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# Credit Risk Tools

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[1]The opinions expressed here are mine and do not necessarily reflect the views and opinions of AIB.

# Contents

# Outline

In this work, we solve a specific risk measurement problem, which involves both credit and market risk. The intention here is to construct several tools which can handle the task of producing a risk measure for a complex financial exposure. We will gloss over the mathematics and the statistics of modeling, concentrating on the computational aspect of the problem. We refer to the cited bibliography for an overview, cfr. e.g. [2], [4], [5], [6], [13], [14] and [18].

Specifically, we will construct several algorithms and simulation schemes to jointly tackle the problem of the analysis of the credit and market risk of a credit derivative portfolio. We will deal with the problem of pricing a synthetic CDO tranche and with the assessment of the evolution behavior of value of the net income resulting from the exposure to a single credit derivative of this sort. We cope with the pricing problem by constructing algorithms which provide the necessary tools to bake out the key variables. The second problem is solved via Monte Carlo simulation. The calculations, which constitute the main input of the simulation engine, can be easily implemented since they only result in the operations of matrix inversion and numerical integration. The flexibility of the risk evaluation method, which has been achieved through stochastic simulation, allows the system to be easily escalated and extended to a collection of basket credit derivatives. Furthermore, the implemented tools can, in principle, be extended to a wider class of similar risk problems. There are limitations to the setup though, which rely mainly on the working hypothesis that the realized credit conditions remain static through time. This is partly due to the static nature of the copula function through which the random times have been generated. The term static here has not to be intended as unchanged but as referred to the simulation time. The static quantities are the forward intensity rates of the Poisson-like random numbers which generate the random default times and evolve according to a survival probability function that is determined at the start of the simulation. Eventually, the effectiveness of the designed procedures for the purpose of risk management depends on the consistent assessment of the default probabilities along the forecasting horizon. This argument impacts also the framework in the market risk perspective, where the stochastic drivers of the price require appropriate parameterization. However, this work gives only partial attention to the problem of the statistical estimation of the credit and market risk factors and leaves this topic to further works.
The main target here is the construction of the structure capable of producing the estimation of the complex portfolio pay-off distributions. Those distributions generated via Monte Carlo simulation can immediately provide a measure of the portfolio risk in terms of the industry standard: value at risk (VaR).

The work is organized as follows. In part I.Algorithms, we will design two simple linear systems to compute the market implied discount factor curve from market rates and the default probabilities from market credit default swap (CDS) spreads. Furthermore, adopting the copula function approach to construct the default time joint distribution of a basket of credits, we develop a direct convolution algorithm to compute the loss distribution function of the overall portfolio. This procedure allows the introduction of stochastic recovery. The loss distribution is then employed to determine, numerically, the value of a single tranche of a collateralized debt obligation (CDO).

Finally, in part II.Simulations, we develop a stochastic cash−flow stream model that is employed to provide insight into the pay−off profile at maturity of the deal, to evaluate the 1y credit value at risk (CVaR), that is the value at risk which is attributable to the credit risk only and incidentally to test for the correctness of the pricing formula. In the second simulation, an auxiliary module is developed to bring in the system the CDO tranche price dynamics and allow providing an estimate of the total value at risk (VaR), including market risk. Subject to random shifts of the CDS term structures and to stochastic correlation, the evolution of the CDO tranche cash value is assumed to be unwound at a given time in the future, providing an estimation of the distribution of the portfolio future potential exposure.

# 1 Algorithms

## 1.1 The Discount Factor curve

The discount factor curve is the main ingredient to construct the model of a financial claim which is sensitive to the full spectrum of spot rates. In the financial market, it is easy to observe directly the shortest segment of the curve, whereas above 1y or 2y is difficult to find benchmark bullet bonds or extremely liquid financial contracts which are directly expressed in terms of spot rates. Therefore, the discount factor curve is inferred from traded financial instruments which are indirect expression of the underlying term structure. In this work, the discount factor curve is constructed through interest rate swap contracts, from the 2y onward, while LIBOR rates are exploited to derive the first segment. We have to mention, though, that overnight indexed swaps (OIS) are becoming more popular in the process of the evaluation of the first segment of the curve.
Usually, the calculation of the spot rates is performed through a forward induction procedure, which is called "bootstrapping" (see [7], [19], [20]). However, a quick linear system can be arranged.

Let us consider the discount curve for the € market. The discount factors at time $t$ on tenor $\tau$, based on money market rates is

$$B(t,\,\tau) = \frac{1}{(1+\alpha r(t,\,\tau))}$$

where $\alpha$ is the year fraction and the spot rate is quoted yoy. The theoretical price of a forward IRS is

$$\sum_{i=1}^{J} \alpha_i s_{t_k} B(t,\,\tau_i) = B(t,\,\tau_0) - B(t,\,\tau_k)$$

where the expectation operator referred to each discounting factor has been suppressed for ease of notation. The swap rates are assumed to be priced according to the (implied) market risk measure. Notice that when $\tau_0 = t$ then $B(t,\,t) = 1$. The piecewise linear discount factor curve $\mathbf{b}$ constructed on market LIBOR and swap rates paid in annual installments, is the solution of the linear equation

$$(\mathbf{S} \cdot [\mathbf{U},\,\mathbf{Q}] + \mathbf{W}) \cdot \mathbf{b} = \mathbf{u} \tag{1}$$

with

$$\mathbf{S} = \left[\begin{array}{c|cccc} \mathbf{I} & 0 & 0 & \cdots & 0 \\ \hline 0 & s_{t_1} & 0 & \cdots & 0 \\ 0 & 0 & s_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{t_K} \end{array}\right],\ \mathbf{U} = \left[\begin{array}{cccc} \multicolumn{4}{c}{\mathbf{I}} \\ \hline 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$\mathbf{Q}=\left[\begin{array}{cccc}\multicolumn{4}{c}{\mathbf{0}}\\\hline 1&0&\cdots&0\\1&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{array}\right],\ \mathbf{W}=\left[\begin{array}{c|ccc}\mathbf{0}&0&\cdots&0\\\hline 0&&&\\\vdots&&\mathbf{I}&\\0&&&\end{array}\right],\ \mathbf{u}=\left(\begin{array}{c}1\\\left(1+\frac{1}{4}r\left(t,1/4\right)\right)^{-1}\\\left(1+\frac{1}{2}r\left(t,1/2\right)\right)^{-1}\\\left(1+r\left(t,1\right)\right)^{-1}\\\hline\mathbf{i}\end{array}\right)$$

The equation (1) is then solved by inverting the system matrix, which is full rank by construction. When the swap is paid semi-annually, equation (1) is slightly modified to account for the discounted component of the intra-period payments. The piecewise discount factor curve is now the $\mathbf{b}$ which solves

$$\left(\mathbf{S}'\cdot\left([\mathbf{U}',\,\mathbf{Q}]+\tfrac{1}{2}\mathbf{Q}\cdot\mathbf{Z}\right)+\mathbf{W}\right)\cdot\mathbf{b}=\mathbf{u}\tag{2}$$

where

$$\mathbf{S}'=\left[\begin{array}{c|ccccc}\mathbf{I}&0&0&\cdots&0\\\hline 0&\frac{1}{2}s_{t_1}&0&\cdots&0\\0&0&\frac{1}{2}s_{t_2}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\frac{1}{2}s_{t_K}\end{array}\right],\ \mathbf{U}'=\left[\begin{array}{cccc}\multicolumn{4}{c}{\mathbf{I}}\\\hline 0&0&1&1\\\vdots&\vdots&\vdots&\vdots\\0&0&1&1\end{array}\right],$$

$$\mathbf{Z}=\left[\begin{array}{cccccccccc}0&0&0&1&1&0&\cdots&0&0&0\\0&0&0&0&1&1&\cdots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&0&0&0&\cdots&1&1&0\\0&0&0&0&0&0&\cdots&0&1&1\end{array}\right]$$

The added term $\frac{1}{2}\mathbf{Q}\cdot\mathbf{Z}$ is the operator which performs the linear interpolation of the adjacent discount factor points to produce the intra-annual discounting of the swap coupons.

The procedure just presented produces a piecewise linear discount factor curve with vertexes settled at time 0, at the LIBOR and at the swap maturities. In some cases, it might be more convenient to determine directly the discount factor curve which yields the par swap rates and the compounded short term rates and, at the same time, satisfies some smoothness constraints. When using a cubic spline with junction points corresponding to the same set of times as in the piecewise linear solution, the output curve is quite indistinguishable from the former line.
Consider the spline determined by the set of cubic polynomials $\{p_k\}_{k=1,\ldots,K}$, with

$$p_k(t)=a_0^k+a_1^k\left(\tfrac{t-t_{k-1}}{t_k-t_{k-1}}\right)+a_2^k\left(\tfrac{t-t_{k-1}}{t_k-t_{k-1}}\right)^2+a_3^k\left(\tfrac{t-t_{k-1}}{t_k-t_{k-1}}\right)^3$$

The polynomial structure of the spline objects allows to write the general problem of interpolation as a linearly constrained quadratic problem. In fact, let

$\alpha$ be the vector containing the coefficients of the set of polynomials, the swap valuation formula can be rewritten as the linear constraint

$$(\mathbf{S} \cdot \mathbf{U} \cdot \mathbf{H} + \mathbf{Q}) \cdot \alpha = \mathbf{i} \tag{3}$$

where $\mathbf{S}$ is now the diagonal matrix containing the swap rates, $\mathbf{U}$ is the $0-1$ matrix indicating the cash-flow times and $\mathbf{H}$ and $\mathbf{Q}$ are suitable matrices which produce the array of the spline's values, respectively, at the cash-flow dates and at the latest cash-flow date. The equation (3) can accommodate both the cases of yearly or semi-annual payments, rather than different swap payment structure. The smoothness constraints, usually of the first order, describe the continuity and differentiability of the spline at the junction points. The individual constraints are written in the following form

$$\begin{aligned} a_0^k + a_1^k + a_2^k + a_3^k - a_0^{k+1} &= 0 \qquad (4)\\ a_1^k + 2a_2^k + 3a_3^k - \left(\frac{t_k - t_{k-1}}{t_{k+1} - t_k}\right) a_1^{k+1} &= 0. \end{aligned}$$

Furthermore, the discount factor curve must be equal to the LIBOR discounting factors and, obviously, must be equal to 1 in $t = 0$, adding the constraints of type

$$a_0^k = u_0^k. \tag{5}$$

The set of constraints (3), (4) and (5) can be blended into the linear system

$$\mathbf{M} \cdot \alpha = \mathbf{u}.$$

Finally, we need the discount factor curve to be decreasing in time. To obtain this feature with our set of cubic spline, we ought to construct, for instance, a set of non linear constraint of this type:

$$\begin{aligned} a_3^k &\leq 0 \qquad (6)\\ \left(a_2^k\right)^2 - 3a_1^k a_3^k &\leq 0 \end{aligned}$$

which would force the first derivative of the spline to be negative or zero within each interval. We indicate the set of constraint (6) as $\mathbf{F}(\alpha) \leq 0$. Having set the problem's constraints, we are still left with a large subspace of splines which might match the discount factor curve. One undesirable feature of the spline interpolation procedure is having uncontrolled oscillation of the interpolation function that can introduce a further source of noise when figured in the dynamic perspective. In order to reduce the likelihood of such a behavior, we choose to pick the spline of minimal length, that is the spline which minimizes the function

$$\sum_k \frac{1}{t_k - t_{k-1}} \int_0^1 \mathrm{d}x \, \sqrt{1 + {p_k^{(1)}}^2}. \tag{7}$$

It can be verified that the solution of the minimization problem is the same as

$$\begin{aligned} &\min_\alpha \alpha \cdot \mathbf{G} \cdot \alpha \qquad (8)\\ &sub \;\; \mathbf{M} \cdot \alpha = u \\ &\qquad \mathbf{F}(\alpha) \leq 0. \qquad (9) \end{aligned}$$

where the matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{bmatrix} \mathbf{L} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4/3 & 3/2 \\ 0 & 1 & 3/2 & 9/5 \end{bmatrix}$$

In figure 3 we graphically compare the output of the linear system algorithm with the output of the optimal spline interpolation procedure.

## 1.2 The implied Probability of Default

The second major component in the construction of the model is the survival probability term structure of each single cash-flow stream generating item. The death event is associated to the cease of the payments. Because of the unfunded nature of a CDS deal, in the valuation formula only the net cash flows are considered. Financially from the protection seller's perspective, a credit default swap spread agreement synthesizes the operation of borrowing money from a default-free subject at a fixed rate, buying the floating rate note corresponding to the underlying reference credit risk, entering an IRS to offset the floating inflows with the fixed outflows due to debt (cfr. [3], [10] and [21]). In a frictionless world, the net result of this operation would be the premium for the credit protection and the potential loss on the note principal. As a consequence, the fair valuation of the CDS spread is such that, let $\tau_0 = t$, the potential actual income balances the potential actual loss that is

$$\sum_{i=1}^{J} \alpha_i s_{t_k} B(t,\, \tau_i) \mathbb{P}_{\{\tau > \tau_i\}} = (1-R) \left[ 1 - \mathbb{P}_{\{\tau=\tau_0\}} + \sum_{i=1}^{J} B(t,\, \tau_i) \left( \mathbb{P}_{\{\tau > \tau_{i-1}\}} - \mathbb{P}_{\{\tau > \tau_i\}} \right) \right]$$

From the last formula, it is a straightforward operation the construction of an algorithm to estimate the implied PD which is embedded into the market prices. Provided we observe at a given instant in time the term structure of the swap spreads for a given issuer, the vector of the implied default probability at the IMM dates is the vector $\mathbf{p}$ which solves the linear system

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{u} \tag{10}$$

with

$$\mathbf{A} = \left[ \frac{[(1-R)\mathbf{i},\, \mathbf{S} \cdot \mathbf{B}] \,+\, (1-R)\mathbf{B} \cdot \mathbf{D}}{1 \quad 0 \qquad \cdots \qquad 0} \right], \; \mathbf{u} = \left[ \frac{(1-R)\mathbf{i}}{1} \right]$$

and

$$\mathbf{S} = \begin{bmatrix} \alpha_1 s_{t_1} & 0 & \cdots & 0 \\ 0 & \alpha_2 s_{t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{t_K} s_{t_K} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathrm{e}^{-t_1 r_{t_1}} & 0 & \cdots & 0 \\ \mathrm{e}^{-t_1 r_{t_1}} & \mathrm{e}^{-t_2 r_{t_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-t_1 r_{t_1}} & \mathrm{e}^{-t_2 r_{t_2}} & \cdots & \mathrm{e}^{-t_K r_{t_K}} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

Where the last row in $\mathbf{A}$ sets the $\mathbb{P}_{\{\tau=\tau_0\}} = 0$. In order to assume non zero immediate default probability, the final row second column in $\mathbf{A}$ can be changed to minus one, where the last element in $\mathbf{u}$ can be set to zero. In fig. 1 and 2 we show the reference CDS basket and the implied default probabilities on 10/05/2011.

It must be noticed that in order to construct a linear system which can be solved uniquely, we have to augment the $\mathbf{S}$ matrix with linearly interpolated spread contracts at the missing cash-flow dates. As it will be shown graphically, this approach does not produce significantly different survival probability curve than the spline method present in the latter subsection. In fact, the same method used to construct a smooth discount factor curve can be used to produce continuous and differentiable survival probability curves. In this case, the swap par equation is substituted with the first block line of the matrix $\mathbf{A}$ multiplied by a suitable matrix $\mathbf{H}$, which represents the pricing relations implied by the CDS spread market quotes. The corresponding known term is represented by the first block of the $\mathbf{u}$ vector

$$\left[ \; [(1-R)\mathbf{i}, \, \mathbf{S} \cdot \mathbf{B}] + (1-R)\mathbf{B} \cdot \mathbf{D} \; \right] \cdot \mathbf{H} \cdot \alpha = \mathbf{u}'.$$

In the latter system there is no need to augment the $\mathbf{S}$ matrix with linearly interpolated spread quotes for the cash-flow dates at which there is no available contract. The smoothness constraints remain the same, provided that the set of junction and extreme points are suitably adjusted. The only matching point condition is represented by the assumption that the immediate default corresponds to the impossible event, which can be released. We finally obtain, again, the quadratic program (8)

$$\begin{aligned} &\min_{\alpha} \alpha \cdot \mathbf{G} \cdot \alpha \\ sub \;\; &\mathbf{M} \cdot \alpha = u \\ &\mathbf{F}(\alpha) \leq 0. \end{aligned}$$

In figure 4 are shown the survival probability curve constructed with the minimal length spline algorithm and the piecewise linear term structures of survival probabilities.

## 1.3 The Credit Basket Loss Distribution

The next step in building up our credit risk toolbox is the extension of the credit risk considerations to a collection of defaultable items. The focus will still remain on the CDS pool, although the only components that are needed here are the discount factor curve and the set of survival probability term structures. Once the implied default probabilities have been calculated, we obtain the individual probability density function of the credit event associated to each basket component,

$$\mathbb{P}_1,\, \mathbb{P}_2,\, \ldots,\, \mathbb{P}_N.$$

Those probabilities only represent the marginal density functions of the basket joint distribution of default times. In order to obtain a measure of the joint credit risk borne by the basket, we adopt the copula function approach (cfr. [16], [17]). The rationale here is that there exist a set of latent factors, $\{M_1, \ldots, M_s\}$, which interconnect the dynamics of default events; conditioning on these factors, the default of each basket item is independent of the rest of the bunch, formally

$$\mathbb{Q}\{\tau \in \Xi\} = \mathbb{E}_{\mathbb{M}} \left\{ \prod_{i=1}^{N} \mathbb{Q}_i\{\tau_i \in \Xi | M\} \right\}$$

where $\tau$ represent the array of the basket components' default events, $\Xi$ is the $\sigma-$algebra on the space of events. In order to convert the $\mathbb{P}_i$ into the $\mathbb{Q}_i$ the popular approach is to employ Sklar's theorem assuming that the structure of the copula function is Gaussian, cfr. [5], [12], [15]. The mapping is realized through the credit index variables

$$Q_j = \sum_i a_j^i M_i + X_j \sqrt{1 - \sum_i \left(a_j^i\right)^2}$$

with i.i.d. random variables $M_i,\ X_j \sim \phi$, the standard normal distribution. As a consequence, $Q \sim \phi(0,\, C)$ with correlation matrix $C = \left[c_j^k\right]$, such that $c_j^k = \sum_i a_j^i a_k^i,\ j \neq k$ and $\left|c_j^k\right| \leq 1$. The normality of the copula function is convenient for the parameterization of the dependency of the default that is achieved through the correlation matrix of the credit indexes, cfr. [17], [21]. The distribution of each random default time conditional on the latent factors is then

$$\mathbb{Q}_j\{\tau_j \leq T|M\} = \Phi \left( \frac{\Phi^{-1}\left(1 - \mathbb{P}_j\left\{\tau_j > T\right\}\right) - \sum_i a_j^i M_i}{\sqrt{1 - \sum_i \left(a_j^i\right)^2}} \right) \tag{11}$$

with $\Phi = \int_{-\infty}^{\cdot} \phi$. Formula (11) not only determines the joint distribution function of the default times but also provides directly a mean to compute the basket loss distribution function.

Let $W = \sum_{j=1}^{N} W_j$ be the notional basket value and $W_j$ the notional value of the components. The distribution of the value of the $j^{th}$ item conditional on the latent factors is therefore

$$\mathbb{G}_j \left\{ w_T^j \in \Omega | M \right\} = \mathbb{Q}_j\left\{\tau_j > T|M\right\} \delta_{W_j} + \mathbb{Q}_j\left\{\tau_j \leq T|M\right\} \mathbb{F}_j \left\{ w_T^j \in \Omega | \tau_j \leq T \right\} \tag{12}$$

where $\delta_{W_j}$ is the delta functional centered at the notional value $W_j$ and $\mathbb{F}_j$ represents the probability measure of the $j^{th}$ recovery value given default, within time $T$. The structure of the copula function implies that conditioning on latent factors, the notional values of the basket components are independent, letting the conditional distribution of the sum of $W_j$ result in the convolution of the probability measures of the value of each credit risky item. Henceforth, the distribution of the basket notional value within the time horizon $T$ will be

$$\mathbb{G}\left\{w_T \in \Omega\right\} = \mathbb{E}_{\mathbb{M}}\left\{\mathbb{G}_1^m \star \cdots \star \mathbb{G}_N^m\right\} \tag{13}$$

where $\mathbb{M}$ stands for the probability measure on $M$. Particular care must be taken in the drawing of the support of $\mathbb{G}$. For this purpose, it is crucial that the set $\{W_j\}_{j=1,\ldots,N}$ be divisible by the grid step.
Eventually, the credit basket loss is the $l_T = \sum_{i=1}^N W_j - w_T$. Let define here, for successive use, the probability measure $\mathbb{L}_T$ of the credit basket loss $l_T$ within the time horizon $T$.

We are finally in the position to construct the numerical algorithm which yields the distribution of the basket notional value $w_T$. The program consists in two procedures: the computation of the chain of convolutions conditional on $M$, appropriately discretized; the computation of the expectation w.r.t. $\mathbb{M}$. Formally, in the context of the single factor copula model, let $\mathfrak{M} = \{m_{-u}, m_{-u+1}, \ldots, m_0, m_1, \ldots, m_u\}$ be the domain of $M$, because of commutativity of the convolution operation, it is indifferent the order at which the $\mathbb{G}_j^m$ are included in the evaluation of the argument of (13). Hence, the $\mathbb{G}\{\cdot|m \in \mathfrak{M}\} = \mathbb{G}_1^m \star \cdots \star \mathbb{G}_N^m$ is the $\mathbb{H}_N^m$ at the end of the recurrence

$$\begin{cases} \mathbb{H}_1^m &= \mathbb{G}_1^m \\ \mathbb{H}_j^m &= \mathbb{H}_{j-1}^m \star \mathbb{G}_j^m,\ j = 2, \ldots, N \end{cases}$$

The final output is obtained averaging the $\mathbb{H}_N^m$ w.r.t. $\mathbb{M}$, that is

$$\mathbb{G} = \sum_{m \in \mathfrak{M}} \mathbb{H}_N^m \Delta \mathbb{M}^m$$

We are not far away from writing the computer code for the evaluation of the portfolio loss distribution in our favorite programming language. We can combine formulas (11), (12) and (13) to give more characterization to the recurrence formula.
Under the hypothesis of constant recovery rates $R_j$, let $q_j^m = \mathbb{Q}_j\{\tau_j \leq T|M\}$ and we get

$$\mathbb{H}_j^m = \left(1 - q_j^m\right)\left(\mathbb{H}_{j-1}^m \star \delta_{W_j}\right) + q_j^m\left(\mathbb{H}_{j-1}^m \star \delta_{R_j}\right) \tag{14}$$

The calculation of the formula (14) does not involve necessarily the operation of convolution. At each step, the output function is a weighted average of the shifted input function, eventually resulting in a linear combination of appropriately shifted $\delta_{x_k}$. In fact, we can obtain a closed−form solution, in case

$q_j^m = q^m$, $W_j = v$ and $R_i = r$, $\forall j$. Expanding the symbolic power and exploiting the properties of the $\delta$ functional, we get

$$\mathbb{H}_N^m = \left\langle (1 - q^m)\,\delta_v + q^m \delta_r \right\rangle^N = \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} (q^m)^k (1-q^m)^{(N-k)} \delta_{kv+(N-k)r}$$

This last formula must be handled with care when implemented in a computer. There is a physical boundary in representing an integer number, therefore the computation of the Newtonian coefficient for a basket with more than 30/50 components becomes quickly unstable.
If we turn back to (14) and introduce stochastic recovery with probability $\mathbb{F}_j$, the recurrence becomes

$$\mathbb{H}_j^m = \left(1 - q_j^m\right)\left(\mathbb{H}_{j-1}^m \star \delta_{W_j}\right) + q_j^m \left(\mathbb{H}_{j-1}^m \star \mathbb{F}_j\right)$$

In fig. 5 and 6 we show the loss distribution of the reference basket portfolio at different default correlation parameters.

## 1.4 The pricing of a CDO tranche

To conclude part I, we provide the pricing algorithm for a financial claim on the basket of credit items. The CDS on CDO tranche formula builds up on the tools presented in this first part, and constitutes the main input device in the market risk simulation presented in part II.
Let the pool of credit risky items be the collateral of some newly issued obligations. The obligations are sorted w.r.t. seniority in sharing the proceeds and liquidating the assets. The reversal sorting order lists the priority of commitment to absorbing the losses of the reference credit basket up to the full tranche value. The notional asset side is usually cut in *senior*, *mezzanine* and *equity tranche*. The upper and lower boundaries of the basket tranche are called *detachment* and *attachment* points. In a *cash flow* CDO the asset proceeds and losses are effectively distributed between the notes subscribers (see, e.g., [8], [9], [11]). A CDS on *synthetic* CDO looks more like an insurance on a slice of the cumulative loss on the underlying basket notional. More precisely, in exchange of the stream of payments referred to the residual tranche notional, the protection buyer pass on to the seller the risk of the cumulative credit loss of the basket, which is above the attachment $a$ and below the detachment $d$. The fair upfront $u$ and spread payment $s$ of CDS on synthetic CDO are the $u$ and $s$ which offset the expected present value of the generated stream of payments and the expected present loss on the tranche notional. Formally,

$$u + s\sum_{i=1}^{k} \alpha_i B(t,\tau_i)\left(W_A - \mathbb{E}_{\mathbb{L}_{\tau_i}} L_A\right) = \mathbb{E}_{\mathbb{L}_{\tau_0}} L_A + \sum_{i=1}^{k} B(t,\tau_i)\left(\mathbb{E}_{\mathbb{L}_{\tau_i}} L_A - \mathbb{E}_{\mathbb{L}_{\tau_{i-1}}} L_A\right) \tag{15}$$

where $\tau_i$, $i > 0$ is the $i^{th}$ IMM date and $\tau_0 = t$, $W_A = d - a$ is the tranche notional value. The loss affecting the tranche is indicated as the random variable

$L_A = \max(0, \min(W_A, l - a))$.
It can be noted that, in order to evaluate the tranche price, the basket loss distributions at each payment date are needed. The latter feature makes the process a non zero computational cost operation. It is required to take into account this aspect when we simulate the operation of unwinding the residual tranche exposure.

# 2 Simulations

In the simulation study of part II, we aim at analyzing the dynamics of the profit and loss due to the exposure to a synthetic CDO tranche. The perspective is that of a protection seller; the results would be specular by the protection buyer's side. In the development of the simulation engine, we distinguish between the in/outflow process of the protection payments/loss coverage and the unwinding of the CDO exposure at current market conditions during the lifetime of the financial claim. This distinction corresponds to the classification of the embedded risk sources into the categories of credit and market risks. This separation is only functional to model design purposes.

In simulation 1, we develop the main simulation engine according to the structure of the stochastic cash-flow stream model. By this term, we generically mean the probabilistic model of a sequence of payments which are random (or deterministic) variables and are indexed by a random (or deterministic) time. We can formally describe the setup of the Monte Carlo simulation as follows. The mathematical objects we need are: a variable which accumulates the basket losses due to credit events and a variable which accumulates the payments for the credit protection. Formally, we have the basket total loss

$$l_t = \sum_{j=1}^{N} (W_j - \Theta_j)\, \mathbf{1}_{\{\tau_j \leq t\}}$$

where the marginal distribution of each random time can be viewed as an inhomogeneous Poisson distribution, whose implied intensities are given by the the initial survival probability term structure. The default time $\tau_j$ is the random time of the first jump in the associated counting process and $\mathbf{1}_{\{\tau_j \leq t\}}$ is the indicator function of the default time. The random variables $\Theta_j$ indicate the stochastic recovery rate of the portfolio component $j$, given default. This model structure is certainly a reduced form model, which, at a certain extent, could be classified as a Poisson-like model, see [6], [13], [14]. The total tranche loss will be, consequently

$$L_{A_t} = \max\left(0,\, \min\left(W_A,\, l_t - a\right)\right).$$

The random variable which then accumulates the net cash flow is therefore

$$C_t = s \sum_{i=1}^{K} \alpha_i \left(W_A - L_{A_{T_i}}\right) \mathbf{1}_{\{T_i \leq t\}} - L_{A_t}$$

where $\mathbf{1}_{\{T_i \leq t\}}$ is the ordinary Heaviside function (càdlàg) centered at the cash flow date $T_i$. Assuming that no extra funds are employed to cover the credit derivative exposure, but the upfront payment and the premium inflows, the dynamics of the protection seller's net position can be described by the SDE

$$\mathrm{d}V_t = r_t V_t \mathrm{d}t + \mathrm{d}C_t,\ V_0 = u. \tag{16}$$

Technically, the Monte Carlo simulation provides a numerical solution to the former equation.

In simulation 2, at 1y since the settlement date the credit derivative exposure is assumed to be unwound at market prices. The market price at which the engagement is liquidated depends on the current portfolio structure and on market risk factors. For each sample path, the current portfolio structure is priced according to (15) at random market components. The market risk drivers are modeled as one geometric Brownian factor multiplying the spread term structures of the portfolio components, whose volatility is set to the average 5y spread (log-differential) volatility (see fig. 9) and a stochastic correlation random variable, which is generated according to the sample distribution given in fig. 10. We shall also see that generating perfect matching prices at simulation time is quite an intensive computation task, therefore we will turn to some approximations.

## 2.1 Simulation 1.a

In this exercise we provide an estimate of the evolution of the portfolio loss due to the intrinsic credit exposure. The simulation produces the system dynamics described in (16). In financial terms, the simulation generates the evolution of the cash account containing the capitalized value of the P/L of the credit derivative deal. On 10/05/2011, the protection seller enters an unfunded CDO tranche deal on the iTRAXX Europe Xover, 06/16 series 15 - 5y with €10,000 notional. The currency unit is assumed to be the €, though the currency denomination leaves the exercise unaffected and in the sequel it will be suppressed. The bespoke tranche is the equity tranche $0 - 12.5\%$ on the reference basket portfolio, with a notional of $1,250$. At the settlement, the equity holder receives an upfront fee of 68.26% unitary value ($68.26\% \times 1,250 = 853.23$, correlation is 0.4), which is poured into an interest generating cash account. The working assumption is that the investor receives a full upfront fee and a zero spread payment, which facilitates the computation of the unwinding value. The interest is accrued linearly at the implied forward rates fixed at the payment date and constant within the coupon period. The proceeds are compounded at the IMM dates. In order to mimic reality more closely, the payments for new losses within each coupon period are as well assumed to be regulated at the next IMM date. The account is allowed to go negative, which means that the debtor position is forwarded to the maturity of the derivative contract, when the net exposure is liquidated. No additional money is assumed to be employed. At each time $t$, the current account value tracks the spot in/outflow which settles the account to 0. The negative tail of the distribution measures the frequency of the occurrence of paths when a residual payment for previous losses yet is due. In fig.7 we show the histogram of the cash account payout at maturity, 20/06/2016, resulting from 50,000 paths generated according to (16). The odds of a final call to cover the unpaid losses are about 61% with a 50% probability of paying more than 320, that is 1/4 ca. of the tranche notional, about 5% loss per year. With a 10% probability, the investor is requested to pay something between 413 and 459, which represents 1/3 ca. of the tranche notional.

As a by-product, we obtain a test for the fair pricing in the semi-analytical approach. At maturity, if the initial payment is "fair", the expected value of the consistencies should be zero. If the data generating process is held in line with the distributions employed for the fair valuation, we expect that the average portfolio value would be zero. The latter is a direct result of the swap argument: since the credit protection price is such that the payments offsets the expected present value of the portfolio losses, the accumulation of the premium payments in an interest bearing account would balance the outcomes of the portfolio losses covered by the credit protection at the maturity of the contract. The sample mean value is tested with an ordinary z-score, $\frac{\hat{\mu}}{\hat{\sigma}}\sqrt{n-1}$, which yields the value of 1.65 accepting the hypothesis of zero mean, with a 95% confidence.

## 2.2 Simulation 1.b

In simulation 1.b, the sample paths are observed at one year since settlement, on 10/05/2012. The array of the sampled points generates the distribution of the potential future value of the cash account, when only the credit events have affected its evolution, cfr. fig.8. Briefly, the distribution represents the future value of the initial inflow minus accumulated losses because of past defaults. The mass concentrates around the average recovery times the number of defaults that have been experienced, during the time horizon of observation. The stochastic recovery renders the distribution of the values more diffused. The spike to the left side of the figure, shows the account consistencies corresponding to the total loss of the tranche, which has a 1% probability ca. and, of course, is about the amount of incremental capital to cover the full tranche value. The spike to the right side of the chart represents the set of paths which have reported no default, in the 68% of the events. The first and fifth percentile provide an estimate of the 99% and 95% confidence credit value at risk (CVaR) of the exposure to credit risk only. That would look quite a good investment if only with a probability of 3% the investor is requested to margin call, whenever losses have eroded the initial premium, while in the remainder of possibilities the investor makes something out of nothing. Certainly, the latter view is only partial. The protection seller is still committed to the residual loss coverage, which can only be settled with an opposite operation that would pass the residual credit risk onto another obligor [1].

## 2.3 Simulation 2

In simulation 2, the previous sample is integrated with the effect of portfolio market exposure, in order to project a picture of the potential total risk affecting the financial pay−off of the deal on final settlement date. At the same random time as in simulation 1.b, i.e. 10/05/2012, the credit derivative exposure is wiped off. The market perception of the uncertainty affecting the future survival rates is expressed by the credit spreads and default correlation dynamics, which constitute the incremental risk sources. In order to obtain reasonable estimations of the distribution of the future potential exposure, the risk factors are related to market data, favoring simplicity and data availability rather than precision of model calibration. The term structure multiplicative factor

[1] The term obligor has been used here to indicate the counterpart in the credit derivative contract which is obliged to cover the losses in case of default. Actually, the synthetic operation of protection selling corresponds to the buying of a bond issued by the reference entity, the protection seller resulting in the obligee, in this case.

has the effect of blowing or contracting the basket sample spreads and therefore components' default probabilities, with an yearly volatility which is set to the average historical volatility of the portfolio components that is, assuming geometric motion, about 50%, cfr. fig. 9. It is quite more difficult to have a quick "realistic" measure of the default correlation parameter. This is strictly related to the evolution of the price of the specific tranche, and it is even inhomogeneous among the capital structure of the deal on a specific asset pool, cfr. [5]. In order to import correlation risk into the simulation and have an idea of how this variable could behave, we take the only base correlation sample available on Bloomberg, that is, on the 06/07/2011, the record of the base correlation of the iTRAXX Europe, 06/13 series 9 - 5y, for the detachment points 3%, 6%, 12% and 22% and, irrespectively of time, merge the sample obtaining an array of $4,000$ ca. data points. The empirical distribution has then been symmetrized around 50 correlation points and smoothed with a Gaussian kernel (fig.10). The default correlation random number has been generated mapping a uniform random number onto the cumulative sample distribution of the default correlations.

At the unwinding time the residual engagement committed to the protection seller and associated with the reference CDO tranche are evaluated at the current market conditions and written off. At the observation time, the cash account is charged with the possibly due coupon fraction computed with reference to the tranche value at the previous coupon date, the accrued interests and with the effective losses within the period fraction. The unwinding price of the deal is evaluated under the simplifying hypothesis described below, at random market factors. This last point needs some insight. If we were to generate the unwinding price at random time, assume the computation lasts 10 seconds, with the $400,000$ paths sample the simulation would last about 46 days. We shrink this computation time to a few seconds with some tricks. First of all, we do not compute the prices for each single tranche consistency, but we assume that in each case recovery happens at expected recovery. Secondly, and more important, we do not take all the possible default sequences that might happen, the assumption here is that defaults happen in sequence from the least likely$-$to$-$ default item onward. This assumption reduces drastically the number of possible portfolio structures to only nine cases, 0 to 8 defaults, which is the maximum number of defaults that the tranche can sustain. The assumption of the sequence of default, moreover, entails that the unwinding prices are the highest possible, providing a conservative estimate of the unwinding cost[2]. Finally, the previous assumptions would leave the computational time unchanged if the tranche price were computed at the simulation time. The programming strategy consists in constructing nine price surfaces, corresponding to each number of defaults in the assumed default sequence, on the domain of the random variables generating the stochastic factors which affect the price. In figure 12 the CDO tranche unitary price surfaces, while in fig. 11 the distributions of the absolute tranche prices.

Finally, the artificial sample distribution of the portfolio value can provide an estimate of the total $VaR$ of the financial operation. In fig.13 we show the

[2] It has also been experimented the computation of the default sequence from the most likely backward, providing the lowest unwinding prices. This computation would provide an interval for the expected $VaR$, which is actually very tight, in our case study.

estimation of the distribution of the future potential exposure of the basket portfolio. It comprehends the net income effect of the exposure to the credit derivative commitments plus the random cost of selling the residual protection to the market. The distribution is *platicurtica*, i.e. it has a kurtosis less than normality, 2.67, while the skewness is negative, with a coefficient of $-0.33$. Because of the events of the tranche total loss, we find an anomalous spike around the value of 400 (more than 8 defaults), the remainder of the tail is determined by some events with 7 and 8 defaults. The total $VaR$ at 99% confidence within 1y time horizon is $-399$, while by the next €23 only another 1% risk is found, because of the unusual peak; the $VaR(5\%, 1y) = -244$ instead. The average cash account consistency is positive at 165 almost close to the median value of 182. The 23% ca. of the sample outcomes represent the possibility of an additional payment to balance the residual debt.

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# Figures

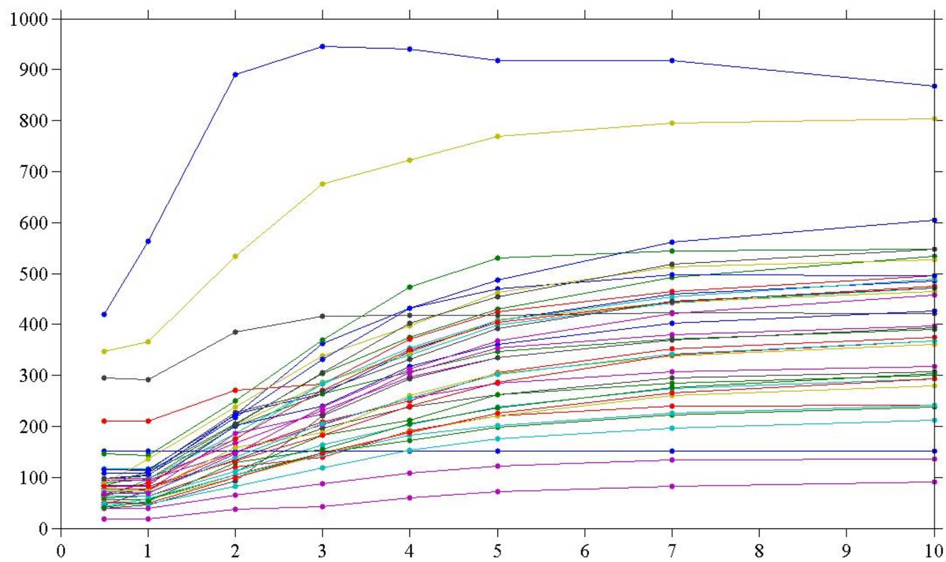


Figure 1: The reference basket spread term structures at the standard maturities: 6m, 1-5y, 7y and 10y. The observation day coincides with the settlement date, which is 10/05/2011. The reference basket is the iTRAXX Europe Xover, series 15.

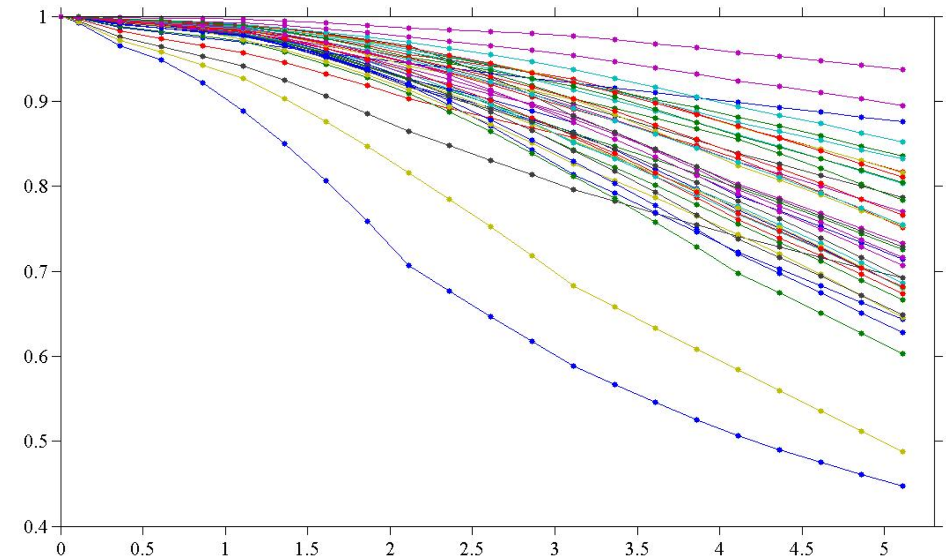


Figure 2: The implied survival probabilities at the IMM dates, on 10/05/2011 for the iTRAXX Europe Xover, series 15.

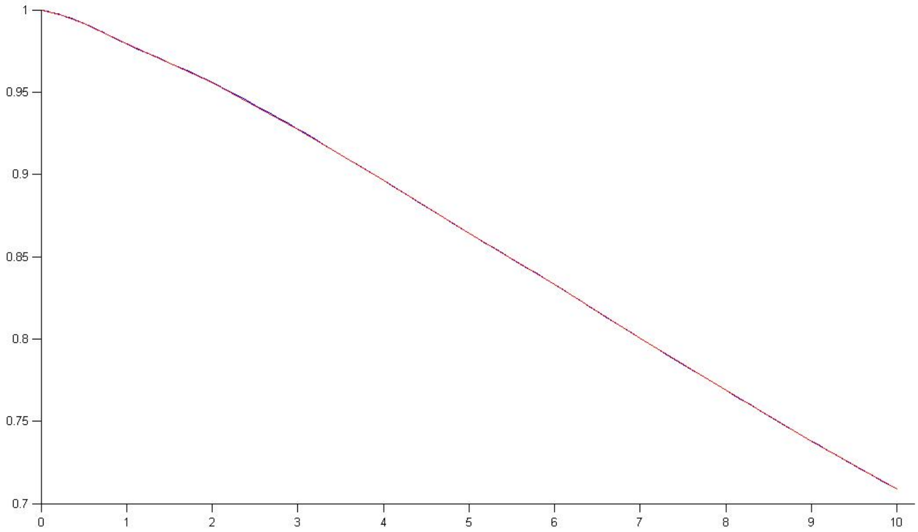


Figure 3: The discount factor curves have been estimated with the linear system method (red curve) and the minimal length cubic spline (blue curve).

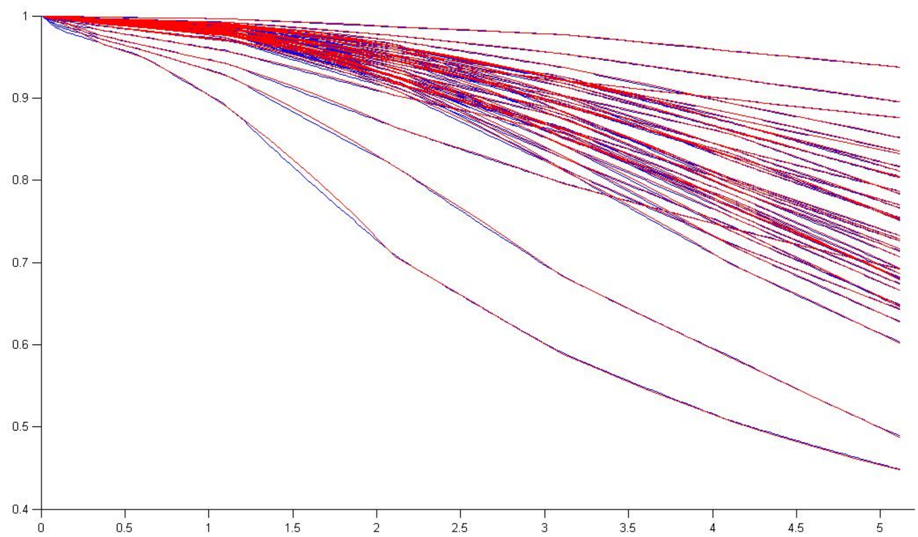


Figure 4: The survival probability curves have been estimated with the linear system method (red curves) and the minimal length cubic spline (blue curves).

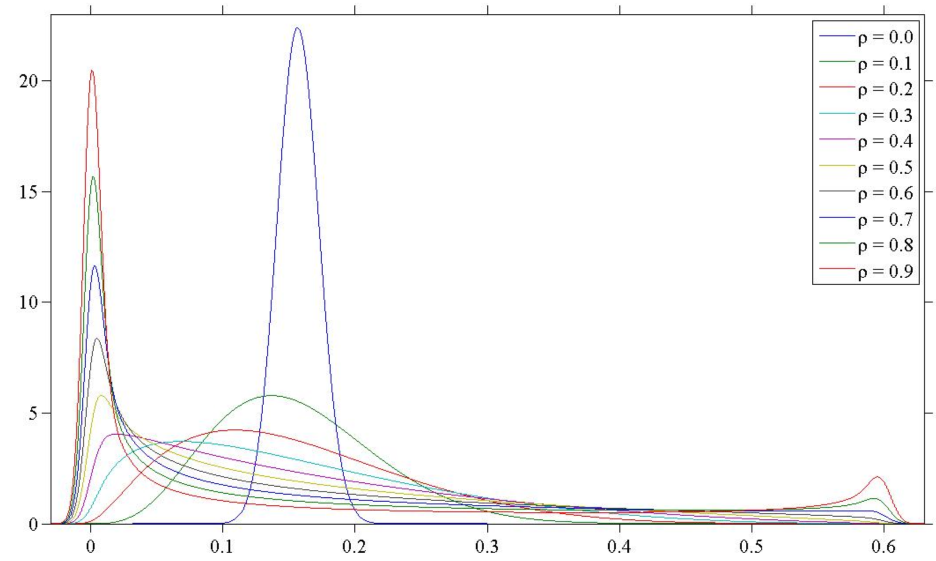


Figure 5: In this figure, we show the output function of the direct convolution algorithm, at different default correlation parameters. The reference deal is the iTRAXX Europe Xover, 06/16 series 15, on 10/05/2011. In order to avoid spiky (although smooth) distributions, the number of basket components have been artificially inflated duplicating the basket 8 times. This has been done for illustration purposes only. The stochastic recovery has a parabolic distribution, centered at 0.4 recovery, ranging from 0.1 to 0.7. The convolution by the recovery distribution has the effect of *regularization* of the argument function. At extreme default correlation parameters, the grid of the latent factor has to be increased to adjust for numerical precision.

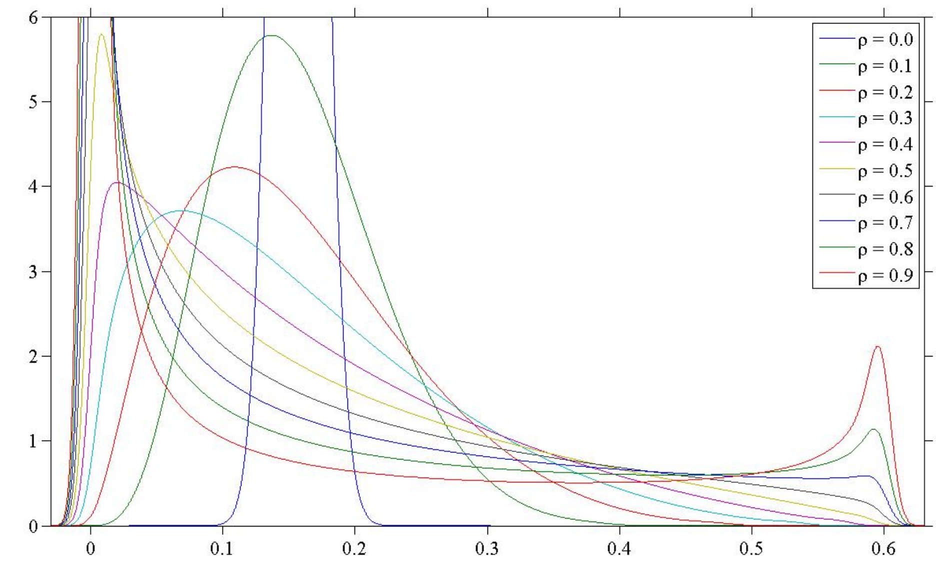


Figure 6: Detail. It is interesting to observe the effect of changes in the correlation parameter. At null correlation, the loss distribution has a central limit like behavior, concentrating around the average loss and with compressed variance. At high correlations, the distribution tends to concentrate around the corner points at zero loss and 0.6 unitary loss, which is the average single item loss. Eventually, at perfect correlation, the loss distribution would split into two deltas centered at the zero loss or 0.6 loss, the whole basket becoming a single item.

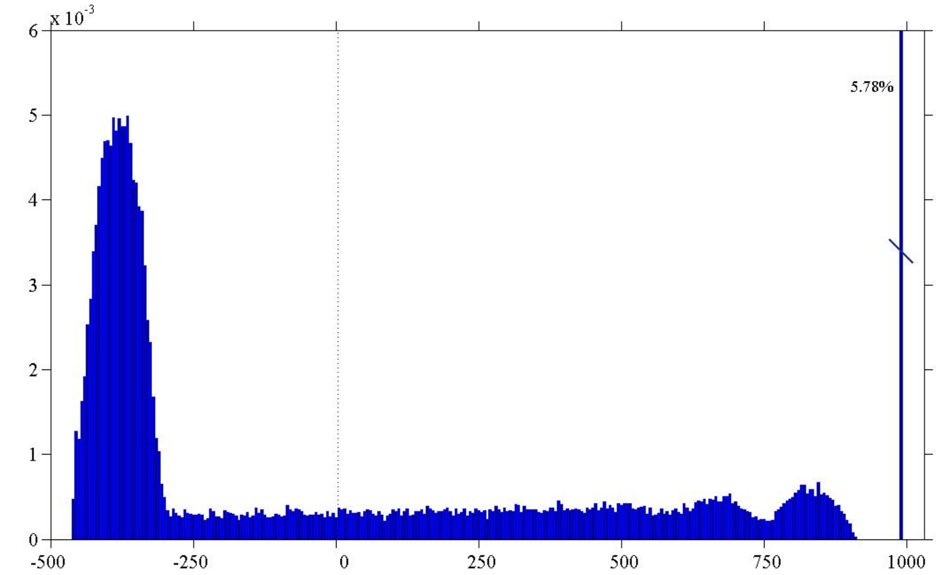


Figure 7: The distribution of the net future value of the reference basket deal (40 items) at maturity, 20/06/2016. The deal is the synthetic CDO equity tranche, $0 - 12.5\%$, which can sustain up to eight defaults. The Monte Carlo generated pay$-$off profile comprises 50,000 iterations. The chart shows the outcome histogram, the estimated probability of zero defaults and the average cash account consistency.

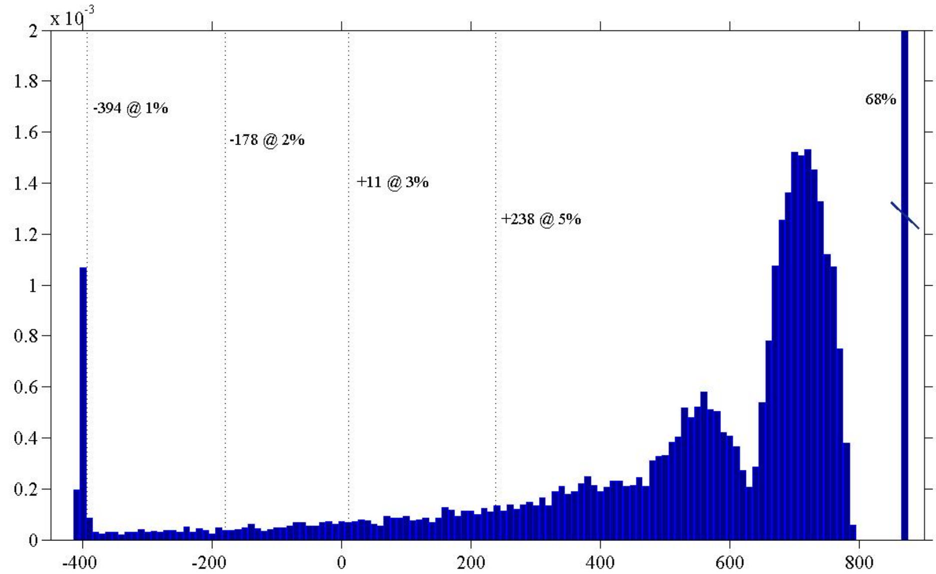


Figure 8: The distribution of the net future value of the reference deal on 10/05/2012. The chart shows the outcome histogram (including accrued interest and intra-period losses), several $CVaR$ levels and the estimated probability of zero defaults.

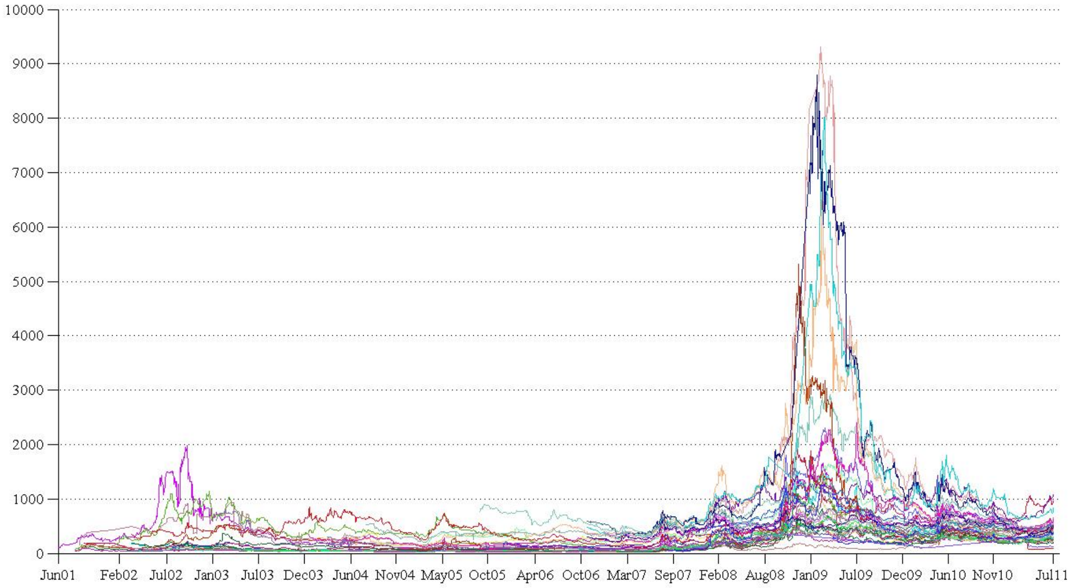


Figure 9: The historical 5y spread sample of the basket components. In the price simulations, the spread factor volatility has been set at the average long run volatility, assuming drift-less geometric motion.

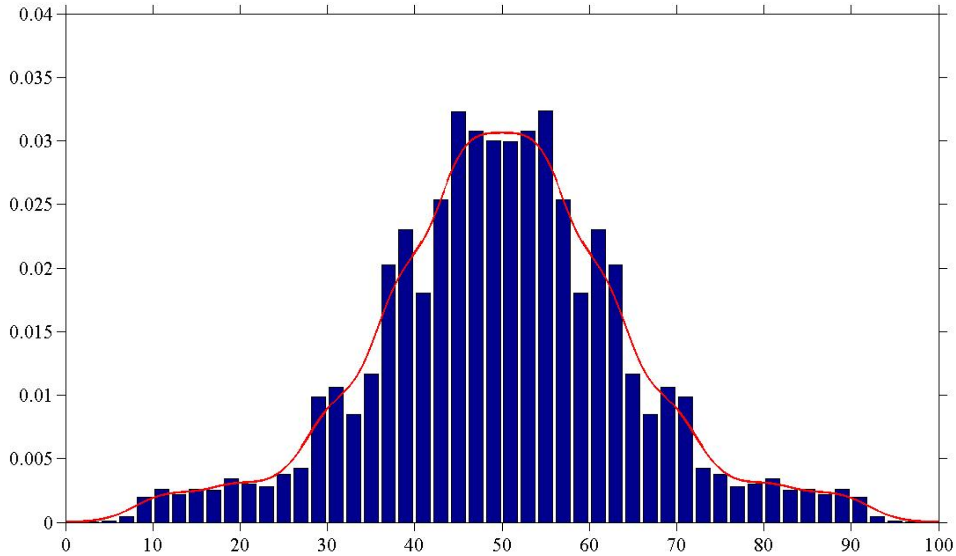


Figure 10: The (symmetrized) distribution of the merged historical base correlation sample of the STCDO of the iTRAXX Europe, 06/13 series 9, detachment points 3%, 6%, 12% and 22%.

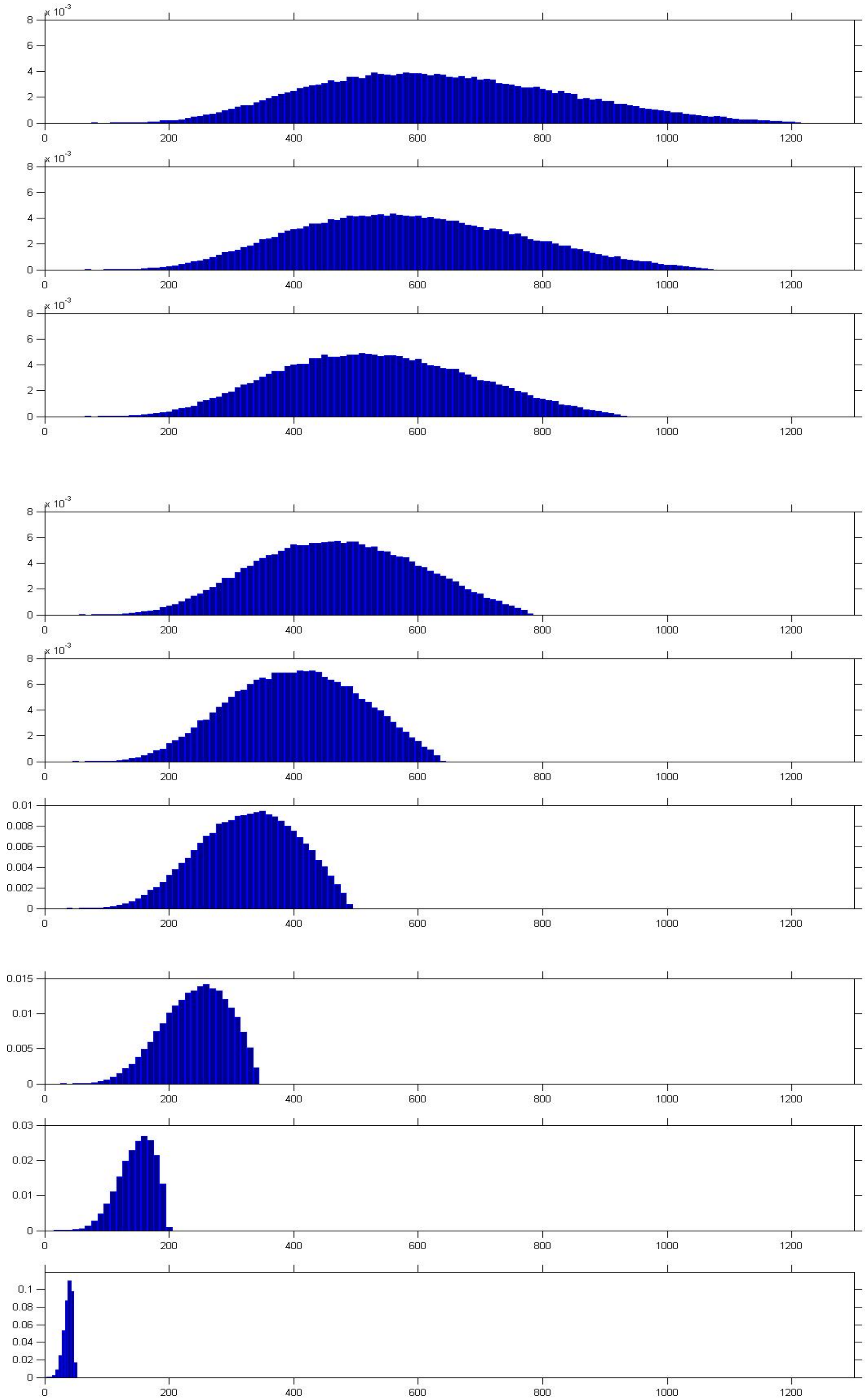




Figure 11: The simulated tranche absolute price distributions, at 0 to 8 defaults.

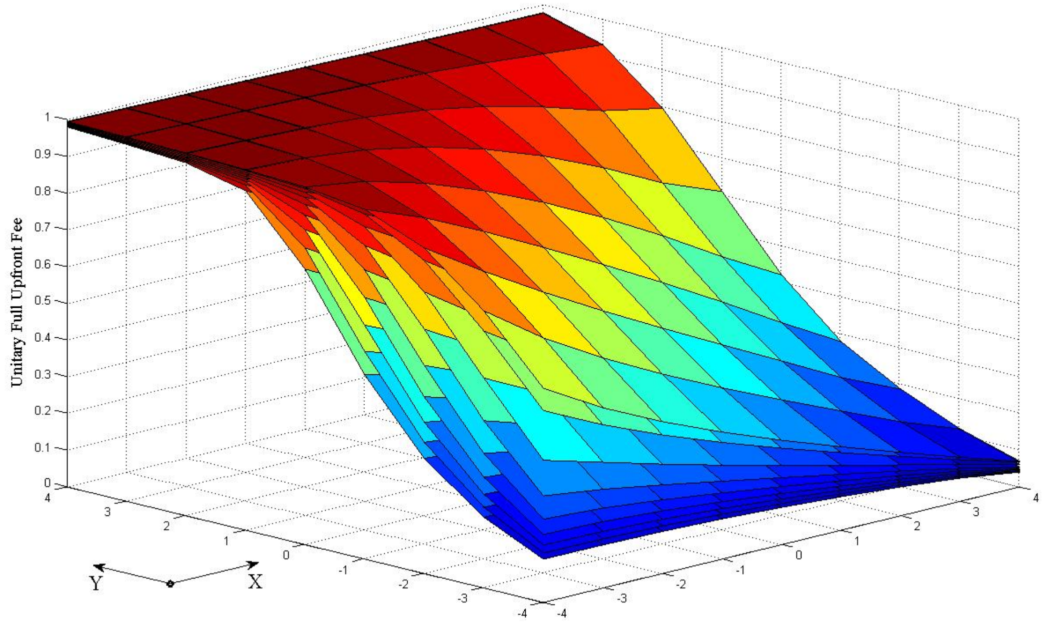


Figure 12: The price surfaces, constructed before simulation time. Each surface, from the bottom to the top, corresponds to 0 to 8 defaults unitary tranche prices. The $X$ variable represents the default correlation parameters, which has been remapped onto the $(-4,\, 4)$ interval. The $Y$ axes supports the term structure standardized random factor.

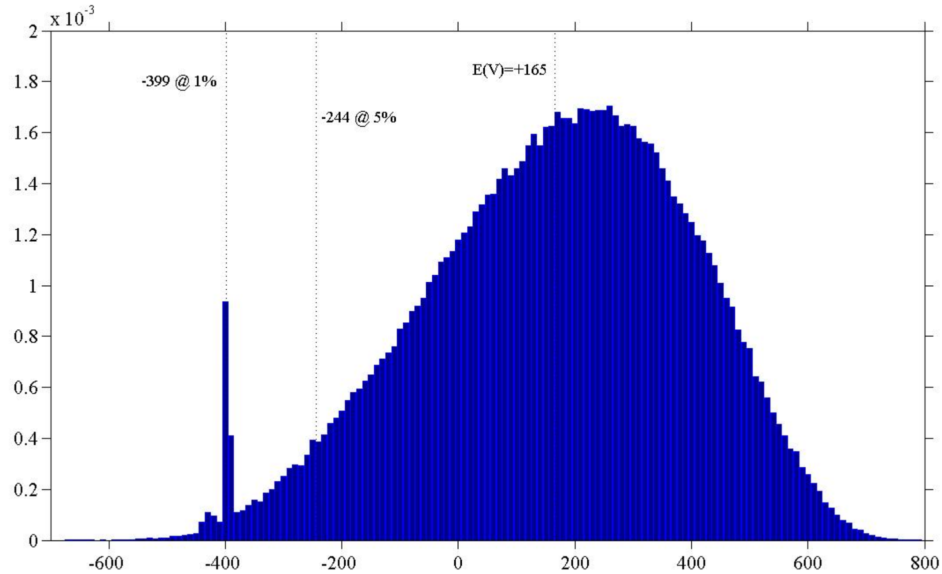


Figure 13: The net future value of the cash account consistencies, after the unwinding. The simulation (400,000 iterations) has been generated with randomized price factors, which have been charged on the cash account future net value, on 10/05/2012, matching the residual basket structure, according to the working hypothesis.