Characterization of monotonic rules in minimum cost spanning tree problems

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Abstract

We characterize, in minimum cost spanning tree problems, the family of rules satisfying monotonicity over cost and population. We also prove that the set of allocations induced by the family coincides with the irreducible core.

Keywords: Cost sharing, minimum cost spanning tree problems, monotonicity, irreducible core.

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*, for short). A group of agents (denoted by N), located at different geographical places, want a particular service which can only be provided by a common supplier, called the source (denoted by 0). Agents will be served through connections which involve some cost. However, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C, where c_{ij} denotes the connection costs between i and j ($i, j \in N \cup \{0\}$).

We assume that agents construct a minimum cost spanning tree (*mcst*). The question is how to divide the cost associated with the *mcst* between the agents. One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help agents/planner to compare different rules and to decide which rule is preferred in a particular situation.

In this paper we focus on two monotonicity properties. *Population monotonicity* claims that if new agents join a "society" no agent from the "initial society" can be worse off; and *cost monotonicity* claims that if connection costs weakly increase, no agent can be better off.

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In the literature there exist two families of rules satisfying both properties. The optimistic weighted Shapley rules studied in Bergantiños and Lorenzo-Freire (2008a, 2008b) and obligation rules studied in Tijs et al. (2006), Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010).

The main objective of this paper is to study the set of rules satisfying population monotonicity and cost monotonicity. We focus on two aspects: to characterize the set of rules satisfying both properties and to characterize the set of allocations induced by these rules.

Given a *mcstp* C, Bird (1976) considers the irreducible matrix C^*, which is obtained from C by reducing the cost of the arcs as much as possible, but without reducing the cost of the *mcst*. Bird (1976) associates with each *mcstp* C a cooperative game with transferable utility (N, v_C). We prove that the set of allocations induced by rules satisfying population monotonicity and cost monotonicity is the core of the game (N, v_{C^*}), the so called *irreducible core*.

A weaker version of population monotonicity is *separability*, which claims that if two groups of agents can connect to the source independently of each other, then we can compute their payments separately. A weaker version of cost monotonicity is *reductionism*, which claims that the rule must depend only on the irreducible matrix. We identify a necessary and sufficient condition for a family of rules to cover all the ones satisfying separability and reductionism. In order to describe this condition, we need to define the so-called, neighborhoods and extra-costs correspondences. A *neighborhood* is a group of agents that are "closer" to each other than to any of the other agents or to the source. An *extra-costs correspondence* is a way of dividing the savings obtained by the agents of a neighborhood when they connect each other through an optimal network. The intuition behind such rules is the following. Initially each agent is connected to the source in the irreducible matrix. Now, agents inside neighborhoods are connected among them. For each neighborhood, the savings are divided among the agents in the neighborhood following the extra-costs correspondence.

We characterize the set of rules satisfying population monotonicity and cost monotonicity, which is a subset of the previous set. We need to select the extra-costs correspondences satisfying the so called *non-decreasing costs property*, which says that the aggregate sum given by the extra-costs correspondence should not decrease when the connection cost between two consecutive neighborhoods is increased. We show how some well-known rules of the literature satisfying both properties can be defined using the extra-costs correspondences.

Our result could be applied for identifying new class of rules satisfying both monotonicity properties. We do it by introducing a class of rules that generalize the obligation rules.

The paper is organized as follows. In Section 2 we introduce the model and the notation. In Section 3 we characterize the set of allocations induced by the rules satisfying population and cost monotonicity. In Section 4 we characterize the set of rules satisfying separability and reductionism. In Section 5 we characterize the set of rules satisfying population monotonicity and cost monotonicity. In Section 6 we apply these results to some known rules in the literature. The proofs are presented in the Appendix.

2 Minimum cost spanning tree problems

We first introduce minimum cost spanning tree problems and some notation used through the paper.

Let $U = \{1, 2, 3, ...\}$ be the (infinite) set of possible agents, and let 0 be a special node called the *source*.

A *minimum cost spanning tree problem* (*mcstp*) is a pair (N_0, C_0) where $N_0 = N \cup \{0\}$, $N \subset U$ is finite and $C_0 = (c_{ij})_{i,j \in N_0}$ is a matrix with $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N_0$.

A *minimum cost connection problem* (*mccp*) is a pair (N, C) where $N \subset U$ is finite and $C = (c_{ij})_{i,j \in N}$ is a matrix with $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N$.

For simplicity, when N is clear, we write C_0 instead of (N_0, C_0) and C instead of (N, C).

Let $\mathcal{C}_0$ be the set of all *mcstp* and let $\mathcal{C}$ be the set of all *mccp*.

Let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of agents in N which come before i in the order given by π, *i.e.*, $Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}$.

As usually, $\mathbb{R}_+$ denotes the set of non-negative real numbers. Given a nonempty set A, let $\Delta(A) = \left\{(x_i)_{i \in A} \in \mathbb{R}_+^A : \sum_{i \in A} x_i = 1\right\}$ be the simplex in $\mathbb{R}^A$.

A *graph* in N_0 is a subset of $\{\{i, j\} : i, j \in N_0, i \neq j\}$. The *cost* of a graph g in (N_0, C_0) is defined as $c(g, C_0) = \sum_{\{i,j\} \in g} c_{ij}$. Analogous definitions can be given for a graph in (N, C).

Given $i, j \in N_0$, a *path* between i and j is a graph of different arcs $\{\{i_{k-1}, i_k\}\}_{k=1}^K$ such that $i_0 = i$ and $i_K = j$. A *spanning tree* in N_0 is a graph in N_0 in which there exists exactly one path between any pair of nodes. Let $\mathbb{G}(N_0)$ (or simply $\mathbb{G}_0$) denote the set of all graphs in N_0 and let $\mathbb{T}(N_0)$ (or simply $\mathbb{T}_0$) denote the set of all spanning trees in N_0. Analogously, we define $\mathbb{G}(N)$ (or simply $\mathbb{G}$) and $\mathbb{T}(N)$ (or simply $\mathbb{T}$) for N.

A *minimum cost spanning tree* (*mcst*) in (N_0, C_0) (respectively, (N, C)) is a spanning tree t in N_0 (respectively, N) with minimum cost, namely $m(t) = \min_{t' \in \mathbb{T}_0} m(t')$ (respectively, $m(t) = \min_{t' \in \mathbb{T}} m(t')$).

A *mcst* is not necessarily unique. However, all *mcst* in C_0 (or in C) have the same cost, that we denote as $m(N_0, C_0)$ (or $m(N, C)$).

Given $S \subset N$, we denote as (S, C_S) the restriction of (N, C) to S, and we denote as $(S_0, (C_S)_0)$ the restriction of (N_0, C_0) to S.

We denote $\max C := \max_{i,j \in N} c_{ij}$ and $\max C_0 := \max_{i,j \in N_0} c_{ij}$.

Given $i, j \in N$, $\alpha \in \mathbb{R}_+$, we denote as αI_{ij} the matrix C given by $c_{kl} = 0$ for all $\{k, l\} \neq \{i, j\}$ and $c_{ij} = \alpha$.

Given $C_0 \in \mathcal{C}_0$, the *irreducible matrix* of C_0, denoted by C_0^*, is defined for each $i, j \in N_0$ as

$$c_{ij}^* = \max_{\{k,l\} \in \tau_{ij}} c_{kl} \tag{1}$$

where τ_{ij} is the (unique) path that connects i and j in some *mcst*. This matrix is well-defined, *i.e.* it does not depend on the chosen *mcst*.

Denote $\mathcal{C}_0^* = \{C_0^* : C_0 \in \mathcal{C}_0\}$. Analogously, $\mathcal{C}^* = \{C^* : C \in \mathcal{C}\}$.

A *rule* is a function f that assigns to each $C_0 \in \mathcal{C}_0$ a vector $f(C_0) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(C_0) = m(C_0)$. As usual, $f_i(C_0)$ represents the payoff assigned to agent $i \in N$.

We now introduce some properties of rules, which we will use in this paper.

Population Monotonicity (PM). For all *mcstp* (N_0, C_0), $S \subset N$, and $i \in S$, we have

$$f_i(N_0, C_0) \leq f_i(S_0, (C_S)_0).$$

This property says that if new agents join a network, no agent from the initial network can be worse off.

Cost Monotonicity (CM). For all *mcstp* (N_0, C_0) and (N_0, C_0') such that $C_0 \leq C_0'$, we have

$$f(N_0, C_0) \leq f(N_0, C_0').$$

This property says that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off. This property is also called solidarity or strong cost monotonicity in some papers such as Bergantiños and Vidal-Puga (2007) and Bergantiños and Kar (2010).

Separability (SEP). For all *mcstp* (N_0, C_0) and $S \subset N$ satisfying $m(N_0, C_0) = m(S_0, (C_S)_0) + m\left((N \backslash S)_0, \left(C_{N \backslash S}\right)_0\right)$, we have

$$f_i(N_0, C) = \begin{cases} f_i(S_0, (C_S)_0) & \text{if } i \in S \\ f_i\left((N \backslash S)_0, \left(C_{N \backslash S}\right)_0\right) & \text{if } i \in N \backslash S. \end{cases}$$

Two subsets of agents, S and $N \backslash S$, can be connected to the source either separately or jointly. If there are no savings when they are jointly connected to the source, this property says that the agents will pay the same in both circumstances. This property is also called decomposition in some papers such as Megiddo (1978) and Granot and Huberman (1981).

Reductionism (RED). For all (N_0, C_0),

$$f(N_0, C_0) = f(N_0, C_0^*).$$

If a rule satisfies this property, then it only depends on irreducible matrices. RED appears in Bogomolnaia and Moulin (2010) and it is introduced in Bergantiños and Vidal-Puga (2007) where it is called independence of irrelevant trees.

PM implies SEP but the reciprocal is false. CM implies RED but the reciprocal is false. See Bergantiños and Vidal-Puga (2007) for details.

3 The irreducible core

Bird (1976) introduces the irreducible core of a *mcstp* (N_0, C_0). We define the *set of monotonic allocations* as the set of allocations induced by rules satisfying CM and PM. In this section we prove that this set coincides with the irreducible core, defined as follows.

A *game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \rightarrow \mathbb{R}$ satisfies $v(\varnothing) = 0$.

The *core* of a *TU* game (N, v) is defined as

$$core\,(N, v) = \left\{ (x_i)_{i \in N} : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \leq v(S)\, \forall S \subset N \right\}.$$

Bird (1976) associates with each *mcstp* (N_0, C_0) the game (N, v_{C_0}). For each coalition $S \subset N$, $v_{C_0}(S) := m(S_0, (C_S)_0)$.

The *irreducible core* of a *mcstp* (N_0, C_0), denoted as $IC(N_0, C_0)$, is the core of the TU game $\left(N, v_{C_0^*}\right)$ where C_0^* is the irreducible matrix associated with C_0.

Given a *mcstp* (N_0, C_0), let $AM(N_0, C_0)$ denote the set of allocations induced by the rules satisfying CM and PM. Namely, $x \in AM(N_0, C_0)$ if and only if there exists a rule f satisfying CM and PM such that $x = f(N_0, C_0)$.

In the next theorem we prove that $AM(N_0, C_0)$ and $IC(N_0, C_0)$ coincide.

Theorem 1. For each *mcstp* (N_0, C_0), $AM(N_0, C_0) = IC(N_0, C_0)$.

Proof. See the Appendix.

As a consequence of Theorem 1, any rule f satisfying CM and PM gives, for any *mcstp* (N_0, C_0), an element $f(N_0, C_0)$ in the irreducible core of (N_0, C_0). Nevertheless, the reciprocal is not true. Given a rule f such that, for each *mcstp* (N_0, C_0), $f(N_0, C_0) \in IC(N_0, C_0)$, it could be the case that f does not satisfy both monotonicity properties.

4 The set of rules satisfying separability and reductionism

In this section we characterize the set of rules satisfying SEP and RED. For doing it we need some new definitions. A *neighborhood* is a group of agents that are "closer" to each other than to any of the other agents or to the source. An *extra-costs correspondence* is a way of dividing the savings obtained by the agents of a neighborhood when they connect among themselves. The rules satisfying both properties could be described as follows. Initially each agent is connected to the source in the irreducible matrix. Now, agents inside neighborhoods are connected among them. For each neighborhood, the savings are divided among the agents in the neighborhood following the extra-costs correspondence.

We first introduce the concepts which will be crucial in our results.

Given $(N_0, C_0) \in \mathcal{C}_0$ and $S \subset N$, $|S| > 1$, we define

$$\delta_S = \min_{i \in S, j \in N_0 \setminus S} c_{ij} - \max_{\{i,j\} \in \tau(S)} c_{ij}$$

where $\tau(S) \in \mathbb{T}(S)$ is a *mcst* in S connecting all the agents in S. Even though $\tau(S)$ is not necessarily unique, it is not difficult to check that $\max_{\{i,j\}\in\tau(S)} c_{ij}$ does not depend on the particular $\tau(S)$ and hence δ_S is well defined. For $S = \{i\}$, we also define $\delta_{\{i\}} = \min_{j\in N_0\backslash\{i\}} c_{ij}$.

Roughly speaking, δ_S may be interpreted, when positive, as some kind of "distance" between S and $N_0\backslash S$.

Definition 1. Let (N_0, C_0) be an *mcstp*. We say that $S \subset N$, $|S| > 1$, is a *neighborhood* in C_0 if $\delta_S > 0$. We denote the set of all neighborhoods in C_0 as $Ne(C_0)$.

Example 1. Let $N = \{1, 2, 3, 4, 5, 6\}$ and C_0 be such that $c_{01} = 50$, $c_{12} = 20$, $c_{13} = 40$, $c_{34} = 10$, $c_{15} = 60$, $c_{36} = 70$, and $c_{ij} > 70$ otherwise. There are exactly two neighborhoods containing node 1: $\{1, 2\}$ because $\delta_{\{1,2\}} = c_{13} - c_{12} = 20$, and $\{1, 2, 3, 4\}$ because $\delta_{\{1,2,3,4\}} = c_{01} - c_{13} = 50 - 40 = 10$. Notice that $\{1, 2, 3\}$ is not a neighborhood because $\delta_{\{1,2,3\}} = c_{34} - c_{13} = -30$.

Some comments about neighborhoods. It is not difficult to check that the neighborhoods of C_0 and C_0^* coincide. Nevertheless, in general, $(C^*)_S \neq (C_S)^*$. Take for example $N = \{1, 2, 3\}$, $c_{12} = c_{13} = 1$, $c_{23} = 2$ and $S = \{2, 3\}$. Then, $c_{23}^* = 1$ and hence $C' = (C^*)_S$ satisfies $c'_{23} = 1$ whereas $C'' = (C_S)^*$ satisfies $c''_{23} = 2$. Later on (Proposition 1.1) we prove that the equality holds when S is a neighborhood.

The next proposition gives some results about neighborhoods.

Proposition 1.

1. $S \subset N$ is a neighborhood in C_0 if and only if S is a neighborhood in C_0^*. Besides, $(C_S)^* = (C^*)_S$ and
$$\delta_S = \min_{i\in S, j\in N_0\backslash S} c_{ij}^* - \max_{i,j\in S} c_{ij}^*.$$
2. If S is a neighborhood in C_0 and $i \in S$, then
$$S = \left\{ j \in N : c_{ij}^* < \min_{k\in S, l\in N_0\backslash S} c_{kl}^* \right\}.$$
3. If S, S' are two neighborhoods in $C_0^* \in \mathcal{C}_0^*$ and $S \cap S' \neq \emptyset$, then either $S \subset S'$ or $S' \subset S$.
4. For each $i \in N$, there exists a unique family of subsets of N, $S_1, S_2, ..., S_q$ with $q \geq 0$[1] such that $\{S_1, ..., S_q\}$ is the set of neighborhoods that contain i, and $S_1 \subset S_2 \subset ... \subset S_q$.
5. There exist no neighborhood in C_0 if and only if $\{\{0, i\}\}_{i\in N}$ is a *mcst* in C_0.

[1] Case $q = 0$ covers the situation in which agent i has no neighborhoods.

Proof of Proposition 1. See the Appendix.

Under Proposition 1.1, for each neighborhood $S \subset N$, we have $(C^*)_S = (C_S)^*$. We denote this matrix as C^*_S.

We now introduce the family of extra cost correspondences, which will be used in the definition of the rules we characterize.

Definition 2. An *extra-costs correspondence* is a function $e : \mathcal{C}^* \times \mathbb{R}_+ \to \mathbb{R}^U_+$ satisfying:

(E1) $e_i(C^*, x) = 0$ for all $(N, C^*) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \notin N$.

(E2) $\sum_{i \in U} e_i(C^*, x) = x$ for all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$.

Definition 3. For each extra-costs correspondence e we define the rule f^e as follows. Given $(N_0, C_0) \in \mathcal{C}_0$ and $i \in N$,

$$f^e_i(N_0, C_0) := c^*_{0i} - \sum_{\substack{S \text{ neighborhood} \\ i \in S}} \left(\delta_S - e_i\left(C^*_S, \delta_S\right)\right).$$

The intuition behind such rules is the following. Initially each agent i pays c^*_{0i}. Now, agents inside neighborhoods are connected among them. For each neighborhood S, the savings are divided among the agents in S following e. The larger is $e_i(C^*_S, \delta_S)$, the smaller is the saving $(\delta_S - e_i(C^*_S, \delta_S))$ corresponding to agent i in neighborhood S.

We compute f^e in two examples.

Example 2. Let $N = \{1, 2\}$, $c_{01} = 10$, $c_{02} = 15$, and $c_{12} = 2$. Then, $c^*_{10} = c^*_{20} = 10$ and $c^*_{12} = 2$. Let e be such that for each C^* and each x, $e_1(C^*, x) = \frac{3x}{4}$ and $e_2(C^*, x) = \frac{x}{4}$. There is a unique neighborhood $S = N$ with $\delta_N = 10 - 2 = 8$. Now,

$$\begin{aligned} f^e_1(C_0) &= c^*_{01} - (\delta_N - e_1(C^*, 8)) = 10 - \left(8 - \frac{3}{4}8\right) = 8 \text{ and} \\ f^e_1(C_0) &= c^*_{02} - (\delta_N - e_2(C^*, 8)) = 10 - \left(8 - \frac{1}{4}8\right) = 4. \end{aligned}$$

Example 1 (continuation). Let e be defined as $e_j(C'^*, x) = \frac{x}{|N'|}$ for all $(N', C'^*) \in \mathcal{C}$ and $j \in N$ ($e_j(C'^*, x) = 0$ otherwise). We compute $f^e_1(C_0)$. There are two neighborhoods containing agent 1: $S_1 = \{1, 2\}$ and $S_2 = \{1, 2, 3, 4\}$. Besides $c^*_{01} = 50$, $\delta_{S_1} = 20$ and $\delta_{S_2} = 10$.

Then,

$$\begin{aligned} f^e_1(C_0) &= 50 - \left(\delta_{S_2} - e_2\left(C^*_{S_2}, 10\right)\right) - \left(\delta_{S_1} - e_1\left(C^*_{S_1}, 20\right)\right) \\ &= 50 - (10 - 2.5) - (20 - 10) = 32.5. \end{aligned}$$

It is not difficult to check that f^e can also be defined as

$$f_i^e(C_0) = c_{0i}^* - \sum_{\substack{S \text{ neighborhood} \\ i \in S}} \left(\sum_{j \in S \setminus \{i\}} e_j(C_S^*, \delta_S) \right).$$

In Proposition 2 we prove that each f^e is a rule, namely, $\sum_{i \in N} f_i^e(N_0, C_0) = m(N_0, C_0)$.

Proposition 2. For each extra-costs correspondence e, f^e is a rule.
Proof of Proposition 2. See the Appendix.

In Theorem 2 we characterize this family of rules.

Theorem 2. A rule f satisfies Separability and Reductionism if and only if $f = f^e$ for some extra-costs correspondence e.
Proof of Theorem 2. See the Appendix.

5 The set of rules satisfying population monotonicity and cost monotonicity

In this section we characterize the set of rules satisfying both monotonicity properties. Since PM implies SEP and CM implies RED, this set of rules will be a subset of the set characterized in the previous section. We will prove that such set of rules coincides with the set of rules induced by extra-costs correspondences satisfying a non-decreasing property.

We first introduce the concepts we will use.

Given $(N^1, C^1), (N^2, C^2) \in \mathcal{C}$, $N^1 \cap N^2 = \emptyset$, and $a \in \mathbb{R}_+$, we define

$$\left(N^1 \cup N^2, C^1 \oplus_a C^2\right)$$

as the *mccp* C given by $c_{ij} = c_{ij}^\alpha$ if $i, j \in N^\alpha$ for some $\alpha \in \{1, 2\}$, and $c_{ij} = a + \max C^1$ for all $i \in N^1$, $j \in N^2$.

For convenience, we write $C^1 \oplus_a C^2 \oplus_b C^3$ instead of $(C^1 \oplus_a C^2) \oplus_b C^3$, and so on.

Given $a = (a_1, ..., a_\Gamma) \in \mathbb{R}_+^\Gamma$, $\left(C^1, ..., C^\Gamma\right) \in \mathcal{C}^\Gamma$, and $\gamma \leq \Gamma$ we denote

$$C^\gamma(a) = C^1 \oplus_{a_1} C^2 \oplus_{a_2} ... \oplus_{a_{\gamma-1}} C^\gamma.$$

Notice that, given $\gamma > 1$,

$$C^\gamma(a) = C^{\gamma-1}(a) \oplus_{a_{\gamma-1}} C^\gamma. \tag{2}$$

Definition 4. We say that an extra-costs correspondence e satisfies the *Non-Decreasing Costs* (NDC) property if for all disjoint sequences $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^\Gamma \subset \mathcal{C}^*$, $\Gamma \geq 1$, $i \in N^{\gamma_i}$ with

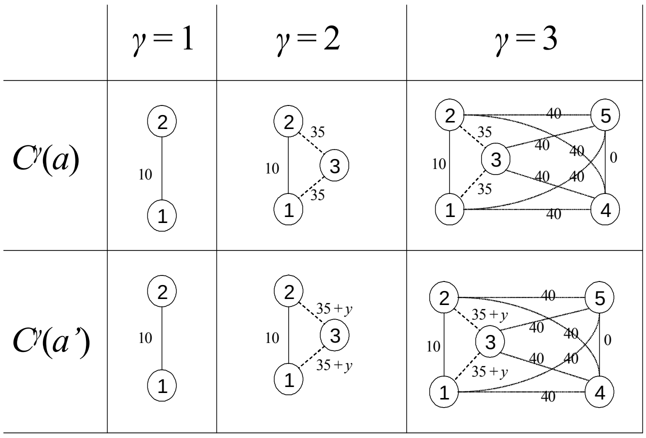

Figure 1: Minimum cost connection problems $C^{\gamma}(a), C^{\gamma}(a')$ for $\gamma = 1, 2, 3$. The NDC property requires the aggregate assignment of extra costs for players 1, 2, 4 and 5 to be not higher with a than with a'.

$\gamma_i \neq 2$, $a \in \mathbb{R}^{\Gamma}_{+}$ with $a_{\gamma} \geq \max C^{\gamma+1} - \max C^{\gamma}$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$ ($y \geq 0$ when $\Gamma = 1$), we have

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a'), a'_{\gamma}\right) \geq \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}(a), a_{\gamma}\right)$$

where $a' = (a_1 + y, a_2 - y, a_3, ..., a_{\Gamma})$ ($a' = (a_1 + y)$ when $\Gamma = 1$).

In the next example we give an intuition of this technical property.

Example 3. Let $\Gamma = 3$, $N^1 = \{1, 2\}, c^1_{12} = 10, N^2 = \{3\}, N^3 = \{4, 5\}$ and $c^3_{45} = 0$. Then, $a = (25, 5, 20)$ and $a' = (25 + y, 5 - y, 20)$ with $y \in [0, 5]$ satisfy the conditions imposed on the definition of NDC: $a_1 = 25 \geq 0 - 10 = \max C^2 - \max C^1$, $a_2 = 5 \geq 0 - 0 = \max C^3 - \max C^2$. $C^{\gamma}(a)$ and $C^{\gamma}(a')$ are described in Figure 1.

Given $i \in N^1$, the NDC property says that

$$\begin{aligned} & e_i\left(C^1(a'), 25 + y\right) + e_i\left(C^2(a'), 5 - y\right) + e_i\left(C^3(a'), 20\right) \\ \geq \ & e_i\left(C^1(a), 25\right) + e_i\left(C^2(a), 5\right) + e_i\left(C^3(a), 20\right). \end{aligned}$$

Given $i \in N^2$, the NDC property says nothing (since we assume $\gamma_i \neq 2$).

Given $i \in N^3$, the NDC property says that

$$e_i\left(C^3(a'), 20\right) \geq e_i\left(C^3(a), 20\right).$$

We now present the characterization.

Theorem 3. A rule f satisfies PM and CM if and only if $f = f^e$ for some extra-costs correspondence e satisfying the NDC property.

Proof of Theorem 3. See the Appendix.

In the literature some authors studied families of rules satisfying both monotonicity properties. The *Equal Remaining Obligations* (*ERO*) was originally introduced by Feltkamp et al. (1994) and later studied in Brânzei et al. (2004) and Bergantiños and Vidal-Puga (2007), among others[2]. The *optimistic weighted Shapley rules* are a family of rules defined by Bergantiños and Lorenzo-Freire (2008a, 2008b). *Obligation rules* were introduced by Tijs et al. (2006) and studied later in Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010). The ERO rule is a optimistic weighted Shapley rule. Besides, optimistic weighted Shapley rules are a subset of obligation rules.

We now show how these rules can be included in our family.

Proposition 3.

1. Obligation rules are the rules f^e where for each (C^*, x) and each $i \in N$,

$$e_i (C^*, x) = o_i (N) x$$

where o is a function o that assigns to each N a vector $o \in \Delta (N)$ such that $o_i (S) \geq o_i (N)$ for all $i \in S \subset N$.

2. Optimistic weighted Shapley rules are the rules f^e such that for each (C^*, x) and each $i \in N$,

$$e_i (C^*, x) = \frac{\omega_i}{\sum_{i \in N} \omega_i} x.$$

where $\omega \in \mathbb{R}^U_+$.

3. The ERO rule is the rule f^e where for each (C^*, x) and each $i \in N$

$$e_i (C^*, x) = \frac{1}{|N|} x.$$

Proof of Proposition 3. See the Appendix.

It is clear, from Proposition 3, that the ERO rule is a particular case of an optimistic weighted Shapley rule, and those are also obligation rules. Hence, our paper provides a unified framework for all these rules.

Theorem 3 can also be used for identifying classes of rules satisfying PM and CM different from the class of rules studied in Proposition 3. We do it in the following. Let $\{o^x\}_{x \in \mathbb{R}_+}$ be a parametric family of obligation functions, i.e. for each $x \in \mathbb{R}_+$, $o^x (N) \in \Delta (N)$ and $o^x_i (S) \geq o^x_i (N)$ for all $i \in S \subsetneq N$. We assume $o^x_i (N)$ is an integrable function of x for all $i \in N$ and

$$\int_a^{a+c} o^x_i (S) \, dx \geq \int_b^{b+c} o^x_i (N) \, dx \tag{3}$$

[2] It is also known as the *folk rule*.

for all $i \in S \subsetneq N$ and $a, b, c \in \mathbb{R}_+$.

Proposition 4. Let $\{o^x\}_{x \in \mathbb{R}_+}$ be defined as before. The rule f^e with e defined as

$$e_i(C^*, x) = \int_0^x o_i^t(N)\, dt$$

for all (C^*, x) and $i \in N$, satisfies CM and PM.

Proof of Proposition 4. See the Appendix.

Clearly, this family contains the obligation rules (simply take $o^x = o$ for all x). Moreover, not all the obligation rules can be defined in this way. Take for example $\hat{o} = \{\hat{o}^x\}_{x \in \mathbb{R}_+}$ defined as follows:

$$\hat{o}_i^x(N) := \begin{cases} \frac{1}{|N|} & \text{if } |N| \neq 2 \\ \frac{1+1_{x \leq 1}}{3} & \text{if } |N| = 2 \text{ and } i = \min_{j \in N} j \\ \frac{2-1_{x \leq 1}}{3} & \text{if } |N| = 2 \text{ and } i = \max_{j \in N} j \end{cases}$$

for all $i \in N$, where $1_{x \leq 1} = 1$ if $x \leq 1$ and $1_{x \leq 1} = 0$ otherwise. The resulting rule $f^{\hat{o}}$ satisfies all the previous properties. It is similar to the ERO rule, but it charges a higher obligation to nodes with low index when the costs are higher (and vice-versa). It is not a obligation rule. For example, take $N = \{1, 2\}$ and, for $z \in \{1, 2\}$, let $\left(N, C_0^{(z)}\right)$ be defined as $c_{01}^{(z)} = c_{02}^{(z)} = z$ and $c_{12}^{(z)} = 0$. Then,

$$\begin{aligned} f_1^{\hat{o}}\left(C_0^{(1)}\right) &= 1 - \left(1 - \int_0^1 \frac{2}{3} dx\right) = \frac{2}{3} \\ f_2^{\hat{o}}\left(C_0^{(1)}\right) &= 1 - \left(1 - \int_0^1 \frac{1}{3} dx\right) = \frac{1}{3} \end{aligned}$$

whereas

$$\begin{aligned} f_1^{\hat{o}}\left(C_0^{(2)}\right) &= 2 - \left(2 - \int_0^1 \frac{2}{3} dx - \int_1^2 \frac{1}{3} dx\right) = 1 \\ f_2^{\hat{o}}\left(C_0^{(2)}\right) &= 2 - \left(2 - \int_0^1 \frac{1}{3} dx - \int_1^2 \frac{2}{3} dx\right) = 1. \end{aligned}$$

Since $f^{\hat{o}}\left(C_0^{(2)}\right) \neq 2 f^{\hat{o}}\left(C_0^{(1)}\right)$, we deduce that $f^{\hat{o}}$ does not satisfy additivity (Brânzei et al., 2004, Bergantiños and Vidal-Puga, 2006). Since all the obligation rules are additive (Lorenzo and Lorenzo-Freire, 2009), we conclude that $f^{\hat{o}}$ is not an obligation rule.

6 Concluding remarks

In this section we summarize the main findings of the paper. Our main objective is to study in *mcstp* the rules satisfying PM and CM.

Given a *mcstp*, its irreducible problem is obtained by reducing the cost of the arcs as much as possible, but without changing the total cost associated with any *mt*. The irreducible core is the core of the irreducible problem and it is an non-empty subset of the core. Our first result says that the set of allocations induced by the rules satisfying PM and CM coincides with the irreducible core.

We introduce the concept of neighborhood. We say that a group of agents S are in a neighborhood if any connection cost between any agent of the neighborhood and any agent outside the neighborhood is larger than any connection cost between any pair of agents in the neighborhood. We define δ_S as the difference between the previous amounts. This δ_S can be interpreted as the extra cost of connecting the agents in S with the agents outside S. An extra cost correspondence specifies how to divide the extra cost δ_S among the agents in S.

Our second result says that the set of rules satisfying SEP and RED coincides with the set of rules induced by extra cost correspondences.

Our third result says the set of rules satisfying PM and CM coincides with the set of rules induced by extra cost correspondences satisfying the NDC property.

We also explain how some rules of the literature satisfying PM and CM can be expressed in terms of extra cost correspondences. Besides, with the help of our result, we identify a new class of rules satisfying PM and CM.

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7 Appendix

We prove the results of the paper.

7.1 Proof of Theorem 1

Let (N_0, C_0) be a *mcstp*. We first prove that $IC(N_0, C_0) \subset AM(N_0, C_0)$. It is well known that $v_{C_0^*}$ is a concave game. Thus, the core of $v_{C_0^*}$ is the convex hull of the family of vector of marginal contributions.

Hence, given $x = (x_i)_{i \in N} \in IC(N_0, C_0)$, there exists $w = (w_\pi)_{\pi \in \Pi_N} \in \Delta(\Pi_N)$ such that for each $i \in N$,

$$x_i = \sum_{\pi \in \Pi_N} w_\pi \left[v_{C_0^*}(Pre(i,\pi) \cup \{i\}) - v_{C_0^*}(Pre(i,\pi)) \right].$$

Let $\pi \in \Pi_N$. We define the rule f^π such that for each $S \subset N$ and each $i \in S$,

$$f_i^\pi(S_0, (C_S)_0) = v_{C^*}(Pre(i,\pi_S) \cup \{i\}) - v_{C^*}(Pre(i,\pi_S))$$

where π_S denotes the order induced by π among the agents in S.

This rule f^π is well defined because

$$\sum_{i\in N} f_i^\pi (N_0, C_0) = v_{C_0^*} (N) = m (N_0, C_0^*) = m (N_0, C_0).$$

For each $w = (w_\pi)_{\pi\in\Pi_N} \in \Delta(\Pi_N)$, we define the rule $f^w = \sum_{\pi\in\Pi_N} w_\pi f^\pi$. Thus, for each $i \in N$

$$x_i = \sum_{\pi\in\Pi_N} w_\pi f_i^\pi (N_0, C_0) = f_i^w (N_0, C_0).$$

It only remains to prove that f^w satisfies PM and CM. Using Proposition 3.3 in Bergantiños and Vidal-Puga (2007) it is not difficult to prove that for each $S \subset N$, $i \notin S$

$$v_{C_0^*} (S \cup \{i\}) - v_{C_0^*} (S) = \min_{k\in S\cup\{0\}} \{c_{ik}^*\}.$$

We prove that for each $\pi \in \Pi_N$, f^π satisfies PM. Let $S \subset T \subset N$, and $i \in S$. Under (1), it is straightforward to check that $((C_T)^*)_S \leq (C_S)^*$. Since $Pre(i, \pi_S) \subset Pre(i, \pi_T)$,

$$f_i^\pi (T_0, (C_T)_0) = \min_{k\in Pre(i,\pi_T)\cup\{0\}} \{c_{ik}^*\} \leq \min_{k\in Pre(i,\pi_S)\cup\{0\}} \{c_{ik}^*\} = f_i^\pi (S_0, (C_S)_0).$$

We prove that for each $\pi \in \Pi_N$, f^π satisfies CM. Let (N_0, C_0) and (N_0, C_0') be such that $C_0 \leq C_0'$. Bergantiños and Vidal-Puga (2007, Lemma 4.2) prove that $C_0^* \leq C_0'^*$. Now,

$$f_i^\pi (N_0, C_0) = \min_{k\in Pre(i,\pi)\cup\{0\}} \{c_{ik}^*\} \leq \min_{k\in Pre(i,\pi)\cup\{0\}} \{c_{ik}'^*\} = f_i^\pi (N_0, C_0').$$

Since for each $\pi \in \Pi_N$, f^π satisfies PM and CM, it is not difficult to check that for each $w = (w_\pi)_{\pi\in\Pi_N} \in \Delta(\Pi_N)$, $f^w = \sum_{\pi\in\Pi_N} w_\pi f^\pi$ satisfies PM and CM.

Finally, we prove that $AM(N_0, C_0) \subset IC(N_0, C_0)$. Let $x \in AM(N_0, C)$. There exists a rule f satisfying CM and PM such that $x = f(N_0, C_0)$. It is not difficult to check that if f satisfies PM then $f(N_0, C_0) \in core(N, v_{C_0})$. Besides, CM implies RED. Therefore, $f(N_0, C_0) = f(N_0, C_0^*) \in core(N, v_{C_0^*}) = IC(N_0, C_0)$. ■

7.2 Proof of Proposition 1

(1) Assume that S is a neighborhood in C_0. Because of the definition of the irreducible matrix, we have that $\min_{i\in S, j\in N_0\setminus S} c_{ij} = \min_{i\in S, j\in N_0\setminus S} c_{ij}^*$. Let $\tau_S \in \mathbb{T}(S)$ be an *mcst* in (S, C_S). Since S is a neighborhood in C_0, τ_S is also an optimal tree in $(S, (C_S)^*)$. Let $C^1 = (C_S)^*$ and let $C^2 = (C^*)_S$. Given $i, j \in S$, let $\tau_{ij} \subset \tau_S$ be the (unique) path from i to j. Then,

$$c_{ij}^1 = \max_{\{k,l\}\in\tau_{ij}} c_{kl} = c_{kl}^* = c_{ij}^2$$

and hence$(C_S)^* = (C^*)_S$.

Because of the definition of C^* we have that $\max_{(i,j)\in\tau_S} c_{ij} = \max_{(i,j)\in\tau_S} c^*_{ij} = \max_{(i,j)\in S} c^*_{ij}$. Now,

$$\begin{aligned}\delta^*_S &= \min_{i\in S, j\in N_0\setminus S} c^*_{ij} - \max_{\{i,j\}\in\tau_S} c^*_{ij} \\ &= \min_{i\in S, j\in N_0\setminus S} c_{ij} - \max_{\{i,j\}\in\tau_S} c_{ij} = \delta_S\end{aligned}$$

which means that S is a neighborhood in C^*_0.

The reciprocal is similar and we omit it.

(2) "$\supset$" Let $j \in N$ be such that $c^*_{ij} < \min_{k\in S, l\in N_0\setminus S} c^*_{kl}$. If $j \notin S$, then $c^*_{ij} \geq \min_{k\in S, l\in N_0\setminus S} c^*_{kl}$, which is a contradiction. Hence, $j \in S$.

"$\subset$": Let $j \in N$ be such that $c^*_{ij} \geq \min_{k\in S, l\in N_0\setminus S} c^*_{kl}$. If $j \in S$, then

$$\delta_S = \min_{k\in S, l\in N_0\setminus S} c^*_{kl} - \max_{k,l\in S} c^*_{kl} \leq c^*_{ij} - c^*_{ij} = 0$$

which cannot be true because S is a neighborhood. Hence, $j \notin S$.

(3) Let $i \in S \cap S'$. If $\min_{k\in S, l\in N_0\setminus S} c^*_{kl} \leq \min_{k\in S', l\in N_0\setminus S'} c^*_{kl}$ then it follows from Proposition 1.2 that $S \subset S'$. If $\min_{k\in S', l\in N_0\setminus S'} c^*_{kl} \leq \min_{k\in S, l\in N_0\setminus S} c^*_{kl}$ then it follows from Proposition 1.2 that $S' \subset S$.

(4) It follows from Proposition 1.3.

(5) Assume $\{(0,i)\}_{i\in N}$ is not an *mcst*. Let $\{k,l\} \subset N$ be such that $c_{kl} = \min_{i,j\in N} c_{ij}$. Thus, $c_{kl} < \min_{i\in N} c_{0i}$. Then, $S = \{k\} \cup \left\{ i \in N : \max_{\{j,j'\}\in\tau_{ik}} c_{jj'} \leq c_{kl} \right\}$ is a neighborhood in C_0.

Assume $\{(0,i)\}_{i\in N}$ is an *mcst*. Then, given any $S \subset N$, we have $\min_{i\in S, j\in N_0\setminus S} c_{ij} = \min_{i\in S} c_{0i}$ and $\max_{\{i,j\}\in\tau(S)} c_{ij} \geq \min_{i\in S} c_{0i}$. Hence

$$\delta_S = \min_{i\in S, j\in N_0\setminus S} c_{ij} - \max_{\{i,j\}\in\tau(S)} c_{ij} \leq 0$$

and S is not a neighborhood. ■

7.3 Proof of Proposition 2

Let $(N_0, C_0) \in \mathcal{C}_0$. Then,

$$\begin{aligned}\sum_{i\in N} f^e_i(N_0, C_0) &= \sum_{i\in N} c^*_{0i} - \sum_{i\in N} \sum_{\substack{S \text{ neighborhood} \\ i\in S}} \left(\delta_S - e_i\left(C^*_S, \delta_S\right)\right) \\ &= \sum_{i\in N} c^*_{0i} - \sum_{S \text{ neighborhood}} \left(\sum_{i\in S} \left(\delta_S - e_i\left(C^*_S, \delta_S\right)\right)\right) \\ &= \sum_{i\in N} c^*_{0i} - \sum_{S \text{ neighborhood}} \left(|S| - 1\right)\delta_S.\end{aligned}$$

Thus, it is enough to prove that for each *mcstp* (N_0, C_0),

$$m(C_0) + \sum_{S \text{ neighborhood}} \left(|S| - 1\right)\delta_S = \sum_{i\in N} c^*_{0i}.$$

Assume first there exists no neighborhood. Under Proposition 1.5, $\{\{0,i\}\}_{i\in N}$ is an *mcst* in (N_0, C_0). Hence, $\{\{0,i\}\}_{i\in N}$ is also an *mcst* in (N_0, C_0^*) and the result is easily checked.

Assume now that there are exactly $k > 0$ neighborhoods and the result is true when there exists less than k neighborhoods. Let S' be a minimal neighborhood (there is no neighborhood S such that $S \subsetneq S'$). Let $\tau_{S'}$ denote a *mcst* in S'. Since S' is minimal, there exists $\alpha \geq 0$ such that $c_{ij} = \alpha$ for all $(i,j) \in \tau_{S'}$.

Let t be a *mcst* in (N_0, C_0). We define C_0' as $c'_{ij} = \alpha + \delta_{S'}$ if $\{i,j\} \subset S'$ and $c'_{ij} = c_{ij}$ otherwise. It is not difficult to check that:

- t is also an *mcst* in (N_0, C_0');
- $c'^*_{0i} = c^*_{0i}$ for all $i \in N$;
- $m(C_0') = m(C_0) + (|S'| - 1)\,\delta_{S'}$; and
- $\{S : S \text{ is a neighborhood in } C_0'\} = \{S : S \text{ is a neighborhood in } C_0\} \setminus \{S'\}$.

Now, applying the induction hypothesis, we have

$$\begin{aligned} & m(C_0) + \sum_{S \text{ neighborhood in } C_0} (|S| - 1)\,\delta_S \\ = \; & m(C_0') - (|S'| - 1)\,\delta_{S'} + \sum_{S \text{ neighborhood in } C_0} (|S| - 1)\,\delta_S \\ = \; & m(C_0') + \sum_{S \text{ neighborhood in } C_0'} (|S| - 1)\,\delta_S \\ = \; & \sum_{i\in N} c'^*_{0i} = \sum_{i\in N} c^*_{0i}. \; \blacksquare \end{aligned}$$

7.4 Proof of Theorem 2

Let e be any extra-costs correspondence and f^e be the associated rule. It is obvious that f^e satisfies RED.

Given $(N_0, C_0) \in \mathcal{C}_0$, let $Ne(N_0, C_0)$ denote the set of neighborhoods in (N_0, C_0).

In order to prove that f^e also satisfy SEP, let $S \subset N$ such that $m(N_0, C_0) = m(S_0, (C_S)_0) + m\left((N \backslash S)_0, (C_{N\backslash S})_0\right)$. Given $i \in S$, it is straightforward to check that $Ne(N_0, C_0) = Ne(S_0, (C_S)_0) \cup Ne\left((N\backslash S)_0, (C_{N\backslash S})_0\right)$. Hence, $f_i^e(N_0, C_0) = f_i^e(S_0, (C_S)_0)$ and this proves that f is separable.

We now prove that if f satisfies SEP and RED, then $f = f^e$ for some extra-costs correspondence e. Let f be such a rule.

Given $(N, C^*) \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$, we define $\left(N_0, C_0^{*(a)}\right) \in \mathcal{C}_0$ as the *mcstp* given by $c_{ij}^{*(a)} = c^*_{ij}$ for all $i,j \in N$ and $c_{0i}^{*(a)} = a$ for all $i \in N$. It is straightforward to check that $C_0^{*(a)} \in \mathcal{C}_0^*$ when $a \geq \max C^*$.

For all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$ we define

$$e_i(C^*, x) = f_i\left(C_0^{*(\max C^* + x)}\right) - f_i\left(C_0^{*(\max C^*)}\right).$$

Given $i \notin N$ we define $e_i(C^*, x) = 0$.

We first prove that e is an extra-costs correspondence. By definition, $e_i(C^*, x) = 0$ for all $(N, C^*) \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, $i \notin N$. Besides,

$$\begin{aligned}
\sum_{i \in U} e_i(C^*, x) &= \sum_{i \in N} e_i(C^*, x) \\
&= m\left(C_0^{*(\max C^* + x)}\right) - m\left(C_0^{*(\max C^*)}\right) \\
&= m(C^*) + \max C^* + x - m(C^*) - \max C^* \\
&= x.
\end{aligned}$$

Hence, e is an extra-costs correspondence.

We need to prove that $f = f^e$. We proceed by induction on the number of neighborhoods $Ne(C_0)$. Assume $|Ne(C_0)| = 0$.

Under Proposition 1.5, $\{(0,i)\}_{i \in N}$ is a *mcst* in C_0. Since f satisfies SEP, $f_i(C_0) = f_i\left(\{i\}_0, \left(C_{\{i\}}\right)_0\right) = c_{0i}$. Besides, since $\{(0,i)\}_{i \in N}$ is an *mcst* in C_0, we have $c_{0i} = c_{0i}^*$ for all $i \in N$ and hence $f^e(C_0) = f(C_0)$.

Assume now the result is true for *mcstp* with less than $|Ne(C_0)|$ neighborhoods.

Assume first that $\max C^* \geq \max_{i \in N} c_{0i}^*$. It is not difficult to check that N is separable, namely, there exists $S \subset N$, $S \neq \varnothing$, and $S \neq N$ such that $m(N_0, C_0) = m(S_0, (C_S)_0) + m\left((N \backslash S)_0, \left(C_{N \backslash S}\right)_0\right)$. Under SEP, $f_i(N_0, C_0) = f_i(S_0, (C_S)_0)$ for all $i \in S$ and $f_i(N_0, C_0) = f_i\left((N \backslash S)_0, \left(C_{N \backslash S}\right)_0\right)$ for all $i \in N \backslash S$. Repeating this argument we can find a partition $\{S_1, ..., S_p\}$ of N satisfying that for each $k = 1, ...p$ $\max C_{S_k}^* < \max_{i \in S_k} c_{0i}^*$ and $f_i(N_0, C_0) = f_i\left((S_k)_0, (C_{S_k})_0\right)$ for each $i \in S_k$.

Hence, we can assume $\max C^* < \max_{i \in N} c_{0i}^*$. Since C^* is irreducible, $\max_{i \in N} c_{0i}^* = c_{0i}^*$ for all $i \in N$. Hence, $N \in Ne(C_0)$ and $\delta_N = \max_{i \in N} c_{i0}^* - \max C^*$. Since f satisfies RED, $f(C_0) = f(C_0^*)$. Now, given $i \in N$,

$$\begin{aligned}
f_i(C_0) &= f_i(C_0^*) = f_i\left(C_0^{*(\max C^* + \delta_N)}\right) \\
&= e_i(C^*, \delta_N) + f_i\left(C_0^{*(\max C^*)}\right).
\end{aligned}$$

Let $C_0' = C_0^{*(\max C^*)}$. It is straightforward to check that C_0' is irreducible. Besides, $Ne(C_0^*) = Ne(C_0') \cup \{N\}$. For each $S \in Ne(C_0')$, $\delta_S = \delta_S'$, and $c_{i0}'^* = c_{i0}^* - \delta_N$. Hence, applying the induction hypothesis, for each $i \in N$,

$$\begin{aligned}
f_i(C_0) &= e_i(C^*, \delta_N) + f_i(C_0') \\
&= e_i(C^*, \delta_N) + c_{0i}'^* + \sum_{S \in Ne(C_0')} (e_i(C_S^*, \delta_S) - \delta_S) \\
&= e_i(C^*, \delta_N) + c_{0i}^* - \delta_N + \sum_{S \in Ne(C_0')} (e_i(C_S^*, \delta_S) - \delta_S) \\
&= c_{0i}^* + \sum_{S \in Ne(C_0^*)} (e_i(C_S^*, \delta_S) - \delta_S) \\
&= f_i^e(C_0). \blacksquare
\end{aligned}$$

7.5 Proof of Theorem 3

We start the proof with the following Lemma.

Lemma 1. (i) Given $(N', C'), (N'', C'') \in \mathcal{C}^*$ and $a \in \mathbb{R}_+$ with $N' \cap N'' = \emptyset$ and $a \geq \max C'' - \max C'$, then $C' \oplus_a C'' \in \mathcal{C}^*$.

(ii) Given a disjoint sequence $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $\Gamma > 1$, $a \in \mathbb{R}_+^\Gamma$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$, then $C^\gamma(a) \in \mathcal{C}^*$ and $C^\gamma(a') \in \mathcal{C}^*$ for all $\gamma = 1, ..., \Gamma$, where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$.

Proof of Lemma 1. (i) Let $C = C' \oplus_a C''$. It is easily checked that $a + \max C' = \max C$. Hence, we can find a *mcst* t in C and C^* such that $t = t^1 \cup t^2 \cup \{(k^1, k^2)\}$ where t^1 is a *mcst* in C', t^2 is a *mcst* in C'', $k^1 \in N^1$ and $k^2 \in N^2$. Since $c_{k^1k^2} = \max C \geq c_{ij}$ for all $(i, j) \in t^1 \cup t^2$ we can deduce, using the definition of irreducible matrix, that $C = C^*$.

(ii) We assume $\gamma > 1$, since the case $\gamma = 1$ is trivial. We proceed by induction on Γ. For $\Gamma = 2$, the result follows from (i) because $a_1' = a_1 + y \geq a_1 \geq \max C^2 - \max C^1$. Assume the result is true for sequences with less than Γ *mcstp*'s, $\Gamma \geq 3$. Under the induction hypothesis, we have $C^\gamma(b)$, $C^\gamma(b') \in \mathcal{C}^*$ where $\gamma = 1, ..., \Gamma - 1$, $b = (a_1, ..., a_{\Gamma-1})$ and $b' = (a_1 + y, a_2 - y, a_3, ..., a_{\Gamma-1})$. Now, it is clear that $C^\gamma(a) = C^\gamma(b)$ and $C^\gamma(a') = C^\gamma(b')$ for all $\gamma = 1, ..., \Gamma - 1$. Hence, the result holds for any $\gamma < \Gamma$. Assume now $\gamma = \Gamma$. We have

$$C^\Gamma(a) \overset{(2)}{=} C^{\Gamma-1}(a) \oplus_{a_{\Gamma-1}} C^\Gamma(a) \overset{(i)}{\in} \mathcal{C}^*$$

and

$$C^\Gamma(a') \overset{(2)}{=} C^{\Gamma-1}(a') \oplus_{a'_{\Gamma-1}} C^\Gamma(a').$$

In order to apply (i) to this last expression (so that $C^\Gamma(a') \in \mathcal{C}^*$) we have to prove that

$$a'_{\Gamma-1} \geq \max C^\Gamma(a') - \max C^{\Gamma-1}(a'). \tag{4}$$

It is straightforward to check that $\max C^\gamma(a') = \max C^\gamma(a)$ for all $\gamma \neq 2$, whereas $\max C^2(a') = \max C^2(a) + y$. Hence, for $\Gamma > 3$,

$$\max C^\Gamma(a') - \max C^{\Gamma-1}(a') = \max C^\Gamma(a) - \max C^{\Gamma-1}(a) \leq a_{\Gamma-1} = a'_{\Gamma-1}$$

and for $\Gamma = 3$,

$$\max C^3(a') - \max C^2(a') = \max C^3(a) - \max C^2(a) - y \leq a_2 - y = a_2'. \blacksquare$$

We now prove that if $f = f^e$ with e satisfying NDC, then f satisfies CM and PM.

Following Norde *et al* (2004), we define the set Σ_{N_0} of linear orders on the arcs of C_0 as the set of all bijections $\sigma : \left\{1, ..., \binom{n+1}{n}\right\} \to \{\{i,j\} : i,j \in N_0\}$. For each *mcstp* (N_0, C_0), there exists at least one linear order $\sigma \in \Sigma_{N_0}$ such that $c_{\sigma(1)} \leq c_{\sigma(2)} \leq ... \leq c_{\sigma\left(\binom{n+1}{n}\right)}$. For any $\sigma \in \Sigma_{N_0}$, we define the set

$$K^\sigma = \left\{C_0 \in \mathcal{C}_0^N : c_{\sigma(k)} \leq c_{\sigma(k+1)} \text{ for all } k = 1, 2, ...\right\},$$

which we call the *Kruskal cone* with respect to σ. One can easily see that $\bigcup_{\sigma \in \Sigma_{N_0}} K^\sigma = \mathcal{C}_0^N$.

We say that a nonempty set $S \subset N$ is a *quasi-neighborhood* in C_0 if $\delta_S \geq 0$. Let $qNe(C_0) = \{S \subset N, S \neq \emptyset : \delta_S \geq 0\}$ denote the set of quasi-neighborhoods in C_0. Clearly, $Ne(C_0) \subset Ne(C_0)$.

We now prove that f satisfies CM. It is enough to prove that $f(N_0, C_0) \leq f(N_0, C_0')$ when there exists $\{k,l\} \subset N_0$ such that $c'_{kl} > c_{kl}$ and $c'_{ij} = c_{ij}$ otherwise. Let (k,l), C_0 and C_0' be defined in this way.

For any $t \in [0,1]$, the *mcstp* (N_0, C_0^t) defined as $c_{ij}^t = (1-t)\,c_{ij} + tc'_{ij}$ satisfies $c'_{kl} \geq c_{kl}^t \geq c_{kl}$ and $c_{ij}^t = c_{ij}$ otherwise. Since Σ_{N_0} is a finite set, there exist a sequence $\{t^1, t^2, ...t^p\} \subset [0,1]$ with $t^1 = 0$ and $t^p = 1$ such that, for all r, we have $t^r < t^{r+1}$ and C^{t^r} and $C^{t^{r+1}}$ belong to the same Kruskal cone.

Hence, it is enough to prove that $f(N_0, C_0) \leq f(N_0, C_0')$ when both C_0 and C_0' belong to the same Kruskal cone. An immediate consequence is that there exists a common *mcst* t in both C_0 and C_0'.

Since f satisfies RED, $f(N_0, C_0) = f(N_0, C_0^*)$. If $\{k,l\} \notin t$, then $C_0^* = C_0'^*$. Thus

$$f(N_0, C_0) = f(N_0, C_0^*) = f(N_0, C_0'^*) = f(N_0, C_0').$$

Hence, we assume $\{k,l\} \in t$. This implies $c_{kl} = c_{kl}^*$ and $c'_{kl} = c'^*_{kl}$. Let $\alpha = c'^*_{kl} - c^*_{kl} > 0$.

Another consequence of C_0, C_0' being in the same Kruskal cone is that, for any $S \subset N$, $|S| > 1$, there exist $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \backslash S$ with $\{i^2, j^2\} \in \tau(S)$ such that

$$\begin{aligned} \delta_S &= \min_{i' \in S, j' \in N_0 \backslash S} c_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c_{i'j'} = c_{i^1 j^1} - c_{i^2 j^2} \text{ and} \\ \delta'_S &= \min_{i' \in S, j' \in N_0 \backslash S} c'_{i'j'} - \max_{\{i',j'\} \in \tau(S)} c'_{i'j'} = c'_{i^1 j^1} - c'_{i^2 j^2}. \end{aligned}$$

Thus δ_S and δ'_S cannot have opposite sign. Namely, $\delta_S > 0$ implies $\delta'_S \geq 0$. From this, it is straightforward to check that $Ne(C_0) \subset qNe(C_0')$ and, analogously, $Ne(C_0') \subset qNe(C_0)$.

Given any $X \subset 2^N$ with $Ne(C_0) \subseteq X \subseteq qNe(C_0)$ and $i \in N$ we have

$$f_i(N_0, C_0) = c_{i0}^* - \sum_{i \in S \in X} \left(\delta_S - e_i(C_S^*, \delta_S)\right). \tag{5}$$

The reason is that for any $S \in qNe(C_0) \setminus Ne(C_0)$, $\delta_S = 0$ and hence $\delta_S - e_i(C_S^*, \delta_S) = 0 - e_i(C_S^*, 0) = 0$.

We define $X = Ne(C_0) \cup Ne(C_0')$. Clearly, $Ne(C_0) \subseteq X \subseteq qNe(C_0)$ and $Ne(C_0') \subseteq X \subseteq qNe(C_0')$.

Fix $i \in N$. We need to prove that $f_i(N_0, C_0) \leq f_i(N_0, C_0')$. Under (5), we have

$$\begin{aligned} f_i(N_0, C_0) &= c_{0i}^* - \sum_{i \in S \in X} (\delta_S - e_i(C_S^*, \delta_S)) \\ f_i(N_0, C_0') &= c_{0i}'^* - \sum_{i \in S \in X} (\delta_S' - e_i(C_S'^*, \delta_S')) . \end{aligned}$$

We have seen above that

$$\delta_S = c_{i^1 j^1} - c_{i^2 j^2} \text{ and } \delta_S' = c'_{i^1 j^1} - c'_{i^2 j^2}$$

for some $i^1, i^2, j^2 \in S$, $j^1 \in N_0 \setminus S$ with $\{i^2, j^2\} \in t_S$.

By hypothesis, $c_{jj'} = c'_{jj'}$ for all $\{j, j'\} \neq \{k, l\}$. Hence, $\delta_S = \delta_S'$ unless $\{i^1, j^1\} = \{k, l\}$ or $\{i^2, j^2\} = \{k, l\}$.

Given $S \in X$ and $\delta_S \neq \delta_S'$ we study both cases:

1. If $\{i^1, j^1\} = \{k, l\}$, then $\delta_S' = \delta_S + \alpha$. Besides, there can be at most two such S. One of them contains node k (if any) and the other contains node l (if any). Assume, on the contrary, that there exist two $S' \in X, S \neq S'$ with $k \in S \cap S'$ (the case for $l \in S$ is analogous). Hence,

   $$c'_{kl} = c'^*_{kl} = \min_{i' \in S, j' \in N_0 \setminus S} c'^*_{i'j'} = \min_{i' \in S', j' \in N_0 \setminus S'} c'^*_{i'j'}.$$

   Since $k \in S \cap S'$, under Proposition 1.4, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

   $$\begin{aligned} c'^*_{kl} &= \min_{i' \in S, j' \in N_0 \setminus S} c'^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c'^*_{i'j'} \\ &\leq \max_{i', j' \in S'} c'^*_{i'j'} \leq \min_{i' \in S', j' \in N_0 \setminus S'} c'^*_{i'j'} = c'^*_{kl} \end{aligned}$$

   which implies that no inequality is strict. In particular, $\max_{i', j' \in S'} c'^*_{i'j'} = c'^*_{kl}$. Since $\{k, l\} \nsubseteq S'$, $\max_{i', j' \in S'} c'^*_{i'j'} = \max_{i', j' \in S'} c^*_{i'j'}$ and hence

   $$\delta_{S'} = \min_{i' \in S', j' \in N_0 \setminus S'} c^*_{i'j'} - \max_{i', j' \in S'} c^*_{i'j'} = c^*_{kl} - c'^*_{kl} = -\alpha < 0,$$

   which is a contradiction.

2. If $\{i^2, j^2\} = \{k, l\}$, then $\delta_S' = \delta_S - \alpha$. Besides, there can be at most one such S. Assume, on the contrary, that there exists $S' \in X$, $S \neq S'$, $k, l \in S \cap S'$, and

   $$c_{kl} = c^*_{kl} = \max_{i', j' \in S} c^*_{i'j'} = \max_{i', j' \in S'} c^*_{i'j'}.$$

Since $k \in S \cap S'$, under Proposition 1.4, $S \subsetneq S'$ or $S' \subsetneq S$. Assume w.l.o.g. $S \subsetneq S'$. Then,

$$c^*_{kl} = \max_{i',j' \in S} c^*_{i'j'} \leq \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} \leq \min_{i' \in S, j' \in S' \setminus S} c^*_{i'j'} \leq \max_{i',j' \in S'} c^*_{i'j'} = c^*_{kl}$$

which implies that no inequality is strict. Thus, $\min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} = c^*_{kl}$ and hence

$$\delta_S = \min_{i' \in S, j' \in N_0 \setminus S} c^*_{i'j'} - \max_{i',j' \in S} c^*_{i'j'} = c^*_{kl} - c^*_{kl} = 0,$$

which implies $\delta'_S = \delta_S - \alpha = -\alpha < 0$, which is a contradiction.

Let $S^k = \left\{ j \in N_0 : c'^*_{kj} < c'^*_{kl} \right\}$ and let $S^l = \left\{ j \in N_0 : c'^*_{kj} < c'^*_{kl} \right\}$. Both S^k and S^l are nonempty (because $k \in S^k$ and $l \in S^l$) and disjoint (it follows from $\{k, l\} \in t$). Since they are disjoint, we can assume w.l.o.g. $0 \notin S^k$. Let $S_1 = S^k$. If $|S_1| > 1$, then

$$\begin{aligned} l &\notin S_1, \\ c'^*_{kl} &= \min_{i' \in S_1, j' \in N_0 \setminus S_1} c'^*_{i'j'}, \\ \delta'_{S_1} &= c'^*_{kl} - \max_{i',j' \in S} c'^*_{i'j'} > 0 \end{aligned}$$

and hence either $S_1 \in Ne\left(C'_0\right)$ or $S_1 = \{k\}$.

Assume that $S_1 \in Ne\left(C'_0\right)$. Since C_0 and C'_0 are in the same Kruskal cone, $\delta_{S_1} = c^*_{i^1 j^1} - c^*_{i^2 j^2}$ and $\delta'_{S_1} = c'^*_{i^1 j^1} - c'^*_{i^2 j^2}$. Since $\delta'_{S_1} > 0$ we deduce that $\delta_{S_1} \geq 0$. Hence $S_1 \in qNe\left(C_0\right)$. Now, it is not difficult to check that S_1 satisfies condition 1. Hence $\delta'_{S_1} = \delta_{S_1} + \alpha$ when $|S_1| > 1$.

Let $S_2 = \left\{ j \in N_0 : c^*_{kj} \leq c^*_{kl} \right\}$. Clearly, $\{k, l\} \subset S_2$. Notice that if $0 \in S_2$ then $S_2 \notin X$. It is straightforward to check that if $0 \notin S_2$ then $S_2 \in X$. Besides $S_1 \subsetneq S_2$ and there is no $S \in X$, $S \neq S_1$, such that $S_1 \subsetneq S \subsetneq S_2$.

In case $0 \notin S_2$, it is not difficult to check that S_2 satisfies condition 2. Hence $\delta'_{S_2} = \delta_{S_2} - \alpha$.

Let $F = \{S \in Ne\left(C_0\right) : S_1 \subset S, \delta_S = \delta'_S\}$ and let $F' = \{S \in Ne\left(C'_0\right) : S_1 \subset S, \delta_S = \delta'_S\}$. It is not difficult to check that $F = F'$ ($F = F' = \emptyset$ is also possible) and $S_1, S_2 \notin F$. By Proposition 1.3 we can assume $F = \{S_3, S_4, ..., S_\Gamma\}$ for some $\Gamma \geq 2$ ($\Gamma = 2$ when $F = \emptyset$) and $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 3, ..., \Gamma - 1$.

Let $G = \{S \in X : S_1 \subset S\}$. Clearly, either $G = \{S_1, ..., S_\Gamma\}$ (when $S_1 \in Ne\left(C'_0\right)$) or $G = \{S_2, ..., S_\Gamma\}$ (when $S_1 = \{k\}$). Besides, $S_\gamma \subsetneq S_{\gamma+1}$ for all $\gamma = 1, 2, ..., \Gamma - 1$.

If $i \notin S_\Gamma$, it is straightforward to check that $f_i\left(N_0, C_0\right) = f_i\left(N_0, C'_0\right)$. We assume $i \in S_\gamma$ for some $\gamma \in \{1, ..., \Gamma\}$. Let γ_i be the minimum of these γ's. We have two cases:

Case 1: $\Gamma = 1$. This means $S_2 \notin X$. Since $\delta_{S_2} \geq 0$, we have $0 \in S_2$, which implies $c^*_{0k} \leq c^*_{kl}$ and also $c'^*_{0k} \leq c'^*_{kl}$.

Subcase 1.1: $S_1 = \{k\} = \{i\}$. This implies $X = \emptyset$ and hence

$$f_i\left(N_0, C'_0\right) - f_i\left(N_0, C_0\right) = c'^*_{0i} - c^*_{0i} \geq 0.$$

Subcase 1.2: $S_1 \in X$. This implies $c'^*_{0k} \geq c'^*_{kl}$ and hence $c'^*_{0k} = c'^*_{kl}$. Thus $c'^*_{i0} - c^*_{i0} = \alpha$ and $C^*_{S_1} = C'^*_{S_1}$. Hence,

$$\begin{aligned}
& f_i(N_0, C'_0) - f_i(N_0, C_0) \\
= \;& c'^*_{0i} - \left(\delta'_{S_1} - e_i\left(C'^*_{S_1}, \delta'_{S_1}\right)\right) - c^*_{0i} + \left(\delta_{S_1} - e_i\left(C^*_{S_1}, \delta_{S_1}\right)\right) \\
= \;& c'^*_{0i} - c^*_{0i} - \left(\delta_{S_1} + \alpha - e_i\left(C^*_{S_1}, \delta_{S_1} + \alpha\right)\right) + \left(\delta_{S_1} - e_i\left(C^*_{S_1}, \delta_{S_1}\right)\right) \\
= \;& e_i\left(C^*_{S_1}, \delta_{S_1} + \alpha\right) - e_i\left(C^*_{S_1}, \delta_{S_1}\right) \geq 0
\end{aligned}$$

where the last inequality comes from applying NDC to $\left\{\left(S_1, C^*_{S_1}\right)\right\}$ with $\Gamma = 1$, $a_1 = \delta_{S_1}$ and $y = \alpha$.

Case 2: $\Gamma > 1$. This means that $S_2 \in X$ and hence $0 \notin S^l$. Thus we can take $S_1 = S^k$ or $S_1 = S^l$. It is not difficult to check that $S_2 = S^k \cup S^l$. If $i \in S_2$ we choose S_1 such that $i \in S_1$. Thus, $\gamma_i \neq 2$ which implies $c'^*_{0i} = c^*_{0i}$.

In this case,

$$\begin{aligned}
& f_i(N_0, C'_0) - f_i(N_0, C_0) \\
= \;& c'^*_{0i} - c^*_{0i} - \sum_{i \in S \in X} \left(\delta'_S - \delta_S - e_i\left(C'^*_S, \delta'_S\right) + e_i\left(C^*_S, \delta_S\right)\right).
\end{aligned}$$

For any $S \in X \backslash G$ with $i \in S$, we have $C^*_S = C'^*_S$, which also implies $\delta_S = \delta'_S$. Hence,

$$\begin{aligned}
& f_i(N_0, C'_0) - f_i(N_0, C_0) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} \left(-\delta'_{S_\gamma} + \delta_{S_\gamma} + e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) - e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right)\right) \\
= \;& \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(\delta'_{s_\gamma} - \delta_{s_\gamma}\right).
\end{aligned}$$

The last term is zero, because $\delta'_{S_1} = \delta_{S_1} + \alpha$, $\delta'_{S_2} = \delta_{S_2} - \alpha$, and $\delta'_{S_\gamma} = \delta_{S_\gamma}$ otherwise (remember that $\gamma_i \neq 2$). Hence,

$$f_i(N_0, C'_0) - f_i(N_0, C_0) = \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right)\right) - \sum_{\gamma=\gamma_i}^{\Gamma} \left(e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right)\right).$$

We now define $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma}$, $a \in \mathbb{R}^{\Gamma}_+$ and $y \in [0, a_2]$ so that $e_i\left(C'^*_{S_\gamma}, \delta'_{S_\gamma}\right) = e_i\left(C^\gamma(a'), a'_\gamma\right)$ and $e_i\left(C^*_{S_\gamma}, \delta_{S_\gamma}\right) = e_i\left(C^\gamma(a), a_\gamma\right)$ for all γ. Under NDC, this will prove that the above expression is nonnegative.

Let $N^1 = S_1$, $C^1 = C^*_{N^1}$, and $a_1 = \delta_{S_1}$. In general, for any $\gamma = 2, ..., \Gamma$, $N^\gamma = S_\gamma \backslash S_{\gamma-1}$, $C^\gamma = (C^*)_{N^\gamma}$, and $a_\gamma = \delta_{S_\gamma}$. We also define $y = \alpha$. Since $c'^*_{kl} = c^*_{kl} + \alpha$, it is straightforward to check that $\alpha \leq a_2$ and hence $y \in [0, a_2]$.

Clearly, $C'^*_{S_1} = C^1$. Now, we prove that $C'^*_{S_2} = C^1 \oplus_{a_1+\alpha} C^2 = C^2(a')$. Let $C^\alpha = C'^*_{S_2}$ and $C^\beta = C^1 \oplus_{a_1+\alpha} C^2$. Clearly, $C^\alpha = (C_{S_2} + \alpha I_{kl})^*$.

It is straightforward to check that $c_{ij}^{\alpha} = c_{ij}^{\beta}$ for all $i, j \in N^1$ and all $i, j \in N^2$. Let $k^1 \in N^1$ and $k^2 \in N^2$. Then,

$$\begin{aligned} c_{k^1k^2}^{\beta} &= \max C^1 + a_1 + \alpha = \max C^1 + \delta_{S_1} + \alpha = \min_{\substack{i \in N^1 \\ j \in N_0 \setminus N^1}} c_{ij} + \alpha \\ &= c_{kl} + \alpha = c_{k^1k^2}^{\alpha}. \end{aligned}$$

Analogously, $C_{S_3}^{\prime *} = (C_{S_3} + \alpha I_{kl})^* = (C^1 \oplus_{a_1+\alpha} C^2) \oplus_{a_2-\alpha} C^3 = C^3(a')$. In general, $C_{S_\gamma}^{\prime *} = \left(C_{S_\gamma} + \alpha I_{kl}\right)^* = C^1 \oplus_{a_1+\alpha} C^2 \oplus_{a_1-\alpha} C^3 \oplus_{a_3} ... \oplus_{a_{\gamma-1}} C^\gamma = C^\gamma(a')$ for all $\gamma = 3, ..., \Gamma$.

Similarly, we can prove that $C_{S_\gamma}^{*} = C^\gamma(a)$ for all $\gamma = 1, ..., \Gamma$.

Hence, by applying NDC, we have

$$f_i(N_0, C_0') - f_i(N_0, C_0) \geq 0.$$

We now prove that f satisfies PM. Under Theorem 2, we know that f satisfies SEP. We must prove that for each $mcstp$ (N_0, C_0) and $j \in N$, $f_i(N_0, C_0) \leq f_i((N\setminus\{j\})_0, C_0)$ for all $i \in N\setminus\{j\}$. Let (N_0, C_0') be defined as $c_{ii'}' = c_{ii'}$ for all $i, i' \in N\setminus\{j\}$ and $c_{ij}' = \max C_{N_0\setminus\{j\}}$ for all $i \in N_0\setminus\{j\}$. Clearly, $m(N_0, C_0') = m\left((N\setminus\{j\})_0, \left(C_{N\setminus\{j\}}'\right)_0\right) + m\left(\{j\}_0, \left(C_{\{j\}}'\right)_0\right)$. Under SEP, $f_i(N_0, C_0') = f_i\left((N\setminus\{j\})_0, \left(C_{N\setminus\{j\}}'\right)_0\right)$ for all $i \in N\setminus\{j\}$. Given $i \in N\setminus\{j\}$, under CM,

$$f_i(N_0, C_0) \leq f_i(N_0, C_0') = f_i((N\setminus\{j\})_0, C_0') = f_i((N\setminus\{j\})_0, C_0).$$

We now prove that if f satisfies CM and PM, then $f = f^e$ for some e satisfying NDC.

We define e as in the proof of Theorem 2. Namely, for all $C^* \in \mathcal{C}^*$, $x \in \mathbb{R}_+$, and $i \in N$,

$$e_i(C^*, x) = f_i\left(C_0^{*(\max C^* + x)}\right) - f_i\left(C_0^{*(\max C^*)}\right)$$

and $e_i(C^*, x) = 0$ for all $i \notin N$. We already proved (proof of Theorem 2) that e is an extra-costs correspondence and $f = f^e$.

Hence, we only need to check that e satisfies NDC. Let $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset C^*$ be a disjoint sequence with $\Gamma \geq 1$, $i \in N^{\gamma_i}$ with $\gamma_i \neq 2$, $a \in \mathbb{R}_+^{\Gamma}$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$ and $y \in [0, a_2]$ (or simply $y \geq 0$, when $\Gamma = 1$).

Assume first that $\Gamma = 1$. We need to prove

$$e_i\left(C^1, a_1 + y\right) - e_i\left(C^1, a_1\right) \geq 0.$$

Let $C = C^1$. By definition,

$$\begin{aligned} & e_i(C, a_1 + y) - e_i(C, a_1) \\ = \; & f_i\left(C_0^{*(\max C^* + a_1 + y)}\right) - f_i\left(C_0^{*(\max C^*)}\right) - f_i\left(C_0^{*(\max C^* + a_1)}\right) + f_i\left(C_0^{*(\max C^*)}\right) \\ = \; & f_i\left(C_0^{*(\max C^* + a_1 + y)}\right) - f_i\left(C_0^{*(\max C^* + a_1)}\right) \geq 0 \end{aligned}$$

where the last inequality comes from the fact that $C_0^{*(\max C^*+a_1+y)} \geq C_0^{*(\max C^*+a_1)}$ and f satisfy CM.

Assume now that $\Gamma > 1$. We need to prove

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) \geq 0$$

where $a' = (a_1 + y, a_2 - y, a_3, ..., a_\Gamma)$ and $C^{\gamma}(b) = C^1 \oplus_{b_1} C^2 \oplus_{b_2} ... \oplus_{b_{\gamma-1}} C^{\gamma}$ for all $\gamma = 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^{\Gamma}$.

By definition,

$$e_i\left(C^*, x\right) = f_i\left(C^* \oplus_x \left(\{0\}, 0\right)\right) - f_i\left(C^* \oplus_0 \left(\{0\}, 0\right)\right).$$

Under SEP, it is straightforward to check that

$$f_i\left(C^{\gamma}(b) \oplus_0 \left(\{0\}, 0\right)\right) = f_i\left(C^{\gamma-1}(b) \oplus_{b_{\gamma-1}} \left(\{0\}, 0\right)\right)$$

for all $\gamma = \gamma_i + 1, ..., \Gamma$ and all $b \in \mathbb{R}_+^{\Gamma}$. Now,

$$\begin{aligned}\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) &= \sum_{\gamma=\gamma_i}^{\Gamma} \left[f_i\left(C^{\gamma}\left(a'\right) \oplus_{a'_{\gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)\right] \\ &= f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) &= \sum_{\gamma=\gamma_i}^{\Gamma} \left[f_i\left(C^{\gamma}\left(a\right) \oplus_{a_{\gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right)\right] \\ &= f_i\left(C^{\Gamma}\left(a\right) \oplus_{a_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) &= f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right) - f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right) \\ &\quad + f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) - f_i\left(C^{\Gamma}\left(a\right) \oplus_{a_{\Gamma}} \left(\{0\}, 0\right)\right).\end{aligned}$$

Under CM, $f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right) \geq f_i\left(C^{\Gamma}\left(a'\right) \oplus_{a'_{\Gamma}} \left(\{0\}, 0\right)\right)$.

We now prove that $f_i\left(C^{\gamma_i}\left(a\right) \oplus_0 \left(\{0\}, 0\right)\right) = f_i\left(C^{\gamma_i}\left(a'\right) \oplus_0 \left(\{0\}, 0\right)\right)$. For $\gamma_i = 1$, $C^1(a) = C^1(a') = C^1$ and the result holds trivially. Assume $\gamma_i > 2$. Then, $N^1 \cup ... \cup N^{\gamma_i - 1}$ and N^{γ_i} are two separable components in both $C^{\gamma_i}(a) \oplus_0 (\{0\}, 0)$ and $C^{\gamma_i}(a') \oplus_0 (\{0\}, 0)$. Besides, the restriction of C^* to N^{γ_i} coincides in both *mcstp*. Under SEP, we obtain the result.

Hence,

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a'\right), a'_{\gamma}\right) - \sum_{\gamma=\gamma_i}^{\Gamma} e_i\left(C^{\gamma}\left(a\right), a_{\gamma}\right) \geq 0. \blacksquare$$

7.6 Proof of Proposition 3

(1) Using an obligation function o we can arrive at a cost allocation as follows. We compute a *mcst* following Kruskal's algorithm (Kruskal, 1956), which consists in to construct a tree by sequentially adding arcs with the lowest cost and without introducing cycles. The cost of each arc selected by Kruskal's algorithm is divided among the agents who benefit from adding this arc. Each of these agents pays the difference between her obligation to two groups, one in which she belonged before the arc was added and the one after. We now define an obligation rule, f^o, formally.

Given a network g we define $P(g) = \{T_k(g)\}_{k=1}^{n(g)}$ as the partition of N_0 in *connected components* induced by g. Namely, $P(g)$ is the only partition of N_0 satisfying the following two properties: Firstly, if $i, j \in T_k(g)$, i and j are connected in g. Secondly, if $i \in T_k$, $j \in T_l$, and $k \neq l$, then i and j are not connected in g. Given a network g, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs to.

Given an *mcstp* (N_0, C_0), let $g^{|N|}$ be a tree obtained applying Kruskal's algorithm to (N_0, C_0), and for each $p = 1, ..., |N|$, (i^p, j^p) is the arc selected by Kruskal's algorithm at Stage p and g^p the set of arcs selected by Kruskal's algorithm at stages $1, ..., p$. For each $i \in N$, we define the obligation rule associated with the obligation function o as

$$f_i^o(N_0, C_0) = \sum_{p=1}^{|N|} c_{i^p j^p} \left(o_i \left(S \left(P \left(g^{p-1} \right), i \right) \right) - o_i \left(S \left(P \left(g^p \right), i \right) \right) \right)$$

where by convention, $o_i(T) = 0$ if $0 \in T$.

Tijs *et al* (2006) prove that f^o is well defined, namely, it is independent of the *mcst* obtained following Kruskal's algorithm.

We prove that if f^o is an obligation rule, then $f^o = f^e$ where $e(C^*, x) = x o_i(N)$ for each (N, C^*) and x.

We proceed by induction on the number of agents. If $|N| = 1$ the result holds trivially. Assume that $f^o = f^e$ when $|N| < q$ and we prove it when $|N| = q$.

Let (N_0, C_0) be an *mcstp*. Since f^o and f^e satisfy CM, it is enough to prove that $f^o(N_0, C_0^*) = f^e(N_0, C_0^*)$.

Let $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^{|N|}$ be an *mcst* in (N_0, C_0^*) as in Proposition 3.1 of Bergantiños and Vidal-Puga (2007). Without loss of generality we assume that $\pi_s = s$ for each $s = 1, ..., |N|$. We consider two cases.

1. There exists $s > 1$ such that $c^*_{s-1,s} \geq c^*_{r-1,r}$ for all $r = 1, ..., |N|$. Let $S = \{1, ..., s-1\}$. Under Propositions 3.1 and 3.3 in Bergantiños and Vidal-Puga (2007) we deduce that $m(N_0, C_0^*) = m(S_0, C_0^*) + m((N \backslash S)_0, C_0^*)$.

   Let $i \in S$. Since f^o and f^e satisfy SEP, we deduce that

   $$f_i^o(N_0, C_0^*) = f_i^o(S_0, C_0^*) \text{ and } f_i^e(N_0, C_0^*) = f_i^e(S_0, C_0^*).$$

   By induction hypothesis $f_i^o(S_0, C_0^*) = f_i^e(S_0, C_0^*)$. Hence, $f_i^o(N_0, C_0^*) = f_i^e(N_0, C_0^*)$.

   Similarly we can prove that $f_i^o(N_0, C_0^*) = f_i^e(N_0, C_0^*)$ when $i \in N \backslash S$.

2. $c_{01}^* > c_{r-1,r}^*$ for all $r = 2, ..., |N|$. Let $\alpha = c_{01}^* - \max_{r=2,...,|N|} \left\{c_{r-1,r}^*\right\}$. Let $C_0'^*$ be the irreducible matrix associated with the tree t and the cost function c' where $c_{01}' = c_{01}^* - \alpha$ and $c_{r-1,r}' = c_{r-1,r}^*$ for all $r = 2, ..., |N|$.

Since $C_0'^*$ is under the conditions of the previous case, we have that $f^o(N_0, C_0'^*) = f^e(N_0, C_0'^*)$. Thus, it is enough to prove that for all $i \in N$,

$$f_i^o(N_0, C_0^*) - f_i^o(N_0, C_0'^*) = f_i^e(N_0, C_0^*) - f_i^e(N_0, C_0'^*).$$

Fix $i \in N$. We first compute $f_i^o(N_0, C_0^*) - f_i^o(N_0, C_0'^*)$. We can apply Kruskal's algorithm to both C_0^* and $C_0'^*$ in such a way that:

- The arc selected at each stage belongs to t. Namely, for each $p = 1, ..., |N|$, $(i^p(C_0^*), j^p(C_0^*)) \in t$ and $(i^p(C_0'^*), j^p(C_0'^*)) \in t$.
- The arc selected at each stage is the same in both problems. Namely, for each $p = 1, ..., |N|$, $(i^p(C_0^*), j^p(C_0^*)) = (i^p(C_0'^*), j^p(C_0'^*))$.
- The last arc selected is $(0, 1)$. Namely, $\left(i^{|N|}(C_0^*), j^{|N|}(C_0^*)\right) = \left(i^{|N|}(C_0'^*), j^{|N|}(C_0'^*)\right) = (0, 1)$.

Thus,

$$\begin{aligned} f_i^o(N_0, C_0^*) - f_i^o(N_0, C_0'^*) &= c_{01}^* o_i(N) - c_{01}'^* o_i(N) \\ &= \alpha o_i(N). \end{aligned}$$

We now compute $f_i^e(N_0, C_0^*) - f_i^e(N_0, C_0'^*)$. It is straightforward to check that if S is a neighborhood of node i in $C_0'^*$, then S is also a neighborhood of i in C_0^*. Besides, N is the unique neighborhood of i in C_0^* which is not a neighborhood of i in $C_0'^*$. Thus,

$$f_i^e(N_0, C_0^*) - f_i^e(N_0, C_0'^*) = c_{0i}^* - (\delta_N - e_i(C_N^*, \delta_N)) - c_{0i}'^*.$$

It is straightforward to check that $\delta_N = \alpha$. Hence,

$$f_i^e(N_0, C_0^*) - f_i^e(N_0, C_0'^*) = e_i(C_N^*, \alpha) = \alpha o_i(N).$$

Using arguments similar to those used above we can prove that if f^e is associated with some e as in the statement, then $f^e = f^o$ where $o(N) = e(C^*, 1)$. Notice that, by hypothesis, $o(N)$ does not depend on C^*.

(2) I is a trivial consequence of part (1) and the definition of optimistic weighted Shapley rules.

(3) I is a trivial consequence of part (1) and the definition of the ERO rule. ■

7.7 Proof of Proposition 4

We prove that the extra-cost correspondence e satisfies the NDC property, which implies, under Theorem 3, that f^e satisfies CM and PM.

Consider a disjoint sequence $\{(N^\gamma, C^\gamma)\}_{\gamma=1}^{\Gamma} \subset \mathcal{C}^*$, $i \in N^{\gamma_i} \subset N$ with $\gamma_i \neq 2$, $a \in \mathbb{R}_+^\Gamma$ with $a_\gamma \geq \max C^{\gamma+1} - \max C^\gamma$ for all $\gamma = 1, ..., \Gamma - 1$, and $y \in [0, a_2]$ ($y \geq 0$ when $\Gamma = 1$). We will prove that

$$\sum_{\gamma=\gamma_i}^{\Gamma} e_i \left(C^\gamma \left(a'\right), a'_\gamma\right) \geq \sum_{\gamma=\gamma_i}^{\Gamma} e_i \left(C^\gamma \left(a\right), a_\gamma\right).$$

If $\Gamma = 1$ the result is straightforward. Assume now that $\Gamma > 1$. Since $a'_\gamma = a_\gamma$ when $\gamma \geq 3$,

$$\begin{aligned} e_i \left(C^\gamma \left(a'\right), a'_\gamma\right) &= \int_0^{a'_\gamma} o_i^x \left(N^1 \cup ... \cup N^\gamma\right) dx \\ &= \int_0^{a_\gamma} o_i^x \left(N^1 \cup ... \cup N^\gamma\right) dx = e_i \left(C^\gamma \left(a\right), a_\gamma\right) \end{aligned}$$

for all $\gamma \geq 3$.

In particular, if $\gamma_i \geq 3$ the inequality holds. Hence, we assume $i \in N^1$. We know that $e_i \left(C^\gamma \left(a'\right), a'_\gamma\right) = e_i \left(C^\gamma \left(a\right), a_\gamma\right)$ for all $\gamma \geq 3$. Thus, it is enough to prove that

$$\sum_{\gamma=1}^{2} e_i \left(C^\gamma \left(a'\right), a'_\gamma\right) \geq \sum_{\gamma=1}^{2} e_i \left(C^\gamma \left(a\right), a_\gamma\right).$$

We make some computations:

$$\begin{aligned} e_i \left(C^1 \left(a'\right), a'_1\right) &= \int_0^{a'_1} o_i^x \left(N^1\right) dx = \int_0^{a_1+y} o_i^x \left(N^1\right) dx \\ e_i \left(C^2 \left(a'\right), a'_2\right) &= \int_0^{a'_2} o_i^x \left(N^1 \cup N^2\right) dx = \int_0^{a_2-y} o_i^x \left(N^1 \cup N^2\right) dx \\ e_i \left(C^1 \left(a\right), a_1\right) &= \int_0^{a_1} o_i^x \left(N^1\right) dx, \text{ and} \\ e_i \left(C^2 \left(a\right), a_2\right) &= \int_0^{a_2} o_i^x \left(N^1 \cup N^2\right) dx. \end{aligned}$$

Thus, the inequality holds if and only if

$$\int_0^{a_1+y} o_i^x \left(N^1\right) dx + \int_0^{a_2-y} o_i^x \left(N^1 \cup N^2\right) dx \geq \int_0^{a_1} o_i^x \left(N^1\right) dx + \int_0^{a_2} o_i^x \left(N^1 \cup N^2\right) dx.$$

Equivalently,

$$\int_{a_1}^{a_1+y} o_i^x \left(N^1\right) dx \geq \int_{a_2-y}^{a_2} o_i^x \left(N^1 \cup N^2\right) dx$$

which is a particular case of the condition given in (3). ■