An analytical and numerical search for bifurcations in open economy New Keynesian models

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AN ANALYTICAL AND NUMERICAL SEARCH FOR BIFURCATIONS IN OPEN ECONOMY NEW KEYNESIAN MODELS

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ABSTRACT

We explore bifurcation phenomena in the open-economy New Keynesian model developed by Gali and Monacelli (2005). We find that the open economy framework brings about more complex dynamics, along with a wider variety of qualitative behaviors and policy responses. Introducing parameters related to the open economy structure affects the values of bifurcation parameters and changes the location of bifurcation boundaries. As a result, the stratification of the confidence region, as previously seen in closed-economy New Keynesian models, remains an important research and policy risk to be considered in the context of the open-economy New Keynesian functional structures. In fact, econometrics and optimal policy design become more complex within an open economy. Dynamical inferences need to be qualified by the risk of bifurcation boundaries crossing the confidence regions. Policy design needs to take into consideration that a change in monetary policy can produce an unanticipated bifurcation, without adequate prior econometrics research.

Keywords: stability; bifurcation; open economy; New Keynesian; determinacy; macroeconomics; dynamic systems

JEL-codes: C14, C22, C52, C61, C62, E32, E37, E61, L16

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1. Introduction

Dynamical economic systems are subject to bifurcations. As Grandmont (1985) has shown, even simple dynamic economic systems may exhibit various types of dynamic behaviors within the same functional structure, with the parameter space stratified into bifurcation regions associated with the different dynamical solution-path behaviors. Therefore, analyzing bifurcation boundaries is required to understand the dynamic properties of an economic system. Barnett and He (1999) investigated the stability of the Bergstrom, Nowman, and Wymer (1992) continuous time macroeconometric model of the UK economy and found both transcritical and Hopf bifurcations. Barnett and He (2006) more recently detected a singularity bifurcation in the Leeper and Sims' (1994) Euler equations macroeconometric model of the U.S. economy. Barnett, Banerjee, Duzhak, and Gopalan (2011) found that including industrial organization features into a Zellner's Marshallian macroeconomic model, permitting entry and exit of firms, does not decrease the relevancy of bifurcation phenomena. Barnett and Duzhak (2008, 2010) analyzed bifurcation using a closed economy New Keynesian model, based on Walsh (2003), and found both Hopf and period doubling bifurcations within the parameter space.

Occurrence of bifurcation boundaries stratifies the parameter space. As observed by Barnett and He (1999, 2002, 2006) and Barnett and Duzhak (2008, 2010), the existence of bifurcation bundaries in parameter space indicates the presence of different solution types corresponding to parameter values close to each other, but on different sides of the bifurcation boundary. Dynamic properties of the system can change dramatically on different sides of a bifurcation boundary. As a result, robustness of inferences about dynamical solution properties can be damaged, if parameter values are close enough to a bifurcation boundary so that the parameters' confidence regions cross the boundary.

2. Model

We investigate the possibility of bifurcations in the open-economy New Keynesian model developed by Gali and Monacelli (2005). We thereby extend the conclusions of Barnett and Duzhak (2008, 2010) to the open economy case. Barnett and Duzhak (2008, 2010) analyze bifurcation with a closed-economy New Keynesian model and found both Hopf and period doubling bifurcations.

Gali and Monacelli (2005) define a small open economy to be “one among a continuum of infinitesimally small economies making up the world economy”. Thus, domestic policy does not affect the other countries and the world economy. In their model, each economy is assumed to have identical preferences, technology, and market structure, although the economies might encounter different, imperfectly correlated productivity shocks. In the Gali and Monacelli model, both consumers and firms are assumed to behave optimally. Consumers maximize expected present value of utility, while firms maximize profits.

The utility maximization problem yields the following dynamic intertemporal IS curve, which is a log-linear approximation to the Euler equation:

$$x_t = E_t x_{t+1} - \frac{1+\alpha(\omega-1)}{\sigma}\left(r_t - E_t\pi_{t+1} - \overline{r}_t\right), \tag{1}$$

where x_t is the gap between actual output and flexible-price equilibrium output, $\overline{r}_t$ is the small open economy's natural rate of interest, and $\sigma_\alpha = \sigma(1-\alpha+\alpha\omega)^{-1}$ and $\omega = \sigma\gamma + (1-\alpha)(\sigma\eta - 1)$ are composite parameters. The lowercase letters denote the logs of the respective variables, $\rho = \beta^{-1} - 1$ denotes the time discount rate, and a_t is the log of labor's average product.

The maximization problem of the representative firm yields, after some algebra, the aggregate supply curve, often called the New Keynesian (NK) Philips curve in log-linearized form:

$$\pi_t = \beta E_t \pi_{t+1} + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)x_t \ , \tag{2}$$

where $\mu = \frac{(1-\beta\theta)(1-\theta)}{\theta}$ and $\omega = \sigma\gamma + (1-\alpha)(\sigma\eta - 1)$.

As stated in Gali and Monacelli (2005), while the closed economy model is nested in the small open economy model as a limiting case, both versions differ in two aspects. First, some coefficients of the open economy model depend on the parameters that are exclusive to the open economy framework, such as the degree of openness, terms of trade, and substitutability among domestic and foreign goods. Second, the natural levels of output and interest rate depend upon both domestic and foreign disturbances, in addition to openness and terms of trade.

The model is closed by adding a simple (non-optimized) monetary policy rule, conducted by the monetary authority, such as:

$$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t \ , \tag{3}$$

where the coefficients $\phi_x > 0$ and $\phi_\pi > 0$ measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In this form, the policy rule (3) is called the Taylor rule (Taylor (1993)). Various versions of the Taylor rule are often employed to design monetary policy in empirical DSGE models. Equations (1) and (2), in combination with a monetary policy rule such as equation (3), constitute a small open economy model in the New Keynesian tradition.

To determine whether a Hopf bifurcation exists in the Gali and Monacelli model, our methodology is that of Gandolfo (1996) and Barnett and Duzhak (2008, 2010). We first evaluate the Jacobian of the system at the equilibrium point, $\pi_t = x_t = 0$, for all $t = 1,2,...$, and then we check whether the conditions of the Hopf Bifurcation Theorem are satisfied. For two dimensional systems, we apply the existence part of the Hopf Bifurcation Theorem given in Gandolfo (1996, page 492):

Theorem 1: *Consider the class of two-dimensional first-order difference equation systems produced by the map* $\mathbf{y} \to \mathbf{f}(\mathbf{y}, \phi)$, $\mathbf{y} \in \mathbb{R}^2$, *with vector of parameters,* $\boldsymbol{\phi} \in \mathbb{R}^N$. *Assume for each* $\boldsymbol{\phi}$, *there exists a local fixed point,* $\mathbf{y}^* = \mathbf{y}^*(\boldsymbol{\phi})$, *in the relevant interval at which the eigenvalues of the Jacobian matrix, evaluated at* $(\mathbf{y}^*(\boldsymbol{\phi}), \boldsymbol{\phi})$, *are complex conjugates,* $\lambda_{1,2} = a \pm ib$, *and satisfy the following properties:*

(i) $|\lambda_1| = |\lambda_2| = +\sqrt{a^2 + b^2} = 1$, *with* $\lambda_i \neq 1$ *for* $i = 1,2$,

where $|\lambda_i|$ *is the modulus of the eigenvalue* λ_i. *Also assume there exists* j *= 1, 2, ..., N such that*

(ii) $\left.\dfrac{\partial |\lambda_i(\boldsymbol{\phi})|}{\partial \phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0$ *for i = 1,2.*

Then, there exists a Hopf bifurcation at the equilibrium point $(\mathbf{y}^*(\boldsymbol{\phi}^*), \boldsymbol{\phi}^*)$.

Note that condition (*ii*) applies for any one value of *j*, and not necessarily for all *j*, so the search for bifurcation can proceed with one parameter at a time, conditionally upon fixed values of the other parameters. But since Theorem 1 is valid only for two dimensional systems, the following theorem from Wen, Xu, and Han (2002) is employed for three dimensional dynamic systems.

Theorem 2: *Consider the class of three-dimensional first-order difference equation systems produced by a map* $\mathbf{y} \to \mathbf{f}(\mathbf{y}, \phi)$, *with* $\mathbf{y} \in \mathbb{R}^3$, *and vector of parameters,* $\boldsymbol{\phi} \in \mathbb{R}^N$. *Let the 3×3 matrix* ***C*** *be the Jacobian of the system, having a third order characteristic polynomial in the following form:*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

Assume that for an equilibrium point $(\mathbf{y}^*(\boldsymbol{\phi}), \boldsymbol{\phi})$, *there exists* $j = 1, 2, \ldots, N$ *such that the following transversality condition holds*

$$\left.\frac{\partial|\lambda_i(\boldsymbol{\phi})|}{\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0$$

for i = 1, 2, where $|\lambda_i|$ *is the modulus of the eigenvalue; and the following eigenvalue conditions hold*

(i) $|a_0| < 1,$

(ii) $|a_0 + a_2| < 1 + a_2,$

(iii) $a_1 - a_0 a_2 = 1 - a_0^2.$

Then, there exists a Hopf bifurcation at the equilibrium point $(\mathbf{y}^*(\boldsymbol{\phi}^*), \boldsymbol{\phi}^*)$.

Regarding the general relationship between Theorem 2 and the eigenvalues, see Barnett and Duzhak (2010, pp. 107-108) and Wen, Xu, and Han (2002, p. 351). For Hopf bifurcation to arise with the class of dynamical systems relevant to the Gali and Monacelli model, Theorem 2 requires a pair of complex conjugate eigenvalues on the unit circle and one real-valued eigenvalue lying outside the unit circle.

For the numerical analysis, we follow the methodology developed by Govaerts, Kuznetsov, Khoshsiar, and Meijer (2008) and use the CL MatCont software within MatLab. We follow Barnett and Duzhak (2008, 2010) to derive the conditions for the existence of Hopf bifurcation. In computations we always use CL MatCont for Hopf and all other forms of bifurcation that the program can detect. We provide Theorems 1 and 2 for Hopf bifurcation, and not the theorems relevant to other forms of bifurcation, primarily because Hopf bifurcation has been the most important and most commonly found in economics as well as in this research, but we do not constraint CL MatCont to search for bifurcation consistent with Theorems 1 and 2. Indeed we do find other types, in some cases, such as period doubling (flip) bifurcation.

We consider contemporaneous, forward, and backward looking policy rules, as well as their hybrid combinations. We summarize analytical results and discuss numerical results for each case. We use the calibration values of the parameters as given in Gali and Monacelli (2005), which are $\beta = 0.99$, $\alpha = 0.4$, $\sigma = \omega = 1$, $\varphi = 3$, $\mu = 0.086$; and for the $N = 3$ policy parameters, we use $\phi_x = 0.125$, $\phi_\pi = 1.5$, and $\phi_r = 0.5$. In our applications, we use the subscripts $j = \pi, x, r$, rather than 1, 2, 3, to designate the subscripts of the three parameters of the New Keynesian policy rules, as defined in equation (9) below.

2.1. Current-Looking Taylor Rule

Consider the following model, in which the first two equations describe the economy, while the third equation is the monetary policy rule followed by the central bank with $N = 2$ policy parameters:

$$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi \right) x_t \ , \tag{4}$$

$$x_t = E_t x_{t+1} - \frac{1 + \alpha(\omega - 1)}{\sigma} \left(r_t - E_t \pi_{t+1} - \overline{r}_t \right) \ , \tag{5}$$

$$r_t = \overline{r}_t + \phi_\pi \pi_t + \phi_x x_t \ . \tag{6}$$

Rearranging the terms, the system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C} \mathbf{y}_t$,

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{\mu}{\beta}+\left(1+\alpha\left(\omega-1\right)\right)\left(\frac{\beta\phi_x+\varphi\mu}{\beta\sigma}\right) & \frac{\left(\beta\phi_\pi-1\right)\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma} \\ -\frac{\mu}{\beta}\left(\varphi+\frac{\sigma}{1+\alpha\left(\omega-1\right)}\right) & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \tag{7}$$

We assume the eigenvalues of the system (7) are complex conjugates. Using Theorem 1, the conditions for the existence of Hopf bifurcation in the system (7) are presented in the following Proposition.

Proposition 1: *Let Δ be the discriminant of the characteristic equation. Then system (7) undergoes a Hopf bifurcation, if and only if $\Delta < 0$ and*

$$\phi_x^* = \frac{\sigma\left(\beta-1\right)}{1+\alpha\left(\omega-1\right)} - \mu\left(\frac{\sigma}{1+\alpha\left(\omega-1\right)}+\varphi\right)\phi_\pi . \tag{8}$$

In the closed economy case, the corresponding value of the bifurcation parameter is $\phi_x^* = \sigma\left(\beta-1\right)-\kappa\phi_\pi$, as given by Barnett and Duzhak (2008). For $\alpha = 0$, Proposition (1) gives the same result as the closed economy counterpart.

We numerically find a period doubling bifurcation at $\phi_x = -2.43$ and a Hopf bifurcation at $\phi_x = -0.52$. Decreasing the value of ω results in a higher absolute value of the bifurcation parameter value, except when $\alpha = 0$. Then changes in ω do not make any difference. On the other hand, decreasing the value of α results in a lower absolute value of the bifurcation parameter, except when $\omega = 1$. Then, changes in α do not make a difference.

Numerical computations indicate that the monetary policy rule equation (6) should have $\phi_x^* < 0$ for a Hopf or period doubling bifurcation to occur. That negative coefficient for the output gap in equation (6) would indicate a procyclical monetary policy: rising interest rates when the output gap is negative or vice versa. Schettkat and Sun (2009) identify situations, such as exchange rate stabilization or an underestimation of the potential output level, which can produce such a result; but otherwise it is difficult to rationalize a negative policy parameter on the output gap.

There is a large literature seeking to explain procyclicality in monetary policy. Demirel (2010), for example, shows that the existence of country spread can explain how optimal fiscal and monetary policies can be procyclical. Leith, Moldovan, and Rossi (2009) argue that with superficial habits, the optimal simple rule might exhibit a negative response to the output gap. Such a perverse policy response to output gap or to inflation can induce instability in the model. A countercyclical monetary policy, on the other hand, would be bifurcation-free and would yield more robust dynamical inferences with confidence regions not crossing a bifurcation boundary.

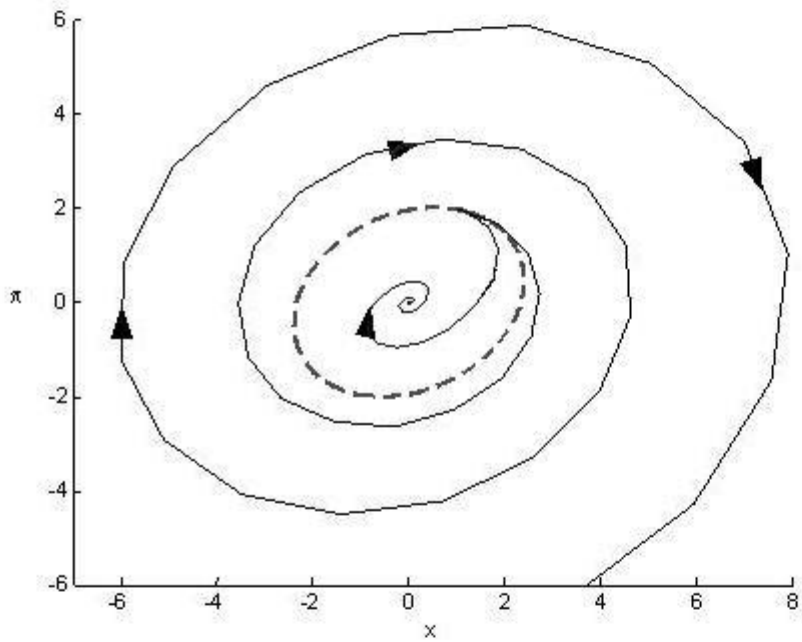


Figure 1: Phase diagram displaying Hopf bifurcation under the current-looking Taylor Rule.

The phase diagram in Figure 1 illustrates a Hopf bifurcation under the current-looking Taylor Rule. There is only one periodic solution, while the other solutions diverge from the periodic solution as $t \to \infty$. The periodic solution is called an unstable limit cycle.

In conclusion, by assuming $\phi_x > 0$ and $\phi_\pi > 0$, the Gali and Monacelli Model with current-looking Taylor rule is not subject to bifurcation within the feasible parameter space, although bifurcation is possible within the more general functional structure of system (7).

2.2. Current-Looking Taylor Rule With Interest Rate Smoothing

Consider the model consisting of equations (4) and (5), along with the following policy rule having $N = 3$ policy parameters:

$$r_t = \overline{r}_t + \phi_\pi \pi_t + \phi_x x_t + \phi_r r_{t-1} \ . \tag{9}$$

We can write that system of three equations in the form $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t + \mathbf{d}_t$:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t r_{t+1} \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} + \begin{bmatrix} -\dfrac{1-\alpha+\alpha\omega}{\sigma}\overline{r}_t \\ 0 \\ E_t \overline{r}_{t+1} - \phi_x \overline{r}_t \dfrac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix} \tag{10}$$

where

$$\mathbf{C} = \begin{bmatrix} \dfrac{\mu}{\beta}\left(1+\varphi\dfrac{1-\alpha+\alpha\omega}{\sigma}\right)+1 & -\dfrac{1-\alpha+\alpha\omega}{\beta\sigma} & \dfrac{1-\alpha+\alpha\omega}{\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \dfrac{1}{\beta} & 0 \\ \phi_x + \dfrac{\mu}{\beta}\left(1+\varphi\dfrac{1-\alpha+\alpha\omega}{\sigma}\right)\left(\phi_x \dfrac{1-\alpha+\alpha\omega}{\sigma} - \phi_\pi\right) & -\dfrac{1}{\beta}\left(\phi_x \dfrac{1-\alpha+\alpha\omega}{\sigma} - \phi_\pi\right) & \phi_r + \phi_x \dfrac{1-\alpha+\alpha\omega}{\sigma} \end{bmatrix}.$$

Assuming the system (10) has a pair of complex conjugate eigenvalues and a real-valued eigenvalue outside the unit circle, the following proposition states the conditions for the system to undergo a Hopf bifurcation.

Proposition 2: *The system (10) undergoes a Hopf bifurcation, if and only if the following transversality condition holds*

$$\left.\frac{\partial\left|\lambda_i(\phi)\right|}{\partial\phi_j}\right|_{\phi_j=\phi_j^*} \neq 0$$

and also

(i) $\phi_r - \beta < 0$, (11)

(ii) $\phi_r\left(\dfrac{\sigma(2+\mu+2\beta)}{1-\alpha+\alpha\omega}+\varphi\mu\right)+\phi_x(\beta+1)+\mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(\phi_\pi+1)+\dfrac{2\sigma}{1-\alpha+\alpha\omega}<0$, (12)

(iii) $\phi_r^2\xi_4 + \phi_r\xi_3 + (\phi_x\phi_r + \phi_x)\xi_2 + \phi_\pi\xi_1 + \xi_0 = -1$. (13)

Since condition (12) in Proposition (2) does not hold, Hopf bifurcation cannot occur in the Gali and Monacelli Model under the current-looking Taylor rule with interest rate smoothing.

We also analyze the system (10) for the existence of a period doubling bifurcation. Keeping the structural parameters and policy parameters, ϕ_π and ϕ_r, constant at their baseline values, while varying the policy parameter ϕ_x over a feasible range, we numerically find period doubling bifurcation at $\phi_x = 0.83$. Lowering ω and raising α increase the value of the bifurcation parameter. There is no bifurcation of any type at $(\omega, \alpha) = (0, 1)$.

Airaudo and Zanna (2005), using a non-separability, money-in-utility-function model, show that cyclical and chaotic dynamics become more likely as the openness of the economy increases and as the exchange rate pass-through into import prices increases. Airaudo and Zanna also show that the existence of cyclical and chaotic dynamics depends upon open economy features and is robust to different timings in the policy rule.

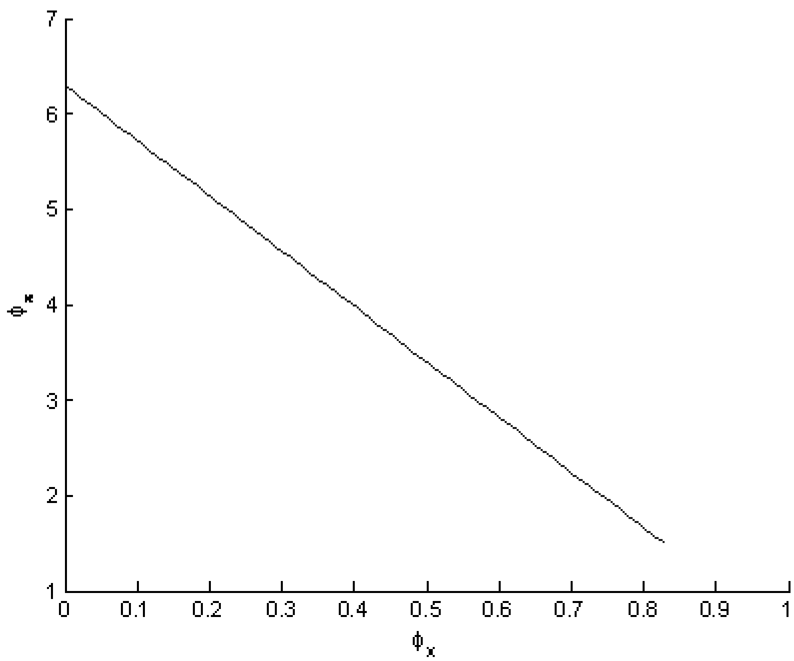


Figure 2: Period doubling bifurcation boundary at $\phi_x = 0.827$ for model (10).

Figure 2 illustrates the period-doubling bifurcation boundary for the parameter ϕ_x. Note that along the bifurcation boundary, which is the set of bifurcation points of the same type, the values of the bifurcation parameter ϕ_x lie between 0 and 0.83. As the magnitude, ϕ_π, of the reaction of central bank to inflation increases, small values of the parameter ϕ_x would be sufficient to induce period doubling bifurcation.

When we consider ϕ_π as the bifurcation parameter, we numerically find a period doubling bifurcation at $\phi_\pi = 5.57$ and a branching point at $\phi_\pi = -1.5$. Lowering ω and raising α increase the value of the bifurcation parameter. There is no bifurcation of any type at $(\omega, \alpha) = (0, 1)$.

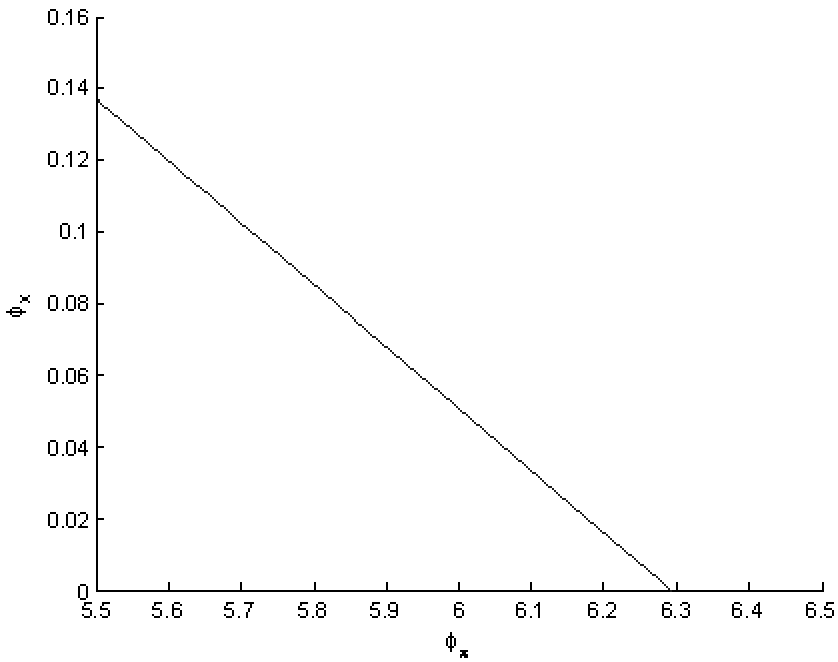


Figure 3: Period doubling bifurcation boundary at $\phi_\pi = 5.57$ for model (10).

Figure 3 displays the bifurcation boundary for the parameter ϕ_π. Along the bifurcation boundary, the values of bifurcation parameter ϕ_π lie between 5.5 and 6.3. This is a relatively small interval for bifurcation to emerge. As the magnitude, ϕ_x, of the reaction of central bank to output gap increases, lower values of the parameter ϕ_π would be sufficient to cause a period doubling bifurcation.

2.3. Forward-Looking Taylor Rule

Consider the model consisting of equations (4) and (5) along with the following policy rule:

$$r_t = \overline{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x E_t x_{t+1} \ . \tag{14}$$

Rearranging terms, we have the reduced system in normal form, $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t$:

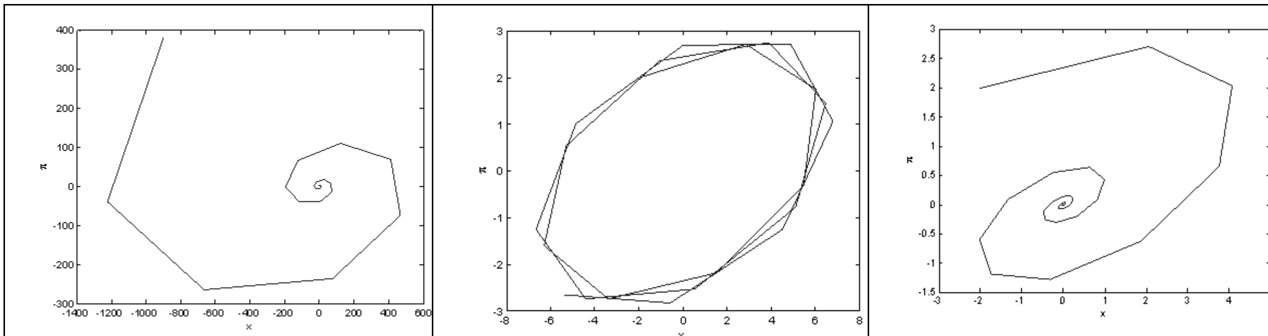
$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \dfrac{\beta\sigma - \left(\mu\sigma + \mu\varphi\left(1+\alpha\left(\omega-1\right)\right)\right)\left(\phi_\pi - 1\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha\left(\omega-1\right)\right)} & \dfrac{\left(\phi_\pi - 1\right)\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma - \beta\phi_x\left(1+\alpha\left(\omega-1\right)\right)} \\ -\dfrac{\mu\sigma + \mu\varphi\left(1+\alpha\left(\omega-1\right)\right)}{\beta + \alpha\beta\left(\omega-1\right)} & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} . \tag{15}$$

Assuming a pair of complex conjugate eigenvalues, and using Theorem 1, we provide the conditions for the existence of a Hopf bifurcation in the following proposition.

Proposition 3: *The system (15) undergoes a Hopf bifurcation, if and only if* $\Delta < 0$ *and*

$$\phi_x^* = \frac{\left(\beta - 1\right)}{\beta} \frac{\sigma}{1+\alpha\left(\omega-1\right)} . \tag{16}$$

Figure 4 provides several phase diagrams displaying Hopf bifurcation in model (15) .



Diverging trajectory at $\phi_\pi = 2.8$ and $\phi_x = 0.4$

Limit cycle at $\phi_\pi = 2.8$ and $\phi_x = 0$

Converging trajectory at $\phi_\pi = 2.8$ and $\phi_x = -0.4$

Figure 4: Phase diagrams showing Hopf bifurcation in model (15).

Numerical analysis with CL MatCont indicates a period doubling bifurcation at $\phi_x = 1.913$ and a Hopf bifurcation at $\phi_x = -0.01$. Given the baseline values of the parameters, Hopf bifurcation occurs outside the feasible set of parameter values. Decreasing the value of ω

results in a higher value of the bifurcation parameter in absolute value, except when $\alpha = 0$. Then changes in ω do not make a difference. But decreasing the value of α results in a lower value of the bifurcation parameter in absolute value, except when $\omega = 1$. Then changes in α do not make any difference. All bifurcations disappear, when $\alpha = 1$ and $\omega = 0$.

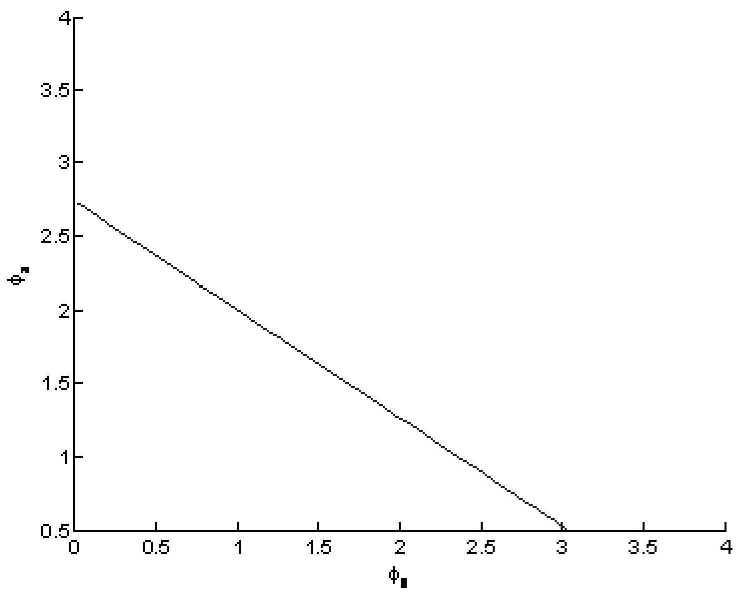


Figure 5: Period doubling bifurcation boundary for ϕ_x in model (15).

Figure 5 displays the boundaries of period doubling bifurcation under a forward looking Taylor rule. Along the bifurcation boundary, the values of the bifurcation parameter, ϕ_x, lie between 0 and 2.8. As the weight, ϕ_π, of central bank reaction to expected inflation increases, smaller values of parameter ϕ_x would be sufficient to cause period doubling bifurcation.

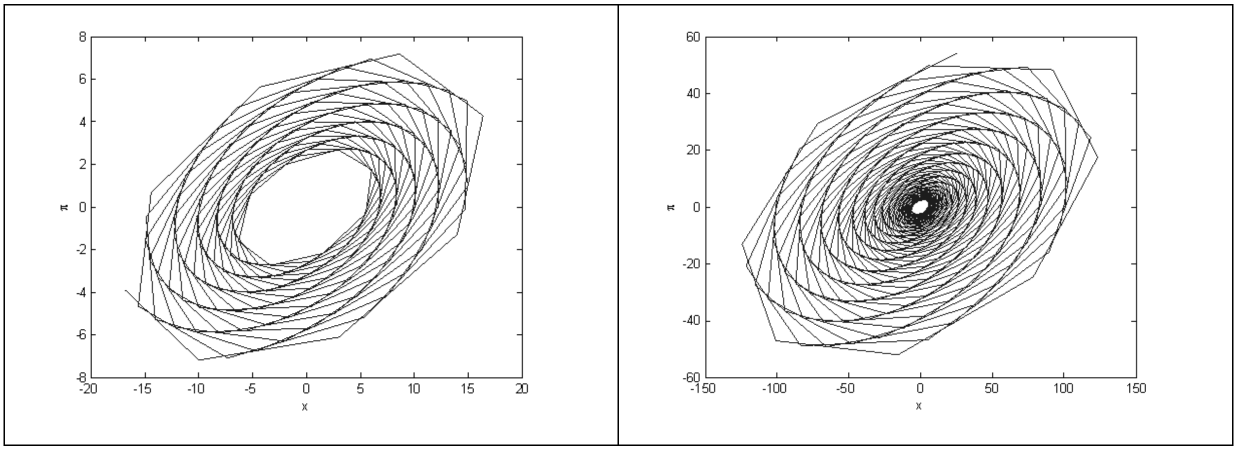


Figure 6: Phase diagrams showing a periodic solution in model (15).

Figure 6 illustrates the phase diagrams, constructed at $\phi_\pi = 2.8$ and $\phi_x = 0$ for two different numbers of iterations. The system has a periodic solution at these parameter values. The origin is a stable spiral point. Any solution that starts around the origin in the phase plane will spiral toward the origin. Since the trajectories spiral inward, the origin is a stable sink.

2.4. Pure Forward Looking Inflation Targeting

Consider the model consisting of equations (4) and (5) along with the following policy rule:

$$r_t = \overline{r}_t + \phi_\pi E_t \pi_{t+1}. \tag{17}$$

Rearranging the terms, we have the following reduced system in normal form $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t$:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \left(\dfrac{\mu}{\beta} + \dfrac{\varphi\mu\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma} \right)\left(\phi_\pi - 1\right) & \dfrac{\left(\phi_\pi - 1\right)\left(1+\alpha\left(\omega-1\right)\right)}{\beta\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha\left(\omega-1\right)} + \varphi \right) & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \tag{18}$$

Figure 7 illustrates a solution path for $\beta = 1$ and $\phi_\pi = 8$. The solution path is periodic and oscillates around the origin without converging or diverging. The origin is a stable center.

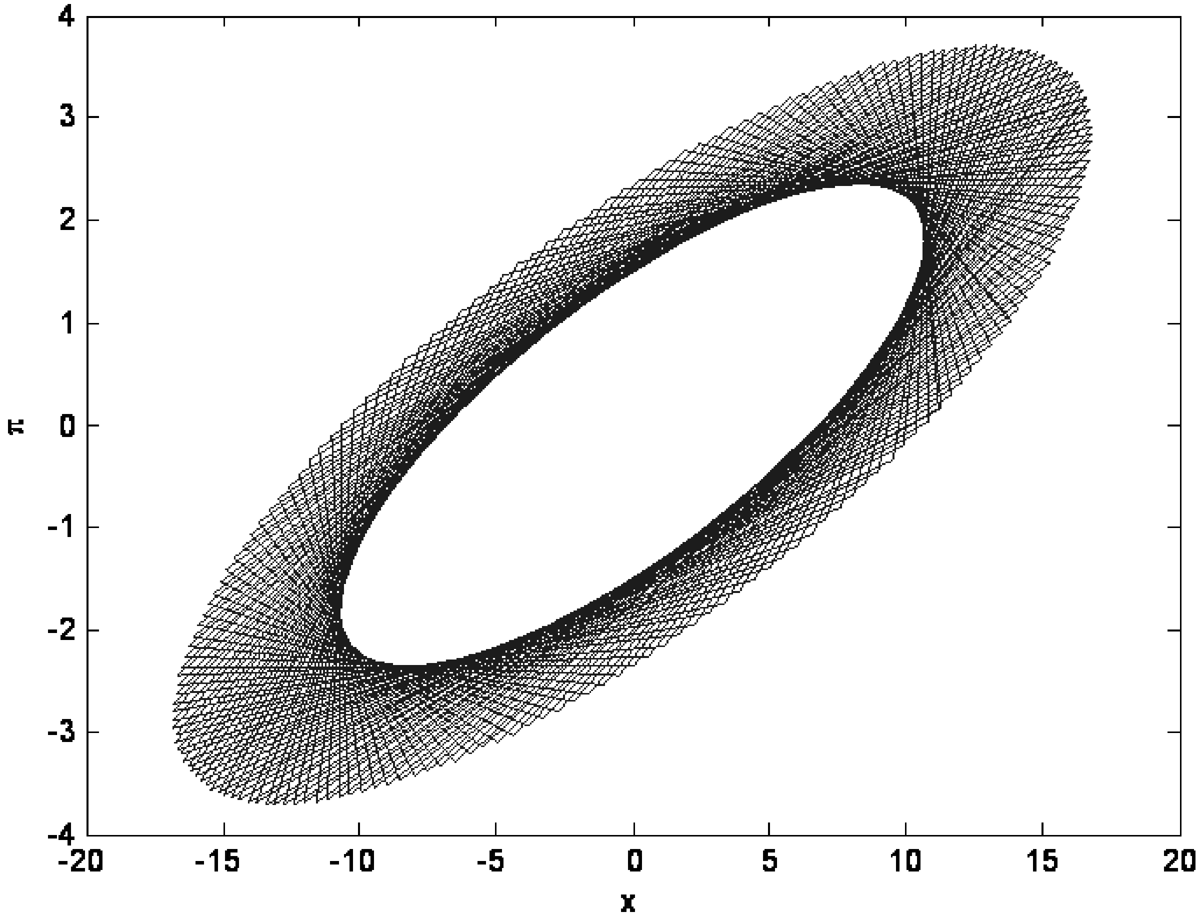


Figure 7: Phase space plot for $\beta = 1$ and $\phi_{\pi} = 8$ in model (18).

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Assuming the presence of a pair of complex conjugate eigenvalues, Hopf bifurcation can occur, if the transversality conditions are satisfied. Using Theorem 1, the conditions for the existence of a Hopf bifurcation are presented in the following Proposition.

Proposition 4: *The system (18) undergoes a Hopf bifurcation, if and only if*

$$\Delta < 0 \; and \;\; \beta^{*} = 1. \tag{19}$$

This result shows that setting the discount factor equal to 1 puts the system on the Hopf bifurcation boundary and creates instability. We also numerically find a period doubling bifurcation at $\beta = -0.91$. But that point is outside the feasible parameter space subset. Furthermore, Hopf bifurcation appears at $\beta = 1$ regardless of the values of α and ω.

Bifurcation analysis in an open economy framework yields the same results as in the closed economy case under forward-looking inflation targeting. Barnett and Duzhak (2010) report a Hopf bifurcation at $\beta = 1$ for the closed economy case. But setting the discount factor at 1 is not justifiable for a New Keynesian model, whether within an open or closed economy framework.

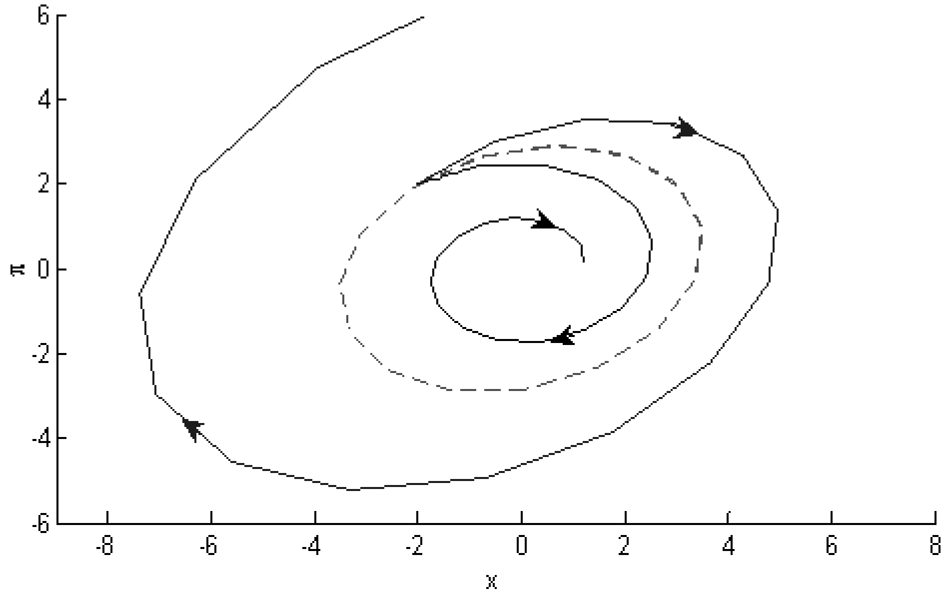

Figure 8: Phase plots for various values of the parameter β in (x, π)-space in model (18).

Phase plots in Figure 8 display Hopf bifurcation. There is only one periodic solution, and other solutions diverge from the periodic solution as $t \to \infty$. The periodic solution is an unstable limit cycle.

If we vary the policy parameter ϕ_π, while setting $\beta = 1$ and keeping the other parameters constant at their baseline values, we numerically find a Hopf bifurcation at $\phi_\pi = 1.0176$, a period doubling bifurcation at $\phi_\pi = 12.76$, and a branching point at $\phi_\pi = 1$.

Decreasing the value of ω results in a higher value of the period doubling bifurcation parameter ϕ_π in absolute value, except when $\alpha = 0$. Then changes in ω have no effect. On the other hand, decreasing the value of α results in a lower value of the bifurcation parameter ϕ_π in absolute value, except when $\omega = 1$. Then changes in α have no effect. Hopf bifurcation at $\beta = 1$ appears independent of the values of α and ω.

2.5. Backward-Looking Taylor Rule

Consider the model consisting of equations (4), (5), and the following policy equation:

$$r_t = \overline{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} \ . \tag{20}$$

We can write the system in the standard form $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t + \mathbf{d}_t$:

$$E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t + \begin{bmatrix} -\dfrac{1+\alpha(\omega-1)}{\sigma}\overline{r}_t \\ 0_t \\ E_t\overline{r}_{t+1} \end{bmatrix}, \tag{21}$$

where $\mathbf{C} = \begin{bmatrix} \dfrac{\mu}{\beta}\left(1+\dfrac{\varphi(1+\alpha(\omega-1))}{\sigma}\right)+1 & -\dfrac{1+\alpha(\omega-1)}{\beta\sigma} & \dfrac{1+\alpha(\omega-1)}{\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) & \dfrac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & 0 \end{bmatrix}$.

In order for a 3-dimensional system to exhibit a Hopf bifurcation, the system should have a real root and a pair of complex conjugate roots on the unit circle. The following proposition states the conditions for the system (21) to exhibit a Hopf bifurcation.

Proposition 5: *The system (21) undergoes a Hopf bifurcation, if and only if the following transversality condition holds,*

$$\left.\frac{\partial\left|\lambda_i(\boldsymbol{\phi})\right|}{\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0 \text{ for some } j,$$

and the following conditions also are satisfied:

$$(i)\ \phi_x + \phi_\pi\mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right) - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0, \tag{22}$$

$$(ii)\ \phi_x(\beta-1) + \mu\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)(1-\phi_\pi) < 0, \tag{23}$$

$$(iii)\ \left(\phi_x+\phi_\pi\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\mu\right)^2+\left(\phi_x+\phi_\pi\left(\frac{\sigma}{1+\alpha(\omega-1)}+\varphi\right)\mu\right)\xi_1-\phi_x\xi_2=\xi_3. \qquad (24)$$

We numerically detect a period doubling bifurcation at $\phi_x = 1.91$. Lowering ω and raising α increase the value of the bifurcation parameter. Starting from the point $\phi_x = 1.91$, we construct the period doubling bifurcation boundary by varying ϕ_x and ϕ_π simultaneously, as shown in Figure 9. Note that along the bifurcation boundary, the positive values of the bifurcation parameter ϕ_x lie between 0 and 13. As the magnitude, ϕ_π, of the central bank reaction to inflation increases, smaller values of parameter ϕ_x would be sufficient to cause period doubling bifurcation under a backward-looking Taylor rule.

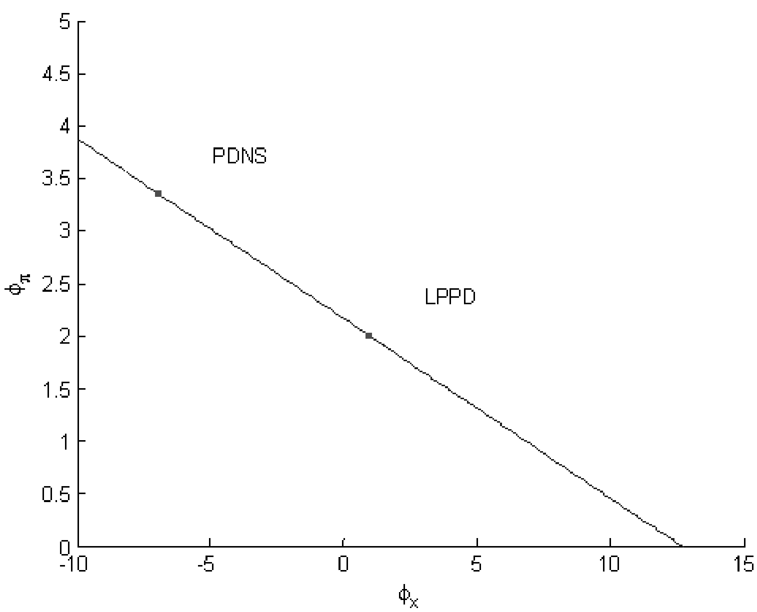

Figure 9: Period doubling bifurcation boundary for ϕ_x in model (21).

While varying both parameters ϕ_x and ϕ_π simultaneously, our numerical analysis with CL MatCont detects a codimension-2 fold-flip bifurcation (LPPD) at $(\phi_x, \phi_\pi) = (0.94, 2.01)$ and a flip-Hopf bifurcation (PDNS) at $(\phi_x, \phi_\pi) = (\text{-}6.98, 3.36)$.

But treating the policy parameter ϕ_π as the potential source of bifurcation, while keeping the other parameters constant at their benchmark values, our numerical analysis with CL MatCont indicates a period doubling bifurcation at $\phi_\pi = 11.87$ .We find period doubling bifurcation at relatively large values of the parameter ϕ_π , but still within the subset of the parameter space defined to be feasible by Bullard and Mitra (2002). Lowering ω and raising α increase the value of the bifurcation parameter ϕ_π .

2.6. Backward-Looking Taylor Rule with Interest Rate Smoothing

Consider the following model, consisting of equations (4) and (5) and the following policy rule:

$$r_t = \overline{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1} + \phi_r r_{t-1} \tag{25}$$

The system can be written in the form $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t + \mathbf{d}_t$,

$$E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t + \begin{bmatrix} -\dfrac{1-\alpha+\alpha\omega}{\sigma}\overline{r}_t \\ 0 \\ E_t \overline{r}_{t+1} \end{bmatrix}, \tag{26}$$

where $\mathbf{C} = \begin{bmatrix} \mu\left(\dfrac{1}{\beta} + \varphi\dfrac{1-\alpha+\alpha\omega}{\beta\sigma}\right) + 1 & -\dfrac{1-\alpha+\alpha\omega}{\beta\sigma} & \dfrac{1-\alpha+\alpha\omega}{\sigma} \\ -\mu\left(1+\varphi\dfrac{1-\alpha+\alpha\omega}{\sigma}\right) & \dfrac{1}{\beta} & 0 \\ \phi_x & \phi_\pi & \phi_r \end{bmatrix}$.

Based on Theorem 2, the following Proposition states the conditions for the system (26) to exhibit a Hopf bifurcation.

Proposition 6: *The system (26) undergoes a Hopf bifurcation at* $\boldsymbol{\phi}^*$, *if and only if the transversality condition* $\left.\dfrac{\partial\left|\lambda_i(\boldsymbol{\phi})\right|}{\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0$ *holds and the following conditions are satisfied:*

(i) $$\left|\frac{\phi_x-\phi_r\dfrac{\sigma}{1-\alpha+\alpha\omega}+\phi_\pi\left(\dfrac{\sigma\mu}{1+\alpha(\omega-1)}+\varphi\mu\right)}{\dfrac{\beta\sigma}{1-\alpha+\alpha\omega}}\right|<1,$$

with $\phi_x-\phi_r\xi_2+\phi_\pi\xi_3<\dfrac{\beta\sigma}{1-\alpha+\alpha\omega}$,

and $\phi_r<\phi_x\xi_2+\phi_\pi\xi_1+\beta$,

(ii)

$$\left|\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\mu\left(\frac{1}{\beta}+\varphi\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)-\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)\right|<1+\phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1}{\beta}$$

with $\phi_x\xi_2+\phi_\pi\xi_1-(1+\phi_r)\xi_0<0$,

and $\phi_x\xi_3-\xi_4(\phi_\pi+\phi_r-1)<0$,

(iii)

$$\phi_r\left(\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)-\phi_x\frac{1-\alpha+\alpha\omega}{\sigma}+\frac{1}{\beta}+\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)\left(\phi_r+\frac{1+\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}+1\right)$$
$$=1-\left(\phi_x\frac{1-\alpha+\alpha\omega}{\beta\sigma}-\phi_r\frac{1}{\beta}+\phi_\pi\left(\frac{\mu}{\beta}+\varphi\mu\frac{1-\alpha+\alpha\omega}{\beta\sigma}\right)\right)^2.$$

Given the benchmark values of the parameters and setting $\phi_r=0.5$, a period doubling bifurcation is detected numerically at $\phi_x=3$. When $\phi_r=1$, period doubling bifurcation occurs at $\phi_x=4.09$.

Starting from this bifurcation point, we construct the bifurcation boundary by varying ϕ_x and ϕ_π simultaneously, and then ϕ_x and ϕ_r simultaneously, as shown in Figure 10. Note that in (ϕ_x,ϕ_π)-space, the bifurcation boundary lies between $\phi_x=3$ and $\phi_x=3.25$. As a result, period doubling bifurcation occurs for a very limited set of values of the parameter ϕ_x, regardless of the value of the parameter ϕ_π. This is not the case in (ϕ_r,ϕ_x)-space, as shown in the second part of the Figure 10. The bifurcation parameter ϕ_x varies more elastically in response to changes in parameter ϕ_r along the period-doubling bifurcation boundary.

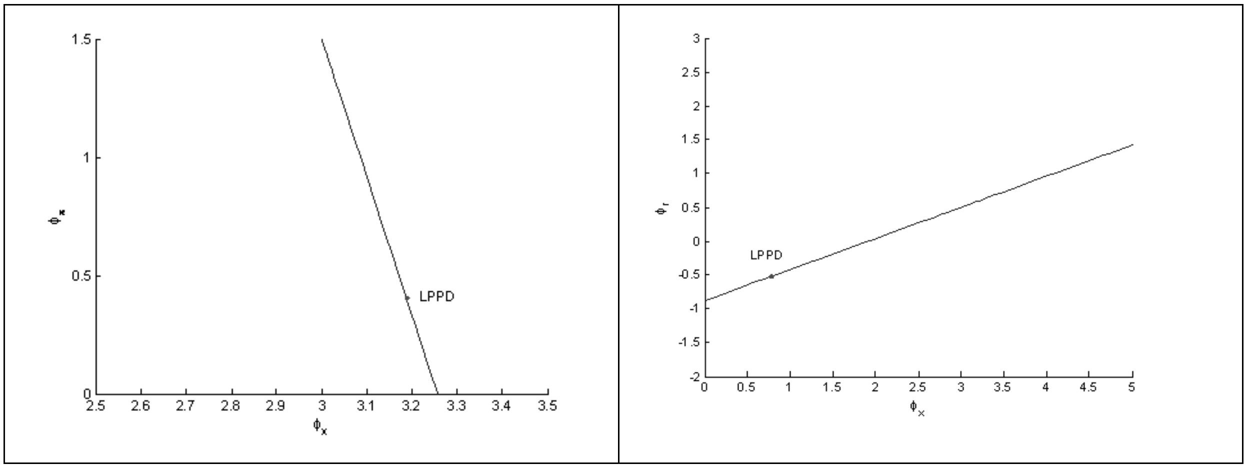

Figure 10: Period-doubling bifurcation boundaries for ϕ_x in (ϕ_r, ϕ_x)-space and (ϕ_x, ϕ_π)-space for model (26).

Our numerical analysis with CL MatCont indicates codimension-2 fold-flip bifurcations at $(\phi_x, \phi_\pi) = (0.41, 3.19)$ and at $(\phi_x, \phi_r) = (0.78, -0.52)$, as well as flip-Hopf bifurcations at $(\phi_x, \phi_\pi) = (-10.44, 5.04)$ and $(\phi_x, \phi_r) = (-0.74, -1.23)$. While treating ϕ_x as the bifurcation parameter, we found that lowering ω or raising α increases the value of the bifurcation parameter. Bifurcation disappears at $(\alpha, \omega) = (1, 0)$.

2.7. Hybrid Taylor Rule

Consider the model consisting of equations (4) and (5), along with the following policy rule:

$$r_t = \bar{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x x_t \tag{27}$$

The model can be written in normal form, $E_t \mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t$, as follows:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \mathbf{C} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \tag{28}$$

where

$$\mathbf{C} = \begin{bmatrix} \dfrac{\beta\phi_x + \mu\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right)(1-\phi_\pi)}{\dfrac{\beta\sigma}{1+\alpha(\omega-1)}} + 1 & \dfrac{(\phi_\pi - 1)\left(1+\alpha(\omega-1)\right)}{\beta\sigma} \\ -\dfrac{\mu}{\beta}\left(\dfrac{\sigma}{1+\alpha(\omega-1)} + \varphi\right) & \dfrac{1}{\beta} \end{bmatrix}.$$

Assuming the system has a pair of complex conjugate eigenvalues, we can expect to find Hopf bifurcation, if certain additional conditions are satisfied. Using Theorem 1 with respect to the policy parameter ϕ_x, the conditions for the existence of a Hopf bifurcation are stated in the following Proposition.

Proposition 7: *The system (28) exhibits a Hopf bifurcation, if and only if* $\Delta < 0$ *and*

$$\phi_x^* = \frac{\sigma(\beta-1)}{1+\alpha(\omega-1)}. \tag{29}$$

Numerical analysis with CL MatCont indicates a period doubling bifurcation at $\phi_x = -1.92$, as well as a Hopf bifurcation at $\phi_x = -0.01$, given the benchmark values of the system parameters. Under the hybrid Taylor rule, values of the bifurcation parameters are outside the feasible region of the parameter space, since the New Keynesian economic theory normally assumes positive values for policy parameters. Therefore, we conclude that the feasible set of parameter values for ϕ_x does not include a bifurcation boundary.

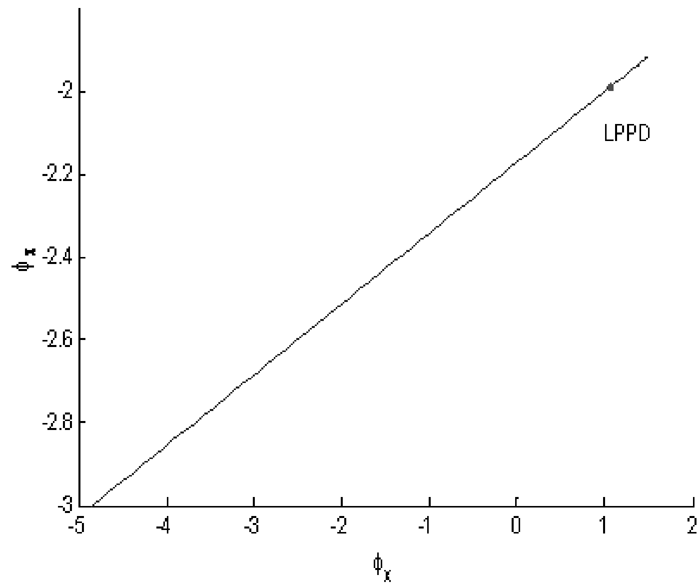

Figure 11: Period doubling bifurcation boundary for ϕ_x in model (7).

Figure 11 illustrates the values of parameters ϕ_x and ϕ_π along the bifurcation boundary. Notice that in (ϕ_π, ϕ_x)-space, the bifurcation parameter ϕ_x varies in the same direction as ϕ_π along the period-doubling bifurcation boundary. As the policy maker's choice for ϕ_π increases, higher values of ϕ_x are required to cause a period doubling bifurcation.

Decreasing the value of ω results in a higher absolute value of the period doubling bifurcation parameter, except when $\alpha = 0$. Then changes in ω have no effect. On the other hand, decreasing the value of α results in a lower absolute value of the bifurcation parameter, except when $\omega = 1$. Then changes in α have no effect.

Figure 12 illustrates solution paths from model (28) with stability properties indicating Hopf bifurcation. The inner spiral trajectory is converging to the equilibrium point, while the outer spiral is diverging. The limit cycle is unstable

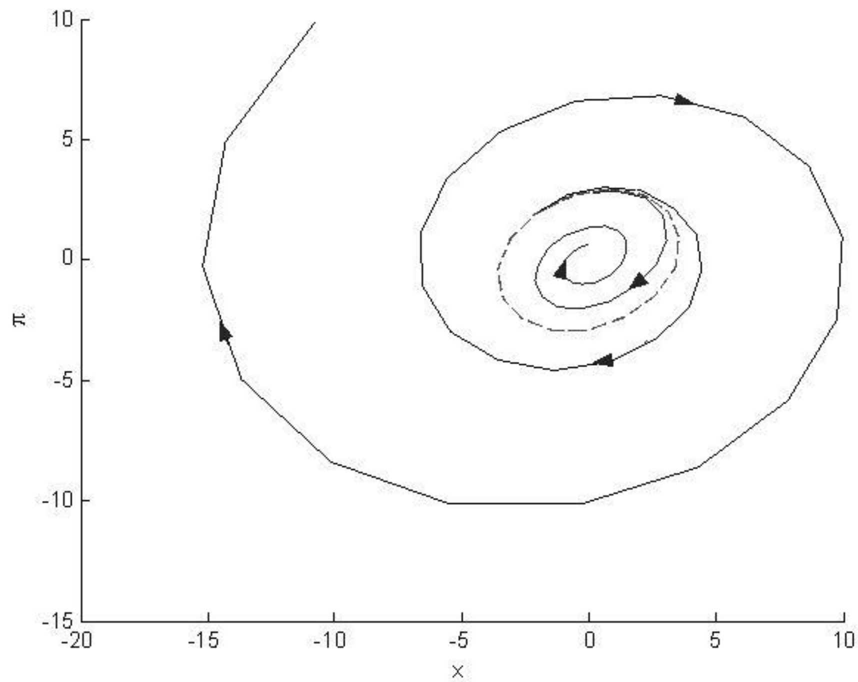

Figure 12: Phase diagram indicating a Hopf bifurcation under the hybrid Taylor rule.

3. Conclusion

We ran bifurcation analyses on the open-economy New Keynesian model developed by Gali and Monacelli (2005). We have shown that in a broad class of open-economy New Keynesian models, the degree of openness has a significant role in equilibrium determinacy and emergence of bifurcations. We acquired that result with various forms and timings of monetary policy rules. The open economy framework brings about more complex dynamics along with a wider variety of qualitative behaviors and policy responses. We established the conditions for Hopf bifurcation with each model, based on the Hopf Bifurcation Theorem. Numerical analyses are performed using our theoretical results and also to search for other types of bifurcation. Limit cycles and periodic behaviors are found, but in some cases only for unrealistic parameter values. Our numerical analyses with CL MatCont also identify the existence of the period doubling bifurcations. In each case, we then numerically constructed corresponding bifurcation boundary diagrams.

The most important findings of this study regard the effects of the openness of economy on the values of bifurcation parameters. Under the monetary policy rules, the degree of openness in New Keynesian models changes the value of bifurcation parameters. But the bifurcation stratification of the confidence regions remains a serious issue. Inferences from New Keynesian models and policy designs using those models should be qualified by the risk that the simulations and inferences could have been produced with parameter settings on the wrong side of a nearby bifurcation boundary. Stratification of the confidence regions, as found in the closed-economy New Keynesian models examined by Barnett and Duzhak (2008, 2010), remains equally as problematic to open economy New Keynesian functional structures.

Comparing the results from Barnett and Duzhak's (2010) closed economy analysis with our open economy cases does not provide clear conclusions about whether openness makes the New Keynesian model more sensitive to bifurcations. One reason is the Gali and Monacelli model's broader set of parameters, including deep parameters relevant to the open economy. The fact that the studies use different sets of benchmark values for the parameters makes direct comparison harder. While the bifurcation phenomena exist in both open and closed economy New Keynesian models, we do not find evidence that open economies are more vulnerable to the problem than closed economies.

Our analysis is restricted to special cases within the framework of open-economy New Keynesian structures, closely following Gali and Monacelli (2005). Therefore, generalizing our results to real economies would require more results with other open-economy New Keynesian models. While a large change in policy parameters can produce unanticipated bifurcation, our research is not about endogenous bifurcations. Our model's parameters, including the policy rule parameters, are fixed and do not move on their own.

REFERENCES

Airaudo, M. and L.-F. Zanna (2005). Interest rate rules, endogenous cycles and chaotic dynamics in open economies. Collegio Carlo Alberto Working Papers, No: 171, December.

Barnett W.A. and E.A. Duzhak (2008). Non-robust dynamic inferences from macroeconometric models: Bifurcation of confidence regions. Physica A, Vol 387, No 15, June , 3817-3825,.

Barnett W.A. and E.A. Duzhak (2010). Empirical assessment of bifurcation regions within New Keynesian models. Economic Theory, Vol. 45, 99–128.

Barnett, W.A. and Y. He (1999). Stability analysis of continuous time macroeconometric systems. Studies in Nonlinear Dynamics and Econometrics, 3(4), 169-188.

Barnett, W.A. and Y. He,(2002). Stabilization policy as bifurcation selection: Would stabilization policy work if the economy really were unstable? Macroeconomic Dynamics, 6, 713–747.

Barnett, W.A. and Y. He (2006). Singularity bifurcations. Journal of Macroeconomics, Vol 28, Issue 1, March , 5-22,.

Barnett, W. A., S. Banerjee, E. Duzhak, and R. Gopalan (2011). Bifurcation analysis of Zellner's Marshallian macro model, Journal of Economic Dynamics and Control 9(35), 1577-1585.

Bergstrom, A.R., Nowman, K.B., and C.R. Wymer (1992). Gaussian estimation of a second order continuous time macroeconometric model of the United Kingdom. Economic Modelling 9, 313-352.

Bullard, J. and K. Mitra (2002). Learning about monetary policy rules. Journal of Monetary Economics, Vol. 49, Issue 6, September, 1105-1129.

Demirel, U. D. (2010). Macroeconomic stabilization in developing economies: Are optimal policies procyclical? European Economic Review, Vol. 54, Issue 3, April, 409-428.

Galí, J. and T. Monacelli (2005). Monetary policy and exchange rate volatility in a small open economy. Review of Economic Studies, Vol. 72, No 3, July.

Gandolfo, G. (1996). Economic Dynamics. 3rd Edition, New York and Heidelburg, Springer-Verlag.

Govaerts W., Kuznetsov Y. A., R. Khoshsiar G. and H.G.E Meijer (2008). CL_MatContM: A toolbox for continuation and bifurcation of cycles of maps, March. http://www.matcont.ugent.be/doc_cl_matcontM.pdf

Grandmont, J.M. (1985). On endogenous competitive business. Econometrica, Vol.53, 995-1045.

Leeper, E. and C. Sims (1994). Toward a modern macro model usable for policy analysis. NBER Macroeconomics Annual, 81-117.

Leith, C., Moldovan, I. and R. Rossi (2009). Optimal monetary policy in a New Keynesian model with habits in consumption. European Central Bank Working Paper Series, No 1076, July.

Schettkat, R. and R. Sun (2009). Monetary policy and European unemployment. Oxford Review of Economic Policy, Vol. 25, Issue 1, 94-108.

Taylor , John B. (1993). Discretion versus policy rules in practice. Carnegie-Rochester Conferences Series on Public Policy, 39, December, 195-214.

Walsh, E. Carl (2003). Monetary Theory and Policy, Cambridge, MA, MIT Press, Second Ed.

Wen G., Xu, D., and X. Han (2002). On creation of Hopf bifurcations in discrete-time nonlinear systems. Chaos, Vol. 12, Issue 2, 350-355.