Non-Robust Dynamic Inferences from Macroeconometric Models: Bifurcation Stratification of Confidence Regions

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Abstract: Grandmont (1985) found that the parameter space of the most classical dynamic models are stratified into an infinite number of subsets supporting an infinite number of different kinds of dynamics, from monotonic stability at one extreme to chaos at the other extreme, and with all forms of multiperiodic dynamics between. The econometric implications of Grandmont's findings are particularly important, if bifurcation boundaries cross the confidence regions surrounding parameter estimates in policy-relevant models. Stratification of a confidence region into bifurcated subsets seriously damages robustness of dynamical inferences.

But Grandmont provided his result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. As a result, Barnett and He (1999, 2001, 2002) investigated a Keynesian structural model, and found results supporting Grandmont's conclusions within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That highly regarded, prototypical Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant.

Criticism of Keynesian structural models by the Lucas critique have motivated development of Euler equations models having policy-invariant deep parameters, which are invariant to policy rule changes. Hence, Barnett and He (2006) chose to continue the investigation of policy-relevant bifurcation by searching the parameter space of the best known of the Euler equations macroeconometric models: the Leeper and Sims (1994) model. The results further confirm Grandmont's views.

Even more recently, interest in policy in some circles has moved to New Keynesian models. As a result, in this paper we explore bifurcation within the class of New Keynesian models. We develop the econometric theory needed to locate bifurcation boundaries in log-linearized New-Keynesian models with Taylor policy rules or inflation-targeting policy rules. Empirical implementation will be the subject of a future paper, in which we shall solve numerically for the location and properties of the bifurcation boundaries and their dependency upon policy-rule parameter settings. Central results needed in this research are our theorems on the existence and location of Hopf bifurcation boundaries in each of the cases that we consider. We provide the proofs of those propositions in this paper. One surprising result from these proofs is the finding that a common setting of a parameter in the future-looking New-Keynesian model can put the model directly onto a Hopf bifurcation boundary.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, then to the Euler equation macroeconomic models, and now to the New Keynesian models. So far, all of our results suggest that Barnett and He's initial findings with the path-breaking policy-relevant Bergstrom-Wymer model appear to be generic.

Keywords:

Bifurcation, inference, dynamic general equilibrium, Pareto optimality, Hopf bifurcation, Euler equations, New Keynesian macroeconometrics, Bergstrom-Wymer model.

JEL Codes:

C14, C22, E37, E32.

1. Introduction

1.1. The History

Grandmont (1985) found that the parameter space of even the simplest, classical models are stratified into bifurcation regions. This result changed prior views that different kinds of economic dynamics can only be produced by different kinds of structures. But he provided that result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. Years of controversy followed, as evidenced by papers appearing in Barnett, Deissenberg, and Feichtinger (2004) and Barnett, Geweke, and Shell (2005). The econometric implications of Grandmont's findings are particularly important, if bifurcation boundaries cross the confidence regions surrounding parameter estimates in policy-relevant models. Stratification of a confidence region into bifurcated subsets seriously damages robustness of dynamical inferences.

The dramatic transformation of views precipitated by Grandmont's paper was criticized for lack of policy relevance. As a result, Barnett and He (1999, 2001, 2002) investigated a continuous-time traditional Keynesian structural model (the Bergstrom-Wymer model), and found results supporting Grandmont's conclusions. Barnett and He found transcritical, codimension-two, and Hopf bifurcation boundaries within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That highly regarded Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant. See Bergstrom and Wymer (1976), Bergstrom (1996), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom, Nowman, and Wymer (1992), and Bergstrom and Nowman (2006). Barnett and He found that bifurcation boundaries cross confidence regions of parameter estimates in that model, such that both stability and instability are possible within the confidence regions.

The Lucas critique has motivated development of Euler equations models. Hence, Barnett and He (2006) chose to continue the investigation of policy relevant bifurcation by searching the parameter space of the best known of the policy relevant Euler-equations macroeconometric models: the Leeper and Sims (1994) model. The results further confirm Grandmont's views, but with the finding of an unexpected form of bifurcation: singularity bifurcation. Although known in engineering, singularity bifurcation has not previously been encountered in economics. Barnett and He (2004) have made clear the mathematical nature of singularity bifurcation and why it is likely to be common in the class of modern Euler equation models rendered important by the Lucas critique.

Recently, interest in policy in some circles has moved away from Euler equations models to New Keynesian models, which have become common in monetary policy formulations. As a result, in this paper we explore bifurcation within the class of New Keynesian models. We study forward-looking and current-looking models and hybrid models having both forward and current looking features. We find the possibility of Hopf bifurcation, with the setting of the policy parameters influencing the existence and location of the bifurcation boundary. No other form of bifurcation is possible in the three equation log-linearized New Keynesian models that we consider. In a future paper, we shall report on our results solving numerically for the location and properties of the bifurcation boundaries and their dependency upon policy-rule parameter settings. A central result used in this research is our proof of the propositions needed to establish the existence of Hopf bifurcation and locate the bifurcation boundaries in the particular models that we consider. One surprising result from these proofs is the finding that a common setting of a parameter in the future-looking New-Keynesian model can put the model directly onto a Hopf bifurcation boundary.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, then to the Euler equation macroeconometric models, and now to New Keynesian models. At this stage of our research, we believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest.

So far, our finding suggest that Barnett and He's initial findings with the path-breaking policy-relevant Bergstrom-Wymer model appear to be generic.

1.2. Bifurcation Background

During the past 30 years, the literature in macroeconomics has moved from comparative statics to dynamics, with many such dynamic models exhibiting nonlinear dynamics. The core of dynamics is bifurcation theory, which becomes especially rich in its possibilities, when the dynamics are nonlinear. The parameter space is stratified into subsets, each of which supports a different kind of dynamic solution. Since we do not know the parameters with certainty, knowledge of the location of the bifurcation boundaries is of fundamental importance. Without knowledge of the location of such boundaries, there is no way to know whether the confidence region about the parameters' point estimates might be crossed by such a boundary, thereby stratifying the confidence region itself and damaging inferences about dynamics.

There are different types of bifurcations, such as flip, fold, transcritical, and Hopf. Hopf bifurcation is the most commonly seen type among economic models, since the existence of a Hopf bifurcation boundary is accompanied by regular oscillations in an economic model, where the oscillations may damp to a stable steady state or may never damp, depending upon which side of the bifurcation boundary the point estimate of the parameters might lie.[1]

The first theoretical work on Hopf bifurcation is in Poincaré (1892). The first specific study and formulation of a theorem on Hopf bifurcation appeared in Andronov (1929), who, with his coauthors, developed important tools for analyzing nonlinear dynamical systems. A general theorem on the existence of Hopf bifurcation was proved by Hopf (1942). While the work of Poincaré and Andronov was concerned with two-dimensional vector fields, the theorem of Hopf is valid in *n* dimensions. When parameters cross a bifurcation boundary such that the solutions change from stable to limit cycles, it is common in mathematics to refer to the resulting bifurcation as Poincaré–Andronov-Hopf bifurcation.

[1] See, e.g., Benhabib and Nishimura (1979), Kuznetsov (1998), and Seydel (1994).

Hopf bifurcations have been encountered in many economic models, such as Benhabib and Nishimura (1979). Historically, optimal growth theory received the most attention as the subject of bifurcation analysis. Hopf bifurcations were also found in overlapping generations models.[2] These studies show that the existence of a Hopf bifurcation boundary results in the existence of closed curves around the stationary state, with the solution paths being stable or unstable, depending upon which side of the bifurcation boundary contains the parameter values.

New Keynesian models have become increasingly popular in policy analysis. The usual New-Keynesian log-linearized model consists of a forward-looking IS-curve, describing consumption smoothing behavior, and a New Keynesian Phillips curve, derived from price optimization by monopolistically competitive firms in the presence of nominal rigidities. This paper pursues a bifurcation analysis of New Keynesian functional structure. We study the system using eigenvalues of the linearized system of difference equations and find existence of a Hopf bifurcation. We also investigate different monetary policy rules relative to bifurcation. In each case, we solve numerically for the location and properties of the bifurcation boundary and its dependency upon policy rule parameter settings.

2. Model

Our analysis is centered on the New Keynesian functional structure described in this section. The main assumption of New Keynesian economic theory is that there are nominal price rigidities preventing prices from adjusting immediately and thereby creating disequilibrium unemployment. Price stickiness is often introduced in the manner proposed by Calvo (1983). The model below, used as the theoretical background for our log linearized bifurcation analysis, is based closely upon Walsh (2003), section 5.4.1, pp. 232 - 239, which in turn is based upon the monopolistic competition model of Dixit and Stiglitz (1977).[3]

It is assumed that there is a continuum of firms of measure 1, and firm $j \in [0,1]$ produces good c_j at price p_j. Since all goods are differentiated in the monopolistically

[2] Aiyagari (1989), Benhabib and Day (1982), Benhabib and Rustichini (1991), Gale (1973).
[3] Other relevant references include Shapiro (2006) and Woodford (2003).

competitive manner, each firm has pricing power over the good it sells. The composite good that enters the consumers' utility functions is $C_t = \left(\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}$, and its dual price aggregator function is $P_t = \left[\int_0^1 p_{jt}^{1-\theta} dj \right]^{\frac{1}{1-\theta}}$, where θ > 1 is the price elasticity of demand for each individual good, assumed to be the same for each good j.[4] As $\theta \to \infty$, the individual goods become closer and closer substitutes, and as a consequence, individual firms have less market power.

Price rigidity faced by the firm is modeled as follows: a random fraction, $0 < \xi < 1$, of firms does not adjust price in each period. The remaining firms adjust prices to their optimal levels p^*_{jt}, $j \in [0,1]$. Accordingly, it follows from the formula for the price aggregator function that the aggregate price in period t satisfies the equation:

$$P_t^{1-\theta} = (1-\xi)(P_t^*)^{1-\theta} + \xi P_{t-1}^{1-\theta}, \tag{2.1}$$

where ξ is the probability that a price will remain unchanged in any given period and $P_t^* = \left[\int_0^1 p_{jt}^{*1-\theta} dj \right]^{\frac{1}{1-\theta}}$ is the optimal aggregate price at time *t*.

Therefore, the aggregate price level in period *t* is determined by the fraction, $1-\xi$, of firms that adjust and charge a new optimal price p^*_{jt} and by the remaining fraction of firms that charge the previous period's price.

2.1. Households

Consumers derive utility from the composite consumption good, C_t, real money balances, and leisure. We define the following variables for period t:

[4] The duality proof can be found in Warsh (2003, p. 233)

M_t = money balances,

N_t = labor quantity,

B_t = real balances of one-period bonds,

W_t = wage rate,

i_t = interest rate,

Π_t = total profits earned by firms.

Consumers supply their labor in a competitive labor market and receive labor income, $W_t N_t$. Consumers own the firms producing consumption goods and receive all profits, Π_t. The representative consumer can allocate wealth to money and bonds and choose the aggregate consumption stream by solving the following problem:

$$max\left\{ E_t \sum_{i=0}^{\infty} \beta^i \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} + \frac{\gamma}{1-b}\left(\frac{M_{t+i}}{P_{t+i}}\right)^{1-b} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right) \right\} \tag{2.2}$$

$$\text{subject to } C_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} = \left(\frac{W_t}{P_t}\right) N_t + \frac{M_{t-1}}{P_t} + (1+i_{t-1})\left(\frac{B_{t-1}}{P_t}\right) + \Pi_t \tag{2.3}$$

with scaling parameters γ and χ along with parameters:

β = time-discount factor,

σ = degree of relative risk aversion,

b^{-1} = interest elasticity of money demand,

η^{-1} = wage elasticity of labor supply.

In practice, the decision of a "representative consumer" is for per capita values of all quantities. The households' first order conditions are given by

$$C_t^{-\sigma} = \beta(1+i_t)E_t\left(\frac{P_t}{P_{t+1}} C_{t+1}^{-\sigma}\right), \tag{2.4}$$

$$\gamma \frac{\left(\frac{M_t}{P_t}\right)^{-\sigma}}{C_t^{-\sigma}} = \frac{i_t}{1+i_t}, \tag{2.5}$$

$$\chi \frac{N_t^{\eta}}{C_t^{-\sigma}} = \frac{W_t}{P_t}. \tag{2.6}$$

Equations (2.4), (2.5), and (2.6) are Euler equations for consumption, money, and labor supply respectively. Following solution of (2.2) subject to (2.3), the representative consumer, in a second stage decision, allocates chosen aggregate consumption, C_t, over the continuum of goods, c_{jt}, $j \in [0,1]$, to minimize the cost of consuming C_t.[5]

Let π_t be the inflation rate at time t. Following Walsh (2003, p. 244), we log-linearize the households' first order condition (2.4) around the steady state inflation rate, $\pi = 0$. With aggregate output by firms equaling aggregate consumption, C_t, in the steady state, we get

$$\hat{y}_t = E_t \hat{y}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - E_t \pi_{t+1}), \tag{2.7}$$

where $\hat{y}_t$ is the percentage deviation of output from its steady state.

Writing (2.7) in terms of output gap, we get

$$x_t = E_t x_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - E_t \pi_{t+1}) + u_t, \tag{2.8}$$

where $x_t = (\hat{y}_t - \hat{y}_t^f)$ is the gap between actual output percentage deviation, $\hat{y}_t$, and the flexible-price output percentage deviation, $\hat{y}_t^f$, and where $u_t \equiv E_t \hat{y}_{t+1}^f - \hat{y}_t^f$. Equation (2.8) can be viewed as describing the demand side of the economy, in the sense of an expectational, forward-looking IS curve.

2.2. Firms

[5] The first stage decision allocating over individual goods, conditionally upon composite goods demand, can be found in Warsh (2003, p. 233)

Firms hire labor and produce and sell consumption goods in a monopolistically competitive market. The production functions for goods c_{jt}, $j \in [0,1]$ have the following form:

$$c_{jt} = Z_t N_{jt},$$

where N_{jt} is time spent on production of good c_{jt} during period t, and Z_t is labor's average product, assumed to be random with mean $E(Z_t) = 1$. Labor's average product is drawn once for all industries, so has no subscript j.

Firms make their production and price-setting decisions by solving the following two problems:

Cost minimization problem:

For each period t, firm j selects labor employment, N_{jt}, to minimize labor cost, $(W_t/P_t)N_{jt}$, subject to the production functions' constraints on technology. The resulting Lagrangian, with Lagrange multipliers φ_j, is

$$\left(\frac{W_t}{P_t}\right) N_{jt} + \varphi_j (\mathrm{C}_{jt} - Z_t N_{jt}), \qquad \mathrm{j} = 1, \ldots, \mathrm{J} \tag{2.9}$$

which is minimized to solve for N_{jt}. The first order condition to solve (2.9) is:

$$\varphi_t = \frac{W_t / P_t}{Z_t}. \tag{2.10}$$

As is usual for Lagrange multipliers, (2.10) can be interpreted as a shadow price. In this case, φ_j is the shadow price, or equivalently the real marginal cost, of producing C_{jt}.

Pricing decision:

Each firm j maximizes the expected present value of its profits by choosing price p_{jt}. Recall that θ is the price elasticity of demand and is the parameter in the consumer quantity and price aggregator functions. Since that elasticity of demand is the same for all goods, the following relationship exists between consumption of each good and aggregate consumption: $c_{jt} p_{jt}^{\theta} = C_t P_t^{\theta}$. Using that result, the profit maximization problem for firm j can be written as[6]:

$$\max \left\{ E_t \sum_{i=0}^{\infty} w^i \Delta_{i,t+1} \left[\left(\frac{p_{jt}}{P_{t+i}} \right)^{1-\theta} - \varphi_{t+i} \left(\frac{p_{jt}}{P_{t+i}} \right)^{-\theta} \right] C_{t+i} \right\}, \tag{2.11}$$

where $\Delta_{i,t+1} = \beta^i (C_{t+1} \backslash C_t)^{-\sigma}$ is the discount factor; and the consumer price indexes, P_{t+i}, are taken as given by the firm for all i = 0, ... ,∞.

This yields the following first order condition, which shows how adjusting firms set their prices, conditional on the current price level:

$$\frac{p_t^*}{P_t} = \left(\frac{\theta}{\theta - 1} \right) \frac{E_t \sum_{i=0}^{\infty} \xi^i \beta^i C_{t+i}^{1-\sigma} \varphi_{t+i} \left(\frac{P_{t+i}}{P_t} \right)^{\theta}}{E_t \sum_{i=0}^{\infty} \xi^i \beta^i C_{t+i}^{1-\sigma} \left(\frac{P_{t+i}}{P_t} \right)^{\theta - 1}}. \tag{2.12}$$

As in Walsh (2003, p. 237), we log-linearize (2.1) and (2.12) around the zero-inflation steady state equilibrium to get the following expression for aggregate inflation:

$$\pi_t = \beta E_t \pi_{t+1} + \tilde{\kappa} \hat{\varphi}_t, \tag{2.13}$$

where $\tilde{\kappa} = \frac{(1-\xi)(1-\beta\xi)}{\xi}$ and where $\hat{\varphi}_t = (\hat{w}_t - \hat{p}_t) - (\hat{y}_t - \hat{n}_t) = \gamma(\hat{y}_t - \hat{y}_t^f)$ is real marginal cost, expressed in terms of percentage deviations around the steady state. In particular, $\hat{w}_t$ is the percentage deviation of the wage rate around its steady state, $\hat{p}_t$ is the percentage deviation of P_t around its steady state, and $\hat{n}_t$ is the percentage deviation of N_t around its steady state.

We can rewrite the relation for inflation, in terms of the output gap. Then (2.13) becomes

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t, \tag{2.14}$$

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6 See, e.g., Walsh (2003, p. 235).

where $x_t = \hat{y}_t - \hat{y}_t^f$ is the gap between actual output-percentage-deviation from steady state and the flexible-price output percentage deviation from steady state, with $\kappa = \gamma\tilde{\kappa} = \gamma \frac{(1-\xi)(1-\beta\xi)}{\xi}$.

We now have two equations. The first equation, (2.8), provides the demand side of the economy. It is a forward-looking IS curve that relates the output gap to the real interest rate. Equation (2.14) is the New-Keynesian Phillips curve, which represents the supply side, by describing how inflation is driven by the output gap and expected inflation. The resulting system of two equations has three unknown variables: inflation, output gap, and nominal interest rate. We need one more equation to close the model.

The remaining necessary equation will be a monetary policy rule, in which the central bank uses a nominal interest rate as the policy instrument. Numerous types of monetary policy rules have been discussed in the economics literature. Two main policy classes are targeting rules and instrument rules.

A simple instrument rule relates the interest rate to a few observable variables. The most famous such rule is Taylor's rule. Taylor demonstrated that a simple reaction function, with a short-term interest-rate policy instrument responding to inflation and output gap, follows closely the observed path of the Federal Funds rate. His original work was followed by a large literature, in which researchers have tried to modify Taylor's rule to get a better fit to the data.[7] We initially center our analysis on the following specification of the current-looking Taylor rule:

$$i_t = a_1\pi_t + a_2 x_t , \qquad (2.15)$$

where a_1 is the coefficient of the central bank's reaction to inflation and a_2 is the coefficient of the central bank's reaction to the output gap. We also consider the forward-looking and the hybrid Taylor rule.

Among targeting rules, the recent literature proposes many ways to define an inflation target.[8] We consider inflation targeting policies of the form:

[7]See, e.g., Clarida, Gali and Gertler (1999); Gali and Gertler (1999); McCallum (1999); and Taylor (1999).
[8] See Bernanke et al. (1999), Svensson (1999), and Gavin (2003).

$$i_t = a_1 \pi_t, \tag{2.16}$$

which is a current-looking inflation targeting rule. Forward-looking inflation targeting will also be considered.

When we use the current-looking Taylor rule, we are left with these three equations.

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}),$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,$$
$$i_t = a_1 \pi_t + a_2 x_t.$$

This 3-equation system constitutes a New Keynesian model.

2.1. Stability Analysis

Continuing with the current-looking Taylor rule, we reduce the system of three equations to a system of two log-linearized equations by substituting Taylor's rule into the consumption Euler equation. The resulting system of expected difference equations has a unique and stable solution, if the number of eigenvalues outside the unit circle equals the number of forward looking variables (see Blanchard and Kahn (1980)). That system of two equations has the following form:

$$\begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -k & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

which can be written as

$$\mathbf{A} E_t \mathbf{x}_{t+1} = \mathbf{B} \mathbf{x}_t,$$

where $\mathbf{x}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\ -k & 1 \end{bmatrix}$.

Premultiply the system by the inverse matrix $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -\frac{1}{\beta\sigma} \\ 0 & \frac{1}{\beta} \end{bmatrix},$$

we get

$$E_t \mathbf{x}_{t+1} = \mathbf{C}\mathbf{x}_t$$

or

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{a_2\beta+k}{\sigma\beta} & \frac{a_1\beta-1}{\sigma\beta} \\ -\frac{k}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix},$$

where **C** = **A**$^{-1}$**B**.

We have two forward-looking variables, x_{t+1} and π_{t+1}. Therefore uniqueness and stability of the solution require both eigenvalues to be outside the unit circle. The eigenvalues of **C** are the roots of the characteristic polynomial

$$p(\lambda) = det\left(\mathbf{C} - \lambda\mathbf{I}\right) \tag{2.1.1}$$

$$= \lambda^2 - \lambda\left[1 + \frac{a_2\beta+k}{\sigma\beta} + \frac{1}{\beta}\right] + \frac{\sigma\beta + a_2\beta + ka_1\beta}{\sigma\beta^2}. \tag{2.1.2}$$

Defining D as

$$D = \left[1 + \frac{a_2\beta+k}{\sigma\beta} + \frac{1}{\beta}\right]^2 - 4\frac{\sigma\beta + a_2\beta + ka_1\beta}{\sigma\beta^2}, \tag{2.1.3}$$

we can write the eigenvalues as

$$\lambda_1 = 0.5(1 + \frac{a_2\beta+\kappa}{\sigma\beta} + \frac{1}{\beta} + 4\sqrt{D})$$

and (2.1.4)

$$\lambda_2 = 0.5(1 + \frac{a_2\beta+\kappa}{\sigma\beta} + \frac{1}{\beta} - 4\sqrt{D}).$$

It can be shown that both eigenvalues will be outside the unit circle, if and only if

$$(a_1 - 1)\gamma + (1-\beta)a_2 > 0. \tag{2.1.5}$$

Equivalently, (2.1.5) holds if $a_1 > 1$. Interest rate rules that meet this criterion are called active. This relationship is also known as Taylor's principle, which prescribes that the interest rate should be set higher than the increase in inflation. Monetary policy satisfying the Taylor's principle is thought to eliminate equilibrium multiplicities. Assuming uniqueness of solutions, the dynamical properties of the system can be explored through bifurcation analysis.

3. Bifurcation Analysis

The New Keynesian model has both a continuous time and a discrete form. To define our notation for the discrete form, we consider a continuously differentiable map

$$\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha}), \tag{3.1}$$

where $\mathbf{x} \in \Re^n$ is a vector of state variables, $\boldsymbol{\alpha} \in \Re^m$ is a parameter vector, and $\mathbf{f}: \Re^n \times \Re^m \to \Re^n$ is continuously differentiable. We will study the dynamic solution behavior of $\mathbf{x}$ as $\boldsymbol{\alpha}$ varies. System (3.1) undergoes a bifurcation, if its parameters pass through a critical (bifurcation) point, defined as follows.

Definition 3.1: Appearance of a topologically nonequivalent phase portrait under variation of parameters is called *a bifurcation*.

At the bifurcation point the structure may change its stability, split into new structures, or merge with other structures. There are two possible bifurcation analyses: local and global. We look at small neighborhoods of a fixed point, $\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*, \boldsymbol{\alpha})$, to conduct local bifurcation analysis.

Definition 3.2: A *local bifurcation* is a bifurcation that can be analyzed purely in terms of a change in the linearization around a single invariant set or attractor.

The bifurcations of a map (3.1) can be characterized by the eigenvalues of the Jacobian of the first derivatives of the map, computed at the bifurcation point. Let $\mathbf{J} = \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha})_{\mathbf{x}}$ be the Jacobian matrix. The eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, of the Jacobian are also referred to as multipliers. Bifurcation will occur, if there are eigenvalues of **J** on the unit circle that violate the following hyperbolicity condition.

Definition 3.3: The equilibrium is called *hyperbolic*, when the Jacobian **J** has no eigenvalues on the unit circle.

Non-hyperbolic equilibria are not structurally stable and hence generically lead to bifurcations as a parameter is varied. There are three possible ways to violate the hyperbolicity condition. They give rise to three codimension-1 types of bifurcations.

Definition 3.4: Bifurcation associated with the appearance of λ_i=1 is called *a fold* (or tangent) bifurcation.

Definition 3.5: Bifurcation associated with the appearance of λ_i=-1 is called *flip* (period-doubling) bifurcation.

Definition 3.6: Bifurcation corresponding to the presence of a pair of complex conjugate eigenvalues, $\lambda_1 = e^{+i\theta_0}$ and $\lambda_2 = e^{-i\theta_0}$, for $0 < \theta_0 < \pi$, is called a *Hopf* bifurcation.

In the 2-dimensional case, we shall need the following theorem, based upon the version of the Hopf Bifurcation Theorem in Gandolfo (1996, ch. 25, p. 492).

Theorem 3.1: (Existence of Hopf Bifurcation in 2 Dimensions) Consider a map $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\alpha})$, where **x** has 2 dimensions; and for each $\boldsymbol{\alpha}$ in the relevant region, suppose that there is a continuously differentiable family of equilibrium points $\mathbf{x}^*=\mathbf{x}^*(\boldsymbol{\alpha})$ at which the eigenvalues of the Jacobian are complex conjugates, $\lambda_1 = \theta(\mathbf{x},\boldsymbol{\alpha}) + i\omega(\mathbf{x},\boldsymbol{\alpha})$

and $\lambda_2 = \theta(\mathbf{x},\boldsymbol{\alpha}) - i\omega(\mathbf{x},\boldsymbol{\alpha})$. Suppose that for one of those equilibria, **(x*,α*),** there is a critical value α_c for one of the parameters, α_i^*, in **α*** such that:

(a) The modulus of the eigenvalues becomes unity at **α** = **α***, but the eigenvalues are not roots of unity. Formally, $\lambda_1, \lambda_2 \neq 1$ and mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2 + \omega^2} = 1$. Also suppose that

(b) $$\left.\frac{\partial\left|\lambda_j(\mathbf{x}^*,\boldsymbol{\alpha}^*)\right|}{\partial \alpha_i^*}\right|_{\alpha_i^* = \alpha_c} \neq 0 \text{ for j = 1,2.}$$

Then there is an invariant closed curve Hopf-bifurcating from **α***.[9]

Condition (b) implies that the eigenvalue crosses imaginary axes with non-zero speed. This theorem only works for a 2x2 Jacobian. We use it for the analysis of the reduced 2x2 model, $\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t$. The more general case requires the rest of the eigenvalues to have a real part less than zero. The 3x3 case requires different tools of investigation. In the three equation case with current-looking or forward looking policy rules, it can be shown that the only form or bifurcation that is possible with the linearized model is Hopf bifurcation. The broader range of possible types of bifurcation possible with the nonlinear model will be the subject of future research. Also we do not consider backwards looking models in this paper, since there currently is more interest in the current looking and future looking models. But in future research, we shall consider the backwards looking models, since they raise the possibility of codimension-2 bifurcation, even with the log-linearized model.

3.1. Current-Looking Taylor Rule

The matrix **C** was the Jacobian of the New Keynesian model presented above. We now change the notation so that the Jacobian is:

[9] Note that we use the notations mod (λ_j) and $|\lambda_j|$ interchangeably to designate modulus of a complex variable.

$$\mathbf{J} = \begin{bmatrix} 1+\dfrac{a_2\beta+k}{\sigma\beta} & \dfrac{a_1\beta-1}{\sigma\beta} \\ -\dfrac{k}{\beta} & \dfrac{1}{\beta} \end{bmatrix}.$$

We apply the Hopf bifurcation existence theorem (3.1) to the Jacobian of the log-linearized New Keynesian model, $\mathbf{A}E_t\mathbf{x}_{t+1} = \mathbf{B}\mathbf{x}_t$. The characteristic equation of the Jacobian is:

$$\lambda^2 - b\lambda + c = 0,$$

where

$$b = \left[1+\frac{a_2\beta+k}{\sigma\beta}+\frac{1}{\beta}\right],$$

$$c = \left[\frac{\sigma\beta+a_2\beta+ka_1\beta}{\sigma\beta^2}\right].$$

In order to get a pair of complex conjugate eigenvalues, the discriminant D must be strictly negative:

$$D = b^2 - 4c = \left[1+\frac{a_2\beta+k}{\sigma\beta}+\frac{1}{\beta}\right]^2 - 4\frac{\sigma\beta+a_2\beta+ka_1\beta}{\sigma\beta^2} < 0.$$

Given the sign of the parameters, the discriminant could be either positive or negative. We assume that the discriminant is negative, so that the roots of the characteristic polynomial are complex conjugate:[10]

$$\lambda_1 = \theta + i\omega \text{ and } \lambda_2 = \theta - i\omega,$$

where $\theta = \frac{1}{2}b$ is the real part, $i\omega$ is the imaginary part, and $\omega = \frac{1}{2}\sqrt{D} = \frac{1}{2}\sqrt{b^2-4c}$.

We need to choose a bifurcation parameter to vary while holding other parameters constant. The model is parameterized by:

[10] This assumption can be satisfied during the numerical procedure for locating the bifurcation regions.

$$\boldsymbol{\alpha} = \begin{pmatrix} \beta \\ \sigma \\ k \\ a_1 \\ a_2 \end{pmatrix}.$$

Candidates for a bifurcation parameter are coefficients for the monetary policy rule, a_1 and a_2. We prove the following proposition.

Proposition 3.1: The new Keynesian model with current-looking Taylor rule, (2.8), (2.14), and (2.15), undergoes a Hopf bifurcation, if and only if the discriminant of the characteristic equation is negative and $a_2^c = \sigma\beta - \kappa a_1 - \sigma$.

Proof: Assume that the system, (2.8), (2.14), (2.15), produces a Hopf bifurcation. By definition, Hopf bifurcation is characterized by the appearance of a pair of complex conjugate multipliers that lie on the unit circle. Since the multipliers are complex conjugate, the discriminant has to be strictly negative.

By condition (a) for Hopf bifurcation, mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2 + \omega^2} = 1$. Substituting $\theta = \frac{1}{2}b$ and $\omega = \frac{1}{2}\sqrt{D}$ into that equation, we get

$$\sqrt{\frac{1}{2}(1 + \frac{a_2\beta + \kappa}{\sigma\beta} + \frac{1}{\beta}) + \frac{1}{4}\left[\frac{4(\sigma\beta + a_2\beta + ka_1\beta)}{\sigma\beta^2} - \left(1 + \frac{a_2\beta + k + \sigma}{\sigma\beta}\right)^2\right]} = 1. \qquad \text{(i)}$$

After solving for a_2, we find that the critical value for the parameter is $a_2^c = \sigma\beta - \kappa a_1 - \sigma$.

Conversely, assume that the discriminant, D, is negative and that $a_2 = \sigma\beta - \kappa a_1 - \sigma$. Substituting for a_2 into the left hand side of equation (i), we find immediately that $\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, thereby satisfying condition (a) for Hopf bifurcation.

It can be shown as follows that the derivative of the modulus with respect to a_2 is a non-zero expression:

$$\left.\frac{\partial|\lambda_1|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \left.\frac{\partial|\lambda_2|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \left.\frac{1}{2\sigma\beta}\left(\frac{\beta\sigma}{\sigma+a_2+\kappa a_1}\right)^{\frac{1}{2}}\right|_{a_2=\alpha_2^c} = \frac{1}{2\sigma\beta}\neq 0,$$

which is condition (b) for Hopf bifurcation. Hence, both conditions of the Hopf bifurcation theorem are satisfied. □

We can combine the critical value for a_2 with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary. The bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma+\sigma\beta-\kappa a_1\beta+\kappa}{\sigma\beta^2} < 1.$$

3.2 Forward-looking Taylor Rule

A forward-looking Taylor rule sets the interest rate according to expected future inflation rate and output gap, in accordance with the following equation:

$$i_t = a_1 E_t \pi_{t+1} + a_2 E_t x_{t+1}. \tag{3.2}$$

The resulting Jacobian has the form

$$\mathbf{J} = \begin{bmatrix} \dfrac{\sigma}{\sigma-a_2}+\dfrac{\kappa\sigma(1+a_1)}{(1-a_2)\beta} & -\dfrac{\sigma(1+a_1)}{(1-a_2)\beta} \\ \dfrac{-\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}.$$

The characteristic equation is

$$\lambda^2 - b\lambda + c = 0, \tag{3.3}$$

where $b = \dfrac{\sigma}{\sigma-a_2}+\dfrac{1}{\beta}+\dfrac{\kappa\sigma(1+a_1)}{(1-a_2)\beta}$ and $c = det(\mathbf{J})$.

In order to get a pair of complex conjugate eigenvalues, the discriminant D must be strictly negative:

$$D = \left(\frac{\sigma\beta + \sigma - a_2}{\beta(\sigma - a_2)} + \frac{\kappa\sigma(1 + a_1)}{(1 - a_2)\beta} \right)^2 - \frac{4\sigma}{\beta(\sigma - a_2)} < 0 .$$

Given the sign of the parameters, the discriminant could be either positive or negative. We assume that the discriminant is negative, so that the roots of the characteristic polynomial are complex conjugate:

$\lambda_1 = \theta + i\omega$ and $\lambda_2 = \theta - i\omega,$

where $\theta = \frac{1}{2}b$ is the real part, $i\omega$ is the imaginary part, and $\omega = \frac{1}{2}\sqrt{D}$.

We need to choose a bifurcation parameter to vary while holding other parameters constant. The model is parameterized by:

$$\boldsymbol{\alpha} = \begin{pmatrix} \beta \\ \sigma \\ k \\ a_1 \\ a_2 \end{pmatrix} .$$

Candidates for a bifurcation parameter are coefficients, a_1 and a_2, for the monetary policy rule. We prove the following proposition.

Proposition 3.2: The New Keynesian model with forward-looking Taylor Rule, (2.8), (2.14), and (3.2), undergoes a Hopf bifurcation, if and only if the discriminant of the characteristic equation is negative and $a_2^c = -\frac{\sigma}{\beta} + \sigma$.

Proof: Assume that a system consisting of (2.8), (2.14), and (3.2) produces a Hopf bifurcation. According to the definition, Hopf bifurcation is characterized by the appearance of the pair of complex conjugate multipliers that lie on the unit circle. Since the multipliers are complex conjugate, the discriminant has to be strictly negative.

By condition (a) for Hopf bifurcation, mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2 + \omega^2} = 1$.

Substituting $\theta = \frac{1}{2}b$ and $\omega = \frac{1}{2}\sqrt{D}$ into that equation, we get

$$\sqrt{\frac{1}{4}\left(\frac{(1-a_2)(\sigma\beta+\sigma-a_2)+\kappa\sigma(1+a_1)(\sigma-a_2)}{(\sigma-a_2)(1-a_2)\beta}\right)^2+\frac{1}{4}\left(4\frac{\sigma}{\beta(\sigma-a_2)}-\left(\frac{\sigma}{\sigma-a_2}+\frac{1}{\beta}+\frac{\kappa\sigma(1+a_1)}{(1-a_2)\beta}\right)^2\right)}=1. \qquad \text{(ii)}$$

Solving for a_2, we find that the critical value for the parameter is $a_2^c = -\dfrac{\sigma}{\beta}+\sigma$.

Conversely assume that the discriminant of characteristic equation (3.3) is negative and that $a_2^c = -\dfrac{\sigma}{\beta}+\sigma$. According to Theorem 3.1, Hopf bifurcation will arise if there is a pair of complex conjugate roots of (3.3) and if conditions (a) and (b) of Theorem 3.1 are satisfied.

Since the discriminant has a negative sign, roots of (3.3) have to be complex conjugate. Condition (a) of Theorem 3.1 states that for a Hopf bifurcation to arise, the modulus of the eigenvalues should be equal to unity. To show that condition (a) holds, substitute $a_2 = -\dfrac{\sigma}{\beta}+\sigma$ into the left hand side of equation (ii) to find mod (λ_1) =1. Since characteristic roots are complex conjugate, it follows that mod (λ_1) = mod (λ_2) = 1, thereby satisfying condition (a) for Hopf bifurcation

It can be shown as follows that the derivative of the modulus with respect to a_2 is a non-zero expression:

$$\left.\frac{\partial|\lambda_1|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \left.\frac{\partial|\lambda_2|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \frac{\beta}{\sigma}\neq 0,$$

which is condition (b) for Hopf bifurcation. Hence, both conditions of the Hopf bifurcation theorem are satisfied. □

We can combine the critical value for a_2 with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary. The bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1<\frac{\sigma\beta+\sigma-a_2}{4\sigma\beta}+\frac{\kappa(1+a_1)(\sigma-a_2)}{4(1-a_2)}<1.$$

3.3. Hybrid Taylor Rule:

Consider the Taylor rule of the following form:

$$i_t = a_1 E_t \pi_{t+1} + a_2 x_t\,, \tag{3.4}$$

where the interest rate is set according to forward-looking inflation and current-looking output gap. A rule of that form was proposed in Clarida, Gali and Gertler [2000]. This form of the rule is intended to capture the central bank's existing policy. Substituting equation (2.4) into the consumption Euler equation (2.8), we acquire the Jacobian,

$$\mathbf{J} = \begin{bmatrix} 1 + \frac{a_2}{\sigma} + \frac{\kappa(1-a_1)}{\beta\sigma} & -\frac{1-a_1}{\sigma\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix},$$

with the associated characteristic polynomial

$$\lambda^2 - b\lambda + c = 0, \tag{3.5}$$

where $b = \frac{1}{\beta} + 1 + \frac{\beta a_2 - \kappa(a_1 - 1)}{\sigma\beta}$

and $c = det(\mathbf{J}) = \frac{1}{\beta} + \frac{a_2}{\sigma\beta}$.

In order to get a pair of complex conjugate eigenvalues, the discriminant D must be strictly negative:

$$D = \left(\frac{\sigma(1+\beta) + \beta a_2 - \kappa(a_1 - 1)}{\sigma\beta} \right)^2 - \frac{4(\sigma + a_2)}{\sigma\beta} < 0\,.$$

Given the sign of the parameters, the discriminant could be either positive or negative. We assume that the discriminant is negative, so that the roots of the characteristic polynomial are complex conjugate:

$\lambda_1 = \theta + i\omega$ and $\lambda_2 = \theta - i\omega,$

where $\theta = \frac{1}{2} b$ is the real part, $i\omega$ is the imaginary part, and $\omega = \frac{1}{2}\sqrt{D}$.

We need to choose a bifurcation parameter to vary while holding other parameters constant. The model is parameterized by:

$$\boldsymbol{\alpha} = \begin{pmatrix} \beta \\ \sigma \\ k \\ a_1 \\ a_2 \end{pmatrix}.$$

Candidates for a bifurcation parameter are coefficients for the monetary policy rule, a_1 and a_2. We shall need the following proposition.

Proposition 3.3: The new Keynesian model with Hybrid-Taylor rule, equations (2.8), (2.14), (3.4), undergoes a Hopf bifurcation, if and only if the discriminant of the characteristic polynomial (3.5) is negative and $a_2^c = \beta\sigma - \sigma$.

Proof: Assume that a system consisting of (2.8), (2.14), and (3.4) produces a Hopf bifurcation. According to the definition, Hopf bifurcation is characterized by the appearance of the pair of complex conjugate multipliers that lie on the unit circle. Since the multipliers are complex conjugate, the discriminant has to be strictly negative.

By condition (a) for Hopf bifurcation, mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2 + \omega^2} = 1$. Substituting $\theta = \frac{1}{2}b$ and $\omega = \frac{1}{2}\sqrt{D}$ into that equation, we get

$$\sqrt{\frac{1}{4}\left(\frac{\sigma + \beta\sigma + \kappa + \beta a_2 - \kappa a_1}{\sigma\beta}\right)^2 + \frac{1}{4}\left(4\frac{\sigma\beta + a_2}{\beta\sigma} - \left(1 + \frac{1}{\beta} + \frac{a_2\beta + \kappa(1 - a_1)}{\sigma\beta}\right)^2\right)} = 1. \qquad \text{(iii)}$$

Solving for a_2, we find that the critical value for the parameter is $a_2^c = \beta\sigma - \sigma$.

Conversely assume that the discriminant of the characteristic polynomial (3.5) is negative and $a_2^c = \beta\sigma - \sigma$. According to Theorem 3.1, Hopf bifurcation will arise if there is a pair of complex conjugate roots of (3.5) and if conditions (a) and (b) of Theorem 3.1 are satisfied.

Since the discriminant has a negative sign, roots of (3.5) have to be complex conjugate. Condition (a) of Theorem 3.1 states that for a Hopf bifurcation to arise the modulus of the eigenvalues should be equal to unity. To show that it holds, substitute $a_2 = \beta\sigma - \sigma$ into the left hand side of equation (iii) to find mod (λ_1) =1. Since characteristic roots are complex conjugate, it follows that mod (λ_1) = mod (λ_2) = 1, thereby satisfying condition (a) for Hopf bifurcation.

It can be shown as follows that the derivative of the modulus with respect to a_2 is a non-zero expression:

$$\left.\frac{\partial|\lambda_1|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \left.\frac{\partial|\lambda_2|}{\partial a_2}\right|_{a_2=\alpha_2^c} = \frac{1}{2\beta\sigma} \neq 0,$$

which is condition (b) for Hopf bifurcation. Hence, both conditions of the Hopf bifurcation theorem are satisfied. □

We can combine the critical value for a_2 with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary. The bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \left[\frac{\sigma(1+\beta^2) - \kappa(a_1 - 1)}{4\sigma\beta}\right] < 1.$$

3.4. Current-Looking Inflation Targeting

Using the inflation targeting equation

$$i_t = a_1 \pi_t, \tag{3.6}$$

instead of the Taylor rule, as the third equation for New Keynesian model, produces the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} \frac{\sigma\beta + k}{\sigma\beta} & \frac{a_1}{\sigma} - \frac{1}{\sigma\beta} \\ -\frac{k}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

with characteristic equation

$$\lambda^2 - b\lambda + c = 0, \tag{3.7}$$

where

$$b = 1 + \frac{\sigma + \kappa}{\beta\sigma},$$

$$c = \frac{1}{\beta}\left(\frac{\beta\sigma + \kappa}{\sigma\beta}\right) + \frac{\kappa}{\sigma\beta}(a_1 - \frac{1}{\beta}).$$

In order to get a pair of complex conjugate eigenvalues, the discriminant D must be strictly negative:

$$D = \left(\frac{\sigma\beta + \sigma + \kappa}{\sigma\beta}\right)^2 - \frac{4(\sigma\beta + \kappa a_1 \beta)}{\sigma\beta^2} < 0.$$

Given the sign of the parameters, the discriminant could be either positive or negative. We assume that the discriminant is negative, so that the roots of the characteristic polynomial are complex conjugate:

$\lambda_1 = \theta + i\omega$ and $\lambda_2 = \theta - i\omega,$

where $\theta = \frac{1}{2}b$ is the real part, $i\omega$ is the imaginary part, and $\omega = \frac{1}{2}\sqrt{D}$.

We need to choose a bifurcation parameter to vary while holding other parameters constant. The model is parameterized by:

$$\boldsymbol{\alpha} = \begin{pmatrix} \beta \\ \sigma \\ k \\ a_1 \end{pmatrix}.$$

A candidate for a bifurcation parameter is the coefficient, a_1, of the monetary policy rule.

We have the following proposition about the current-looking inflation targeting New Keynesian model.

Proposition 3.4: The New Keynesian model with current-looking inflation targeting, equations, (2.8), (2.14), and (3.6), produces a Hopf bifurcation, if and only if the discriminant of the characteristic equation (3.7) is negative and $a_1^c = \dfrac{\sigma\beta - \sigma}{\kappa}$.

Proof: Assume that a system consisting of (2.8), (2.14), and (3.6) produces a Hopf bifurcation. According to the definition, Hopf bifurcation is characterized by the appearance of the pair of complex conjugate multipliers that lie on the unit circle. Since the multipliers are complex conjugate, the discriminant has to be strictly negative.

By condition (a) for Hopf bifurcation, mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2 + \omega^2} = 1$. Substituting $\theta = \dfrac{1}{2}b$ and $\omega = \dfrac{1}{2}\sqrt{D}$ into that equation, we get

$$\sqrt{\left(\frac{\sigma(\beta+1)+\kappa}{4\sigma\beta}\right)^2 + \frac{1}{4}\left(4\left(\frac{\kappa a_1\beta + \sigma\beta}{\sigma\beta^2}\right)^2 - \left(\frac{(\beta+1)}{\beta} + \frac{\sigma+\kappa}{\sigma\beta}\right)^2\right)} = 1 \quad . \tag{iv}$$

Solving for a_1, we find that the critical value for the parameter is $a_1^c = \dfrac{\sigma\beta - \sigma}{\kappa}$.

Conversely assume that the discriminant of the characteristic polynomial (3.7) is negative and $a_1^c = \dfrac{\sigma\beta - \sigma}{\kappa}$. According to Theorem 3.1, Hopf bifurcation will arise if there is a pair of complex conjugate roots of (3.7) and if conditions (a) and (b) of Theorem 3.1 are satisfied.

Since the discriminant has a negative sign, roots of (3.7) have to be complex conjugate. Condition (a) of Theorem 3.1 states that for a Hopf bifurcation to arise, the modulus of the eigenvalues should be equal to unity. To show that it holds, substitute

$a_1^c = \dfrac{\sigma\beta - \sigma}{\kappa}$ into the left hand side of equation (iv) to find mod (λ_1) =1. Since characteristic roots are complex conjugate, it follows that mod (λ_1) = mod (λ_2) = 1, thereby satisfying condition (a) for Hopf bifurcation.

It can be shown as follows that the derivative of the modulus with respect to a_1 is a non-zero expression:

$$\left.\frac{\partial|\lambda_1|}{\partial a_1}\right|_{a_1=\alpha_1^c} = \left.\frac{\partial|\lambda_2|}{\partial a_1}\right|_{a_1=\alpha_1^c} = \frac{\kappa}{2\beta\sigma} \neq 0.$$

which is condition (b) for Hopf bifurcation. Hence, both conditions of the Hopf bifurcation theorem are satisfied. □

We can combine the critical value for a_1 with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary. The bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\sigma + \kappa}{\sigma\beta} < 1.$$

3.5. Forward-Looking Inflation Target Rule

Using the following forward-looking inflation targeting rule,

$$i_t = a_1 E_t \pi_{t+1} \tag{3.8}$$

instead of the current-looking rule, as the third equation for New Keynesian model, (2.8), (2.14), (3.8), produces the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} 1 - \dfrac{\kappa}{\beta\sigma}(a_1 - 1) & \dfrac{1}{\sigma\beta}(a_1 - 1) \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}$$

with characteristic equation

$$\lambda^2 - b\lambda + c = 0, \tag{3.9}$$

where

$$b = \frac{1+\beta}{\beta} - \frac{\kappa}{\sigma\beta}(a_1 - 1),$$
$$c = \frac{\beta\sigma - \kappa(a_1 - 1)}{\sigma\beta^2} - \frac{\kappa(a_1 - 1)}{\sigma\beta^2}.$$

In order to get a pair of complex conjugate eigenvalues, the discriminant D must be strictly negative:

$$D = \left(\frac{\sigma(\beta+1) - \kappa(a_1 - 1)}{\sigma\beta}\right)^2 - \frac{4}{\beta} < 0\,.$$

Given the sign of the parameters, the discriminant could be either positive or negative. We assume that the discriminant is negative, so that the roots of the characteristic polynomial are complex conjugate:

$\lambda_1 = \theta + i\omega$ and $\lambda_2 = \theta - i\omega,$

where $\theta = \frac{1}{2}b$ is the real part, $i\omega$ is the imaginary part, and $\omega = \frac{1}{2}\sqrt{D}$.

We need to choose a bifurcation parameter to vary, while holding other parameters constant. The model is parameterized by:

$$\boldsymbol{\alpha} = \begin{pmatrix} \beta \\ \sigma \\ k \\ a_1 \end{pmatrix}.$$

We have the following proposition about the forward-looking inflation-targeting New Keynesian model. Surprisingly this result does not require separate setting of a_1 to attain Hopf bifurcation. Under the conditions of this proposition, no freedom remains to select a_1 independently.

Proposition 3.5: The New Keynesian model, (2.8), (2.14), (3.8), with forward-looking inflation targeting produces a Hopf bifurcation, if and only if the discriminant of the characteristic equation (3.9) is negative and $\beta^c = 1$.

Proof: Assume that a system consisting of (2.8), (2.14), and (3.8) produces a Hopf bifurcation. According to the definition, Hopf bifurcation is characterized by the appearance of the pair of complex conjugate multipliers that lie on the unit circle. Since the multipliers are complex conjugate, the discriminant has to be strictly negative.

By condition (a) for Hopf bifurcation, mod (λ_1) = mod (λ_2) = $+\sqrt{\theta^2+\omega^2}=1$. Substituting $\theta=\frac{1}{2}b$ and $\omega=\frac{1}{2}\sqrt{D}$ into that equation, we get

$$\sqrt{\left(\frac{\sigma\beta-\kappa(a_1-1)+\sigma}{2\beta\sigma}\right)^2+\left(\frac{1}{4}\right)\left(\left(1+\frac{1-\kappa(a_1-1)}{\sigma\beta}\right)^2-\frac{4}{\beta}\right)^2}=1. \qquad \text{(v)}$$

Solving for β we find that the critical value for the parameter is $\beta^c = 1$.

Conversely assume that the discriminant of the characteristic polynomial (3.9) is negative and $\beta^c = 1$. According to Theorem 3.1, Hopf bifurcation will arise if there is a pair of complex conjugate roots of (3.9) and if conditions (a) and (b) of Theorem 3.1 are satisfied.

Since the discriminant has a negative sign, roots of (3.9) have to be complex conjugate. Condition (a) of Theorem 3.1 states that for a Hopf bifurcation to arise the modulus of the eigenvalues should be equal to unity. To show that it holds, substitute $\beta^c = 1$ into the left hand side of equation (v) to find mod (λ_1) =1. Since characteristic roots are complex conjugate, it follows that mod (λ_1) = mod (λ_2) = 1, thereby satisfying condition (a) for Hopf bifurcation.

It can be shown as follows that the derivative of the modulus with respect to β is a non-zero expression:

$$\left.\frac{\partial|\lambda_1|}{\partial\beta}\right|_{\beta=\beta^c} = \left.\frac{\partial|\lambda_2|}{\partial\beta}\right|_{\beta=\beta^c} = -\frac{1}{2} \neq 0,$$

which is condition (b) for Hopf bifurcation. Hence, both conditions of the Hopf bifurcation theorem are satisfied. □

We can combine the critical value for β with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary. The bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\kappa(a_1 - 1)}{2\sigma} < 1.$$

Parameter β is the discount factor from the firm's optimization problem. It is also a coefficient in the Phillips curve scaling the impact of expected inflation. Some authors assume for simplicity that $\beta = 1$.[11] Surprisingly we find that that setting can put the New Keynesian model with forward-looking inflation targeting directly on top of a Hopf bifurcation boundary, and hence can induce instability. This conclusion is conditional upon the assumption that the log-linearized New Keynesian model is a good approximation to the economy and that the discriminant of the characteristic equation (3.9) is negative. In such cases, setting the discount factor β equal to unity is not appropriate.

3.6. Numerical Methods

Now we can describe our methodology for numerical bifurcation analysis. We want to detect a bifurcation boundary, defined to be a set of bifurcation points of the same type. In our cases, those bifurcation boundaries consist of combinations of parameter values that give rise to Hopf bifurcation, if the boundary is crossed. The

[11] See Roberts (1995),Clarida, Gali, and (1999).

numerical procedure we will apply is based on the Existence Theorem for Hopf bifurcation, Theorem 3.1.

We find the bifurcation region by solving the following system numerically for values of $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c$ at which:

$$\begin{cases} \mathbf{f}(\mathbf{x},\boldsymbol{\alpha}) - \mathbf{x} = \mathbf{0} \\ \left(\dfrac{tr(\mathbf{J})}{2}\right)^2 < det(\mathbf{J}) \\ |\lambda_1| = |\lambda_2| = 1. \end{cases}$$

where $\boldsymbol{\alpha}^c$ is a subvector of the complete parameter vector, $\boldsymbol{\alpha}$. The resulting numerically computed solutions for $\boldsymbol{\alpha}^c$ define a set of vectors locating a Hopf bifurcation boundary. The set of parameters that we will vary is the private sector parameters,

$$\boldsymbol{\alpha}^c = \begin{pmatrix} \sigma \\ \beta \\ \kappa \end{pmatrix}.$$

When using a Taylor rule, our complete parameter vector $\boldsymbol{\alpha}$ contains five parameters. We will vary three of them, $\boldsymbol{\alpha}^c$, while holding one parameter constant and solving for the fifth parameter. When using inflation targeting, our complete parameter vector $\boldsymbol{\alpha}$ contains four parameters. We will vary three of them while solving for the fourth, when the freedom to select that parameter independently exists.

With Taylor's rule cases, we will choose and hold constant the value of the monetary policy parameter, a_1. For instance it can be set equal to 1.1, which will satisfy Taylor's principle of optimality. Parameter a_2 will take on values according to the formula from our corresponding proposition describing the combination of parameters corresponding to Hopf bifurcation of the system. For example if we are solving numerically for a New Keynesian model with current-looking Taylor's rule, the parameter a_2 will be set according to the formula from proposition 3.1,

$$a_2^c = \sigma\beta - \kappa a_1 - \sigma.$$

With inflation targeting cases, three parameters are varied and we solve for the fourth parameter as the "bifurcation parameter." As in the Taylor rule cases, we choose a parameter of the policy rule to be the bifurcation parameter in the case of current-looking inflation targeting. That parameter is a_1. But with forward looking inflation targeting, we set the private sector parameter β = 1 to attain bifurcation, with the rest of the parameters contrained to satisfy the discriminant negativity condition. Under those conditions, no freedom is left to vary the policy parameter a_1.

4. Conclusion

In dynamical analysis, it is essential to employ bifurcation analysis to detect whether any bifurcation boundaries exist close to the parameter estimates of the model in use. If such a boundary crosses into the confidence region around the parameter estimates, robustness of dynamic inferences is seriously compromised. Our ongoing bifurcation analysis of New Keynesian functional forms is detecting the possibility of Hopf bifurcation. This paper provides the methodology that we have developed and are using. One surprising result from the proofs in this paper is the theoretical finding that a common setting of the parameter β in the future-looking New-Keynesian model can put the model directly onto a Hopf bifurcation boundary.

We have been analyzing the reduced log-linearized system. Study of the full system will require different tools, which will be the subject of future research. In cases in which we do not locate Hopf bifurcation within the theoretically feasible region of the log-linearized system, we cannot conclude that Hopf or other types of bifurcation might not arise in the original non-linearized system. In short, when we find Hopf bifurcation with the linearized system, the result is sufficient but not necessary for existence of a bifurcation boundary.

In this paper, we develop the formulas, prove the propositions, and provide the numerical procedures we currently are using to detect bifurcation boundaries in the parameter spaces of New Keynesian models. A subsequent papers will provide our empirical results. Future research will include bifurcation analysis of the continuous time models as well as of New Keynesian models with backward-looking monetary policy rules. Different types of New Keynesian Phillips curves also will be considered.

References

Aiyagari, S. R. (1989). Can there be short-period deterministic cycles when people are long lived?. *Quarterly Journal of Economics* 104, 163-185.

Andronov, A. A. (1929). Les Cycles Limits de Poincaré et la Théorie des Oscillations Autoentretenues. *Comptes-rendus de l'Academie des Sciences* 189, 559-561.

Barnett, William A. and Yijun He (1999). Stability Analysis of Continuous-Time Macroeconometric Systems. *Studies in Nonlinear Dynamics and Econometrics* 3, 169-188.

Barnett, William A. and Yijun He (2001). Nonlinearity, Chaos, and Bifurcation: A Competition and an Experiment. In Takashi Negishi, Rama Ramachandran, and Kazuo Mino (Eds.), *Economic Theory, Dynamics and Markets: E*ssays *in Honor of Ryuzo Sato* (pp. 167-187). Amsterdam: Kluwer Academic Publishers.

Barnett, William A. and Yijun He (2002). Stabilization Policy as Bifurcation Selection: Would Stabilization Policy Work if the Economy Really Were Unstable?. *Macroeconomic Dynamics* 6, 713-747.

Barnett, William A. and Yijun He (2004). Bifurcations in Macroeconomic Models. In Steve Dowrick, Rohan Pitchford, and Steven Turnovsky (Eds), *Economic Growth and Macroeconomic Dynamics: Recent Developments in Economic Theory* (pp. 95-112). Cambridge, UK: Cambridge University Press.

Barnett, William A. and Yijun He (2006). Robustness of Inferences to Singularity Bifurcation. *Proceedings of the Joint Statistical Meetings of the 2005 American Statistical Society* 100. American Statistical Association, February.

Benhabib, J., Day, R. H. (1982). A Characterization of Erratic Dynamics in the Overlapping Generations Model. *Journal of Economic Dynamics and Control* 4, 37-55.

Benhabib J., Nishimura K. (1979). The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth. *Journal of Economic Theory* 21, 421-444.

Benhabib, J., Rustichini, A. (1991). Vintage Capital, Investment and Growth. *Journal of Economic Theory* 55, 323-339.

Bergstrom, A. R. (1996). Survey of Continuous Time Econometrics. In W. A. Barnett, G. Gandolfo, and C. Hillinger (Eds.). *Dynamic Disequilibrium Modeling* (pp, 3-26). Cambridge, UK: Cambridge University Press.

Bergstrom, A. R., K. B. Nowmann, and S. Wandasiewicz (1994). Monetary and Fiscal Policy in a Second-order Continuous Time Macroeconometric Model of the United Kingdom. *Journal of Economic Dynamics and Control* 18, 731-761.

Bergstrom, A. R., K. B. Nowman, and C. R. Wymer (1992). Gaussian Estimation of a Second Order Continuous Time Macroeconometric Model of the United Kingdom. *Economic Modelling* 9, 313-352.

Bergstrom, A.R., and C.R. Wymer (1976). A Model of Disequilibrium Neoclassic Growth and its Application to the United Kingdom. In A.R. Bergstrom (Ed.), *Statistical Inference in Continuous Time Economic Models*. Amsterdam: North Holland, 267-327.

Bergstrom, A. R. and K. B. Nowman (2006). *A Continuous Time Econometric Model of the United Kingdom with Stochastic Trends*. Cambridge, UK: Cambridge University Press, forthcoming.

Bernanke, Ben S., Thomas Laubach, Frederic S. Mishkin, and Adam S. Posen (1999).

Inflation Targeting: Lessons from the International Experience. Princeton, NJ: Princeton University Press.

Calvo, G. (1983). Staggered Prices in a Utility-Maximizing Framework, *Journal of Monetary Economics* 12, 383-398.

Clarida, Richard, Jordi Galí, and Mark Gertler (1999). The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature* 37, 1661–1707.

Dixit, Avinash and Joseph E. Stiglitz (1977). Monopolistic Competition and Optimum Product Diversity. *American Economic Review* 67, 297-308.

Gali, J., and M. Gertler (1999). Inflation Dynamics: a Structural Econometric Analysis. *Journal of Monetary Economics* 44, 195-222.

Gale, D. (1973). Pure Exchange Equilibrium of Dynamic Economic Models. *Journal of Economic Theory* 6, 12-36.

Gandolfo, Giancarlo 1996. *Economic Dynamics*. Third edition. New York and Heidelburg: Springer.

Gavin, William T (2003), Inflation Targeting: Why It Works and How To Make It Work Better, Federal Reserve Bank of Saint Louis Working Paper 2003-027B.

Grandmont, J. M. (1985). On Endogenous Competitive Business Cycles. *Econometrica* 53, 995-1045.

Hopf, E. (1942). Abzweigung Einer Periodischen Lösung von Einer Stationaren Lösung Eines Differetialsystems. *Sachsische Akademie der Wissenschaften Mathematische-Physikalische*, Leipzig 94, 1-22.

Kuznetsov,Yu.A (1998). *Elements of Applied Bifurcation Theory.* New York: Springer - Verlag.

Leeper, E. and C. Sims (1994). Toward a Modern Macro Model Usable for Policy Analysis. *NBER Macroeconomics Annual*, 81-117.

McCallum, B.T. (1999). Issues in the design of monetary policy rules. In J.B. Taylor and M. Woodford (Eds.). *Handbook of Macroeconomics*. Amsterdam: North-Holland Pub. Co.

Poincaré, H. (1892). *Les Methodes Nouvelles de la Mechanique Celeste*. Paris: Gauthier-Villars.

Roberts, J. M. (1995). "New Keynesian Economics and the Phillips Curve." *Journal of Money, Credit and Banking* 27. 975-984.

Seydel, R. (1994). *Practical bifurcation and stability analysis*. New York: Springer-Verlag.

Shapiro, A. H. (2006). Estimating the New Keynesian Phillips Curve: A Vertical Production Chain Approach. Federal Reserve Bank of Boston Working Paper No. 06-11.

Svensson, Lars E. O. (1999). Inflation Targeting as a Monetary Policy Rule. *Journal of Monetary Economics* 43, 607-54.

Taylor, John B. (1999). A Historical Analysis of Monetary Policy Rules. In John B. Taylor (Ed.), *Monetary Policy Rules* (pp. 319-40). Chicago: University of Chicago Press for NBER.

Walsh, Carl E. (2003). *Monetary Theory and Policy*. 2nd edition. Cambridge MA: MIT Press.

Woodford, Michael (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, NJ: Princeton University Press.