {} & \frac{(1-\alpha)}{2} F_\varepsilon(\varepsilon^*_{SS})(v_t + E[\varepsilon_{t+1}|\varepsilon_{t+1} \leq \varepsilon^*_{SS}] + L - C^{s*}_{ST}(v_t + \varepsilon^*_{SS}, -L)) \\ & + \frac{\alpha}{2} F_\varepsilon(\varepsilon^*_{SS} - \delta^*)(v_t + E[\varepsilon_{t+1}|\varepsilon_{t+1} \leq \varepsilon^*_{SS} - \delta^*] + L - C^{s*}_{ST}(v_t + \varepsilon^*_{SS}, -L)) \end{aligned} \tag{38}$$

which yields the following equilibrium limit order profit function

$$V^{LO*}_{ST} = C^{s*}_{ST}(v_t, -L) - (v_t - L) = \max_\varepsilon \frac{p_{ST}(\varepsilon)}{1 + p_{ST}(\varepsilon)} [2L + AS_{ST}(\varepsilon)] \tag{39}$$

where $AS_{ST}(\varepsilon) = \frac{\frac{(1-\alpha)}{2} F_\varepsilon(\varepsilon) E[\varepsilon_{t+1} - \varepsilon | \varepsilon_{t+1} \leq \varepsilon] + \frac{\alpha}{2} F_\varepsilon(\varepsilon - \delta^*) E[\varepsilon_{t+1} - \varepsilon | \varepsilon_{t+1} \leq \varepsilon - \delta^*]}{p_{ST}(\varepsilon)}$ and $p_{ST}(\varepsilon) = \frac{(1-\alpha)F_\varepsilon(\varepsilon) + \alpha F_\varepsilon(\varepsilon - \delta^*)}{2}$.

Similarly, suppose that the equilibrium quotation strategy of fast traders is to post bid prices $(B^*_{t,FT}, B^{+*}_{t,FT})$. Then, there exists some ε^*_{FF} such that $B^*_{t,ST} = C^{s*}_{FT}(v_t + \varepsilon^*_{FF}, -L)$ and consequently $p^b_t(B^*_{t,FT}|\theta_{t+1} = 1) = F_\varepsilon(\varepsilon^*_{FF})/2$. From Lemma 1 (which is also still valid), it is immediate that $p^b_t(B^{+*}_{t,FT}|\theta_{t+1} = 0) = 1/2$. Using $C^{s*}_{ST}(v_t, -L) = C^{s*}_{FT}(v_t, -L) - \delta^*$, we have

$$\begin{aligned} C^{s*}_{FT}(v_t, -L) - (v - L) = {} & \frac{(1-\alpha)}{2}(v_t + L - C^{s*}_{FT}(v_t, -L) + \delta^*) \\ & + \frac{\alpha}{2} F_\varepsilon(\varepsilon^*_{FF})(v_t + E[\varepsilon_{t+1}|\varepsilon_{t+1} \leq \varepsilon^*_{FF}] + L - C^{s*}_{FT}(v_t + \varepsilon^*_{FF}, -L)) \end{aligned} \tag{40}$$

which implies

$$V_{FT}^{LO*} = C_{FT}^{s*}(v_t, -L) - (v_t - L) = \max_{\varepsilon} \frac{p_{FT}(\varepsilon)}{1 + p_{FT}(\varepsilon)} \left[2L + AS_{FT}(\varepsilon)\right] \tag{41}$$

where $AS_{FT}(\varepsilon) = \frac{\frac{(1-\alpha)}{2}\delta^* + \frac{\alpha}{2}F_\varepsilon(\varepsilon)E[\varepsilon_{t+1} - \varepsilon | \varepsilon_{t+1} \leq \varepsilon]}{p_{FT}(\varepsilon)}$ and $p_{FT}(\varepsilon) = \frac{(1-\alpha)+\alpha F_\varepsilon(\varepsilon)}{2}$.

While the model does not admit a closed-form solution under the new distributional assumption, it is easy to show that an equilibrium always exists[10]. The procedure to compute the equilibrium numerically is as follows. First, we start with some initial guess δ_0^* and numerically maximize the profit functions V_k^{LO*}. This yields a new estimate $\delta_1^* = V_{FT}^{LO*}(\delta_0^*) - V_{ST}^{LO*}(\delta_0^*)$, which then is used again in the maximization procedure. We continue iterating this relationship until the distance $\left|\delta_t^* - \delta_{t-1}^*\right|$ satisfies some formal convergence criterion. As it is customary, we employ a wide range of different starting values to ensure that our maximization procedure finds the global optimum. We always find a unique equilibrium. As before, we normalize $L = 1$ and compute the equilibrium for several values of σ over the entire range of α. Incorporating the extension of Section 6 is straightforward, as it suffices to modify equations (39) and (41) by replacing the terms $(1 - \alpha)$ by $(1 - \alpha)\eta$ everywhere.

Figures 1-5 in Appendix C plot the equivalents to Figures 2-6 of Appendix B, where we fix $L = 1$ as before end employ three different values for σ (0.05, 1, and 5). For the sake of parsimony, we just state the results that relate to Propositions 3-5 and 7. Notice that it is straightforward to prove analytically that $V_{FT}^{LO*} > V_0^{LO*} > V_{ST}^{LO*}$ for all $\alpha \in (0, 1)$ when ε is normally distributed.

Numerical Results

a) $TR_{ST}^* > TR_0^* > TR_{FT}^*$ for all $\alpha \in (0, 1)$.

b) There exists some threshold $\hat{\sigma}$ such that $TR^* > TR_0^*$ for all $\sigma > \hat{\sigma}$ and $\alpha \in (0, 1)$.

c) $E(\tau_{ST}^*) > E(\tau^*) > E(\tau_0^*) > E(\tau_{FT}^*)$ for all $\alpha \in (0, 1)$.

d) $W_{ST}^*(\alpha) < W^*(0)$ for all $\alpha \in (0, 1)$ and thus $W^*(\alpha^*) < W^*(0)$ for $\alpha^* > 0$. Equilibrium investment is excessive.

e) For a fixed level of $\sigma > 0$ there exist thresholds $\eta^*, \eta^{**} \in [0, 1]$ with $\eta^* < \eta^{**}$ such that we have $W^*(\alpha^*) > W^*(0)$ $(W^*(\alpha^*) < W^*(0))$ for $\eta < \eta^*$ $(\eta > \eta^{**})$.

[10] Notice that the $AS(\varepsilon)_k$ are continuous and strictly decreasing in ε, with $\lim_{\varepsilon \to -\infty} AS(\varepsilon)_k = 0$ and $\lim_{\varepsilon \to \infty} AS(\varepsilon)_k = -\infty$. Hence the exists some finite ε such that $V_k^{LO} > 0$. Moreover, notice that $\delta^* = V_{FT}^{LO*}(\delta^*) - V_{ST}^{LO*}(\delta^*)$ with $\delta^* \in [0, L]$ such that Brouwer's fixed point theorem implies the existence of an equilibrium.

Points a) and b) closely mirror the results in Proposition 3: the presence of FTs increases the likelihood that STs submit market orders and leads to an increase in overall trading volume for a sufficiently high level of σ, while we may observe a lower trading volume for low σ. Notice from Figure 2 in Appendix C that even for a low level of σ we have $TR^* > TR_0^*$ for α sufficiently large in the case of ε being normally distributed. This occurs because there is always some order shading as ε is unbounded, such that once there is a sufficient proportion of FTs the reduction in order shading will always lead to an increase in trading volume. For the same reason, STs are always missing out on some picking off profits when trading against FTs' limit orders as it is not possible to condition on a "worst case scenario" where limit orders cannot become stale. Point c) states that this effect is always strong enough to offset potentially more aggressive quotes by other STs (also see the discussion following Proposition 4). Hence the presence of fast traders always increases the average and STs' expected trading costs, that is $E(\tau_{ST}^*) > E(\tau^*) > E(\tau_0^*)$, which is not always the case for the closed-form solution as shown in Proposition 4. Notice that a departure from the arms race assumption $\eta = 1$ can readily generate $E(\tau_{ST}^*) < E(\tau^*) < E(\tau_0^*)$, which is more in line with the empirical evidence cited in Section 4.3. Finally, points d) and e) are the continuous equivalents of Propositions 5 and 7 and thus do not require further commenting. Overall, the results obtained from solving the model numerically are qualitatively almost identical to those obtained in closed form under Assumption 1, in particular for the main results regarding the effect of investments in trading technology and social welfare.

8 Conclusion

This paper contributes to the ongoing controversy on the pros and cons of the increasing automation of the trading process, in particular in relation to the enourmous growth in algorithmic and high-frequency trading during the past decade. We show that a pure arms race where technology only helps to outpace other market participants leads to a welfare loss. Although trade may increase, the equilibrium level of investment is excessive from a social welfare perspective as slow traders are exposed to negative externalities. This result may reverse completely if technology additionally helps to improve the overall level of market efficiency (e.g by aggregating dispersed trading information in a fragmented market). Sufficiently large efficiency gains may lead to an increase in social welfare even though the equilibrium level of investment may fall short of the social optimum. In either case, welfare may be improved by policy intervention via taxes or subsidies on technology investments.

In sum, our model relates simultaneously to the widespread concern that speed advantages are being used to extract rents from slower market participants as well as to the main argument put forth by HFT supporters that technology helps to improve market efficiency. Our results suggest that

there is a trade-off between both effects, and the direction into which the pendulum ultimately swings is determined by the pre-investment level of market efficiency. While it is hard to deny that today's electronic markets are more efficient than the dealer markets of the past, it has become equally demanding to term further reductions in latency with anything else but an arms race with little or no gain for society as a whole. Given that there is scope for intervention from either side (taxes or subsidies), the implementation of adequate policies crucially hinges on the correct evaluation of the current level of market efficiency, a task that is likely to be complicated by measurement problems.

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9 Appendix A: Proofs

9.1 Proof of Lemma 1

Consider a fast buyer that has placed a bid at time t. If he observes the innovation ε_{t+1} and can still modify his order, he knows that the next agent (provided he is a seller) is a ST with sell cutoff price $C^{s*}_{ST}(v_t + \varepsilon_{t+1}, -L)$. Clearly, the optimal bid price is slightly above this cutoff price, as a lower (higher) bid has a zero (the same) execution probability. A symmetric argument holds for fast sellers. ■

9.2 Proof of Lemma 2

We prove only the statements for sell cutoff prices. Symmetry then establishes the corresponding arguments for buy cutoff prices.

a) Suppose that, in equilibrium, a slow buyer posts a buy limit order that executes only if the asset value decreases. Then, a fast trader can always do better by posting the same bid price and revise his order according to Lemma 1 in case the asset value increases and the next trader is a ST. Similarly, if a slow buyer posts a limit order that executes irrespectively of the asset value innovation, a fast trader can do better by revising his order according to Lemma 1 and obtaining higher profits by incorporating the latest innovation into his limit price whenever possible (i.e. when a slow seller follows). Hence we conclude that $C^{s*}_{FT}(v_t, -L) > C^{s*}_{ST}(v_t, -L)$.

b) By definition, we have $C^{s*}_k(v_t, y_t) - (v_t + y_t) = V^{LO*}_k(y_t)$ for $k \in \{ST, FT\}$ and $y_t \in \{-L, +L\}$. As the execution probability of any limit order is no greater than 1/2, we have $L \geq V^{LO*}_k(y_t) \geq 0$, which implies $v_t + y_t + L \geq C^{s*}_k(v_t, y_t) \geq v_t + y_t$. Hence we can write $v_t + 2L \geq C^{s*}_{ST}(v_t, +L) \geq v_t + L$ and $v_t \geq C^{s*}_{FT}(v_t, -L) \geq v_t - L$, which establishes the result.

c) First, suppose that that $\sigma \geq L/2$. From the previous step, we know that $v_t + \sigma \geq C^{s*}_{ST}(v_t + \sigma, -L) \geq v_t + \sigma - L$ and $v_t - \sigma \geq C^{s*}_{FT}(v_t - \sigma, -L) \geq v_t - \sigma - L$, which directly implies $C^{s*}_{ST}(v_t + \sigma, -L) \geq C^{s*}_{FT}(v_t - \sigma, -L)$. Now assume that $\sigma < L/2$ and consider a fast buyer's decision regarding his quotation strategy. He will opt for a high fill rate in equilibrium iff $\frac{\alpha}{4}[v - \sigma + L - C^{s*}_{FT}(v_t + \sigma, -L)] + \frac{\alpha}{4}[v + \sigma + L - C^{s*}_{FT}(v_t + \sigma, -L)] \geq \frac{\alpha}{4}[v - \sigma + L - C^{s*}_{FT}(v_t - \sigma, -L)]$, which is satisfied in our case as $v - \sigma + L \geq v + \sigma \geq C^{s*}_{FT}(v_t + \sigma, -L)$. Now consider a slow buyer and suppose he posts a buy limit order with bid price equal to $C^{s*}_{FT}(v_t + \sigma, -L)$. As this is not necessarily his equilibrium strategy we have that $V^{LO*}_{ST}(y_h) \geq \frac{1}{2}[v + L - C^{s*}_{FT}(v_t + \sigma, -L)]$. But we just concluded that $V^{LO*}_{FT}(y_h) = \frac{\alpha}{2}[v + L - C^{s*}_{FT}(v_t + \sigma, -L)] + \frac{1-\alpha}{4}[v - \sigma + L - C^{s*}_{ST}(v_t - \sigma, -L)] + \frac{1-\alpha}{4}[v + \sigma + L - C^{s*}_{ST}(v_t + \sigma, -L)]$, and therefore $V^{LO*}_{FT}(y_h) - V^{LO*}_{ST}(y_h) \leq \frac{1-\alpha}{4}[C^{s*}_{FT}(v_t + \sigma, -L) - C^{s*}_{ST}(v_t - \sigma, -L)] + \frac{1-\alpha}{4}[C^{s*}_{FT}(v_t + \sigma, -L) - C^{s*}_{ST}(v_t + \sigma, -L)] = \frac{1-\alpha}{2}[C^{s*}_{FT}(v_t, -L) - C^{s*}_{ST}(v_t, -L)] + \frac{1-\alpha}{2}\sigma$. Symmetry between buyers

and seller implies $V_{FT}^{LO*}(y_h) - V_{ST}^{LO*}(y_h) = C_{FT}^{s*}(v_t, -L) - C_{ST}^{s*}(v_t, -L)$, such that we conclude $C_{FT}^{s*}(v_t, -L) - C_{ST}^{s*}(v_t, -L) \leq \frac{1-\alpha}{1+\alpha}\sigma$, which finally leads us to $C_{ST}^{s*}(v_t + \sigma, -L) - C_{FT}^{s*}(v_t - \sigma, -L) = C_{ST}^{s*}(v_t, -L) - C_{FT}^{s*}(v_t, -L) + 2\sigma > 0$ as desired. ■

9.3 Proof of Proposition 1

For each type of equilibrium, the proof procedes in three steps:

1) Conjecture an ordering of cutoff prices.

2) Conjecture equilibrium strategies and solve for the equilibrium cutoff prices.

3) Verify that

   a) the assumed strategies are best replies (i.e. deviations are not profitable) and

   b) the cutoff prices satisfy the assumed ordering.

Lemma 2 implies that it suffices to consider the following four orderings of cutoff prices.

Ordering 1: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 2: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 3: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

Ordering 4: $C_{ST}^{s*}(v_t - \sigma, -L) \leq C_{FT}^{s*}(v_t - \sigma, -L) \leq C_{ST}^{s*}(v_t + \sigma, -L) \leq C_{FT}^{s*}(v_t + \sigma, -L) \leq C_{ST}^{s*}(v_t - \sigma, +L) \leq C_{FT}^{s*}(v_t - \sigma, +L) \leq C_{ST}^{s*}(v_t + \sigma, +L) \leq C_{FT}^{s*}(v_t + \sigma, +L)$

For each ordering of sell cutoff prices, there is a corresponding ordering (due to symmetry) of buy cutoff prices. For example, for Ordering 1, we have $C_{ST}^{b*}(v_t + \sigma, +L) \geq C_{FT}^{b*}(v_t + \sigma, +L) \geq C_{ST}^{b*}(v_t + \sigma, -L) \geq C_{FT}^{b*}(v_t + \sigma, -L) \geq C_{ST}^{b*}(v_t - \sigma, +L) \geq C_{FT}^{b*}(v_t - \sigma, +L) \geq C_{ST}^{b*}(v_t - \sigma, -L) \geq C_{FT}^{b*}(v_t - \sigma, -L)$

The following four tables contain the conditional and unconditional execution probabilities of buy limit orders according to the position of the limit price relative to the cutoff sell prices, separately for each employed ordering.

Bid Price (Ordering 1)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(2-\alpha)/4$	$(2-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$1/2$	1	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-\alpha)/4$	1	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 2)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$(2-\alpha)/4$	$(2-\alpha)/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-2\alpha)/4$	$(2-\alpha)/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-\alpha)/4$	1	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 3)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$(2-\alpha)/2$	$1/2$	$(1-\alpha)/2$
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(3-2\alpha)/4$	$(2-\alpha)/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-\alpha)/4$	$(2-\alpha)/2$	$1/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Bid Price (Ordering 4)	Execution Probability	Execution Probability conditional on $\varepsilon_{t+1} = -\sigma$	Execution Probability conditional on $\varepsilon_{t+1} = +\sigma$
$\leq C^{s*}_{ST}(v_t - \sigma, -L)$	0	0	0
$\in (C^{s*}_{ST}(v_t - \sigma, -L), C^{s*}_{FT}(v_t - \sigma, -L)]$	$(1-\alpha)/4$	$(1-\alpha)/2$	0
$\in (C^{s*}_{FT}(v_t - \sigma, -L), C^{s*}_{ST}(v_t + \sigma, -L)]$	$1/4$	$1/2$	0
$\in (C^{s*}_{ST}(v_t + \sigma, -L), C^{s*}_{FT}(v_t + \sigma, -L)]$	$(2-\alpha)/4$	$1/2$	$(1-\alpha)/2$
$\in (C^{s*}_{FT}(v_t + \sigma, -L), C^{s*}_{ST}(v_t - \sigma, +L)]$	$1/2$	$1/2$	$1/2$
$\in (C^{s*}_{ST}(v_t - \sigma, +L), C^{s*}_{FT}(v_t - \sigma, +L)]$	$(3-\alpha)/4$	$(2-\alpha)/2$	$1/2$
$\in (C^{s*}_{FT}(v_t - \sigma, +L), C^{s*}_{ST}(v_t + \sigma, +L)]$	$3/4$	1	$1/2$
$\in (C^{s*}_{ST}(v_t + \sigma, +L), C^{s*}_{FT}(v_t + \sigma, +L)]$	$(4-\alpha)/4$	1	$(2-\alpha)/2$
$> C^{s*}_{FT}(v_t + \sigma, +L)$	1	1	1

Type 1 equilibrium:

Let $\sigma_1^* = 4L/(5-\alpha)$ and $\alpha_1^* = \sqrt{5} - 2$.

Case A:

Step 1: Assume Ordering 1.

Step 2: Conjecture the following equilibrium strategies: Slow buyers submit a buy limit order with a bid price slightly above $C^{s*}_{ST}(v_t - \sigma, -L)$, which has a probability of execution of $(1-\alpha)/4$ (see Table A.1, Panel 1). Fast buyers submit a buy limit order with a bid price slightly above $C^{s*}_{FT}(v_t - \sigma, -L)$. If the next trader is not a FT, they cancel this order after observing the innovation in the fundamental value and set a new bid price slightly above $C^{s*}_{ST}(v_t - \sigma, +L)$ ($C^{s*}_{ST}(v_t + \sigma, +L)$) if $\varepsilon_{t+1} = -\sigma$

($\varepsilon_{t+1} = +\sigma$). The probability of execution for this strategy is $(1-\alpha)/4+(1-\alpha)/4+\alpha/4 = (2-\alpha)/4$ (see Table A.1, Panel 1). Moreover, conjecture the analogous strategies for slow and fast sellers, e.g. a slow seller submits a sell limit order with ask price slightly below $C^{b*}_{ST}(v_t+\sigma,+L)$ with probability of execution equal to $(1-\alpha)/4$. Thus, cutoff prices have to satisfy the following system of equations.

$$\begin{aligned}
v_t + L - C^{b*}_{ST}(v_t,+L) &= \frac{1-\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{ST}(v_t-\sigma,-L)\right] \\
C^{s*}_{ST}(v_t,-L) - (v_t - L) &= \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t+\sigma,+L) - (v_t+\sigma-L)\right] \\
v_t + L - C^{b*}_{FT}(v_t,+L) &= \frac{1-\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{ST}(v_t-\sigma,-L)\right] \\
&\quad + \frac{1-\alpha}{4}\left[v_t + \sigma + L - C^{s*}_{ST}(v_t+\sigma,-L)\right] \\
&\quad + \frac{\alpha}{4}\left[v_t - \sigma + L - C^{s*}_{FT}(v_t-\sigma,-L)\right] \\
C^{s*}_{FT}(v_t,-L) - (v_t - L) &= \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t+\sigma,+L) - (v_t+\sigma-L)\right] \\
&\quad + \frac{1-\alpha}{4}\left[C^{b*}_{ST}(v_t-\sigma,+L) - (v_t-\sigma-L)\right] \\
&\quad + \frac{\alpha}{4}\left[C^{b*}_{FT}(v_t+\sigma,+L) - (v_t+\sigma-L)\right]
\end{aligned}$$

Straightforward algebra yields the following cutoff prices:

$$\begin{aligned}
C^{s*}_{ST}(v_t,-L) &= v_t - L + (2L)\frac{1-\alpha}{5-\alpha} \\
C^{s*}_{FT}(v_t,-L) &= v_t - L + (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)} \\
C^{b*}_{ST}(v_t,+L) &= v_t + L - (2L)\frac{1-\alpha}{5-\alpha} \\
C^{b*}_{FT}(v_t,+L) &= v_t + L - (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}
\end{aligned}$$

Step 3: Due to symmetry, it suffices to analyze the strategies of buyers. As mentioned, the optimal bid price must be chosen such that it is slightly higher than the lower bound of any of the proposed intervals, because a higher bid can be decreased without reducing the execution probability. Moreover, it is easy to see that Ordering 1 implies that it is not optimal to post a bid price $B \in (C^{s*}_{ST}(v_t-\sigma,+L), C^{s*}_{ST}(v_t+\sigma,-L)]$. Such a limit order is only executed in the case of a price decrease, and therefore $B > C^{s*}_{ST}(v_t-\sigma,+L) \geq v_t - \sigma + L$. But then, the execution of this limit order cannot be profitable, because the bid price is above the reservation price of the trader posting the limit order. Moreover, it is clear that a bid price $B > C^{s*}_{ST}(v_t+\sigma,+L) \geq v_t + \sigma + L$ cannot be optimal, because it is higher the maximum valuation of any trader.[11] Thus, the proposed strategy

[11] Using the same kind of reasoning, we can reduce the possible equilibrium bid prices for other orderings of cutoff prices as well.

for slow buyers is a best reply iff:

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

Similarly, fast buyers have no incentives to deviate iff:

$$\begin{aligned}
\frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{2}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

Brute-force algebra reveals that these inequalities and the assumed ordering of cutoff prices are satisfied if and only $\alpha \leq \alpha_1^*$ and $\sigma \geq \frac{24+\alpha(1-\alpha)}{(5-\alpha)(4+\alpha)}L$.

Case B:

Step 1: Assume Ordering 2.

Step 2: Conjecture the same equilibrium strategies as in Case A, which implies identical cutoff prices.

Step 3: The proposed strategies are best replies (for buyers) iff

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{2-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{1}{2}\left[v_t-\sigma+L-C_{ST}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
\frac{\alpha}{4}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{2}\left[v_t-\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C_{FT}^{s*}(v_t+\sigma,-L)\right]
\end{aligned}$$

These inequalities together with the conjectured ordering of cutoff prices are satisfied iff $\alpha \leq \alpha_1^*$ and $\frac{24+\alpha(1-\alpha)}{(5-\alpha)(4+\alpha)}L > \sigma \geq L$.

Case C:

Step 1: Assume Ordering 3.

Step 2: Conjecture the same equilibrium strategies as in Case A, which implies identical cutoff prices.

Step 3: The proposed strategies are best replies (for buyers) iff

$$\begin{aligned}
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C^{s*}_{FT}(v_t-\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t-\sigma,-L)\right] &\geq \frac{1}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1-\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t+\sigma,-L)\right] \\
\frac{1-\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t-\sigma,-L)\right] &\geq \frac{2-\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{ST}(v_t+\sigma,-L)\right] \\
&\quad +\frac{1}{4}\left[v_t-\sigma+L-C^{s*}_{FT}(v_t+\sigma,-L)\right] \\
\frac{\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{FT}(v_t-\sigma,-L)\right] &\geq \frac{\alpha}{4}\left[v_t-\sigma+L-C^{s*}_{FT}(v_t+\sigma,-L)\right] \\
&\quad +\frac{\alpha}{4}\left[v_t+\sigma+L-C^{s*}_{FT}(v_t+\sigma,-L)\right]
\end{aligned}$$

These inequalities together with the conjectured ordering of cutoff prices are satisfied iff $\alpha \leq \alpha_1^*$ and $L > \sigma \geq \sigma_1^*$. Combining Cases A, B and C, we conclude that the following quotation strategy and the associated order choice strategy constitute an equilibrium iff $\alpha \leq \alpha_1^*$ and $\sigma \geq \sigma_1^*$

$$\begin{aligned}
B^*_{t,HT} &= C^{s*}_{ST}(v_t-\sigma,-L) \qquad & B^*_{t,AT} = C^{s*}_{FT}(v_t-\sigma,-L) \\
A^*_{t,HT} &= C^{b*}_{ST}(v_t+\sigma,+L) \qquad & A^*_{t,AT} = C^{b*}_{FT}(v_t+\sigma,+L)
\end{aligned}$$

The proof for the remaining equilibrium types follows exactly the same logic. In order to conserve space, we will simply indicate the orderings that give rise to those equilibria and give the equilibrium bid quotes in terms of the equilibrium cutoff prices.

Type 2 equilibrium: Orderings 1 – 4

Type 3 equilibrium: Orderings 3 and 4

Type 4 equilibrium: Ordering 4

Type 5 equilibrium: Ordering 4

The remaining cutoff variables are defined as

$\alpha_2^* = \frac{\sqrt{33}-5}{2}$ $\quad \sigma_2^* = L\frac{2\alpha(1+\alpha)}{3-4\alpha}$ $\quad \sigma_3^* = L\frac{4(4+\alpha)}{26-\alpha^2}$ $\quad \sigma_4^* = L\frac{2(1-\alpha)(4+\alpha)}{7+3\alpha}$
$\sigma_5^* = L\frac{4(1+\alpha)}{7+3\alpha}$

The following table gives closed form solutions for the sellers' sell cutoff prices in each type of equilibrium.

Equilibrium Type	$C^{s*}_{ST}(v_t, -L)$	$C^{s*}_{FT}(v_t, -L)$
1	$v_t - L + (2L)\frac{1-\alpha}{5-\alpha}$	$v_t - L + (2L)\frac{8-\alpha(3+\alpha)}{(5-\alpha)(4+\alpha)}$
2	$v_t - L + (2L)\frac{1+\alpha}{7+3\alpha}$	$v_t - L + (2L)\frac{3-\alpha}{7+3\alpha}$
3	$v_t - L + (2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$v_t - L + (2L)\frac{8-\alpha(2+\alpha)}{(6-\alpha)(4+\alpha)} + \frac{4(1-\alpha)}{(6-\alpha)(4+\alpha)}\sigma$
4	$v_t - L + (2L)\frac{2-\alpha}{6-\alpha} - \frac{2}{6-\alpha}\sigma$	$v_t - L + (2L)\frac{4+\alpha(2-\alpha)}{(6-\alpha)(2+\alpha)} + \frac{2-\alpha(8-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$
5	$v_t - L + (2L)\frac{1}{3} - \frac{2}{3(1+\alpha)}\sigma$	$v_t - L + (2L)\frac{1}{3} + \frac{1-3\alpha}{3(1+\alpha)}\sigma$

Finally, it is straightforward but tedious to verify that there exist no other equilibria. ■

9.4 Proof of Proposition 2

It follows from equations (9) and (10) that $V^{LO*}_{ST} = C^{s*}_{ST}(v_t, -L) - (v_t - L)$ and $V^{LO*}_{FT} = C^{s*}_{FT}(v_t, -L) - (v_t - L)$. Then $V^{LO*}_{FT} > V^{LO*}_{ST}$ is a direct consequence of Lemma 2. Moreover, we know from Foucault (1999) that $V^{LO*}_0 = 2L/5$ for $\sigma \geq \sigma^*_1(0)$ and $V^{LO*}_0 = (2L - \sigma)/3$ for $\sigma < \sigma^*_1(0)$. Using the sell cutoff prices obtained in the Proof of Proposition 1, it is straightforward to verify that we have $V^{LO*}_{FT} > V^{LO*}_0 > V^{LO*}_{ST}$ for all $\alpha \in (0, 1)$. ■

9.5 Proof of Proposition 3

For each type of equilibrium, the transitions from one state to another follow a Markov chain with transition matrix P^i, $i \in \{1, 2, 3, 4, 5\}$. We have

$$P^1 = \begin{bmatrix} \frac{3(1-\alpha)}{4} & \frac{1-\alpha}{4} & \alpha & 0 \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha & 0 \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} \frac{3(1-\alpha)}{4} & \frac{1-\alpha}{4} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix} \qquad P^4 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{3\alpha}{4} & \frac{\alpha}{4} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{2} \\ 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

Given these transition matrices, the stationary probability distribution $\varphi^i = (\varphi_1^i, \varphi_2^i, \varphi_3^i, \varphi_4^i)$ is given by the left eigenvector associated with the unit eigenvalue. Straightforward calculations reveal

$$\begin{aligned}
\varphi^1 &= (\frac{4(1-\alpha)(4-\alpha)}{(4+\alpha)(5-\alpha)}, \frac{(1-\alpha)(4+5\alpha-\alpha^2)}{(4+\alpha)(5-\alpha)}, \frac{4\alpha}{4+\alpha}, \frac{\alpha^2}{4+\alpha}) \\
\varphi^2 &= (\frac{4(1-\alpha)(4-\alpha)}{20-\alpha+\alpha^2}, \frac{(1-\alpha)(4+3\alpha+\alpha^2)}{20-\alpha+\alpha^2}, \frac{16\alpha}{20-\alpha+\alpha^2}, \frac{\alpha(4-\alpha+\alpha^2)}{20-\alpha+\alpha^2}) \\
\varphi^3 &= (\frac{(1-\alpha)(4-\alpha)}{6-\alpha}, \frac{2(1-\alpha)}{6-\alpha}, \frac{\alpha(5-\alpha)}{6-\alpha}, \frac{\alpha}{6-\alpha}) \\
\varphi^4 &= (\frac{8(1-\alpha)}{12+\alpha-\alpha^2}, \frac{(1-\alpha)(4+\alpha-\alpha^2)}{12+\alpha-\alpha^2}, \frac{2\alpha(5-\alpha)}{12+\alpha-\alpha^2}, \frac{\alpha(2+3\alpha-\alpha^2)}{12+\alpha-\alpha^2}) \\
\varphi^5 &= (\frac{2(1-\alpha)}{3}, \frac{1-\alpha}{3}, \frac{2\alpha}{3}, \frac{\alpha}{3})
\end{aligned}$$

Using the definition of the trading rates in equations (13), (15) and (17), we obtain

$$\begin{array}{lll}
TR^1 = \frac{4+\alpha(1-\alpha)}{(4+\alpha)(5-\alpha)} & TR^1_{ST} = \frac{4+\alpha(5-\alpha)}{(4+\alpha)(5-\alpha)} & TR^1_{FT} = \frac{\alpha}{4+\alpha} \\
TR^2 = \frac{4+3\alpha(1-\alpha)}{20-\alpha+\alpha^2} & TR^2_{ST} = \frac{4+\alpha(3+\alpha)}{20-\alpha+\alpha^2} & TR^2_{FT} = \frac{4-\alpha(1-\alpha)}{20-\alpha+\alpha^2} \\
TR^3 = \frac{2-\alpha}{6-\alpha} & TR^3_{ST} = \frac{2}{6-\alpha} & TR^3_{FT} = \frac{1}{6-\alpha} \\
TR^4 = \frac{4-\alpha(1-\alpha)}{12+\alpha-\alpha^2} & TR^4_{ST} = \frac{4+\alpha(1-\alpha)}{12+\alpha-\alpha^2} & TR^4_{FT} = \frac{2+\alpha(3-\alpha)}{12+\alpha-\alpha^2} \\
TR^5 = \frac{1}{3} & TR^5_{ST} = \frac{1}{3} & TR^5_{FT} = \frac{1}{3}
\end{array}$$

It is immediate that we have $TR^i_{ST} \geq TR^i_{FT}$ for all i, such that $TR^*_{ST} \geq TR^*_{FT}$ follows. Foucualt (1999) shows that $TR^*_0 = 1/5$ for $\sigma \in \Sigma_2$ and $TR^*_0 = 1/3$ for $\sigma \in \Sigma_1$. If $\sigma \in \Sigma_2$, a type-1 or a type-2 equilibrium may arise and it is straightforward to verify that in this case we have $TR^i_{ST} > TR^i > 1/5 > TR^i_{FT}$ for all $\alpha \in (0,1)$ and $i = 1,2$. Similarly, if $\sigma \in \Sigma_1$, a type-3, type-4 and type-5 equilibrium may arise, and it is easy to check that in this case, we have $TR^i_{ST} \geq 1/3 \geq TR^i \geq TR^i_{FT}$ for all $\alpha \in (0,1)$ and $i = 3,4,5$ as required. ■

9.6 Proof of Proposition 4

Using the closed-form equilibrium quotes (see the proof of Proposition 1), it is straightforward to calculate the trading costs for each combination of limit order trader and market order trader via equations (19) - (22) and (24) – (27). They are collected in the following table.

Trading Cost	Type 1	Type 2	Type 3	Type 4	Type 5
$\tau^{S,+\sigma}_{ST,ST}$	n/a	n/a	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{ST,ST}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{FT,ST}$	n/a	n/a	n/a	n/a	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{FT,ST}$	n/a	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L - \frac{10-2\alpha}{6-\alpha}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{ST,FT}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{5+\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L + \frac{2}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{ST,FT}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{5+\alpha}{7+3\alpha}L$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L + \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L + \frac{2}{3(1+\alpha)}\sigma$
$\tau^{S,+\sigma}_{FT,FT}$	n/a	n/a	n/a	$\frac{4+\alpha^2}{(6-\alpha)(2+\alpha)}L - \frac{2-\alpha(8-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{1}{3}L - \frac{1-3\alpha}{3(1+\alpha)}\sigma$
$\tau^{S,-\sigma}_{FT,FT}$	$\frac{4+\alpha(7+\alpha)}{(5-\alpha)(4+\alpha)}L$	$\frac{1+5\alpha}{7+3\alpha}L$	$\frac{8+\alpha(6+\alpha)}{(6-\alpha)(2+\alpha)}L - \frac{4(1-\alpha)}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{4+\alpha^2}{(6-\alpha)(2+\alpha)}L - \frac{26-\alpha^2}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{1}{3}L - \frac{7+3\alpha}{3(1+\alpha)}\sigma$

Then, the expected trading costs for ST and FT, respectively, are obtained by straightforward substitution into equations (23) and (28). We obtain

$$E(\tau^1_{ST}) = \frac{3+\alpha}{5-\alpha}L$$

$$E(\tau^1_{FT}) = \frac{4+\alpha(7+\alpha)}{(5-\alpha)(4+\alpha)}L$$

$$E(\tau^2_{ST}) = \frac{(4-\alpha)(1-\alpha)(1+5\alpha)+8\alpha(5+\alpha)}{(7+3\alpha)((4-\alpha)(1-\alpha)+8\alpha)}L$$

$$E(\tau^2_{FT}) = \frac{1+5\alpha}{7+3\alpha}L$$

$$E(\tau^3_{ST}) = \frac{2+\alpha}{6-\alpha}L + \frac{2\alpha(5-\alpha)-(1-\alpha)(4-\alpha)^2}{4(6-\alpha)}\sigma$$

$$E(\tau^3_{FT}) = \frac{(1-\alpha)(4-\alpha)(4+\alpha)(2+\alpha)+\alpha(5-\alpha)(8+\alpha(6+\alpha))}{(4+\alpha)(6-\alpha)}L + \frac{2(5-\alpha)((4+\alpha)(4-\alpha)+2\alpha)}{(4+\alpha)(6-\alpha)}\sigma$$

$$E(\tau^4_{ST}) = \frac{2+\alpha}{6-\alpha}L + \frac{2\alpha(5-\alpha)-(1-\alpha)(4-\alpha)^2}{4(6-\alpha)}\sigma$$

$$E(\tau^4_{FT}) = \frac{8(1-\alpha)(2+\alpha)^2+4\alpha(5-\alpha)(4+\alpha^2)}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}L + \frac{2(5-\alpha)(8(1-\alpha)(2+\alpha)+\alpha(28-8\alpha)}{(2+\alpha)(6-\alpha)(8+4\alpha(3-\alpha))}\sigma$$

$$E(\tau^5_{ST}) = \frac{1}{3}L - \frac{4-6\alpha}{3(1+\alpha)}\sigma$$

$$E(\tau^5_{FT}) = \frac{1}{3}L - \frac{4}{3(1+\alpha)}\sigma$$

Average expected trading costs for each type of equilibrium can then be computed using equation (29). It is easy to see that expected overall trading costs in the absence of AT are given by $E(\tau^*_0) = 3L/5$ for $\sigma \in \Sigma_2$ and $E(\tau^*_0) = (L-2\sigma)/3$ for $\sigma \in \Sigma_1$. It is easily verified that $E(\tau^i_{ST}) > E(\tau^i) > E(\tau^i_{FT})$ for all $\alpha \in (0,1)$ and $i = 1,2,3,4,5$.

i) For $\sigma \in \Sigma_2$, the only equilibria that may arise are of type 1 or 2. It is straightforward to verify that $E(\tau^i_{ST}) < 3L/5$ for all $\alpha \in (0,1)$ and $i = 1,2$. Similarly, for $\sigma \in \Sigma_2$ we are either in a type-3,

type-4 or ype-5 equilibrium and we have $E(\tau^i_{ST}) < (L-2\sigma)/3$ for all $\alpha \in (0,1)$ and $i = 3,4,5$.

ii) Let $\sigma \in \Sigma_2$. Then we are in a type-1 equilibrium for $\alpha < \alpha^*_1$, and some algebra shows that $E(\tau^1_{ST}) > E(\tau^1) > 3L/5$. Similarly, for any $\sigma \in \Sigma_1$ there exist some level of α such that a type-3 or a type-4 equilibrium arises. Some cumbersome computations reveal that in this case $E(\tau^i_{ST}) > E(\tau^i) > (L-2\sigma)/3$ for all $\alpha \in (0,1)$ and $i = 3,4$.

iii) It is tedious but straightforward to verify that $E(\tau^2_{ST}) < 3L/5$ iff $\alpha < \frac{7-\sqrt{33}}{2}$, from which the result follows for $\sigma \in \Sigma_2$ because $\frac{7-\sqrt{33}}{2} > \alpha^*_1$ and we always have $E(\tau^i) < E(\tau^i_{ST})$. Similarly, it can be verified that $E(\tau^5_{ST}) < (L-2\sigma)/3$ iff $\alpha < 1/4$, such that the results also obains for $\sigma < \sigma^*_2(1/4)$ (notice that if $\sigma \geq \sigma^*_2(1/4)$, we do not have a type-5 equilibrium for $\alpha < 1/4$). ■

9.7 Proof of Proposition 5

Clearly we can never have $\alpha^+ = 1$ because $W^*(1) = W^*(0) - c$. Moreover, if $\sigma \in \Sigma_1$ the socially optimal level of investment is $\alpha^+ = 0$ because $TR^* \leq TR^*_0$ for $\alpha \in (0,1)$, such that any positive level of investment implies a loss in welfare. Hence assume that $\sigma \in \Sigma_2$. Notice that in this case we have $\Delta W^*(1) = 0$ which implies that we must have $\alpha^* < 1$. Thus consider an interior equilibrium $\alpha^* \in (0,1)$ and assume that $\alpha^+ > 0$ (otherwise $\alpha^* > \alpha^+$ is trivial). By definition $\left.\frac{\partial W^*(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha^+} = 0$ and $\Delta W^*(\alpha^*) = c$. Now notice that the chain rule implies that $\frac{\partial W^*(\alpha)}{\partial \alpha} = \Delta W^*(\alpha) + (1-\alpha)\frac{\partial W^*_{ST}(\alpha)}{\partial \alpha} + \alpha\frac{\partial W^*_{ST}(\alpha)}{\partial \alpha} - c$ and it is easy (albeit very cumbersome) to verify that $(1-\alpha)\frac{\partial W^*_{ST}(\alpha)}{\partial \alpha} + \alpha\frac{\partial W^*_{ST}(\alpha)}{\partial \alpha} < 0$ for all α, such that we conclude that $\Delta W^*(\alpha^+) > c$. Finally, because for $\sigma \in \Sigma_2$ we have $\frac{\partial \Delta W^*(\alpha)}{\partial \alpha} < 0$ for all α, this implies $\alpha^* > \alpha^+$.

To show that $W^*(\alpha^*) < W^*(0)$ for $\alpha^* > 0$ it suffices to consider the case where $\sigma \in \Sigma_2$ and $\alpha^* < 1$. It is straightforward to verify that $W^*(0) = 2L/5 > W^*_{ST} = \frac{\varphi^*_1}{\varphi^*_1+\varphi^*_2}V^{LO*}_{ST} + \frac{\varphi^*_2}{\varphi^*_1+\varphi^*_2}(L - E(\tau^*_{ST}))$ by substituting the equilibrium expected profits from posting limit orders, stationary probability distributions and trading costs from the Proofs of Propositions 1, 3 and 4 for the type-1 and type-2 equilibrium. ■

9.8 Proof of Proposition 6

The Proof goes exactly as the one for Proposition 1 (notice that Lemma 1 and 2 still hold), except that the execution probabilities of limit orders now need to be adjusted for the fact that slow traders may miss trading opportunities. For example, a bid quote $B = C^{s*}_{ST}(v_t - \sigma, -L)$ is executed if a) the next trader is a slow seller (probability $(1-\alpha)/2$), b) the asset value has decreased (probability 1/2) and c) the opportunity is not missed (probability η). Hence $p^b_t(B) = \eta(1-\alpha)/2$. The cutoff values are given by

$$\sigma_1^* = L\frac{(1-\alpha)\eta}{4+(1-\alpha)\eta} \qquad \sigma_2^* = L\frac{2\alpha(2-(1-\alpha)\eta)}{\bar{\eta}(\bar{\eta}(2+(1-\alpha)\eta)-3\alpha)} \qquad \sigma_3^* = L\frac{4(4+\alpha)}{16+8\bar{\eta}+2(1-\alpha)\eta(\bar{\eta}+\alpha)+\alpha^2}$$

$$\sigma_4^* = L\frac{2(1-\alpha)\eta(4+\alpha)}{\bar{\eta}(8+2\alpha-(1-\alpha)\eta\bar{\eta})} \qquad \sigma_5^* = L\frac{4(2-(1-a)\eta)}{8+2\alpha-(1-\alpha)\eta\bar{\eta}} \qquad \alpha_1^* = \frac{\sqrt{4-\eta(4-5\eta)}+\eta(1-\eta)-2}{\eta(2-\eta)}$$

and α_2^* is defined implicitly by the intersection of σ_2^*, σ_3^* and σ_5^*.

The equilibrium sell cutoff prices can be written as $C_k^{s*}(v_t, -L) = v_t - L + \gamma_k^L(2L) + \gamma_k^\sigma \sigma$ for $k \in \{ST, FT\}$, and the coefficients are given by

Equilibrium Type	γ_{ST}^L	γ_{ST}^σ	γ_{FT}^L	γ_{FT}^σ
1	$\frac{(1-\alpha)\eta}{4+(1-\alpha)\eta}$	0	$\frac{8\eta(1-\alpha)+\alpha(4+(1-\alpha)\eta)}{(4+(1-\alpha)\eta)(4+\alpha)}$	0
2	$\frac{\bar{\eta}(2-(1-\alpha)\eta)}{8+2\alpha-(1-\alpha)\eta\bar{\eta}}$	0	$\frac{(1-\alpha)\eta(4-\bar{\eta})+2\alpha}{8+2\alpha-(1-\alpha)\eta\bar{\eta}}$	0
3	$\frac{2\bar{\eta}-\alpha}{4+2(1-\alpha)\eta+\alpha}$	$-\frac{2\bar{\eta}}{4+2(1-\alpha)\eta+\alpha}$	$\frac{(1-\alpha)\eta(8+2\alpha)+\alpha(4+\alpha)}{(4+\alpha)(4+2(1-\alpha)\eta+\alpha)}$	$\frac{4(1-\alpha)\eta\bar{\eta}}{(4+\alpha)(4+2(1-\alpha)\eta+\alpha)}$
4	$\frac{2\bar{\eta}-\alpha}{4+2(1-\alpha)\eta+\alpha}$	$-\frac{2\bar{\eta}}{4+2(1-\alpha)\eta+\alpha}$	$\frac{4\bar{\eta}+\alpha(2\bar{\eta}-\alpha)}{(4+2(1-\alpha)\eta+\alpha)(2+\alpha)}$	$\frac{2(1-\alpha)\eta\bar{\eta}-4\alpha-2\alpha(1-\alpha)\eta-\alpha^2}{(4+2(1-\alpha)\eta+\alpha)(2+\alpha)}$
5	$\frac{\bar{\eta}(2-(1-a)\eta)}{2(2+\alpha)-(1-\alpha)\eta\bar{\eta}}$	$-\frac{2\bar{\eta}}{2(2+\alpha)-(1-\alpha)\eta\bar{\eta}}$	$\frac{\bar{\eta}(2-(1-a)\eta)}{2(2+\alpha)-(1-\alpha)\eta\bar{\eta}}$	$\frac{(1-\alpha)\eta\bar{\eta}-2\alpha}{2(2+\alpha)-(1-\alpha)\eta\bar{\eta}}$

■

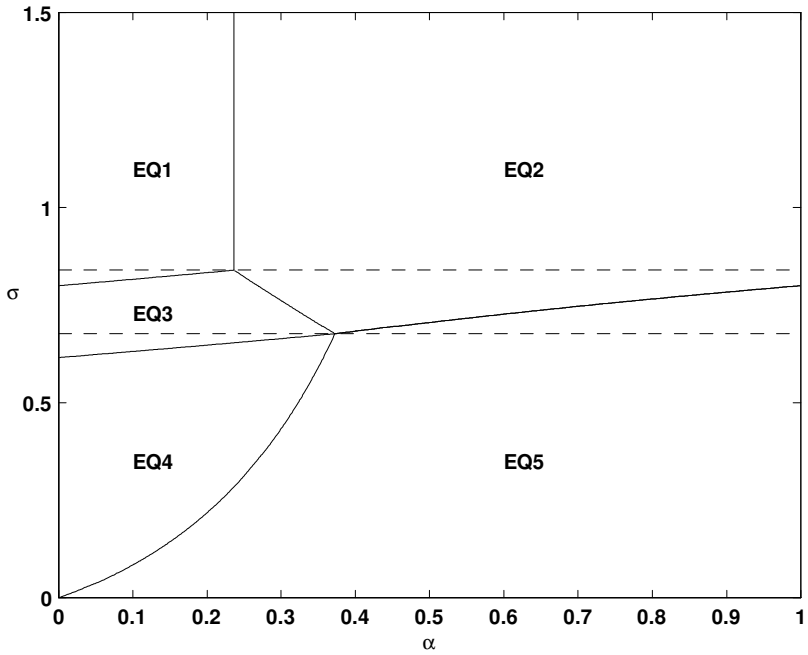
9.9 Proof of Proposition 7

Because there cannot be any trade if limit orders are never found we have $\lim_{\eta\to 0} W^*(0) = 0$. Now notice that for (σ, L) fixed, the range of α that gives rise to a type-2 or a type-5 equilibrium approaches the unit interval as η converges to zero because $\lim_{\eta\to 0} \alpha_1^* = 0$ and $\lim_{\eta\to 0} \alpha_2^* = 0$. For both types of equilibrium we have $\lim_{\eta\to 0} V_{ST}^{LO*} > 0$ (using the cutoff sell prices derived in the proof of Proposition 6) and hence $\lim_{\eta\to 0} W_{ST}^* > 0$ for all $\alpha > 0$, such that the result follows immediately. ■

10 Appendix B: Figures

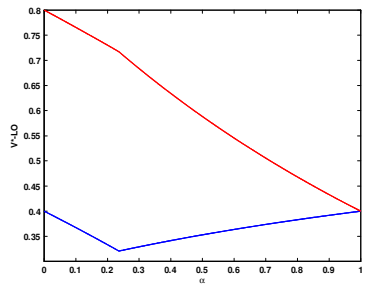
10.1 Figure 1: Equilibrium Map

This graph depicts the different regions in the (α, σ)-space that give rise to the respective equilibria. We have set $L = 1$. The are between the two dashed lines is the interval $[\sigma_2^*(\alpha_2^*), \sigma_1^*(\alpha_1^*))$.

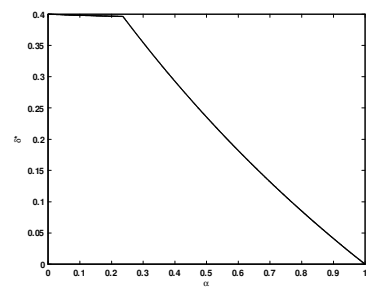


10.2 Figure 2: Equilibrium limit order profits and δ^*

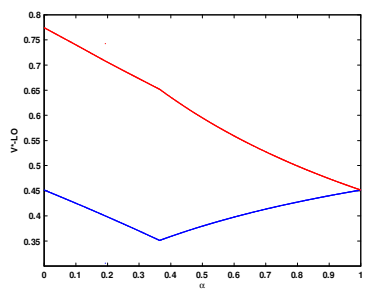
The figures in the left (right) column depict V_{ST}^{LO*} (blue) and V_{FT}^{LO*} (red) (δ^*) for different levels of σ ($\hat{\sigma}_3$, $\hat{\sigma}_2$, and $\hat{\sigma}_1$ are defined in Section 3) as functions of α.



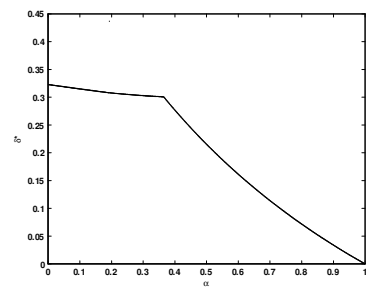
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = \hat{\sigma}_3$



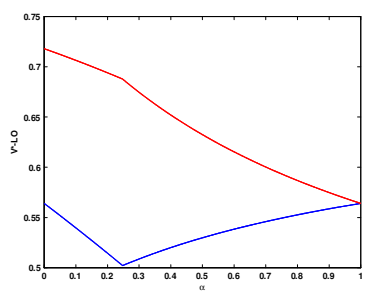
δ^* for $\sigma = \hat{\sigma}_3$



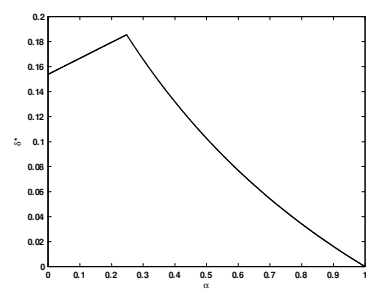
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = \hat{\sigma}_2$



δ^* for $\sigma = \hat{\sigma}_2$



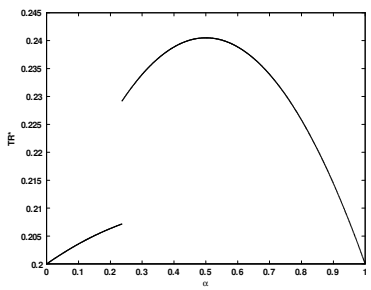
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = \hat{\sigma}_1$



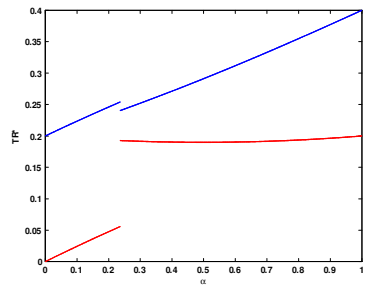
δ^* for $\sigma = \hat{\sigma}_1$

10.3 Figure 3: Trading Rates

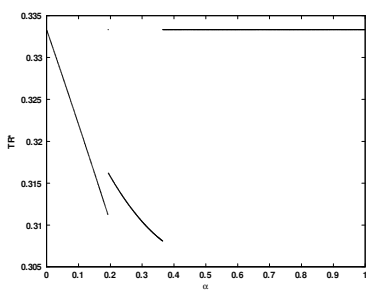
The figures in the left (right) column depict TR^* (TR^*_{ST} (blue) and TR^*_{FT} (red)) for different levels of σ ($\hat{\sigma}_3$, $\hat{\sigma}_2$, and $\hat{\sigma}_1$ are defined in Section 3) as functions of α.



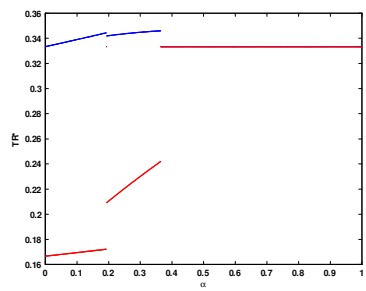
TR^* for $\sigma = \hat{\sigma}_3$



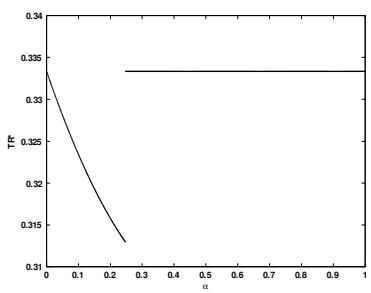
TR^*_{ST} and TR^*_{FT} for $\sigma = \hat{\sigma}_3$



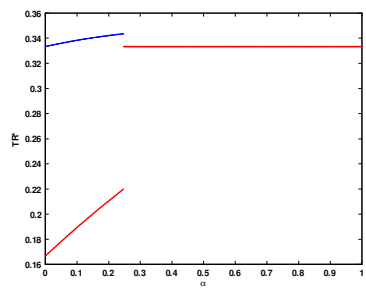
TR^* for $\sigma = \hat{\sigma}_2$



TR^*_{ST} and TR^*_{FT} for $\sigma = \hat{\sigma}_2$



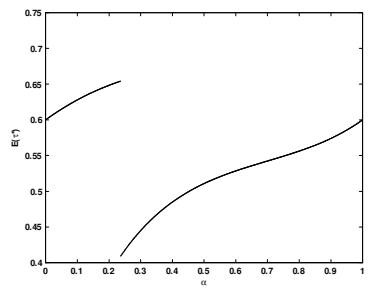
TR^* for $\sigma = \hat{\sigma}_1$



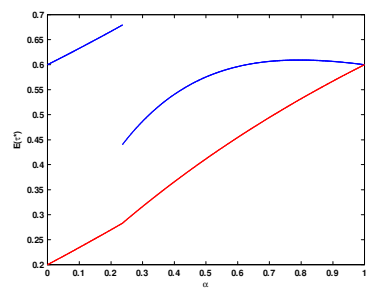
TR^*_{ST} and TR^*_{FT} for $\sigma = \hat{\sigma}_1$

10.4 Figure 4: Trading Costs

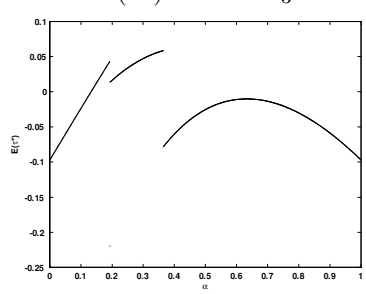
The figures in the left (right) column depict $E(\tau^*)$ ($E(\tau^*_{ST})$ (blue) and $E(\tau^*_{FT})$ (red)) for different levels of σ ($\hat{\sigma}_3$, $\hat{\sigma}_2$, and $\hat{\sigma}_1$ are defined in Section 3) as functions of α.



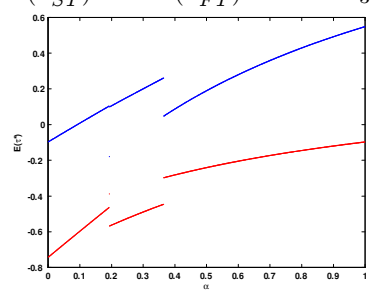
$E(\tau^*)$ for $\sigma = \hat{\sigma}_3$



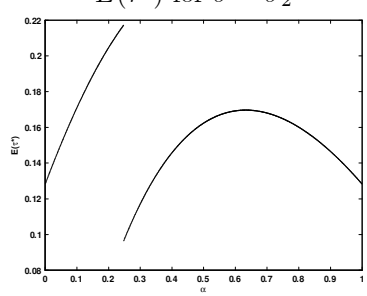
$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = \hat{\sigma}_3$



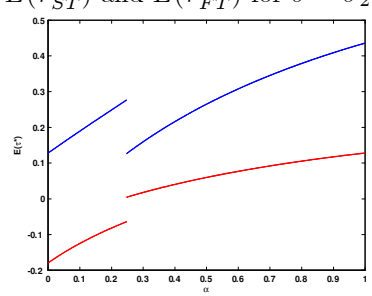
$E(\tau^*)$ for $\sigma = \hat{\sigma}_2$



$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = \hat{\sigma}_2$



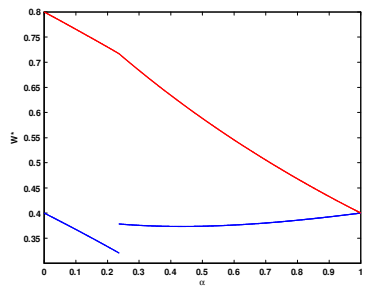
$E(\tau^*)$ for $\sigma = \hat{\sigma}_1$



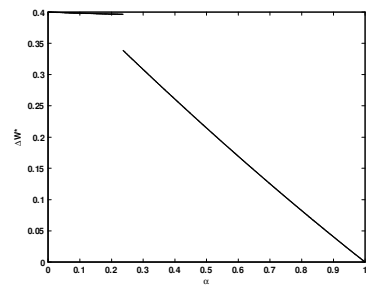
$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = \hat{\sigma}_1$

10.5 Figure 5: Welfare

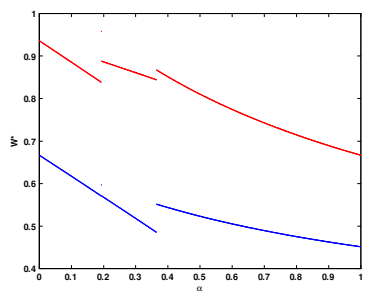
The figures in the left (right) column depict WF^*_{ST} (blue) and WF^*_{FT} (red) (ΔW^*) for different levels of σ ($\hat{\sigma}_3$, $\hat{\sigma}_2$, and $\hat{\sigma}_1$ are defined in Section 3) as functions of α.



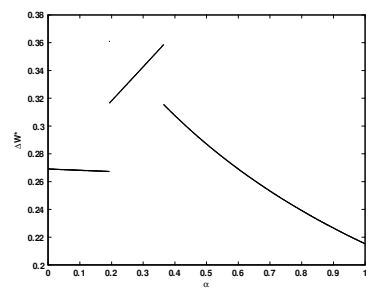
W^*_{ST} and W^*_{FT} for $\sigma = \hat{\sigma}_3$



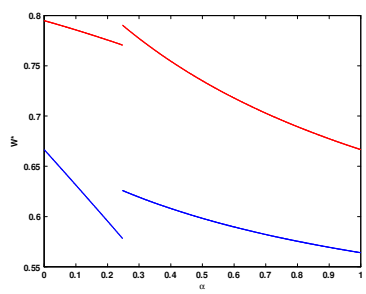
ΔW^* for $\sigma = \hat{\sigma}_3$



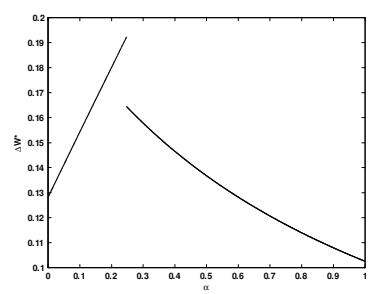
W^*_{ST} and W^*_{FT} for $\sigma = \hat{\sigma}_2$



ΔW^* for $\sigma = \hat{\sigma}_2$



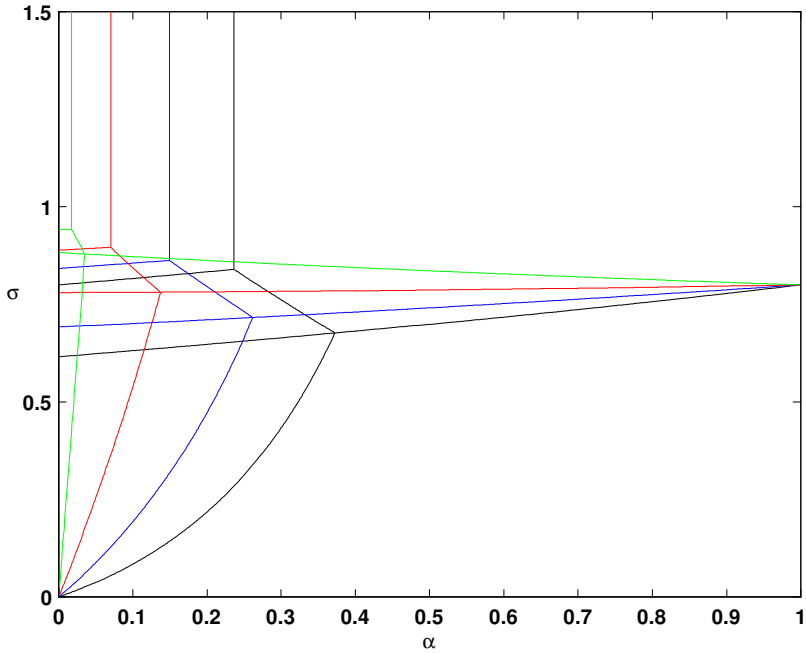
W^*_{ST} and W^*_{FT} for $\sigma = \hat{\sigma}_1$



ΔW^* for $\sigma = \hat{\sigma}_1$

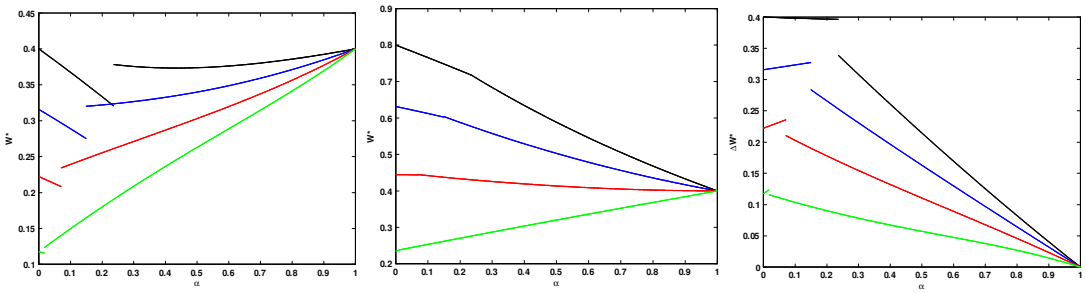
10.6 Figure 6: Equilibrium Map for different values of η

This graph depicts the different regions in the (α, σ)-space that give rise to the respective equilibria for $\eta = 1$ (black), $\eta = 0.75$ (blue), $\eta = 0.5$ (red) and $\eta = 0.25$ (green). We have set $L = 1$.



10.7 Figure 7: Welfare with efficiency gains

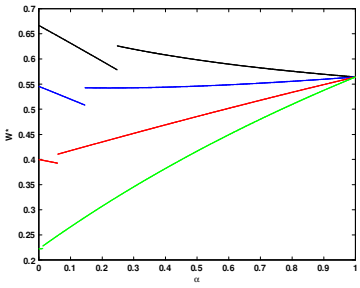
These figures depict W^*_{ST}, W^*_{FT} and W^* (from left to right column) as functions of α for different values of σ ($\hat{\sigma}_1$ and $\hat{\sigma}_3$ defined in Section 3) and η ($\eta = 1$ (black), $\eta = 0.75$ (blue), $\eta = 0.5$ (red) and $\eta = 0.25$ (green)).



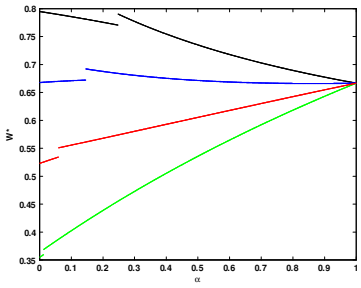
W^*_{ST} for $\sigma = \hat{\sigma}_1$

W^*_{ST} for $\sigma = \hat{\sigma}_1$

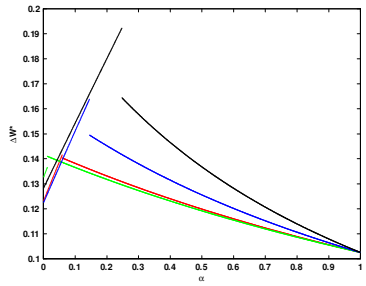
ΔW^* for $\sigma = \hat{\sigma}_1$



W^*_{ST} for $\sigma = \hat{\sigma}_3$



W^*_{FT} for $\sigma = \hat{\sigma}_3$

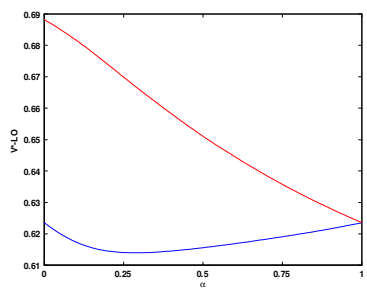


W^* for $\sigma = \hat{\sigma}_3$

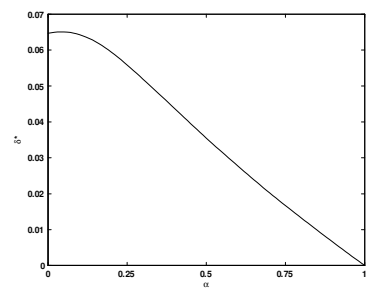
11 Appendix B: Figures

11.1 Figure 1: Equilibrium limit order profits and δ^*

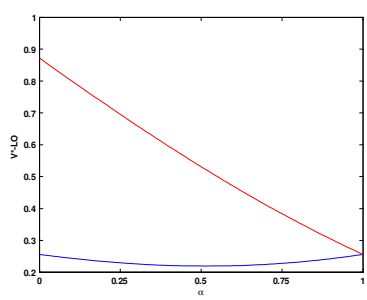
The figures in the left (right) column depict V_{ST}^{LO*} (blue) and V_{FT}^{LO*} (red) (δ^*) as functions of α for $\sigma = 0.05$, $\sigma = 1$ and $\sigma = 5$, respectively.



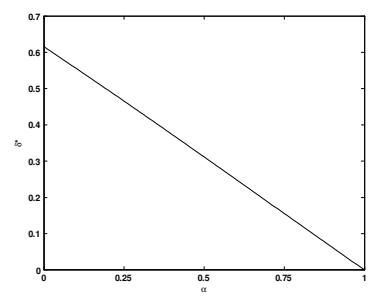
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = 0.05$



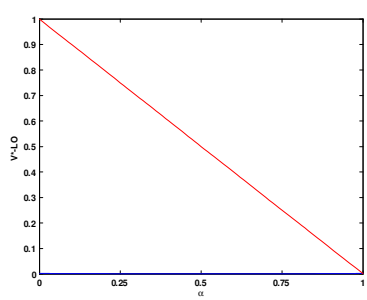
δ^* for $\sigma = 0.05$



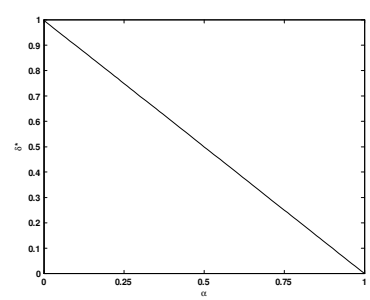
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = 1$



δ^* for $\sigma = 1$



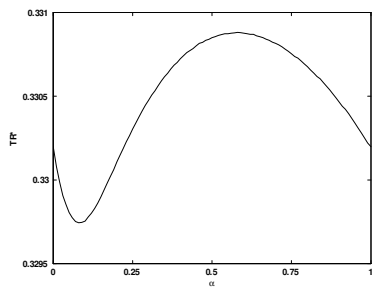
V_{ST}^{LO*} and V_{FT}^{LO*} for $\sigma = 5$



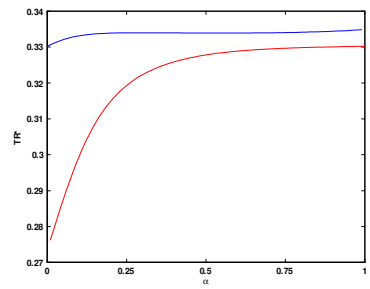
δ^* for $\sigma = 5$

11.2 Figure 2: Trading Rates

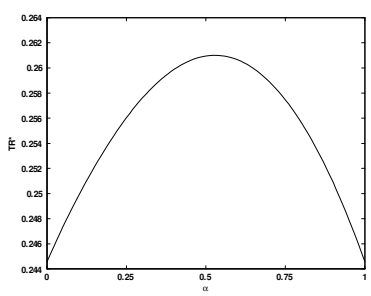
The figures in the left (right) column depict TR^* (TR^*_{ST} (blue) and TR^*_{FT} (red)) as functions of α for $\sigma = 0.05$, $\sigma = 1$ and $\sigma = 5$, respectively.



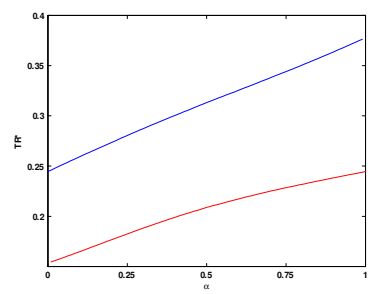
TR^* for $\sigma = 0.05$



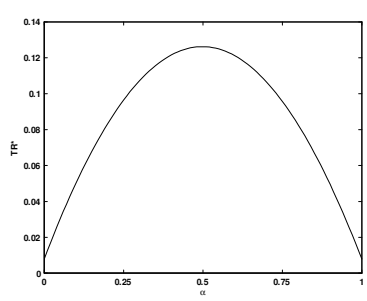
TR^*_{ST} and TR^*_{FT} for $\sigma = 0.05$



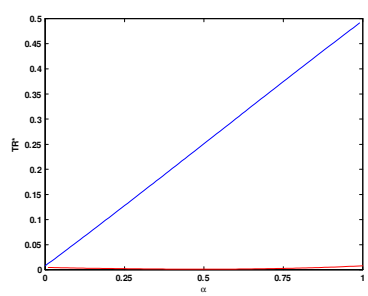
TR^* for $\sigma = 1$



TR^*_{ST} and TR^*_{FT} for $\sigma = 1$



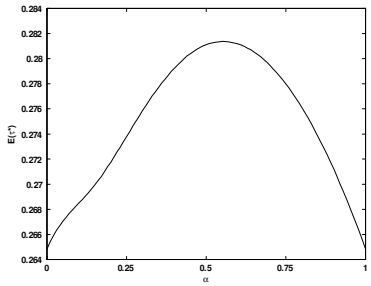
TR^* for $\sigma = 5$



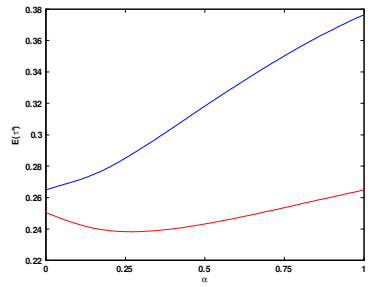
TR^*_{ST} and TR^*_{FT} for $\sigma = 5$

11.3 Figure 3: Trading Costs

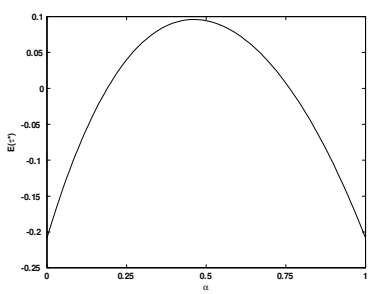
The figures in the left (right) column depict $E(\tau^*)$ ($E(\tau^*_{ST})$ (blue) and $E(\tau^*_{FT})$ (red)) as functions of α for $\sigma = 0.05$, $\sigma = 1$ and $\sigma = 5$, respectively.



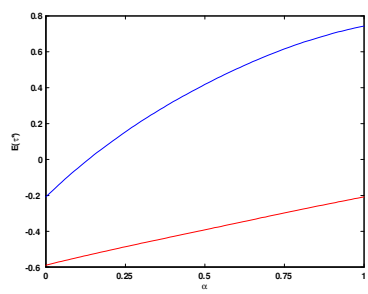
$E(\tau^*)$ for $\sigma = 0.05$



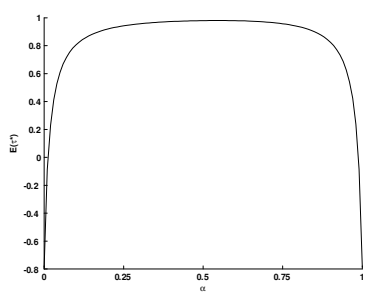
$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = 0.05$



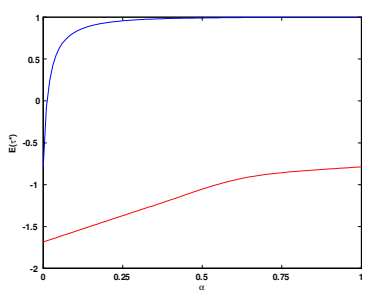
$E(\tau^*)$ for $\sigma = 1$



$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = 1$



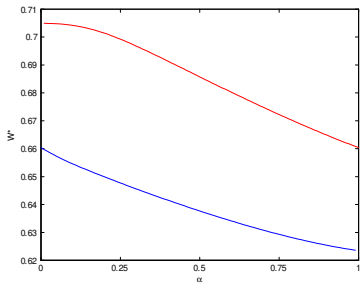
$E(\tau^*)$ for $\sigma = 5$



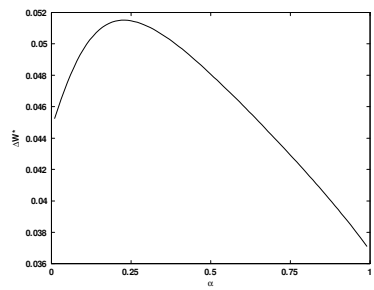
$E(\tau^*_{ST})$ and $E(\tau^*_{FT})$ for $\sigma = 5$

11.4 Figure 4: Welfare

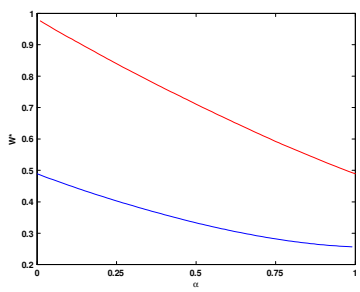
The figures in the left (right) column depict WF^*_{ST} (blue) and WF^*_{FT} (red) (ΔW^*) as functions of α for $\sigma = 0.05$, $\sigma = 1$ and $\sigma = 5$, respectively.



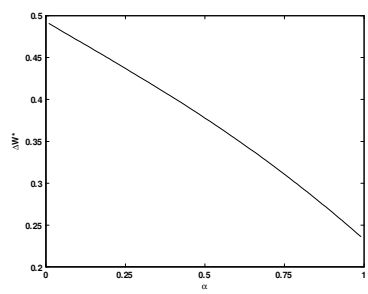
W^*_{ST} and W^*_{FT} for $\sigma = 0.05$



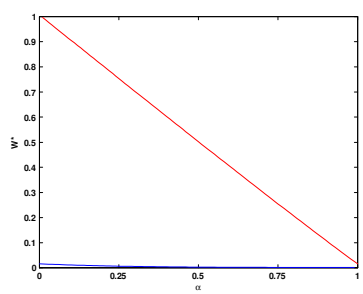
ΔW^* for $\sigma = 0.05$



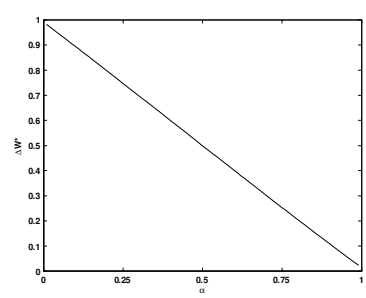
W^*_{ST} and W^*_{FT} for $\sigma = 1$



ΔW^* for $\sigma = 1$



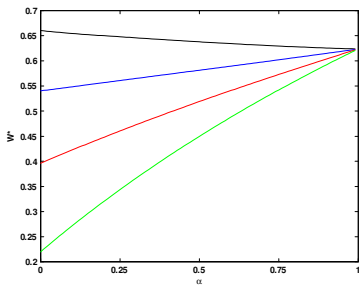
W^*_{ST} and W^*_{FT} for $\sigma = 5$



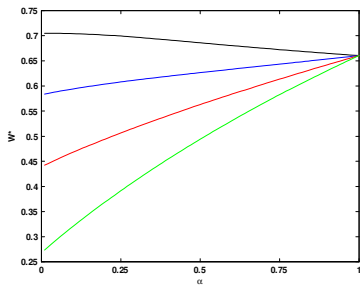
ΔW^* for $\sigma = 5$

11.5 Figure 5: Welfare with efficiency gains

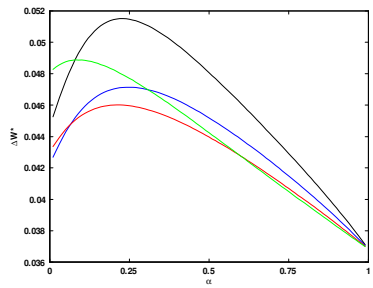
These figures depict W^*_{ST}, W^*_{FT} and W^* (from left to right column) as functions of α for $\sigma = 0.05$, $\sigma = 1$ and $\sigma = 5$, respectively, and different values of η ($\eta = 1$ (black), $\eta = 0.75$ (blue), $\eta = 0.5$ (red) and $\eta = 0.25$ (green)).



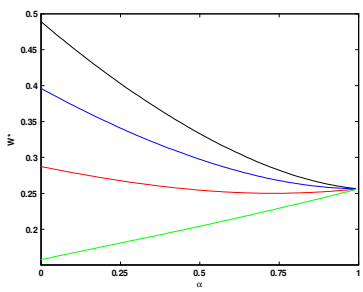
W^*_{ST} for $\sigma = 0.05$



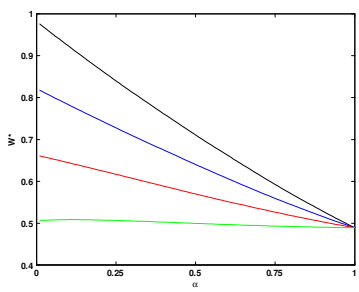
W^*_{FT} for $\sigma = 0.05$



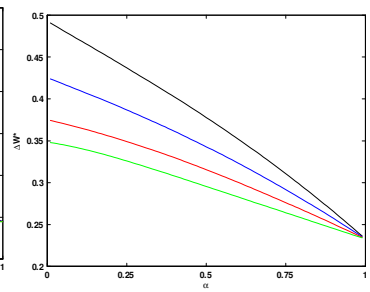
ΔW^* for $\sigma = 0.05$



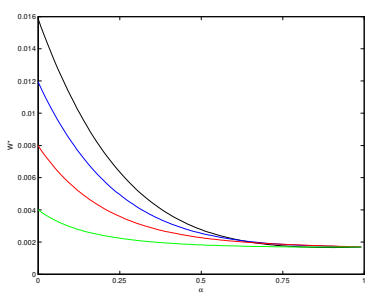
W^*_{ST} for $\sigma = 1$



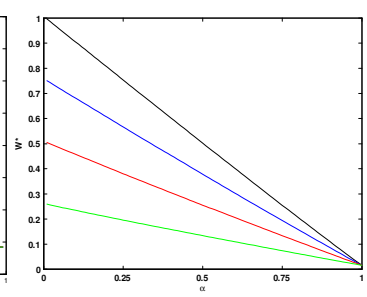
W^*_{FT} for $\sigma = 1$



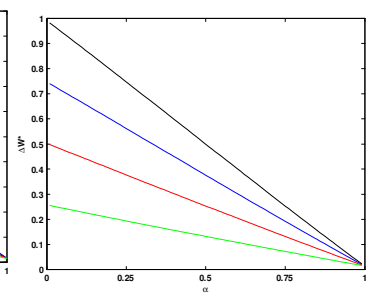
ΔW^* for $\sigma = 1$



W^*_{ST} for $\sigma = 5$



W^*_{FT} for $\sigma = 5$



ΔW^* for $\sigma = 5$