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ABSTRACT

In the classical newsvendor model, when demand is represented by the normal distribution singly truncated at point zero, the standard optimality condition does not hold. Particularly, we show that the probability not to have stock-out during the period is always greater than the critical fractile which depends upon the overage and the underage costs. For this probability we derive the range of its values. Writing the safety stock coefficient as a quantile function of both the critical fractile and the coefficient of variation we obtain appropriate formulae for the optimal order quantity and the maximum expected profit. These formulae enable us to study the changes of the two target inventory measures when the coefficient of variation increases. For the optimal order quantity, the changes are studied for different values of the critical fractile. For the maximum expected profit, its changes are examined for different combinations of the critical fractile and the loss of goodwill. The range of values for the loss of goodwill ensures that maximum expected profits are positive. The sizes of the relative approximation error which result in by using the normal distribution to compute the optimal order quantity and the maximum expected profit are also investigated. This investigation is extended to different values of the critical fractile and the loss of goodwill. The results indicate that it is naïve to suggest for the coefficient of variation a maximum flat value under which the normal distribution approximates well the target inventory measures.

Keywords: Classical newsvendor model; truncated normal distribution; optimality condition; critical fractile; loss of goodwill; relative approximation error.

JEL Codes: C24; C44; M11; M21.

1. Introduction

For the classical newsvendor model, the sufficient optimality condition to determine the order quantity is given by the well-known standard critical fractile formula (Khouja, 1999). This formula states that the probability not to observe stock-out during the demand life-cycle (otherwise called as the period) is equal to a critical fractile which depends upon overage and underage costs. The overage cost equals to unit purchase cost minus salvage value, and the underage cost is the difference between profit margin and loss of goodwill. Setting a-priori the probability not to have stock-out during the period, the optimal order quantity is computed from the inverse cumulative distribution function of demand evaluated at the critical fractile.

When demand is normally distributed, the standard critical fractile formula holds only when the coefficient of variation is sufficiently small. In this case, the probability to take negative demand is negligible. Taking the inverse of the cumulative distribution function evaluated at the critical fractile, the optimal order quantity is equal to the average demand plus the safety stock coefficient times the standard deviation of demand. In this optimal order quantity equation, the safety stock coefficient is a quantile function of the critical fractile. So, the computed order quantity ensures that the requested probability of not having stock-out during the period is eventually attained.

The Normal distribution has been widely used in inventory management to model demand. A first reason is that the theoretical properties of normal distribution enable us to derive exact expressions for target inventory measures such as the optimal order quantity and the maximum expected profit. The second reason is that we can take good approximations for these measures when the coefficient of variation is low. Lau (1997) offered a simple formula to compute the expected cost of the classical newsvendor model when demand is normal with a coefficient of variation less than 0.3.

Perakis and Roels (2008) derived order quantities that maximize the newsvendor's maximum regret and they stated that a normal distribution with small coefficient of variation is robust and is also entropy maximizing when only mean and variance are known. To investigate purchase decision of newsvendor products when demand distribution is unknown, Benzion et al. (2008) conducted an experiment during which demand data were generated from a normal distribution with coefficient of variation equal to one third. The normal distribution with the same coefficient of variation was also used in a similar experiment conducted by Benzion et al. (2010) where half of the participants knew the demand distribution and the other half did not.

When the demand coefficient of variation is large, using again the normal distribution, the probability to take negative demand on a given period is not any more negligible. In such case, and if data for demand are available, Gallego et al. (2007) recommend the fit of the empirical distribution to one of the known non-negative random variables such as the Gamma, or the Negative Binomial, or the Lognormal. For the three distributions, the authors showed that the optimal order quantity first increase and then decrease when the demand standard deviation increases.

To cope with negative values for demand, Strijbosch and Moors (2006) suggested two alternatives. The first alternative is to interpret negative demand as purchases being sent back to stores. However, in markets of newsvendor products (magazines, clothing, perishable food etc.), this explanation could not stand as it is very unlikely customers to have the possibility to return back to stores purchases of such products (unless the product is faulty). The second alternative is to regret negative values. In such case, the demand of the period should be modeled by the normal distribution singly truncated at point zero.

In the current paper, we follow the second alternative and illustrate that when demand follows the singly truncated normal distribution at point zero, the standard critical fractile formula does not hold. Particularly, we show that the probability of not having stock-out during the period is always greater than the critical fractile. Writing the safety stock coefficient as a quantile function of both the critical fractile and the coefficient of variation, we derive appropriate expressions for computing the optimal order quantity and the maximum expected profit. These expressions allow us to study analytically the changes of these two target inventory measures when the coefficient of variation is raising due to demand standard deviation increases. Particularly, these changes are examined at different sizes of the critical fractile. The changes of maximum expected profits are also explored at different values of the loss of goodwill for which the maximum expected profit is positive.

The use of the non-truncated normal distribution for approximating the exact values of the optimal order quantity and the maximum expected profit (when the demand distribution is modeled by the truncated normal) is also investigated analytically. As criterion of investigation we use the approximation error as percentage of the exact values of the two target inventory measures. The derived expressions of the two target inventory measures for the singly truncated normal allow us to evaluate this criterion no matter what values the average demand of the period, the selling price, the purchasing cost and the salvage value take on. So, for both target inventory measures, this relative approximation error is studied at different sizes of the critical fractile. For the maximum expected profit the relative approximation error is also examined for different values of the loss of goodwill. Having established this experimental framework, to the extent of our knowledge, this study using the singly truncated normal at point zero is performed for the first time.

Our work comes closest to the paper of Hu and Munson (2011). Assuming that demand follows the truncated normal distribution at point zero, the authors gave an expression to compute the optimal order quantity, and using this expression they derived the function for the maximum expected profit. Keeping fixed the probability of not having stock-out during the period, they examined the behavior of optimal order quantities and maximum expected profits when the coefficient of variation was getting larger.

However, their study was based on Monte-Carlo simulations and on specific values for the average demand, the selling price, the purchase cost and the salvage value. Besides, their simulation results were generated under the restrictive assumptions that loss of goodwill is zero and the sum of overage and underage costs remains the same when the probability to have stock-out during the period is decreasing. The last two remarks differentiate our work, which is based on an analytic approach without making specific assumptions about the values of the overage and the underage cost.

The paper of Strijbosch and Moors (2006) also handles the case of modeling demand with the normal distribution with large coefficients of variation. For the (R,S) inventory control system, the authors used the censored and the truncated normal distribution to derive safety factors and order-up-to-levels. Halkos and Kevork (2011) used three alternative distributions to construct confidence intervals for the optimal order quantity and the maximum expected profit when the demand follows the singly truncated normal at point zero. These distributions were the non-truncated normal, the lognormal and the exponential. They concluded that only for very few combinations of the critical fractile and the sample size the confidence intervals of the non-truncated normal and the lognormal distribution attained acceptable confidence levels. But these intervals are characterized by low precision and stability. Other works which dealt with the prons and cons of using the normal distribution with small or large coefficients of variation can be found in Janssen et al. (2009).

The aforementioned arguments and discussion lead the rest of the paper to be structured as follows when demand follows the singly truncated normal at point zero: In the next section, we derive the optimality condition and determine the range of values for the probability not to have stock-out during the period. In section 3, we study the changes of the optimal order quantity when the coefficient of variation increases. In section 4, we derive the expression for the maximum expected profit, and determine the range of values for the loss of goodwill where maximum expected profit is positive. In the same section, we derive analytic forms for the relative approximation errors so that to examine how well the use of the non-truncated normal approximates the exact values of the two target inventory measures. Finally, the last section summarizes the most important findings of the current work. In each section, symbols which are used in the analysis are explained when required. Nonetheless, for the reader's convenience, table 1 provides the list of symbols, which are used throughout this paper, with their explanation.

2. Optimality Condition

In the classical newsvendor model, the demand which will occur during the period should be a non-negative random variable X with cumulative distribution function $F(\mathrm{x})$. Having specified at the start of the period the critical fractile R from the equation $\mathrm{R} = (\mathrm{p} - \mathrm{c} + \mathrm{s})/(\mathrm{p} - \mathrm{v} + \mathrm{s})$, when X is continuous the order quantity maximizing the expected profit (or the expected cost) of the period satisfies the sufficient optimality condition $F(\mathrm{Q}^*) = \mathrm{R}$ and thus is determined from the equation $\mathrm{Q}^* = F^{-1}(\mathrm{R})$. To satisfy the demand of the period, the newsvendor has available stock at the start of the period only the optimal order quantity, Q^*. Further, receiving this ordered quantity, he is not charged with any fixed costs.

Ordering Q^*, the critical fractile R expresses the probability the newsvendor not to experience a stock-out during the period. Following the rule of thumb suggested by Schweitzer and Cachon (2000), if $\mathrm{R} > 0.5$ (or alternatively $\mathrm{R} < 0.5$), the newsvendor product is classified as high-profit (or low-profit respectively). This principle implies that among different newsvendor products that one with the largest R has been purchased at the highest cost, it is sold at the highest price, and it yields the largest profit margin.

When X follows the normal distribution with mean μ and variance σ^2, the application of the formula $F\left(Q^*\right) = R$ approximates well the optimal order quantity only when the coefficient of variation ($CV = \sigma/\mu$) is sufficiently small. This happens because the probability of taking negative demand is negligible and so, in the profit function of the newsvendor model (e.g. see Khouja, 1999), it is legitimate to set the lower limit of X at minus infinity instead of zero. The probability of negative demand equals to $1 - \Phi_\theta$, where θ is the inverse of coefficient of variation, and Φ_θ is the cumulative distribution function of the standard normal evaluated at θ. Hence when CV is sufficiently small, the optimal order quantity will be well approximated from the known formula (e.g. see Silver et al., 1998)

$$Q^*_{ap} = \mu + z_R \sigma, \tag{1}$$

and the computed Q^*_{ap} will ensure that the probability to observe a stock-out during the period is almost identical to R.

On the contrary, when CV is not sufficiently small, how well formula (1) approximates the optimal order quantity is under question. This can be explained by considering a realization of values from the normal random variable X for a sufficiently large number of consecutive periods. If CV is large, then a significant number of negative values will appear in this realization. So regretting all the negative values, the remaining “truncated part” of the realization will follow the singly truncated normal distribution at point zero, which has probability density function

$$f^+(x) = \left[\sigma\Phi_\theta\right]^{-1} \cdot (2\pi)^{-1/2} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2},$$

and will be denoted as $X^+ \sim N^+\left(\mu, \sigma^2\right)$.

Table 1: Notation and Terminology

Symbol	Explanation
Q:	Order quantity at the start of the period
p:	Selling price per unit
c:	Purchase cost per unit
v:	Salvage value
s:	Loss of goodwill, where $s = \delta(p - c)$
R:	Requested critical fractile, where $R = (p - c + s)/(p - v + s)$
X:	Demand of the period
$F(x)$:	Cumulative distribution function of demand
$X \sim N(\mu, \sigma^2)$	Demand follows the non-truncated normal distribution
CV:	Coefficient of variation, where $CV = \sigma/\mu$
z_R:	Safety stock coefficient when $X \sim N(\mu, \sigma^2)$ and sufficiently small CV
Q^*_{ap}:	Optimal order quantity when $X \sim N(\mu, \sigma^2)$
ξ^*_{ap}:	Maximum expected profit when $X \sim N(\mu, \sigma^2)$
$X^+ \sim N^+(\mu, \sigma^2)$	Demand follows the normal distribution singly truncated at point zero
θ:	Inverse of coefficient of variation, where $\theta = 1/CV$
ϕ_z:	Probability density function of the standard normal evaluated at $z = (Q - \mu)/\sigma$,
Φ_z:	Cumulative distribution function of the standard normal evaluated at $z = (Q - \mu)/\sigma$,
Φ_θ:	Probability demand of the period to be positive
ξ:	Expected profit per period when $X^+ \sim N^+(\mu, \sigma^2)$
Q^*:	Optimal order quantity maximizing ξ when $X^+ \sim N^+(\mu, \sigma^2)$
ξ^*:	Maximum expected profit when $X^+ \sim N^+(\mu, \sigma^2)$
RAE_{Q^*}	Relative approximation error when Q^*_{ap} is used to approximate Q^*
RAE_{ξ^*}	Relative approximation error when ξ^*_{ap} is used to approximate ξ^*

When the demand which will occur during the period follows X^+, the expected profit function is derived in the Appendix and is given by

$$\xi = (p - c)Q - (p - v)\{(Q - \mu - \sigma \cdot \omega)\} + (p - v + s)\left\{(Q - \mu)\frac{1 - \Phi_z}{\Phi_\theta} - \sigma\frac{\phi_z}{\Phi_\theta}\right\}. \tag{2}$$

The explanations of symbols in (2) are given in Table 1. Maximizing ξ with respect to Q, and using the derivatives, $d\Phi_z/dQ = \sigma^{-1}\phi_z$ and $d\phi_z/dQ = -\sigma^{-1}z\phi_z$, first and second order conditions are given respectively from the following expressions:

$$\frac{d\xi}{dQ} = (p - c + s) - (p - v + s)\left(1 - \frac{1 - \Phi_z}{\Phi_\theta}\right) = 0, \tag{3a}$$

$$\frac{d^2\xi}{dQ^2} = -\frac{(p - v + s)\phi_z}{\sigma\Phi_\theta} < 0. \tag{3b}$$

From (3a) and (3b), the optimal order quantity satisfies the sufficient optimality condition

$$h = \Pr\left(X \le Q^*\right) = \Pr\left(Z \le \frac{Q^* - \mu}{\sigma}\right) = \Pr\left(Z \le z_{1-(1-R)\Phi_\theta}\right) = 1 - (1 - R)\Phi_\theta. \tag{4}$$

From (4), we reach the first important conclusion. Since $(1 - R)(1 - \Phi_\theta) > 0$, it follows that $1 - (1 - R)\Phi_\theta > R$ and hence $z_{1-(1-R)\Phi_\theta} > z_R$. So, if the probability of taking negative demand is not negligible, by ordering the optimal quantity, Q^*, the probability not to have a stock-out during the period is always greater than the critical fractile R. The rate of change of probability $h = \Pr(X \le Q^*)$ when CV is getting larger is studied in the next proposition.

Proposition 1: Given the critical fractile R, the probability $h = \Pr(X \le Q^*)$ is:

(a) increasing in CV with lower limit R and upper limit $(R + 1)/2$,

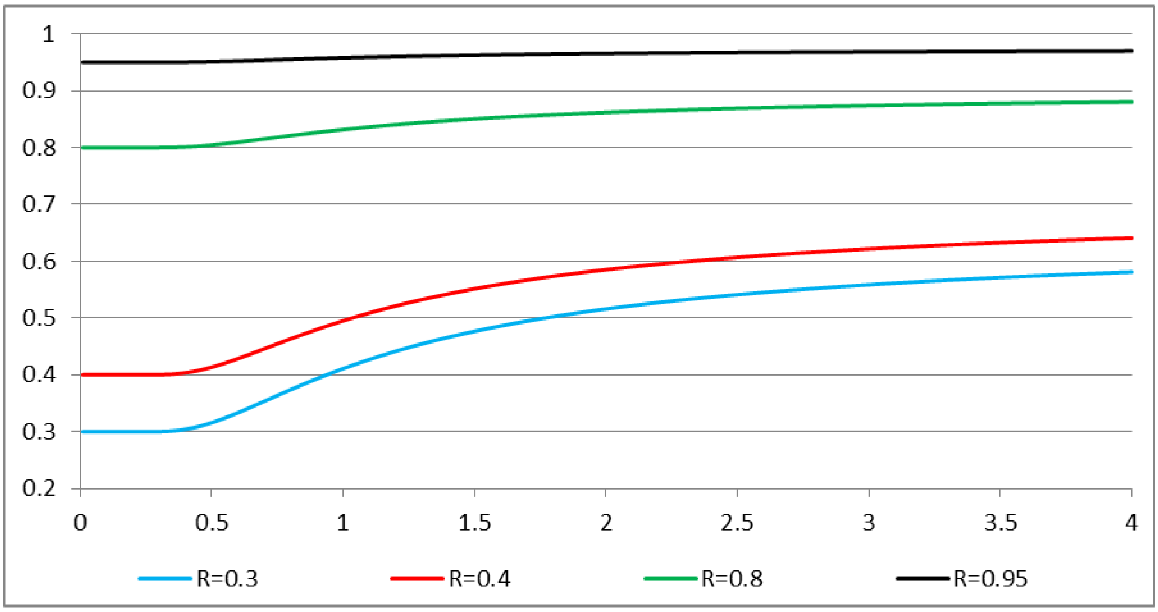
(b) has inflection point at $CV = \sqrt{2}/2$.

Proof: See in the Appendix.

An immediate implication of proposition 1 is the following. Even if the critical fractile R is below 0.5, there will be a range of values of the coefficient of variation (CV) where probability h will exceed 0.5. In this case, the rule of thumb using R to classify the product as low or high-profit should be used with cautiousness. When demand follows distributions of non-negative random variables, like exponential, gamma, etc., this rule of thumb works well since it holds $\Pr(X \le Q^*) = R$. But when demand follows the truncated normal X^+, this equality is not true. So, with R below 0.5, there will be for certain a range of values for CV where R would classify the product as low-profit, while the probability not to have stock-out during the period would classify it as high-profit.

The results of proposition 1 are verified in Figure 1. Further, table 2 displays, for selected combinations of R and CV, the values of probability h. Given CV, the error of approximating probability h with R depends upon the value of R. Setting the approximation error to be less than a certain size, R approximates well probability h when the coefficient of variation is below a critical value which increases as R is getting larger. Setting, for example, the size of the approximation error below 10^{-5}, this critical value ranges between 0.22 and 0.26 when R is between 0.3 and 0.95. This range of R is reasonable form the practice point of view.

Figure 1: Graph of $h = \Pr(X \leq Q^*)$ when CV is increasing



The data of table 2 also verify the problem which we raise for using R to classify the product as low or high-profit when R is below 0.5. As an example we mention the case of R=0.4 and CV equal or greater than 1.5. Using the rule of thumb with R, the product is classified as low-profit. On the contrary, using the probability not to have a stock-out during the period, the product is high-profit. Observe also that when CV is more than 3, this probability is greater than 62%.

Table 2: Values of probability $h = \Pr(X \leq Q^*)$

CV	R = 0.3	R = 0.4	R = 0.8	R = 0.95
0.1	0.30000	0.40000	0.80000	0.95000
0.2	0.30000	0.40000	0.80000	0.95000
0.21	0.30000	0.40000	0.80000	0.95000
0.22	0.30000	0.40000	0.80000	0.95000
0.23	0.30001	0.40000	0.80000	0.95000
0.24	0.30001	0.40001	0.80000	0.95000
0.25	0.30002	0.40002	0.80001	0.95000
0.26	0.30004	0.40004	0.80001	0.95000
0.27	0.30007	0.40006	0.80002	0.95001
0.28	0.30012	0.40011	0.80004	0.95001
0.29	0.30020	0.40017	0.80006	0.95001
0.3	0.30030	0.40026	0.80009	0.95002
0.4	0.30435	0.40373	0.80124	0.95031
0.5	0.31593	0.41365	0.80455	0.95114
1	0.41106	0.49519	0.83173	0.95793
1.5	0.47675	0.55150	0.85050	0.96263
2	0.51598	0.58512	0.86171	0.96543
3	0.55861	0.62167	0.87389	0.96847
4	0.58091	0.64078	0.88026	0.97007

3. Optimal ordering policy

In the current section, we express the optimal order quantity, Q^*, as a function of the coefficient of variation (CV), and then, given the critical fractile, R, we study how Q^* changes when CV is making larger.

Using condition (4), the optimal order quantity is determined from the equation

$$Q^* = \mu + z_h \sigma = \mu(1 + z_h \cdot CV). \tag{5}$$

Given the average demand of period and increasing CV, the rate of change of Q^* depends upon the properties of the function $g(CV) = 1 + z_h \cdot CV$. Differentiating g given R, and using result (A9) of the Appendix, we obtain

$$\frac{\mathrm{d}g}{\mathrm{dCV}} = z_h + \mathrm{CV}\cdot\frac{\mathrm{d}z_h}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{dCV}} = z_h + (1-\mathrm{R})\theta\frac{\phi_\theta}{\phi_{z_h}}.$$

The derivative $\mathrm{d}z_h/\mathrm{d}h = \phi_{z_h}^{-1}$ was obtained using formula (2) of Steinbrecher and Shaw (2008).

When R exceeds 0.5, we showed in the previous section that $z_h > z_\mathrm{R} > 0$, and hence $\mathrm{d}g/\mathrm{dCV}$ is positive for any $\mathrm{CV} > 0$. But, when R is set below 0.5, we showed through proposition 1 that there is a range of values of CV for which $z_h < 0$. In this case, $\mathrm{d}g/\mathrm{dCV}$ is made up of two terms of which the first is negative and the second is positive. Then to determine the final sign of $\mathrm{d}g/\mathrm{dCV}$, we shall investigate its rate of change when CV is increasing. So we proceed to the next proposition:

Proposition 2: Given that R is below 0.5, $\mathrm{d}g/\mathrm{dCV}$ is increasing in CV with lower limit $z_\mathrm{R} < 0$ and upper limit $z_{(1+\mathrm{R})/2} > 0$.

Proof: See in the Appendix.

From proposition 2 we deduce that setting R below 0.5 and increasing CV, there is only one value $\theta_o = 1/\mathrm{CV}_o$ for which the derivative $\mathrm{d}g/\mathrm{dCV}$ becomes zero. This value θ_o satisfies the equation $(1-\mathrm{R})\theta_o\phi_{\theta_o} = -z_{h_o}\phi_{h_o}$.

Summarizing, we conclude that for any $0 < \mathrm{R} < 1$, the optimal order quantity, Q^*, is a convex function of CV since $\mathrm{d}g^2/\mathrm{dCV}^2 > 0$ from (A11) in the appendix. Further, the limiting values of Q^* are

$$\lim_{\mathrm{CV}\to 0}\mathrm{Q}^* = \mu\left(1 + \lim_{\mathrm{CV}\to 0}(z_h\cdot\mathrm{CV})\right) = \mu(1 + z_\mathrm{R}\cdot 0) = \mu,$$

and

$$\lim_{\mathrm{CV}\to \infty}\mathrm{Q}^* = \mu\left(1 + \lim_{\mathrm{CV}\to \infty}(z_h\cdot\mathrm{CV})\right) = \mu\left(1 + z_{(1+\mathrm{R})/2}\cdot\infty\right) = \infty.$$

Additionally, when $R > 0.5$, then Q^* is increasing in CV for any $CV > 0$, while for $R < 0.5$, Q^* has global minimum at $\theta_o = 1/CV_o$, and increases only when $CV > CV_o$.

To illustrate numerically the findings of this section, we consider four hypothetical newsvendor products whose R has been set up at 0.3, 0.4, 0.8, and 0.95 respectively. From the principle of low/high-profit products of Schweitzer and Cachon (2000), a product with higher R has larger profit margin, which follows from higher price and higher purchasing cost. However, with larger R and higher price, p, the average demand of period, μ, should be reduced such that a negative relationship between p and μ to hold. The selected values of μ are displayed in table 3. In the same table we have also included the values of the safety stock coefficient, z_h, and the optimal order quantity, Q^*, computed from (5). For the selected values of μ and R the graphs of $Q^* = \mu \cdot g(CV)$ are presented in Figure 3.

From the data of table 3 and the graphs of figure 3, the remarks which have been made in this section are verified. Observe also that CV_o, from which Q^* is starting to increase, becomes smaller as R is approaching 0.5. So, for R=0.3, CV_o is approximately equal to 0.69, while for R=0.4, CV_o is approximately equal to 0.54.

4. Maximum Expected Profit

When Q takes on its optimal value, using (3a) and $R = (p - c + s)/(p - v + s)$, the following relationship is obtained

$$1 - \frac{1 - \Phi_z}{\Phi_\theta} = 1 - R = \frac{c - v}{p - v + s}. \tag{6}$$

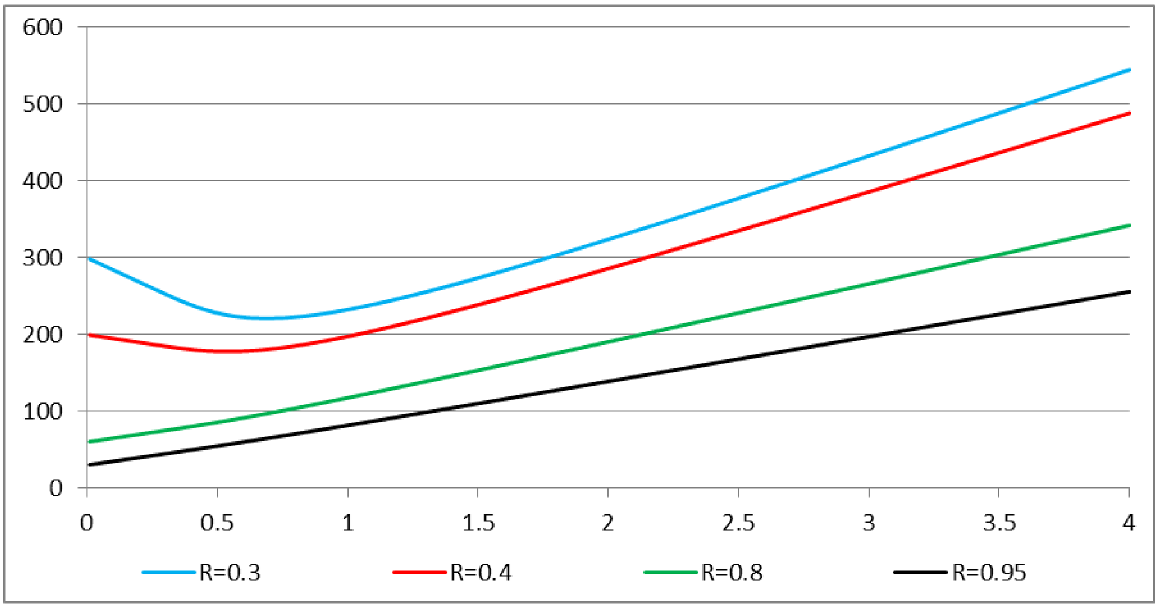
Also from (5) and the definition of the standardized value z, which is given in Table 1, we take

$$z = \frac{Q^* - \mu}{\sigma} = z_h. \tag{7}$$

Table 3: Exact values for the optimal order quantity when demand follows the normal distribution singly truncated at zero

	R = 0.3 , μ =300		R = 0.4 , μ =200		R = 0.8 , μ =100		R = 0.95 , μ =50	
CV	z_h	Q^*	z_h	Q^*	z_h	Q^*	z_h	Q^*
0.2	-0.5244	268.54	-0.2533	189.87	0.8416	70.10	1.6449	39.87
0.3	-0.5235	252.88	-0.2527	184.84	0.8419	75.15	1.6451	44.81
0.4	-0.5119	238.57	-0.2437	180.50	0.8461	80.31	1.6479	49.77
0.53	-0.4657	225.95	-0.2076	177.99	0.8630	87.44	1.6594	56.38
0.54	-0.4610	225.33	-0.2039	177.98	0.8647	88.02	1.6606	56.90
0.55	-0.4561	224.75	-0.2001	177.99	0.8665	88.60	1.6618	57.42
0.6	-0.4304	222.53	-0.1798	178.43	0.8763	91.55	1.6685	60.03
0.68	-0.3867	221.11	-0.1448	180.30	0.8933	96.44	1.6801	64.27
0.69	-0.3812	221.10	-0.1404	180.63	0.8954	97.07	1.6816	64.81
0.7	-0.3756	221.12	-0.1359	180.97	0.8976	97.70	1.6832	65.35
0.8	-0.3214	222.86	-0.0919	185.30	0.9197	104.14	1.6984	70.76
0.9	-0.2708	226.89	-0.0503	190.95	0.9410	110.81	1.7132	76.26
1	-0.2248	232.55	-0.0120	197.59	0.9610	117.66	1.7272	81.82
1.5	-0.0583	273.75	0.1294	238.83	1.0386	153.47	1.7820	110.19
2	0.0401	324.03	0.2150	286.01	1.0880	190.56	1.8175	139.05
3	0.1474	432.70	0.3099	385.91	1.1450	266.09	1.8588	197.29
4	0.2042	545.05	0.3605	488.43	1.1763	342.31	1.8817	255.81

Figure 2: Graph of Q^* when CV is increasing



So replacing Q with Q^* in (2), and using (6) and (7), the maximum expected profit is given

by $$\xi^* = (p-c)(\mu + z_h\sigma) - (p-v)(z_h - \omega)\sigma + (c-v)z_h\sigma - (p-v+s)\sigma\frac{\phi_{z_h}}{\Phi_\theta} =$$

$$= (p-c)\mu + (p-v)\sigma\omega - (p-v+s)\sigma\frac{\phi_{z_h}}{\Phi_\theta} =$$

$$= (p-c+s)\mu + (p-v+s)\sigma\omega - s(\mu+\sigma\omega) - (p-v+s)\sigma\frac{\phi_{z_h}}{\Phi_\theta}, \tag{8}$$

where $\omega = \phi_z / \Phi_\theta$.

Following Lapin (1994), the loss of goodwill, s, is defined as the present value of future profits which are expected to be lost from present unsatisfied customers who will not come back to the store to purchase the same product in the future. So it is legitimate to set $s = \delta(p-c)$, where $\delta \geq 0$. Hence, from $R = (p-c+s)/(p-v+s)$ we obtain

$$p - v + s = \frac{(1+\delta)(p-c)}{R}, \tag{9}$$

and replacing (9) into (8), we take the final expression for the maximum expected profit

$$\xi^* = (p-c)\mu\left\{1 + \frac{CV\cdot\omega}{R}\left(1 - \frac{\phi_{z_h}}{\phi_\theta}\right) - \delta\frac{CV\cdot\omega}{R}\left[\frac{\phi_{z_h}}{\phi_\theta} - (1-R)\right]\right\}. \tag{10}$$

Considering R, $p-c$, and μ as given, the expression within the brackets in (10) defines the size of the maximum expected profit. To study the behavior of ξ^* when CV is increasing, initially we shall determine the range of parameter δ, where maximum expected profits are non-negative. The next proposition gives a prerequisite result to specify this range.

Proposition 3: For $0 < R < 1$, the function $u(CV) = \frac{\phi_{z_h}}{\phi_\theta} - (1-R)$ is decreasing in CV and has

upper limit $\sqrt{2\pi}\phi_{(1+R)/2} - (1-R) > 0$.

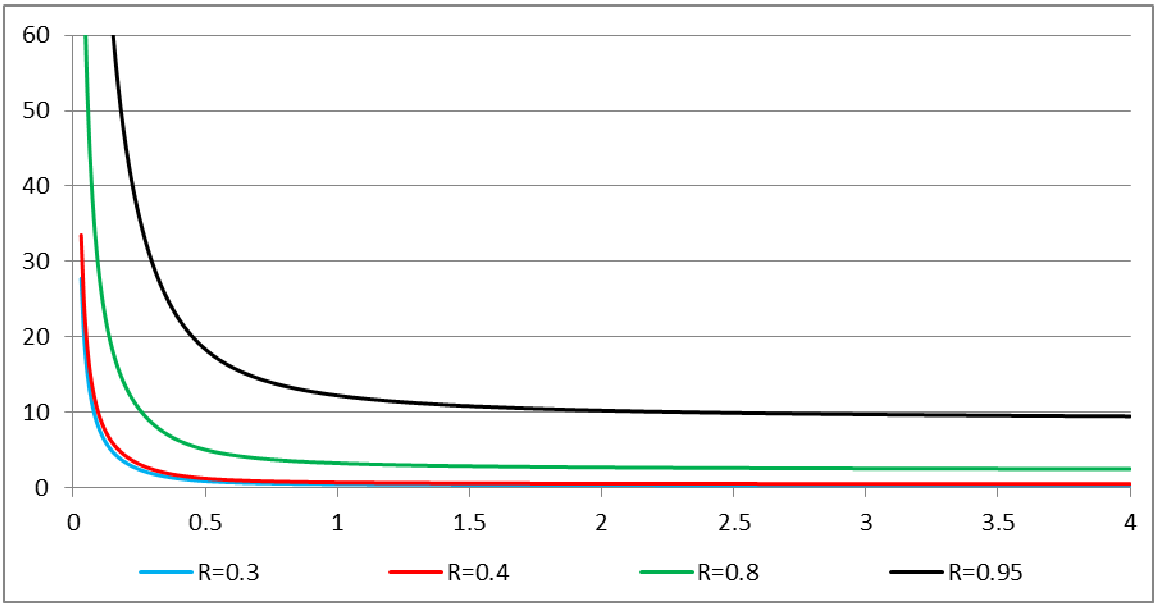
Proof: See in the Appendix.

Hence ξ^* is positive when δ satisfies the inequality

$$\delta \le \delta_o = \frac{\dfrac{R}{CV\cdot\omega} + \left(1 - \dfrac{\phi_{z_h}}{\phi_\theta}\right)}{\dfrac{\phi_{z_h}}{\phi_\theta} - (1-R)}. \tag{11}$$

For different values of R, figure 3 illustrates the graph of δ_o when CV is increasing. Given R, the range $[0,\delta_o]$ is getting narrower as CV is getting larger. If we consider CV as given, then the range $[0,\delta_o]$ becomes wider as R increases.

Figure 3: Graph of δ_o when CV is increasing

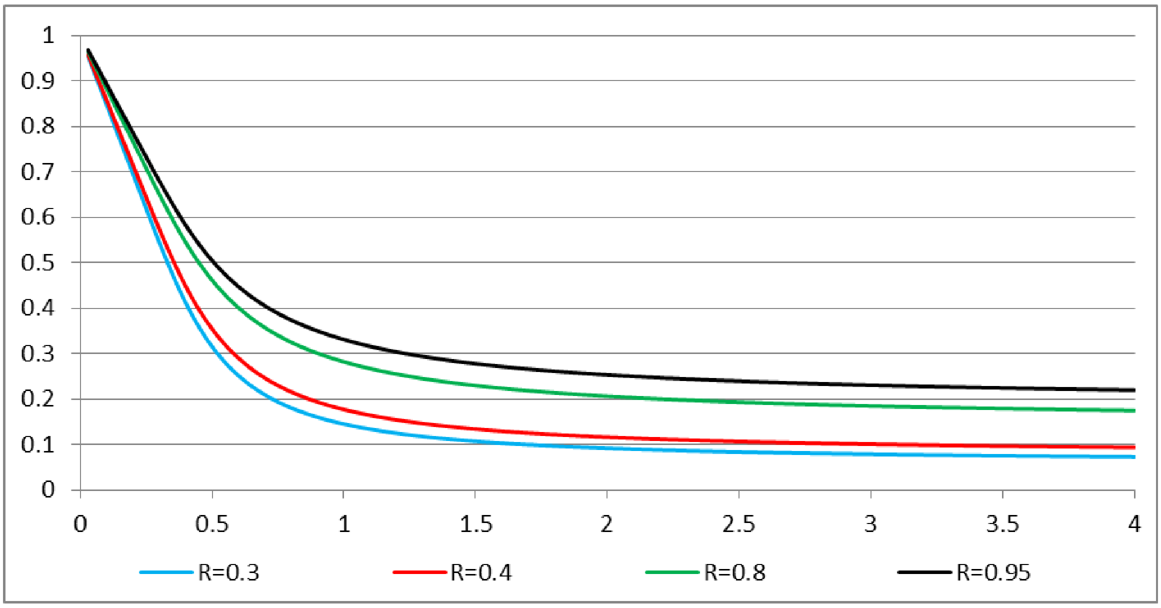


When CV tends to infinity, then for each R, δ_o tends to a corresponding limiting value

$$\delta_\infty = \lim_{CV\to\infty} \delta_o = \frac{1 - \phi_{z_{(1+R)/2}}\sqrt{2\pi}}{\phi_{z_{(1+R)/2}}\sqrt{2\pi} - (1-R)}. \tag{12}$$

This limit is obtained since when $CV \to \infty$ then $\theta \to 0$ and $CV\cdot\omega \to \infty$, $\phi_{z_h} \to \phi_{(1+R)/2}$, and $\phi_\theta \to \left(\sqrt{2\pi}\right)^{-1}$. So, from (11) and (12) we deduce that for any R, if δ is less than or equal to the corresponding limiting value δ_∞, then the maximum expected profit will be positive for any $CV < \infty$. This is verified in Figure 4, where for different R's and with δ to be equal to the corresponding δ_∞, we display the graph of $\xi^* / [(p-c)\mu]$ when CV is increasing. For each R, ξ^* decreases as CV is getting larger, and given CV, the expression within the brackets in (10) takes on larger values when R is increasing.

Figure 4 : Graph of $\xi^* / [(p-c)\mu]$ when CV is increasing



We are closing this section by examining the approximation error in computing both the optimal order quantity and the maximum expected profit if we used the non-truncated normal distribution. For the optimal order quantity, the relative approximation error is defined as $\left(Q^* - Q^*_{ap}\right)/Q^*$, and using (1) and (5), this is given by

$$RAE_{Q^*} = \frac{CV \cdot (z_h - z_R)}{(1 + z_h \cdot CV)}. \tag{13}$$

When demand is modeled by the non-truncated normal distribution, the maximum expected profit is computed from (e.g. see Kevork, 2010),

$$\xi_{ap}^* = (p-c)\mu\left\{1-(1+\delta)\frac{\phi_{z_R}}{R}CV\right\}.$$

So, the corresponding RAE is obtained using (10) from

$$RAE_{\xi^*} = \frac{\xi^* - \xi_{ap}^*}{\xi^*} = \frac{\frac{(1+\delta)CV}{R}\left\{\omega\left(1-\frac{\phi_{z_h}}{\phi_\theta}\right)+\phi_{z_R}\right\}-\delta\cdot\omega\cdot CV}{1+\frac{(1+\delta)\cdot CV\cdot\omega}{R}\left(1-\frac{\phi_{z_h}}{\phi_\theta}\right)-\delta\cdot\omega\cdot CV}. \tag{14}$$

Table 4 displays the values of (13) and (14) when CV is increasing. The relative approximation error of both Q^* and ξ^* depends upon the critical fractile. For ξ^*, its RAE_{ξ^*} depends also on the loss of goodwill. For this reason, the values of δ were restricted on the interval from zero up to δ_∞. This range of δ values ensures that at the corresponding R, maximum expected profits will be positive for any $CV < \infty$. Given the value of CV, both RAE_{Q^*} and RAE_{ξ^*} are decreasing as R is getting larger. Given R and CV, with zero loss of goodwill RAE_{ξ^*} is larger than RAE_{Q^*}. Further, RAE_{ξ^*} is making larger as the loss of goodwill is increasing.

The previous arguments indicate that it seems naive to suggest for the coefficient of variation a maximum flat value under which the optimal order quantity and the maximum expected profit to be well approximated by using the non-truncated normal distribution. Table 4, or alternatively formulae (13) and (14), offer the required information such that, according to the case and the desired size of the approximation error, to be able to decide in favour of the normal distribution singly truncated at point zero.

5. Conclusions

In the classical newsvendor model, the optimality condition for expected profit maximization (or expected cost minimization) states that the probability not to have stock-out during the period equals to a critical fractile whose value depends upon the overage and the underage cost. In the current paper we point out that this condition is true only when demand of the period is a continuous non-negative random variable. If demand distribution is normal but with coefficient of variation not sufficiently small, the probability the stochastic law of generating demand to give negative values is not negligible. In this case, the demand should be modeled by the normal distribution singly truncated at point zero.

With demand to follow the truncated normal distribution, we prove that for any value of the critical fractile, the probability not to experience a stock-out during the period is always greater than the critical fractile. Particularly, we give for the first time, the range of values for this probability at any critical fractile. Furthermore, if the critical fractile is less than 0.5, we emphasize that the rule of thumb to classify the product as low or high-profit should be used with cautiousness. We show that there is a range of values for the coefficient of variation where the critical fractile would classify the product as low profit, while the probability of not observing stock-out during the period would give a high-profit product.

Writing the safety stock coefficient as a quantile function of both the coefficient of variation and the critical fractile, appropriate formulae to compute exactly the optimal order quantity and the maximum expected profit are developed when demand follows the normal distribution singly truncated at point zero. These formulae allows to study the changes of the two target inventory measures when the coefficient of variation is increasing, no matter which values the selling price, the purchasing cost and the salvage value take on. So, the behavior of the optimal order quantity in changes of the coefficient of variation depends upon only the critical fractile. For the maximum expected profit, its behavior depends on the critical fractile

Table 4: Relative approximation error in computing the optimal order quantity and the maximum expected profit using the classical normal distribution instead of the truncated normal

	R = 0.3				R = 0.4				R = 0.8				R = 0.95			
	RAE_{Q^*}	RAE_{ξ^*}			RAE_{Q^*}	RAE_{ξ^*}			RAE_{Q^*}	RAE_{ξ^*}			RAE_{Q^*}	RAE_{ξ^*}		
CV		$\delta = 0$	$\delta = 0.15$	$\delta = 0.3$		$\delta = 0$	$\delta = 0.2$	$\delta = 0.4$		$\delta = 0$	$\delta = 1$	$\delta = 2$		$\delta = 0$	$\delta = 2$	$\delta = 4$
0.2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.25	0.00%	0.01%	0.02%	0.02%	0.00%	0.01%	0.01%	0.01%	0.00%	0.00%	0.01%	0.01%	0.00%	0.00%	0.00%	0.01%
0.26	0.00%	0.03%	0.03%	0.03%	0.00%	0.02%	0.02%	0.03%	0.00%	0.01%	0.01%	0.01%	0.00%	0.01%	0.01%	0.01%
0.27	0.01%	0.04%	0.05%	0.06%	0.00%	0.03%	0.04%	0.05%	0.00%	0.02%	0.02%	0.03%	0.00%	0.01%	0.02%	0.02%
0.28	0.01%	0.08%	0.09%	0.11%	0.01%	0.06%	0.07%	0.08%	0.00%	0.03%	0.03%	0.04%	0.00%	0.02%	0.03%	0.04%
0.29	0.02%	0.12%	0.15%	0.17%	0.01%	0.09%	0.11%	0.13%	0.00%	0.04%	0.05%	0.07%	0.00%	0.03%	0.04%	0.06%
0.3	0.03%	0.19%	0.22%	0.27%	0.02%	0.14%	0.17%	0.20%	0.01%	0.06%	0.08%	0.11%	0.00%	0.05%	0.07%	0.09%
0.31	0.05%	0.28%	0.34%	0.40%	0.03%	0.21%	0.25%	0.30%	0.01%	0.09%	0.12%	0.16%	0.01%	0.07%	0.10%	0.14%
0.32	0.07%	0.40%	0.48%	0.58%	0.05%	0.29%	0.36%	0.43%	0.02%	0.13%	0.18%	0.24%	0.01%	0.10%	0.14%	0.20%
0.33	0.10%	0.56%	0.68%	0.82%	0.07%	0.41%	0.50%	0.61%	0.02%	0.18%	0.25%	0.33%	0.01%	0.14%	0.20%	0.28%
0.34	0.14%	0.76%	0.92%	1.12%	0.09%	0.55%	0.68%	0.83%	0.03%	0.25%	0.33%	0.45%	0.02%	0.19%	0.27%	0.38%
0.35	0.18%	1.01%	1.23%	1.50%	0.13%	0.73%	0.90%	1.11%	0.04%	0.32%	0.44%	0.60%	0.02%	0.26%	0.36%	0.50%
0.4	0.63%	3.18%	3.95%	4.97%	0.43%	2.28%	2.85%	3.61%	0.13%	0.98%	1.35%	1.91%	0.07%	0.76%	1.09%	1.60%
0.5	2.98%	13.4%	17.5%	23.6%	1.97%	9.41%	12.3%	16.5%	0.57%	3.84%	5.56%	8.40%	0.30%	2.96%	4.41%	6.99%
0.6	7.60%	31.6%	43.2%	63.1%	4.95%	21.9%	29.7%	42.6%	1.36%	8.59%	12.9%	20.8%	0.71%	6.53%	10.1%	17.3%
0.7	14.1%	55.5%	79.1%	125.0%	9.08%	38.1%	53.6%	81.5%	2.41%	14.5%	22.5%	38.7%	1.23%	10.9%	17.5%	32.1%
0.8	21.9%	82.5%	121.9%	207.8%	13.9%	56.3%	81.4%	131.1%	3.60%	21.0%	33.4%	60.9%	1.82%	15.7%	25.8%	50.6%
0.9	30.2%	110.4%	168.3%	308.5%	19.1%	75.1%	111.2%	188.7%	4.84%	27.6%	44.8%	86.0%	2.42%	20.5%	34.4%	71.6%
1	38.6%	138.1%	216.0%	423.7%	24.4%	93.6%	141.5%	251.8%	6.09%	34.1%	56.2%	113.0%	3.02%	25.1%	43.0%	94.4%
1.5	76.6%	256.2%	435.8%	1129.9%	48.1%	172.6%	278.7%	592.8%	11.5%	61.0%	106.3%	254.9%	5.60%	44.3%	80.4%	218.0%
2	104.5%	339.1%	603.1%	1914.4%	65.5%	228.2%	381.2%	910.1%	15.5%	79.7%	142.9%	383.5%	7.45%	57.4%	107.6%	335.8%
3	139.7%	440.9%	820.5%	3427.0%	87.6%	296.4%	512.8%	1405.2%	20.5%	102.5%	189.4%	581.3%	9.76%	73.4%	142.1%	528.8%
4	160.4%	499.6%	951.0%	4755.5%	100.5%	335.8%	591.1%	1753.3%	23.5%	115.6%	216.9%	719.2%	11.1%	82.5%	162.5%	672.4%

and the size of the loss of goodwill. To the extent of our knowledge, this study is conducted for the first time in the context of the experimental framework which we are establishing in the current paper.

The behavior of the optimal order quantity in changes of the coefficient of variation is differentiated accordingly if the critical fractile is less or greater than 0.5. When the critical fractile is below 0.5, then as the coefficient of variation is getting larger, the optimal order quantity initially decreases, and then there is a turning point after which it is starting to increase. This turning point corresponds to a value of the coefficient of variation which becomes smaller as the critical fractile is approaching 0.5. If on the other hand the critical fractile is greater than 0.5, the optimal order quantity is an increasing function of the coefficient of variation.

For each critical fractile we showed that there is a range of values of the loss of goodwill where maximum expected profit is positive for any finite value of the coefficient of variation. For different values of the loss of goodwill within this range, we examined the changes of the maximum expected profit for different values of the critical fractile. We found out that no matter if the critical fractile is less or greater than 0.5, maximum expected profits are reducing with increasing the coefficient of variation. On the other hand, given the value of the coefficient of variation, maximum expected profits become larger as the critical fractile is increasing.

Finally, when the size of the coefficient of variation is not very small, we examined how well the use of the non-truncated normal distribution can approximate the values of the optimal order quantity and the maximum expected profit. As criterion of evaluation we used the approximation error as percentage of the exact value of either the optimal order quantity or the maximum expected profit. For both target inventory measures, this relative approximation error depends on the coefficient of variation and the critical fractile. For the maximum expected profit, it depends also upon the loss of goodwill.

From the results which were obtained, we concluded that it is too simple to suggest a maximum flat value for the coefficient of variation under which the use of the non-truncated normal distribution can give accurate approximations for the two target inventory measures. The reason is that the size of the relative approximation error differs among the two target inventory measures. When the loss of goodwill is zero, maximum expected profit gives higher relative approximation errors than the optimal order quantity. When the critical fractile is increasing, the relative approximation error for both target inventory measures is reducing. And, as the loss of goodwill is rising, then the relative approximation error for the maximum expected profit is making larger.

Closing this paper, for using the non-truncated normal distribution, we recommend researchers or practitioners first to specify the size of the approximation error for either the optimal order quantity or the maximum expected profit. For the maximum expected profit, the knowledge of the loss of goodwill is necessary. And, unfortunately, this is not easy to know in real life situations. Then, using the formulae of the relative approximation errors, which we offer, or alternatively from the data of table 4, to determine the maximum value of the coefficient of variation under which the use of the non-truncated normal distribution is accepted.

Appendix

Proof of (1)

Using formula (1) of Khouza (1999), we give the following alternative form of the profit function,

$$\pi = \begin{cases} (p-c)Q-(p-v)(Q-X^{+}) & \text{if } X^{+} \le Q \\ (p-c)Q+s(Q-X^{+}) & \text{if } X^{+} > Q \end{cases}. \tag{A1}$$

The expected value of (A1) is

$$\xi = \int_0^Q \{(p-c)Q-(p-v)(Q-x)\} f^{+}(x)dx + \int_Q^\infty \{(p-c)Q+s(Q-x)\} f^{+}(x)dx =$$

$$= (p-c+s)Q-(p-v+s)Q\int_0^Q f^{+}(x)dx + (p-v+s)\int_0^Q xf^{+}(x)dx - s\int_0^\infty xf^{+}(x)dx. \tag{A2}$$

The first integral in (A2) is given in Hu and Manson (2011) as

$$\int_0^Q f^{+}(x)dx = \frac{\Phi_z-\Phi_{-\theta}}{1-\Phi_{-\theta}} = 1-\frac{1-\Phi_z}{\Phi_\theta}. \tag{A3}$$

To find the second integral, we take the intermediate result by setting $v = u^2/2$,

$$\int_{-\theta}^{z} u\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \int_{-\frac{-\theta^2}{2}}^{-\frac{-z^2}{2}} \frac{1}{\sqrt{2\pi}}e^{v}dv = -(\phi_z-\phi_\theta). \tag{A4}$$

Then, using (A3) and (A4)

$$\int_0^Q xf^{+}(x)dx = \frac{1}{\Phi_\theta}\int_{-\theta}^{z}(\mu+\sigma u)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du =$$

$$= \mu\int_0^Q f^{+}(x)dx + \frac{\sigma}{\Phi_\theta}\int_{-\theta}^{z} u\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \mu\left(1-\frac{1-\Phi_z}{\Phi_\theta}\right)-\sigma\frac{\phi_z-\phi_\theta}{\Phi_\theta}. \tag{A5}$$

The last integral in (A2) is the expected value of X^{+}, which is obtained from formula (13.134) of Johnson et al. (1994, p.156) as

$$\int_0^\infty x f^+(\mathrm{x})\mathrm{dx} = \mu + \sigma\frac{\phi_\theta}{\Phi_\theta}. \tag{A6}$$

Replacing (A3), (A5), and (A6) into (A2),

$$\xi = (\mathrm{p}-\mathrm{c})\mathrm{Q} - (\mathrm{p}-\mathrm{v})\mathrm{Q} + (\mathrm{p}-\mathrm{v}+\mathrm{s})(\mathrm{Q}-\mu)\frac{1-\Phi_\mathrm{z}}{\Phi_\theta} - (\mathrm{p}-\mathrm{v}+\mathrm{s})\mu -$$

$$-(\mathrm{p}-\mathrm{v}+\mathrm{s})\sigma\frac{\phi_\mathrm{z}-\phi_\theta}{\Phi_\theta} - \mathrm{s}\mu - \mathrm{s}\sigma\frac{\phi_\theta}{\Phi_\theta},$$

and finally

$$\xi = (\mathrm{p}-\mathrm{c})\mathrm{Q} - (\mathrm{p}-\mathrm{v})\{(\mathrm{Q}-\mu-\sigma\cdot\omega)\} + (\mathrm{p}-\mathrm{v}+\mathrm{s})\left\{(\mathrm{Q}-\mu)\frac{1-\Phi_\mathrm{z}}{\Phi_\theta} - \sigma\frac{\phi_\mathrm{z}}{\Phi_\theta}\right\}.$$

Proof of proposition 1

(i) $$\lim_{\mathrm{CV}\to 0} h = 1-(1-\mathrm{R})\lim_{\theta\to\infty}\Phi_\theta = \mathrm{R}, \tag{A7}$$

$$\lim_{\mathrm{CV}\to\infty} h = 1-(1-\mathrm{R})\lim_{\theta\to 0}\Phi_\theta = \frac{\mathrm{R}+1}{2}, \tag{A8}$$

and

$$\frac{\mathrm{d}h}{\mathrm{dCV}} = \frac{\mathrm{d}}{\mathrm{dCV}}\left[1-(1-\mathrm{R})\Phi_\theta\right] = -(1-\mathrm{R})\frac{\mathrm{d}\Phi_\theta}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dCV}} = (1-\mathrm{R})\theta^2\phi_\theta > 0. \tag{A9}$$

as $\frac{\mathrm{d}\theta}{\mathrm{dCV}} = -\frac{1}{\mathrm{CV}^2} = -\theta^2$.

(ii) $$\frac{\mathrm{d}^2 h}{\mathrm{dCV}^2} = (1-\mathrm{R})\theta^2\frac{\mathrm{d}\phi_\theta}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dCV}} + (1-\mathrm{R})\phi_\theta\frac{\mathrm{d}\theta^2}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dCV}} =$$

$$= \frac{(1-\mathrm{R})\theta^2\phi_\theta}{\mathrm{CV}^2} - \frac{2(1-\mathrm{R})\theta\phi_\theta}{\mathrm{CV}^2} =$$

$$= \frac{(1-\mathrm{R})\phi_\theta}{\mathrm{CV}^3}\left(\frac{1}{\mathrm{CV}^2} - 2\right).$$

Hence $\dfrac{d^2 h}{dCV^2} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} 0$ when $CV \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \dfrac{\sqrt{2}}{2}$.

Proof of proposition 2

Using (A9), we take $\dfrac{dz_h}{dCV} = \dfrac{dz_h}{dh}\dfrac{dh}{dCV} = \dfrac{(1-R)\theta^2\phi_\theta}{\phi_{z_h}}$ and

$$\frac{d}{dCV}\left(\frac{dg}{dCV}\right) = \frac{dz_h}{dCV} + (1-R)\frac{\phi_\theta}{\phi_{z_h}}\frac{d\theta}{dCV} + (1-R)\theta\frac{d}{dCV}\left(\frac{\phi_\theta}{\phi_{z_h}}\right) =$$

$$= (1-R)\theta\frac{d}{dCV}\left(\frac{\phi_\theta}{\phi_{z_h}}\right).$$

But

$$\frac{d}{dCV}\left(\frac{\phi_\theta}{\phi_{z_h}}\right) = \phi_{z_h}^{-1}\frac{d\phi_\theta}{d\theta}\frac{d\theta}{dCV} + \phi_\theta\frac{d\phi_{z_h}^{-1}}{d\phi_{z_h}}\frac{d\phi_{z_h}}{dz_h}\frac{dz_h}{dh}\frac{dh}{dCV} =$$

$$= \phi_{z_h}^{-1}(-\theta\phi_\theta)(-\theta^2) + \phi_\theta(-\phi_{z_h}^{-2})(-z_h\phi_{z_h})\phi_{z_h}^{-1}(1-R)\theta^2\phi_\theta =$$

$$= \theta^3\frac{\phi_\theta}{\phi_{z_h}} + (1-R)\theta^2 z_h\left(\frac{\phi_\theta}{\phi_{z_h}}\right)^2. \qquad \text{(A10)}$$

Hence

$$\frac{d^2 g}{dCV^2} = (1-R)\theta^3\frac{\phi_\theta^2}{\phi_{z_h}}\left\{\frac{\theta}{\phi_\theta} + (1-R)\frac{z_h}{\phi_{z_h}}\right\} > 0 \qquad \text{(A11)}$$

The positive sign of (A11) is explained as follows. When $h \geq 0.5$, then $z_h \geq 0$ and hence $d^2 g/dCV^2$ is positive. On the other hand, when $h < 0.5$, then $z_h < 0$ and thus the expression inside the brackets of (A11) is made up of two terms with the first one to be positive and the second one negative. Even in this case, the net result of the sum of the two terms in brackets is positive, and this is explained by using the following relationships which are deduced from the properties of the standard normal curve:

(a) As $h > \mathrm{R}$, it holds that $(1-\mathrm{R})|z_h| < |z_h| < |z_{\mathrm{R}}|$.

(b) Since $-2\mathrm{R}^2 + 2\mathrm{R} - 1 < 0$, we take $\frac{1}{2(1-\mathrm{R})} > \mathrm{R}$, and $|z_{\mathrm{R}}| < z_{1/2(1-\mathrm{R})}$.

(c) If $h = 1-(1-\mathrm{R})\Phi_\theta < 0.5$, it follows that $\Phi_\theta = \Pr(Z \le \theta) < \frac{1}{2(1-\mathrm{R})}$ and $z_{1/2(1-\mathrm{R})} < \theta$.

From (a), (b) and (c), we reach the inequalities $(1-\mathrm{R})|z_h| < |z_h| < \theta$ and $\phi_{z_h} > \phi_\theta$ from which we obtain $(1-\mathrm{R})\frac{|z_h|}{\phi_{z_h}} < \frac{\theta}{\phi_\theta}$. Thus the expression inside the brackets of (A11) is also positive when $h < 0.5$.

The proof is completed by taking the following limits using (A7) and (A8):

$$\lim_{cv\to 0}\frac{dg}{d\mathrm{CV}} = z_{\lim\limits_{cv\to 0} h} + (1-\mathrm{R})\left(\lim_{\theta\to\infty}\theta\phi_\theta\right)\frac{1}{\phi_{z_{\lim\limits_{\mathrm{CV}\to 0} h}}} = z_{\mathrm{R}} + (1-\mathrm{R})(\infty\cdot 0)\frac{1}{\phi_{z_{\mathrm{R}}}} = z_{\mathrm{R}} < 0$$

$$\lim_{cv\to \infty}\frac{dg}{d\mathrm{CV}} = z_{\lim\limits_{cv\to \infty} h} + (1-\mathrm{R})\left(\lim_{\theta\to 0}\theta\phi_\theta\right)\frac{1}{\phi_{z_{\lim\limits_{\mathrm{CV}\to \infty} h}}} = +(1-\mathrm{R})\left(\frac{0}{\sqrt{2\pi}}\right)\frac{1}{\phi_{z_{(\mathrm{R}+1)/2}}} = z_{(\mathrm{R}+1)/2} > 0$$

Proof of proposition 3

From (A10) and (A11),

$$\frac{du}{d\mathrm{CV}} = \frac{d}{d\mathrm{CV}}\left(\frac{\phi_{z_h}}{\phi_\theta}\right) = \frac{d}{d\mathrm{CV}}\left(\frac{\phi_\theta}{\phi_{z_h}}\right)^{-1} = (-1)\left(\frac{\phi_\theta}{\phi_{z_h}}\right)^{-2}\frac{d}{d\mathrm{CV}}\left(\frac{\phi_\theta}{\phi_{z_h}}\right) =$$

$$= -\left(\frac{\phi_{z_h}}{\phi_\theta}\right)^2\left\{\theta^3\frac{\phi_\theta}{\phi_{z_h}} + (1-\mathrm{R})\theta^2 z_h\left(\frac{\phi_\theta}{\phi_{z_h}}\right)^2\right\} =$$

$$= -\theta^2\phi_{z_h}\left\{\frac{\theta}{\phi_\theta} + (1-\mathrm{R})\frac{z_h}{\phi_{z_h}}\right\} < 0.$$

Using (A7) and (A8),

$$\lim_{CV\to 0} u = \frac{\lim\limits_{CV\to 0} \phi_{z_h}}{\lim\limits_{\theta\to\infty} \phi_\theta} - (1-R) = \frac{\phi_{z_{\lim\limits_{CV\to\infty} h}}}{\phi_{\lim\limits_{\theta\to\infty}\theta}} - (1-R) = \frac{\phi_{z_R}}{0} - (1-R) = \infty\,,$$

$$\lim_{CV\to\infty} u = \frac{\lim\limits_{CV\to\infty} \phi_{z_h}}{\lim\limits_{\theta\to 0} \phi_\theta} - (1-R) = \frac{\phi_{z_{\lim\limits_{CV\to\infty} h}}}{\phi_{\lim\limits_{\theta\to 0}\theta}} - (1-R) = \sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R).$$

To prove that $\lim\limits_{CV\to\infty} u$ is positive, we take the following limits

$$\lim_{R\to 0}\left\{\sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R)\right\} = \sqrt{2\pi}\phi_{z_{0.5}} - 1 = 0\,,$$

$$\lim_{R\to 1}\left\{\sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R)\right\} = \sqrt{2\pi}\phi_{z_1} = \sqrt{2\pi}\phi_\infty = 0\,.$$

Further,

$$\frac{d}{dR}\left\{\sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R)\right\} = \sqrt{2\pi}\,\frac{d\phi_{z_{\frac{1+R}{2}}}}{dz_{\frac{1+R}{2}}}\,\frac{dz_{\frac{1+R}{2}}}{d\left(\frac{1+R}{2}\right)}\,\frac{d\left(\frac{1+R}{2}\right)}{dR} + 1 = 1 - z_{\frac{1+R}{2}}\sqrt{\frac{\pi}{2}}\,,$$

and

$$\frac{d^2}{dR^2}\left\{\sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R)\right\} = -\sqrt{\frac{\pi}{2}}\,\frac{dz_{\frac{1+R}{2}}}{d\left(\frac{1+R}{2}\right)}\,\frac{d\left(\frac{1+R}{2}\right)}{dR} = -\sqrt{\frac{\pi}{8}}\left(\phi_{z_{\frac{1+R}{2}}}\right)^{-1} < 0\,.$$

Hence $\sqrt{2\pi}\phi_{z_{(1+R)/2}} - (1-R) > 0$, since this function is concave for any R, and its lower and upper limit is zero.

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