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20 April 2012

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April 20, 2012

Abstract

After deriving a model describing the law of evolution of innovators and imitators the article focuses on their relationships under two different scenarios: prey-predator, in which innovators are regarded as preys, and competing species. Analytic results show that among the feasible equilibria the coexistence equilibrium is the only stable equilibrium under the first scenario. We also find conditions on the parameters allowing local stability of the coexistence equilibrium in the second scenario. Such conditions imply the existence of an inverse-U shaped relationship between innovation and imitation.

Keywords: imitation, innovation, intellectual property rights, Lotka-Volterra system
JEL Classification: O31; O33; O34; C6

1 Introduction

The controversial issue of protection of intellectual property rights, IPR, or more generally innovation, has been long debated and actually it does not seem to come at an end. The major reason of controversy emerges because of the presence of a trade-off that the government faces when has to decide the IPR degree of protection. In turn, the trade-off emerges from the public good nature of ideas.

At one extreme of the trade-off there are the alleged benefits that IPR convey to society by preventing possible underinvestment, thereby fostering economic growth. In particular, stronger IPR are said to stimulate innovation by protecting innovators from imitation. Indeed, many countries have put into place more effective or rigorous protection policies, such as the establishment of the Court of Appeals of the Federal Circuit by US Congress and the EU directive 2004/48 on the enforcement of intellectual property rights.
At the other extreme, enhanced protection can have negative effects on growth by creating monopolies. Many works indicates that the relationship between IPR protection and economic growth is actually not so clear. Helpman (1993) allowing for exogenous imitation with an innovative country and an imitative country shows that strengthening IPR in an

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imitative country does not necessarily stimulate innovations in innovative country in the long run. More recently, Horii and Iwaisako (2007) use the average growth rate from 1966 to 2000 of some countries to indicate that it is difficult to find a positive relationship between IPR protection and the growth rate, and Gould and Gruden (1996) show a positive but 'weak' relationship between them. To explain this fact, Horii and Iwaisako (2007) construct a quality ladder model where strengthened protection can depress the incentive to innovate. Koléda (2005) shows that the effect of patent novelty requirements on growth can be inverse-U shaped, which implies that tightening the IPR protection dampens economic growth for a range of stronger novelty requirements. Similarly, Furukawa (2007) presents a mechanism through which stronger IPR protection depresses economic growth by focusing on learning by experience. More specifically, under some circumstances, the rate of innovation has an inverse-U shape as a function of imitation, which is an inverse measure of IPR protection. This theoretical finding is also supported by Aghion et al. (2005) who find strong evidence of an inverted-U relationships between competition and innovation. At its very essence this strand of the literature implies that relaxing IPR can be a growth enhancing policy.

One of the way followed in the literature to model the degree of IPR protection is to introduce imitation via an exogenous imitation rate, competing with innovation (Helpman 1993, Lai 1998, Cysne and Turchick 2012). In this paper we take the stance of modelling the relationships between innovation and imitation from a behavioral point of view. Imitation is regarded as speeding up the rate of innovation diffusion and is an inverse measure of IPR protection. Tse (2002) shows that a longer diffusion lag, while improving the appropriability of R&D investment, lowers the productivity of R&D. There is thus a fundamental conflict between productivity and appropriability of R&D investment. A timely knowledge diffusion is important at least for two reasons. First, consumers can access new or qualitatively improved products. Second, with a substantial lag length firms may only access the less updated knowledge embodied in the older blueprints produced by others. As a result, a slow diffusion hampers the birth rate of innovation itself[1].

In our view imitation can accidentally mitigate the diffusion lag. In the model innovators and imitators are regarded as competing for the same asset and entry the market requires undergoing sunk costs. Expanding the methodology proposed by Dixit and Pindyck (1994) we derive the dynamics of innovators and imitators in the market under two different scenarios: prey-predator, in which innovators are regarded as preys, and competition. More precisely, we derive the joint dynamics of imitators and innovators as a Lotka-Volterra system. The scenarios are then obtained through an appropriate selection of the variation range of the parameters of the system. Analytical results show that among the feasible equilibria the coexistence equilibrium -i.e.: the equilibrium associated to the simultaneous existence of innovators and imitators in the long run - is the only stable under the prey-predator scenario. While, in case of competition, the coexistence equilibrium is the unique locally stable if the IPR degree of protection lies in an interval. The results are in line with the literature claiming that stronger IPR protection is not always the best possible choice and confirm the inverse-U shaped relationship between innovation and competition.

The organization of the paper is as follows. Section 2 presents the set up of the model. The derivation of innovators and imitators entry rates follow in Section 3 and the solution of the system is presented in Section 4. Section 5 and 6 analyze the case of prey-predator and competing species, respectively, analytically and numerically. Finally, Section 7 concludes.

[1] Caballero and Jaffe (1993) estimate that the median lag between a cited patent and the citing patent is 9-10 years.

2 The model

Consider an industry composed of Q firms that can be either innovators or imitators. Any one firm's inverse demand function is of the type:

$$P_t = Y_t D(Q), \qquad t > 0, \tag{1}$$

where P_t is the price faced by the firm at time t, D is the demand function and it is assumed to be decreasing with respect to Q so that the price falls if, *caeteris paribus*, the number of innovators and/or imitators increases, consistently with both the increase in aggregate supply and in competition. Y_t can be interpreted as an idiosyncratic demand shock ultimately capturing a shift to profitability at time t. It can be thought of as reflecting random fluctuations of taste or a technology shock. The important point is that such a shock can be the source of a competitive advantage that allows new entrants to act as innovators, either because they are innovators themselves, or because they imitate innovators, and enter the industry.

By paying an entry cost R, any firm can get an initial draw Y_0 of its demand shock Y_t from a known distribution. Thereafter $\{Y_t\}_{t>0}$ will follow a geometric Brownian motion process that is firm-specific, or independent across firms:

$$dY_t = \alpha Y_t dt + \sigma Y_t dz_t \qquad t > 0, \tag{2}$$

where $\{z_t\}_{t>0}$ is a standard Brownian motion while $\alpha \in \mathbb{R}$ and $\sigma > 0$ represent the drift and the diffusion rate of the stochastic process $\{Y_t\}_{t>0}$, respectively. Furthermore, we will also assume that each firm can start actual operation by paying a further investment cost I.

After the payment of the cost R, a firm observes the value Y_0. If Y_0 exceeds a critical threshold $Y^{(N)}$, the firm pays the investment cost I and becomes an active producer. Otherwise it lets $\{Y_t\}_{t>0}$ evolve and activates if and when $Y^{(N)}$ is reached.

Example 1. *As an illustrative example, R can be representative of a situation where a pharmaceutical company can develop a new drug by incurring the research cost. The firm patents the drug, but unless the profit estimate is sufficiently high, it will not incur the additional investment expenditure I that is necessary to begin production.*

Analogously, imitators can be thought of as incurring a fixed investment cost, K, for research and imitation in order to enter the market and appropriate a share of the innovators' income. The activation decision is made when the expected income randomly fluctuating reaches a threshold, $Y^{(M)}$, that is proportional to K[2].

Let us denote as N_t, M_t the number of innovators and imitators at time t that will reach the activation decision, respectively. Moreover, let us assume that a Poisson death process[3] at rate λ ensures that the number of firms is constant in the long run, i.e. $N + M$ must equal λQ. Following the method elaborated by Dixit and Pindyck (1994) it is possible to determine the rate at which N and M enter the market. This method is based on the idea that for industry equilibrium only the total number of firms in various states matter, i.e. how many are active, and how many are waiting for $Y^{(N)}$ or $Y^{(M)}$.

[2] One of the main results of the real options literature is to prove the inadequacy of the Net Present Value criteria, NPV, for the decision to invest under uncertainty when fixed costs must be incurred. In this case, the hurdle rate is greater than that required by the NPV and the critical entry point in terms of expected future gain is a multiple of fixed costs. Such a critical value increases as uncertainty surrounding the gain from investment increases, σ, and decreases as the expected growth rate α of the gain increases.

[3] By death or mortality rate it is meant the rate at which firms exit the market.

3 Derivation of the entry rates

This section derives the rates at which innovators and imitators activate. We adopt and extend to our case the methodology proposed by Dixit and Pindyck (1994).
Let us define the process $\{w_t\}_{t>0} = \{log(Y_t)\}_{t>0}$ and the threshold $w^{(N)} = \log(Y^{(N)})$, where $\{Y_t\}_{t>0}$ is a geometric Brownian motion as in (2). Applying Ito's lemma it can be easily verified that the dynamics of $\{w_t\}_{t>0}$ follows a Brownian motion of the type

$$dw_t = \theta dt + \sigma dz, \qquad t > 0,$$

with $\theta = \alpha - \frac{\sigma^2}{2}$. Let us start with the innovation problem and consider the case of N would-be innovators. Let $g(w)$ be the density function of the initial value w_0 of $\{w_t\}_{t\geq 0}$ and $G(w)$ the corresponding distribution function, with $w \in \mathbb{R}$.
The agents who are candidates for the innovation are distributed continuously between the (log of the) minimum value of the expected gain -i.e.: 0- and the (log of the) innovation threshold value -i.e.: $Y^{(N)}$. Hence, the agents are distributed in the interval $(-\infty, w^{(N)}]$.
Now, let us introduce the function $\phi : (-\infty, w^{(N)}] \to (0, 1]$ such that $N'_t\phi(w)$ is the density of these agents at location w, where $N'_t = \frac{dN_t}{dt}$, being dN_t and dt small variations of N_t and t, respectively. For notational simplicity we will omit the index t hereafter.
For the density to be stable over time, we must have that the agents leaving (to the right) the location because of payoff increases be exactly counterbalanced by agents arriving in the location because of payoff decreases. In order to formalize this condition, we denote as $dh = \sigma\sqrt{dt}$ a small variation of net gain w. Some of the firms located in dh will die by a proportion λdt. Among the survivors, a fraction p will move to the right, i.e. will activate, and a fraction $1-p$ will move to the left, i.e. will exit the market. The parameter p can be found using the binomial approximation of the Brownian motion, namely $p = \frac{1}{2}\left(1 + \frac{\theta\sqrt{dt}}{\sigma}\right)$.
Therefore, the stability condition can be written as follows:

$$N'\phi(w)dh = N'dt\ g(w)dh + p(1-\lambda dt)\ N'\phi(w-dh)dh + (1-p)(1-\lambda dt)\ N'\phi(w+dh)\ dh \quad (3)$$

Rearranging the terms of (3) and using a Taylor expansion, expression (3) becomes a differential equation with constant coefficients and variable term:

$$\frac{\sigma^2}{2}\phi''(w) - \theta\phi'(w) - \lambda\phi(w) + g(w) = 0. \quad (4)$$

The general solution of equation (4) is of the type:

$$\phi(w) = C_1 \exp(b_1 w) + C_2 \exp(b_2 w) + \phi_0(w) \quad (5)$$

where C_1 and C_2 will be pinned down, as we will see shortly, through the fulfilment of some boundary conditions and b_1 and b_2 are the roots of the characteristic equation

$$\frac{\sigma^2}{2}b^2 - \theta b - \lambda = 0, \quad (6)$$

i.e.:

$$b_1 = \frac{\theta + \sqrt{\theta^2 + 2\lambda\sigma^2}}{\sigma^2} > 0, \qquad b_2 = \frac{\theta - \sqrt{\theta^2 + 2\lambda\sigma^2}}{\sigma^2} < 0$$

while ϕ_0 is a particular solution to the differential equation in (4), and it vanishes as $w \to -\infty$.

For analytical tractability, let us assume now that the distribution of the initial payoff, Y_0, is uniform over a range $(0, Y^*)$. In this case, the logarithm of the initial payoff, w_0, is distributed according to an exponential function: $g(w) = \exp(w - w^*) = G(w)$, with $w^* = log(Y^*)$, meaning that the initial draw of $\{w_t\}_{t>0}$ is taken from an exponential random variable. In this case, it can be easily verified that a specific solution ϕ_0 is given by:

$$\phi_0(w) = \frac{\exp(w - w^*)}{\lambda + \theta - \frac{\sigma^2}{2}} \tag{7}$$

with $\frac{\sigma^2}{2} < \lambda + \theta$ to make economic sense. This condition, along with $\sigma^2 > \theta$, implies the positive root of the quadratic expression to be greater than unity, $b_1 > 1$, a condition that we will always assume. The definition of ϕ provides us with the boundary conditions, that are:

$$\begin{cases} \lim\limits_{w \to -\infty} \phi(w) = 0; \\ \\ \phi(w^{(N)}) = 0. \end{cases} \tag{8}$$

The first line in (8) is a simple restatement of the fact that a probability distribution function takes value zero at its lower limit. The second line is due to the fact that the mass of waiting firms is zero at the entry threshold. By exploiting the boundary conditions in (8) we can determine the two constants as:

$$\begin{cases} C_2 = 0; \\ \\ C_1 = -\frac{\exp\left[w^{(N)}(1-b_1) - w^*\right]}{\left(\lambda + \theta - \frac{\sigma^2}{2}\right)}. \end{cases} \tag{9}$$

By substituting (7) and (9) into (5) we finally get the definite solution of the differential equation, i.e.:

$$\phi(w) = -\frac{\exp\left[w^{(N)}(1 - b_1) - w^* + b_1 w\right] - \exp(w - w^*)}{\left(\lambda + \theta - \frac{\sigma^2}{2}\right)}. \tag{10}$$

The rate of activation of the innovators is the rate at which waiting firms hit $w^{(N)}$. This is given by the fraction $p(1 - \lambda dt)$ of the firms located just at the left of $w^{(N)}$. Using a Taylor expansion and by using (7), (8) and (10), we obtain:

$$\begin{aligned} &\frac{1}{2}N' \left[\phi(w^{(N)}) - \phi'(w^{(N)})dh\right] dh = -\frac{1}{2}N'\phi'(w^{(N)})(dh)^2 = \\ &= -\frac{\sigma^2}{2}N'\phi'(w^{(N)})dt = N'\frac{\frac{\sigma^2}{2}(b_1 - 1)}{\left(\lambda + \theta - \frac{\sigma^2}{2}\right)} \exp(w^{(N)} - w^*), \end{aligned} \tag{11}$$

being the last term equal to the rate of activation, since these firms activate in the time interval dt.

The smaller the entry threshold for innovation, the smaller the number of firms waiting to enter. In the special case in which the threshold is zero, in particular, the rate of entry will be a minimum, since all firms will have already entered and no firm would be to the left of the entry threshold. Conversely, in the case in which the threshold is equal to the upper

bound of the range, the number of waiting firms will be a maximum and equal to N (all firms will be to the left of the threshold).
For a stationary population the total number of exits must equate the number of entries. The latter is given by the sum of those firms who find their w_0 greater than the entry threshold, $N'[(1 - G(w^{(N)})]$, plus the activation flow found in (11). Therefore we must have:

$$\lambda N = N' \left[1 - G(w^{(N)}) - \frac{\sigma^2}{2} \phi'(w^{(N)}) \right] = N' \left[1 - \exp(w^{(N)} - w^*) \frac{\lambda + \theta - \frac{\sigma^2}{2} b_1}{\left(\lambda + \theta - \frac{\sigma^2}{2} \right)} \right]. \quad (12)$$

For notational convenience, we define $\dot{n} \equiv \frac{N'}{N}$ and $\dot{m} \equiv \frac{M'}{M}$. Solving for $\dot{n}$ yields:

$$\dot{n} = \frac{\lambda \left(\lambda + \theta - \frac{\sigma^2}{2} \right)}{\left(\lambda + \theta - \frac{\sigma^2}{2} \right) - e^{(w^{(N)} - w^*)} \left(\lambda + \theta - \frac{\sigma^2}{2} b_1 \right)} = \frac{\lambda \Gamma}{\Gamma - \frac{Y^{(N)}}{Y^*}} \quad (13)$$

with

$$\Gamma \equiv \frac{\left(\lambda + \theta - \frac{\sigma^2}{2} \right)}{\left(\lambda + \theta - \frac{\sigma^2}{2} b_1 \right)} > 1$$

Using a similar method for M imitators, assuming a uniform distribution over the range $w^{(M)} - w^*$ we can conclude that their relative rate of entry is:

$$\dot{m} = \frac{\lambda \left(\lambda + \theta - \frac{\sigma^2}{2} \right)}{\left(\lambda + \theta - \frac{\sigma^2}{2} \right) - \frac{Y^{(M)}}{Y^*} \left(\lambda + \theta - \frac{\sigma^2}{2} b_1 \right)} = \frac{\lambda \Gamma}{\Gamma - \frac{Y^{(M)}}{Y^*}} \quad (14)$$

From (13) and (14) appears clearly that the rates of entry are non-linear positive functions of the thresholds $Y^{(N)}$ and $Y^{(M)}$, respectively of innovators and imitators, $\frac{d\dot{n}}{dY^{(N)}} > 0$; $\frac{d\dot{m}}{dY^{(M)}} > 0$.
It is worth highlighting that the thresholds $Y^{(N)}$ and $Y^{(M)}$ cannot be independent from the number of competitors, both innovators and imitators. Therefore, we can write $Y^{(N)} = Y^{(N)}(N, M)$, $Y^{(M)} = Y^{(M)}(N, M)$. Taking a first order Taylor expansion of $\dot{n}$ and $\dot{m}$ around an arbitrary point (N_0, M_0) equations (13) and (14) can be rewritten as:

$$\begin{cases} \dot{n} = \dot{n} \mid_{M=M_0, N=N_0} + \frac{d\dot{n}}{dN} \mid_{M=M_0, N=N_0} (N - N_0) + \frac{d\dot{n}}{dM} \mid_{M=M_0, N=N_0} (M - M_0) \\ \dot{m} = \dot{m} \mid_{M=M_0, N=N_0} + \frac{d\dot{m}}{dN} \mid_{M=M_0, N=N_0} (N - N_0) + \frac{d\dot{m}}{dM} \mid_{M=M_0, N=N_0} (M - M_0). \end{cases} \quad (15)$$

By applying the chain rule $\frac{d\dot{n}}{dN} = \frac{\partial \dot{n}}{\partial Y^{(N)}} \frac{\partial Y^{(N)}}{\partial N}$, $\frac{d\dot{n}}{dM} = \frac{\partial \dot{n}}{\partial Y^{(N)}} \frac{\partial Y^{(N)}}{\partial M}$ into the Taylor expansion for $\dot{n}$ in (15), we obtain:

$$\dot{n} = \dot{n}_0 + \frac{\partial \dot{n}}{\partial Y^{(N)}} \left[\frac{\partial Y^{(N)}}{\partial N} (N - N_0) + \frac{\partial Y^{(N)}}{\partial M} (M - M_0) \right] =$$

$$= \dot{n}_0 - \frac{\partial \dot{n}}{\partial Y^{(N)}} \left(\frac{\partial Y^{(N)}}{\partial N} N_0 + \frac{\partial Y^{(N)}}{\partial M} M_0 \right) + \frac{\partial \dot{n}}{\partial Y^{(N)}} \left(\frac{\partial Y^{(N)}}{\partial N} N + \frac{\partial Y^{(N)}}{\partial M} M \right), \quad (16)$$

where $\dot{n}_0 \equiv \dot{n}\mid_{M=M_0,N=N_0}$ and recalling that both the partial and the total derivatives are evaluated at the expansion point.
In order to avoid an explosive growth rate, from an economic point of view, it is reasonable to assume that $\frac{d\dot{n}}{dN} \leq 0$. In turn, by (13) this implies $\frac{\partial Y^{(N)}}{\partial N} \leq 0$, and thus we have found that the innovators' entry threshold is negatively related to the total number of active innovators. Put another way: the lower the threshold, the smaller the number of firms waiting to enter. This is consistent with the fact that new entrants are willing to accept a lower perspective profit when competition is more severe. As naturally imitators are detrimental to innovators, we must assume that $\frac{d\dot{n}}{dM} \leq 0$, and it follows that $\frac{\partial Y^{(N)}}{\partial M} \leq 0$.
Similarly, to avoid the imitators growth rate to go off $\frac{d\dot{m}}{dM} \leq 0$ is required, which implies $\frac{\partial Y^{(M)}}{\partial M} \leq 0$. Hereafter, we propose two alternative scenarios. In the first one, we consider that, as imitators thrive imitating, their entry rate must be a positive function of the number of active innovators. Hence, we assume that $\frac{d\dot{m}}{dN} \geq 0$ which implies $\frac{\partial Y^{(M)}}{\partial N} \geq 0$. Thereby, equation (16) can be rewritten as

$$\dot{n} = \underbrace{\dot{n}_0 - \left(\frac{d\dot{n}}{dN}N_0 + \frac{d\dot{n}}{dM}M_0\right)}_{\text{nonnegative}} + \underbrace{\frac{d\dot{n}}{dN}}_{\text{nonpositive}} N + \underbrace{\frac{d\dot{n}}{dM}}_{\text{nonpositive}} M \tag{17}$$

and

$$\dot{m} = \underbrace{\dot{m}_0 - \left(\frac{d\dot{m}}{dN}N_0 + \frac{d\dot{m}}{dM}M_0\right)}_{\text{undetermined}} + \underbrace{\frac{d\dot{m}}{dN}}_{\text{nonnegative}} N + \underbrace{\frac{d\dot{m}}{dM}}_{\text{nonpositive}} M \tag{18}$$

$$\begin{cases} \frac{N'}{N} = r - sN - fM \\ \\ \frac{M'}{M} = g + cN - eM \end{cases} \tag{19}$$

with $r \equiv \dot{n}_0 - \left(\frac{d\dot{n}}{dN}N_0 + \frac{d\dot{n}}{dM}M_0\right) > 0$; $s \equiv -\frac{d\dot{n}}{dN} > 0$; $f \equiv -\frac{d\dot{n}}{dM} > 0$; $e \equiv -\frac{d\dot{m}}{dM} > 0$ and, finally $c \equiv \frac{d\dot{m}}{dN}$ and $g \equiv \dot{m}_0 - \left(\frac{d\dot{m}}{dN}N_0 + \frac{d\dot{m}}{dM}M_0\right)$ that can be either positive or negative.

Equations in (19) can be stated as indicating that the rates are function of both the number of innovators and the number of imitators. Such a dependence shows how the evolution of one of the two populations is intrinsically connected to the dynamics of the other. Therefore, in studying the evolution of the two groups one must take into account how they interact. At a closer look (19) resembles very much to the prey-predator model elaborated by Volterra (1926) and popularized by Lotka (1956).

The canonical Lotka-Volterra system states that in each period innovators increase by a proportion of r and, at the same time, die out by "natural death" by a quadratic proportion $-sN_t^2$. If we consider the number of patents issued at any one time as proxy for the number of innovators, the quadratic term $-sN_t^2$ captures the high mortality rate reported by the literature on patent renewals (Shankerman 1998, Lanjouw et al 1998, Pakes and Simpson 1989, just to cite the most prominent). According to that strand of the literature, about 50% of the patents drop out before they reach age ten and only a negligible part of those remaining reaches the last year of life, the 20th. The last term of the first equation of (19) quantifies the rate of death induced by the coexistence of preys and predators. The greater the number of the preys the higher the possibility of hunting for predators, and the greater the number of predators the greater the number of victims.

According to this interpretation, the law of evolution of the imitators, namely the imitators' growth rate, is characterized by an exogenous mortality or birth rate, according to the sign g takes on, and by a birth rate as a function of the number of innovators, c. The term $-eM_t$ acts as reducing imitators' growth rate and can be thought of as capturing competition among imitators.

In the second scenario, we consider the cases in which innovators can somehow "hunt" imitators. In such cases (19) describes the evolution of two competing species. This occurrence can be easily captured by the term $\frac{d\dot{m}}{dN} \leq 0$, which, in turn, implies that the sign of some parameters changes: $g > 0$ and $c \leq 0$. This situation can be representative of economies in which innovators are endowed with effective private and/or public protection of IPR, or whenever innovations make imitations obsolete. These conditions are more likely to occur in advanced economies. In other words $-c$ can be thought of as representing the degree of IPR protection.

4 The solutions of the system

The Volterra system in (19) can be traced back to the family of systems of ordinary differential equations that can be written in a very general form as:

$$\dot{x}_i = x_i \left[r_i + \sum_{k=1}^{T} a_{ik} x_k \right] \quad \text{with} \quad i = 1, ..., T$$

where, according to the notation commonly used, a_{ik} are the generic coefficients that describe the interactions of the variables x_i. In our particular case $T = 2$.
This case has been graphically studied since Volterra's pioneering works and analytically in many other papers. Given the economic essence of the problem, in the following we will refer to the pioneering work by Goh (1976, 1980), that finds necessary and sufficient conditions for the local stability of a so-called "feasible equilibrium", namely an equilibrium characterized by strictly positive values of the variables.
Let us start by calculating the coordinates of our system equilibria. The system has three kinds of equilibria: the total extinction equilibrium $E_0(0,0)$, the one-category equilibria in which only one kind of agents survives and eventually the coexistence equilibrium. The one-category equilibria are characterized by the following Cartesian coordinates:

$$E_N\left(\frac{r}{s}; 0\right) \quad ; \quad E_M\left(0; \frac{g}{e}\right)$$

where E_N and E_M represent the equilibrium in which either innovators or imitators survive, respectively. In particular E_M makes sense only for $g > 0$.
The coexistence equilibrium is given by:

$$E^* = (N^*; M^*) = \left(\frac{er - gf}{es + cf}; \frac{gs + rc}{es + cf}\right) \tag{20}$$

when the two coordinates are strictly positive.
In order to analyze the property of the system we must study its Jacobian matrix:

$$J : \begin{bmatrix} r - 2sN - fM & -fN \\ cM & g + cN - 2eM \end{bmatrix}$$

calculated at the different equilibrium points. Thus, for the extinction equilibrium the Jacobian becomes:

$$J(E_0) : \begin{bmatrix} r & 0 \\ 0 & g \end{bmatrix}$$

the eigenvalues of which are $\lambda_1^0 = r$ and $\lambda_2^0 = g$. We assume that $r > 0$, so the extinction equilibrium is never locally stable in our model.
For the one-category equilibria the Jacobian matrix are the following:

$$J(E_N) : \begin{bmatrix} -r & -\frac{fr}{s} \\ 0 & g + \frac{rc}{s} \end{bmatrix} \quad \text{and} \quad J(E_M) : \begin{bmatrix} r - \frac{gf}{e} & 0 \\ \frac{cg}{e} & -g \end{bmatrix}$$

characterized by eigenvalues equal to $\lambda_1^N = -r$ and $\lambda_2^N = g + \frac{rc}{s}$ for E_N, the equilibrium with innovators and no imitators, while in the remaining case, E_M, $\lambda_1^M = r - \frac{gf}{e}$ and $\lambda_2^M = -g$.
Finally, to study the local stability of the coexistence equilibrium we refer to the necessary and sufficient conditions given by Goh (1976):

$$\begin{aligned} -sN^* - eM^* < 0 \qquad & \text{(c.1)} \\ N^*M^*(es + fc) > 0 \quad & \text{(c.2)} \end{aligned} \tag{21}$$

where N^* and M^* are the amounts of innovators and imitators corresponding to the coexistence equilibrium (20) when it is feasible, i.e. with positive equilibrium values of the variables.
According to the discussion of the results obtained in the previous Section, we are interested in two different scenarios.

Scen-1 $c, e, f, s, r > 0$ and no restrictions on g;

Scen-2 $e, f, g, s, r > 0$ and $c < 0$;

Scen-1 depicts the situation in which innovators (or more generally innovations) are regarded as preys and imitators (or more generally imitations) as predators. Scen-2 is such that the two kinds of agents compete with each other.

It is immediate to prove that for either scenario the extinction equilibrium is never asymptotically stable. In fact, at least the eigenvalue $\lambda_1^0 = r$ is strictly positive in both cases. The extinction equilibrium is similar in spirit to the shutdown equilibrium in which neither innovators nor imitators have any incentive to enter the market and operate. This first and very simple result envisages that a shutdown equilibrium in the market of ideas, although possible from a purely theoretical point of view, cannot be an enduring situation. In the next Sections we deepen the analysis of the remaining two equilibria, for the time being we study the relative dimension of N^* and M^*, namely the conditions under which M^* is greater that N^*.
Under Scen-1 we can claim that $N^* < M^*$ when the following inequality holds:

$$\frac{r}{s+f}(e - c) < g. \tag{22}$$

Assuming a positive imitators' birth rate, $g > 0$, a sufficient condition requires $e < c$, namely the rate at which imitators die out, e, must be less that the rate at which they thrive hunting innovators, c. Assuming $g < 0$ the condition $e < c$ becomes necessary, no longer sufficient because it can still occur that $e < c$ but $\frac{r}{s+f}(e-c) > g$, being $g < 0$.
To discuss Scen-2 it is useful to rewrite inequality (22) as follows:

$$\frac{r}{s+f} < \frac{g}{e-c} \tag{23}$$

Notice that the parameters on the left-hand-side, LHS, are all related to innovators. In particular, the denominator pertains innovators' death rate due to both natural death, s, or induced by imitators, f. Therefore, we can define the ratio $\frac{r}{s+f}$ as the *relative* birth rate, in the sense that births are compared to deaths. Similarly for the RHS.
To sum up, it is possible to claim that the necessary and sufficient condition in order to have $N^* < M^*$ is that innovators' relative birth rate is less than imitators' one.

5 First scenario: the prey-predator model

The definition of feasible equilibrium requires E^* to have strictly positive coordinates, implying the following inequality to hold:

$$-\frac{c}{s} < \frac{g}{r} < \frac{e}{f} \tag{24}$$

otherwise one equilibrium coordinate, N^* or M^*, would be negative. The relations in (24) states that the ratio g/r rather than the endogenous mortality/growth rates g and r must be considered in order to identify the feasibility conditions of E^*.
The inequalities in (24) assure that if the coexistence equilibrium is feasible, then E_N must necessarily be locally unstable given that in order to have a negative value of the eigenvalue λ_2^N, condition (24) must be violated. Similarly, violation of (24) leads also to a negative value of λ_1^M. Hence, we can state the following proposition:

Proposition 1. *Whenever the coexistence equilibrium is feasible, there are no other equilibria that can be locally asymptotically stable under Scen-1.*

It is worth recalling that both innovation and imitation activities carry out a social task, being innovation the engine of economic growth and being imitation an easy and cheap way to diffuse innovation. Therefore, the lack of one of the two populations will prevent the economy to grow at a substantial rate, differently from a situation of coexistence. In addition, in the specific case of infringements, namely illegal imitations, since fighting crime is a costly activity, as first pointed out by the Nobel laureate Becker (1968), it is economically preferable a situation in which a certain amount of illegal imitations were tolerated. In this respect, the stability of the coexistence equilibrium may be considered as a desirable feature, in that it allows stable economic growth. Hence, Proposition 1 can be rephrased in economic terms as stating that whenever it is socially acceptable that imitators and innovators coexist, this solution is preferable to others in which neither imitators nor innovators exist.

5.1 Stability of the coexistence equilibrium

It is easy to check that, under Scen-1, the inequalities in (24) imply (c.1) and (c.2) in (21) to hold. That is, if the coexistence equilibrium is feasible, then both local stability conditions

apply.
Yet, it is possible to go deeper in the analysis by identifying parameters constellations under which convergence is oscillating or monotone. In mathematical terms we must check if eigenvalues are complex or real, respectively.
Even if it is possible to obtain analytical conditions on the parameters pertaining each of the two cases, their complicated expressions would be hardly interpretable from an economic point of view. Therefore, we prefer to simulate two examples corresponding to the two cases, i.e. oscillating or monotonic convergence. In Figure 1-A the phase plane displays a converging trajectory when the eigenvalues of the feasible equilibrium are complex. We used the following set of parameters: (c, e, f, g, r, s)=$(1.27, 0.28, 2.156, -0.88, 1.9, 0.149)$.

Figure 1. Oscillatory convergence Panel A Panel B

Figure 1-B shows the timeplot of both variables converging oscillating.
Figure 2 corresponds to a case in which the eigenvalues of the feasible equilibrium are real, and the convergence is monotone.
The set of parameters used is (c, e, f, g, r, s)=$(0.73, 3.785, 0.96, 1.64, 1.383, 0.21)$.

Figure 2. Monotone convergence Panel A Panel B Panel C

These two examples show that whenever the parameters of the Volterra system (19) are such to generate oscillating convergence towards a feasible equilibrium, in the transition phase innovators and imitators can dominate the other specie. In this respect the ratio N/M will go through consecutive periods in which it is greater and other lower than unity (Figure 1). Furthermore, whenever parameters generate monotone convergence, over the transition phase, the N/M ratio can be either greater of lower than unity, but in this case the situation $N = M$ can occur at most once (as an example, see Figure 2-C obtained with this set of parameters: $(c, e, f, g, r, s) = (2.62, 3, 0.16, 1.7, 1, 1.34)$).

6 Second scenario: the model of competition

Under the competing species case, namely under Scen-2, the coexistence equilibrium has strictly positive components provided that either:

$$\frac{e}{f} < \frac{g}{r} < -\frac{c}{s} \tag{25}$$

or:

$$-\frac{c}{s} < \frac{g}{r} < \frac{e}{f} \tag{26}$$

are fulfilled. Again, these relations involve the ratio between the exogenous growth/mortality rates g/r.
Conditions (25) and (26) are mutually exclusive. Therefore, we can consider separately the two sub-cases.
When condition (25) applies the coexistence equilibrium must be necessarily unstable given that condition (c.2) in (21) does not hold.
At the same time the first condition of (25) ensures that the eigenvalues of both E_N and E_M are all negative. In other words we have a situation of bistability and, depending on the initial condition, trajectories will converge towards one equilibrium or the other. Figure

3, obtained with the set of parameters (c, e, f, g, r, s)=$(-2, 1, 1, 1.7, 1, 1)$, is an example of the shape of the basins of attraction of the coexisting locally stable equilibria. The values of the point (N_0, M_0) contained in the dark blue area represent the initial condition leading to converge to E_M, while the points in the light blue area represent the initial condition leading to converge to E_N.

Figure 3. Basin of attraction

Figure 3 reveals that even if E^* is locally unstable, it still plays an important role. Its stable manifold separates the two basins of attraction.
This scenario is a quite typical one in cases of competition. It reminds the *competitive exclusion principle* or *Gause's law*, which means that imitators and innovators look like two species competing for the same resources and a little advantage with respect to the coexistence equilibrium values is enough to make one specie survive and the other one die out.
Let us consider now the case in which the coexistence equilibrium is feasible because condition (26) applies. It is easy to check that the positivity of N^*, M^*, s, e and $(es + fc)$ imply the local stability of E^* to hold, according to (21).
The argument put forth so far about the stability of the system in a competing world can be easily summarized and rationalized as follows. By simply rearranging (26) we have that a necessary condition for it is the following:

$$-cf < es \tag{27}$$

On the LHS of inequality (27) we find the product between the rate c at which innovators hunt imitators and the rate f at which the latter hunt the former. Let us define this quantity, cf, as the *cross hunting rate.* On the RHS we find the product of the two natural death rates, es. Let us define this quantity as the *cross natural death rate.* The comparison between the two quantities can be regarded as measuring the degree of competitiveness. Accordingly, a cross hunting rate less than a cross natural death rate can be read as stating that competition must not be too severe. It is also worth noting that condition (27) implies that $-c$ has an upper bound, being $-c < es/f$. Hence, we can claim the following Proposition:

Proposition 2. *The coexistence equilibrium, whenever feasible, is the unique locally stable equilibrium only if condition (27) holds, under Scen-2.*

It is commonly acknowledged that innovation is the true engine of economic growth, but at the same time innovators must not be endowed with too aggressive tools to hamper imitation activity, otherwise the economic system will go through instability (here as a metaphor of an undesirable economic situation). In this respect, the economic rational of Proposition 2 is perfectly in line with that of Proposition 1, but under different economic scenarios. A certain amount of competition between the two sub-populations is desirable, but there must be a limit to the extent to which one population can hamper the others' activity. Accordingly, the upper bound to $-c$ represents an upper bound to the degree of IPR protection. Finally, recall that in order to have feasible and stable coexistence $-c$ must lie in the interval $0 < -c < es/f$. It is worth noting that this interval is consistent with an inverse-U shaped relationship between innovation and competition, as found by Aghion et al. (2005), Furukawa (2007) and many others.

7 Concluding Remarks

Starting from the model of Dixit and Pindyck (1993) set in a context of dynamic uncertainty and irreversible investment, we have derived the law of evolution of innovators and imitators in the market. Their relationships have been studied under two different contexts: prey-predator and competition. Analytic results and numerical simulations show that when the first scenario applies the coexistence equilibrium is the only stable equilibrium. As far as competition is concerned, we find that the conditions for local stability of the coexistence equilibrium bound the degree of IPR protection. As a result, innovation has an inverse-U shape as a function of imitation, namely of competition, entailing that a stronger degree of IPR is not always the best policy to be pursued.

Acknowledgment. Fabio Tramontana has been supported by the funds of the PRIN project "Local interactions and global dynamics in economics and finance:models and tools", MIUR, Italy. The opinions expressed by the authors are their only and do not necessarily reflect the positions of the Institutes.

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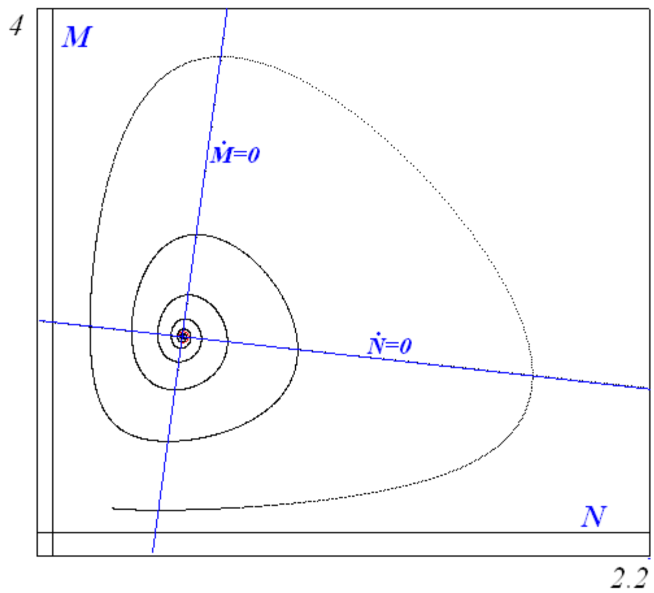


Figure 1(a)

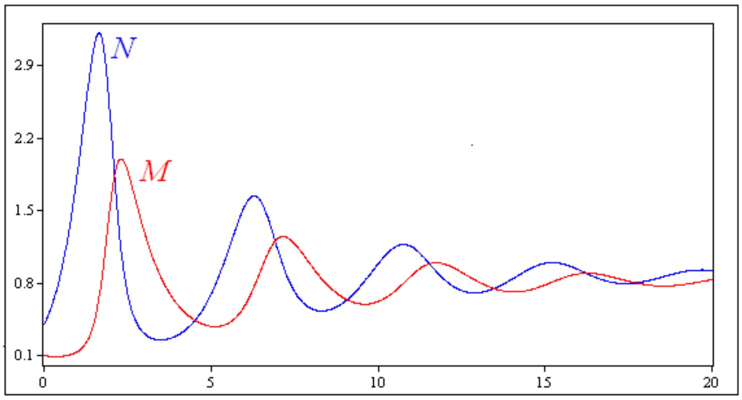


Figure 1(b)

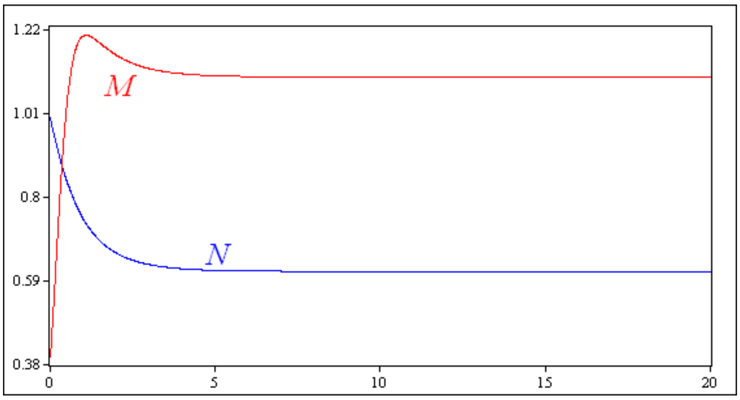


Figure 1(c)

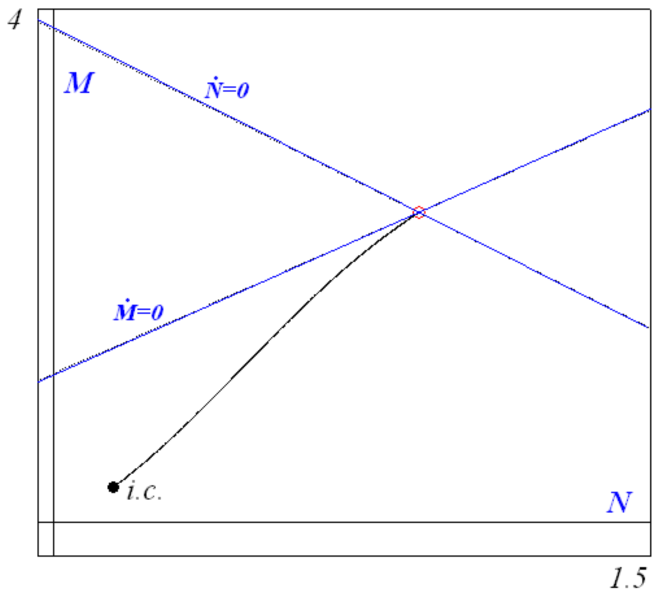
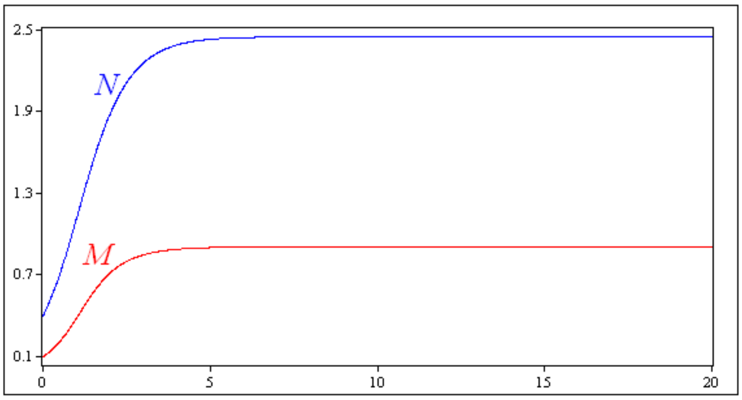


Figure 2(a)



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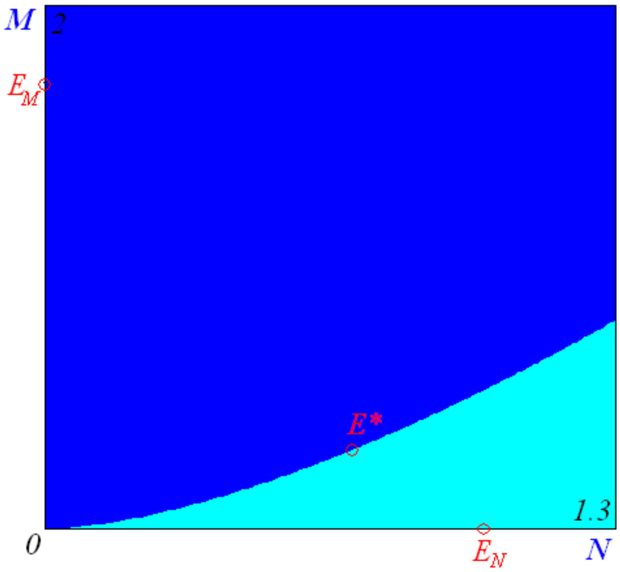

Figure 3