A Two-Country NATREX Model for the Euro/Dollar

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Abstract

This paper develops a NATREX (NATural Real EXchange rate) model for two large economies, the Eurozone and the United States. The NATREX approach has already been adopted to explain the medium-long term dynamics of the real exchange rate in a number of industrial countries. So far, however, it has been applied to a one-country framework where the "rest of the world" is treated as given. In this paper, we build a NATREX model where the two economies are fully specified and allowed to interact. Our theoretical model offers the basis to empirical estimation of the euro/dollar equilibrium exchange rate that will be carried out in future research.

Key words: NATREX; equilibrium exchange rate; euro/dollar; structural approach

JEL classification: F31; F36; F47

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1 Introduction

Since the introduction of the euro as common currency for 11 member states of the EU, the euro-dollar exchange rate has surprised most observers for its highly unexpected dynamics. As figure 1 shows, it has depreciated steadily and almost continuously from January 1999 to February 2002, then it reversed this trend, began to rise and kept doing so until December 2004. Thereafter the euro has substantially stabilized at a high level with respect to the US dollar. The study of such an anomalous behavior has given rise to a growing literature that looks for theoretically coherent explanations. They include black market holdings of euros (Sinn and Westermann, 2001); reactions to fiscal policy (Cohen and Loisel, 2000); responses to fundamental variables such as productivity differentials (e.g. Alquist and Chinn 2002, Bailey et al. 2001), growth rate of money, per capita income, population and capital accumulation (Welfens, 2001); inflation differentials, relative rates of return of US versus Euroland assets, current account (e.g. De Grauwe, 2000, De Grauwe and Grimaldi, 2005. For a survey see Shams, 2005). Despite the large number of studies on the issue, however, the state of the art does not seem to have reached any satisfactory arrival point: the behavior of the euro-dollar between 1999 and 2004 remains mostly puzzling to the economic theory. Yet, understanding what drives the euro-dollar real exchange rate developments is crucial for both theoretical and policy implications. In this paper we look at the issue from a different perspective. Rather than trying to explain the *actual* real exchange rate (RER), we study the determinants of the real *equilibrium* exchange rate (REER) of the euro/dollar in order to provide a yardstick against which the development of the actual real exchange rate is gauged.

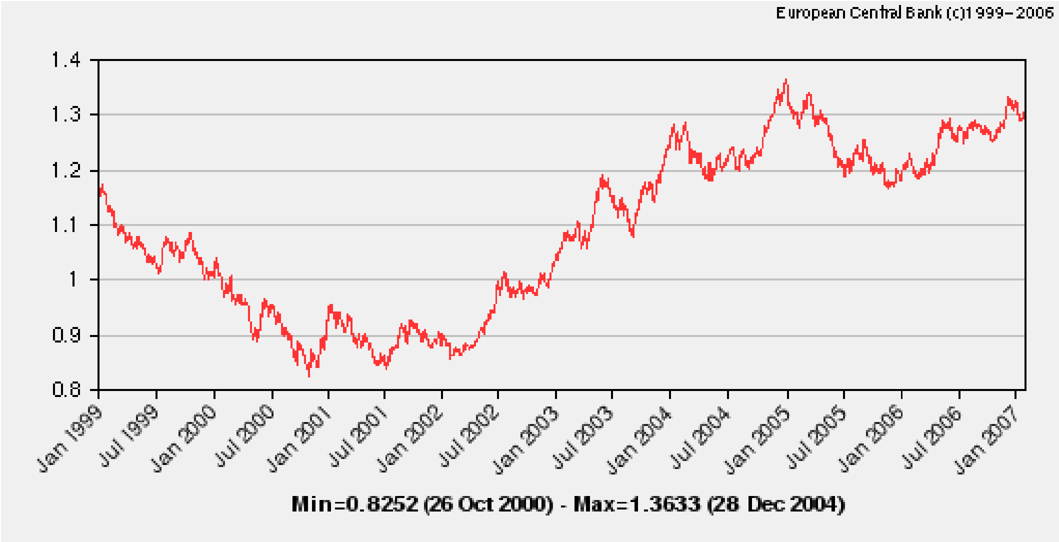

Fig. 1. Source: European Central Bank, 2007

The REER is a sustainable rate that satisfies internal (economy is at capacity output) and external (long-term accounts of balance of payments are in equilibrium) balance. Several approaches have been suggested to determine the real equilibrium exchange rate: monetary model, FEER (Fundamental Equilibrium Exchange Rate), DEER (Desirable Equilibrium Exchange Rate), BEER (Behavioral Equilibrium Exchange Rate), PEER (Permanent Equilibrium Exchange Rate), and the NATREX (NATural Real EXchange rate)[1] to mention the most important ones (see MacDonald and Stein, 1999, for a survey). They intend to answer questions like: What are the determinants and dynamics of the REER? To what extent is it affected by relative prices (PPP) and to what extent by real fundamentals? How can misalignments been measured and what are their causes? How do specific policies affect the equilibrium exchange rate? Responses to these questions and the like are crucial to the ECB to correctly define its monetary policy and to evaluate the effects of the EU enlargements upon the equilibrium value of the euro (see Stein, 2001, 2002).

In this paper, we adopt the NATREX approach (Stein, 1990). It is based on a specific theoretical dynamic stock-flow model to derive the equilibrium real exchange rate. The equilibrium concept ensures simultaneously internal and external equilibrium, and reflects

[1] Obstfeld and Rogoff (2001) use the New Open Economy Macroeconomics (NOEM) to address misalignment issues. For a survey on this approach see Lane (2001).

the behavior of the fundamental variables behind investment and saving decisions in the absence of cyclical factors, speculative capital movements and movements in international reserves. Several notable studies have already adopted the NATREX approach to explain the medium-long term dynamics of the real exchange rate in a number of industrial countries: USA (Stein, 1995, 1997), Australia (Lim and Stein, 1995), Germany (Stein and Sauernheimer, 1996), France (Stein and Paladino, 1999), Italy (Gandolfo and Felettigh, 1998; Federici and Gandolfo, 2002), Belgium (Verrue and Colpaert, 1998), the Eurozone (Detken and Marin, 2002; Duval, 2002), China (Holger et al., 2001), and Hungary (Karadi, 2003). However, in the previous literature, the NATREX approach has been applied to a one-country framework where the "rest of the world" is treated as given. Yet, the recognition of the interdependence of the world economy requires to extend the framework and endogenize the "rest of the world" in a two-country context. This work is the first to build a two-country NATREX model and apply it to the Eurozone and the US. The theoretical model presented in this paper offers the basis for empirical estimation that will be carried out in future research.

The remainder of this paper is organized as follows. Section 2 briefly introduces the theory of NATREX. Section 3 reviews the main alternative approaches to the equilibrium exchange rate provided by the literature and the previous empirical applications to the euro/dollar. Section 4 presents our theoretical model, whereas Sections 5 and 6 respectively illustrate the NATREX derivation and the qualitative analysis of the model.

2 NATREX approach

In this section we briefly recall the theoretical framework of the NATREX approach: for a complete treatment, the reader should consult Allen (1995) and Stein (1995, 2001, 2006).

Following Stein (2001), the NATREX is a moving equilibrium exchange rate responding to continual changes in exogenous real fundamentals, and representing the trajectory of the medium-to-long run equilibrium. Therefore it may be interpreted as a benchmark from which misalignments of the actual real exchange rate are measured. Denoting the real economic fundamentals by Z, foreign debt by F and domestic output by Y, the actual real exchange rate (E) can be written as:

$$E = \{E(Z) + E(F,Z) - E(Z)\} + [E - E(F,Z)]$$

where $E(F,Z)$ is the NATREX. Whence, simplifying the terms within the curly brackets, we have:

$$E = NATREX + \text{deviations}$$

The NATREX represents the intercyclical equilibrium real exchange rate that ensures the balance of payments' equilibrium in the absence of cyclical factors, speculative capital movements, changes in monetary policy, and movements in international reserves. In other words, the NATREX is the equilibrium real exchange rate that would prevail if the above-mentioned factors could be removed and the GNP were at capacity. Since it is an equilibrium concept, the NATREX should guarantee the attainment of both internal and external equilibrium, the focus being on the long run. The long-run internal equilibrium is achieved when the economy is at capacity output, that is when the rate of capacity utilization is at its stationary mean: deflationary and inflationary pressures are excluded. The long-run external equilibrium is achieved when the long-term accounts of the balance of payments are in equilibrium. Short term (speculative) capital movements and movements in official reserves are bound to be short term transactions, since they are unsustainable in the long run. In the long-run equilibrium they must average out at zero; hence, the excess of national (private plus public) investment over national saving must be entirely financed through international long term borrowing. Thus, unlike other models of REER, the NATREX approach adds dynamics to the determination of the REER.

Under these conditions long term capital inflows and excess national investment over saving coincide, so that also the real market long run equilibrium condition and the long term external equilibrium condition coincide. The framework of the approach lies on the national income accounting equation:

$$S - I = CA, \tag{1}$$

where CA is the balance of payments' current account, the private and public sectors having been aggregated into a single one. Relation (1) is intended in real terms: the model assumes neutrality of money and that monetary policy keeps inflation at a level compatible with internal equilibrium (at least in the long run). Therefore, the focus being on the real part of the economy, there is no need to model the money market. Perfect international capital mobility is assumed: the real interest rate is driven by the portfolio equilibrium condition or real interest parity condition, possibly with a risk premium.

The system is assumed to be self-equilibrating (hence the adjective *natural* in the acronym NATREX). Take for example an initial position of full equilibrium ($S - I = CA = 0$) and suppose an exogenous shock leads to a situation where $S - I < 0$. Given the perfect international capital mobility, the interest rate cannot play the role of the adjustment variable; rather, the difference between national investment and national saving originates a corresponding inflow of long-term capital. The RER appreciates accordingly, leading to a deterioration in the current account. The capital inflow also causes an increase in the stock of foreign debt, which in turn determines a decrease in total consumption (household and government) and hence an increase in saving, until equilibrium is restored. In conclusion, the RER is the adjustment variable in equation (1).

The hypothesis of perfect foresight is rejected. Rather, rational agents that efficiently use all the available information will base their intertemporal decisions upon a sub-optimal feedback control (SOFC) rule (Infante and Stein, 1973; Stein, 1995). Basically, SOFC starts from the observation that the optimal solution derived from standard optimization techniques in perfect-knowledge perfect-foresight models has the saddle-path stability property, hence the slightest error in implementing the stable arm of the saddle will put the system on a trajectory that will diverge from the optimal steady state. Actual optimizing agents know that they do not possess the perfect knowledge required to implement the stable arm of the saddle without error, hence it is rational for them to adopt SOFC, which is a closed loop control that only requires current measurements of a variable, not perfect foresight, and will put the economy on a trajectory which is asymptotic to the unknown perfect-foresight stable arm of the saddle.

The consumption and investment functions are derived accordingly, through dynamic programming techniques with feedback control. No difference is made between the private and the public decisional process. The model can be solved for its medium run and long run (steady state) solutions. Any perturbation on the real fundamentals of the system pushes the equilibrium RER on a new medium-to-long-run trajectory. Since cyclical, transitory and speculative factors are considered noise, averaging out at zero in the long run, the actual RER converges to the equilibrium trajectory. The PPP theory turns out to be only a special case of the NATREX approach: "the issue is not whether or not the real exchange rate is stationary over an arbitrary period, but whether it reflects the [real] fundamentals." (Stein, 1995, p. 43).

3 Alternative approaches to the euro/dollar equilibrium exchange rate

3.1 Other than NATREX

Several ways of calculating *equilibrium* exchange rates have been suggested in the literature. While for a complete survey we refer the reader to MacDonald and Stein (1999), in this subsection we only consider empirical works that have been applied to the euro/dollar[2] (for a survey see Stein, 2001, and ECB, 2002. For further considerations on the alternative approaches to the euro/dollar see Williamson 2004). We can distinguish three broad categories: *monetary*, *statistical* and *structural* approaches.

The monetary approach models the nominal exchange rate as the interaction between the relative demand for and supply of two currencies. Accordingly the nominal exchange rate, e.g., of the euro/dollar appreciates (depreciates) in the event of a relative decrease (increase) in the money supply of, or a relative increase (decrease) in the money demand for, euros. It follows that in addition to monetary aggregates and base money, further determinants of the exchange rate are factors that influence the demand for money such as income, interest rates and inflation rate. Monetary models for the euro/dollar exchange rate have been developed and estimated by Chinn and Alquist (2002) and van Aarle et al. (2000).

The BEER (Behavioral Equilibrium Exchange Rate; Clark and MacDonald, 1999) belongs to the second group. Differently from the structural approaches, the BEER relies on a

[2] We remark that, due to the lack of long-span series for the *actual* euro/dollar exchange rate, most of these studies use the *synthetic* euro.

statistical concept of equilibrium. Its calculation involves the estimation of a reduced-form equation that explains the behavior of the actual real exchange rate in terms of a set of long-run (such as terms of trade, net foreign assets, relative government debt, and price indices) and medium-run (such as real interest rate differential) economic fundamentals, and a set of transitory factors affecting the real exchange rate in the short run. A number of empirical studies have applied this methodology to estimate the equilibrium exchange rate of the euro/dollar: Gern et al. (2000), Clostermann and Shnatz (2000), and Lorenzen and Thygesen (2000); and of the euro real effective exchange rate: Koen et al. (2001), and Maeso-Fernandez et al. (2001)[3].

The measured BEER, generated from variables that are highly persistent and often non-stationary, is likely to be a very persistent series as well. Adopting the Johansen cointegration method, the vector of cointegrated variables underlying the BEER can be decomposed into permanent and transitory components (see Clark and MacDonald, 2004). The PEER (Permanent Equilibrium Exchange Rate) is derived considering only the former ones. The PEER has been estimated for the euro/dollar by Alberola et al. (1999) and Hansen and Roeger (2000), and for the euro real effective exchange rate by Alberola et al. (1999), and Maeso-Fernandez et al. (2001).

Among the structural approaches to the real equilibrium exchange rate, the FEER (Fundamental Equilibrium Exchange Rate, Williamson, 1985, 1994) is the main alternative to the NATREX (illustrated in Section (2)). In this approach the equilibrium exchange rate is defined as the real exchange rate that is consistent with macroeconomic balance, which is generally interpreted as when the economy is operating at full employment and low inflation (internal balance), and a current account that is sustainable, *i.e.* reflects underlying net capital flows (external balance). The core of the FEER approach is the identity equating the current account (CA) to the (negative of) the capital account (NFA). The attention of the method is on the determinants of the current account, which is generally explained as a function of home and foreign output or demand and the real effective exchange rate. Given the parameters of the current account, including in particular the sensitivity of current ac-

[3]Bénassy-Quéré et al. (2006) adopt the BEER approach to derive consistent real effective equilibrium exchange rate for 15 countries of the G20.

count flows to the real exchange rate, the FEER is calculated using an exogenously given estimate of sustainable net capital flows. The main difference between the FEER and the NATREX lies on the fact that while the former is a medium-term concept where external and internal balance prevails, the latter explains the determinants of the evolution of the real equilibrium exchange rate in the medium to the longer run, where the net foreign debt is constant and the capital stock is at its steady state level. The FEER approach has been adopted to estimate the real equilibrium exchange rate of the euro/dollar by Wren-Lewis and Driver (1998), and Borowski and Couharde (2000).

Finally two further approaches to the equilibrium exchange rate are the DEER (Desirable Equilibrium Exchange Rate) and the CHEER (Capital Enhanced Equilibrium Exchange Rate). The former (see Bayoumi et al., 1994) is a variant of the FEER and assumes target values for the macroeconomic objectives, such as targeted current account surplus for each country. The latter (see MacDonald, 2000, 2006) exploits the cointegration properties of the interest rate differentials and the real exchange rate.

3.2 NATREX

At the best of our knowledge, there are only two applications of the NATREX approach to the Eurozone, namely Detken et al. (2002)[4] and Duval (2002), both considering the exchange rate of the synthetic euro vis-à-vis the US dollar from early 1970 to 2000. Whereas the former employ a multiple-equation structural model estimation, the latter implements a reduced form single equation estimation as described in what follows.

Duval (2002) builds a dynamic model for the real equilibrium exchange rate of the euro/dollar combining the NATREX and the BEER approaches. First, the long run REER for tradable goods (e_T^{LT} denotes the variable in logarithmic terms) is modelled as a NATREX. Accordingly, the author obtains the reduced-form equation of e_T^{LT} as a function of the time preferences (c_t) and technological progress (a_t):

$$e_T^{LT,i} = e_T^{LT,i}(a_t, c_t) \tag{2}$$

where i stands for either the Eurozone (EU) or the United States (US). Second, (2) is combined with the real equilibrium exchange rate of the euro/dollar for the whole economy

[4]See also Detken and Marin-Martinez (2001).

(e^{LT}), that is:

$$e^{LT} = e_T^{LT} + \alpha \left[\left(p_{NT}^{US} - p_T^{US} \right) - \left(p_{NT}^{EU} - p_T^{EU} \right) \right] \tag{3}$$

where p_T^{EU} (p_T^{US}) is the natural logarithm of the European (US) price index for tradables, p_{NT}^{EU} (p_{NT}^{US}) the corresponding index in natural logarithms for non-tradables, and α the share of non-tradable goods in GDP (the same in both countries, by construction). Therefore (3) turns out to be:

$$e^{LT} = e_T^{LT}(a_t, c_t) + \alpha \left[\left(p_{NT}^{US} - p_T^{US} \right) - \left(p_{NT}^{EU} - p_T^{EU} \right) \right] \tag{4}$$

Following Clark and MacDonald (1998), the short run REER (e^{ST}) is defined as:

$$e^{ST} = e_T^{LT}(a_t, c_t) + \alpha \left[\left(p_{NT}^{US} - p_T^{US} \right) - \left(p_{NT}^{EU} - p_T^{EU} \right) \right] + \theta \left(r^{US} - r^{EU} \right) \tag{5}$$

where r^{EU} (r^{US}) is the European (US) ex-ante real interest rate, which, given that $e^{ST} = e_T^{ST}(a_t, c_t) + \alpha \left[\left(p_{NT}^{US} - p_T^{US} \right) - \left(p_{NT}^{EU} - p_T^{EU} \right) \right]$, becomes:

$$e_T^{ST} = e_T^{LT}(a_t, c_t) + \theta \left(r^{US} - r^{EU} \right) \tag{6}$$

Relying on the described relations, Duval (2002) employs time series and Johansen cointegration analysis of the variables involved (e_T, and its long run fundamentals, a_t, c_t). Then, under the assumption of a single cointegrating vector, estimates a Vector Error Correction Model (VECM hereafter), and finally the long run relationship using the Stock and Watson (1993) methodology. Results suggest that the time preferences and the relative prices explain the most part of the fluctuations of the euro/dollar; whereas the direct effects of the technological progress are modest. Yet, the overall impact of technological progress, including the indirect effect on relative prices of non-tradables, is in magnitude comparable to that of time preferences. Finally, the author is able to measure the misalignment of the observed real exchange rate of the euro/dollar against the equilibrium (both long and short term) one. According to this study, the euro turns out to be undervalued between early 70s and mid-80s, and then from 1999 onwards.

The approach followed by Detken et al. (2002) departs from the previous one as it involves the estimation of the NATREX model in its structural form. The model is defined as follows:

National account identity:

$$\frac{C}{Y}+\frac{I}{Y}+\frac{BT}{Y}=1-\frac{SCN}{Y} \tag{7}$$

Behavioral equations:

$$\frac{I}{Y}=\alpha_1+\alpha_2\widehat{A}-\alpha_3\frac{K}{YR_{-1}}-\alpha_4 r^{LT}-3-\alpha_5 E_{-4} \tag{8}$$

$$\frac{C}{Y}=\alpha_6+\alpha_7\frac{K}{YR}A-\alpha_8\frac{F}{Y}-\alpha_9 r^{ST}+\alpha_{10} i^{LT} \tag{9}$$

$$\frac{BT}{Y}=-\alpha_{11}-\alpha_{12}E-\alpha_{13}\frac{K}{Y_{-4}}+\alpha_{14}\frac{C^{'}}{Y^{'}_{-4}}+\alpha_{15}tot \tag{10}$$

Real uncovered interest parity:

$$E-\widehat{E}=r-r^{'} \tag{11}$$

Fisher equation:

$$\left(i-i^{'}\right)=\left(\pi-\pi^{'}\right)+\left(r-r^{'}\right) \tag{12}$$

Stock accumulation:

$$F-F_{-1} = -CA \tag{13}$$

$$K = (1-\delta)K_{-1}+\frac{I}{P_I} \tag{14}$$

Steady state of stock variables:

$$\frac{F}{Y}=-\left(\frac{1+q}{q}\right)\frac{CA}{Y} \tag{15}$$

$$\frac{K}{YR}=\frac{1+g}{\delta+g}\frac{I}{Y}\frac{P_Y}{P_I} \tag{16}$$

where BT is balance of trade, I is total (private plus public) investment, C is total consumption, SCN variation in stocks. i and r are respectively nominal and real interest rates (LT and ST standing for long and short term), Y and YR are nominal and real GDP; K is real capital stock, A global productivity. F denotes net foreign debt, tot price of exports over price of imports, E the real *effective* exchange rate, π (expected equal to actual) inflation. δ stands for rate of depreciation of the capital stock, q and g stand respectively for the nominal and real rate of GDP growth. Finally P_Y and P_I refer to the GDP and investment price indices. The hat ("^") denotes the equilibrium variables and the prime (" ′ ") the foreign ones.

Equations (8), (9) and (10) are estimated by using a VECM. The coefficients so obtained are substituted into model (7)-(16) that is then solved for the medium and long term equilibrium real exchange rate giving the NATREX. The real effective exchange rate of the synthetic euro is then compared with the benchmark generated by the model simulations to measure the misalignments, if any. It turns out that the synthetic euro was overvalued from the second half of the '70s until the early '80s; periods of over- and under-evaluations alternate thereafter until 1997 when it fell and stayed below the equilibrium value (more remarkably starting from 1999 onwards) for the remaining part of the period considered. Also, Detken et al. (2002) emphasize the main driving forces of the dynamics of the equilibrium exchange rate for the euro resulting from their work, and namely the productivity growth and the terms of trade.

4 Two-country NATREX model

We consider two large economies, the Eurozone (EU) and the United States (US), which are modelled symmetrically. The model is defined by dynamic equations for fundamental variables in each country plus the national account identities. The equations specify the dynamics for, respectively: net social investment, internal social consumption, trade balance (goods and services), interest rate, technology and capital stock. As already observed in Section 2, variables adjust with a certain lag to their desired (partial equilibrium) level, according to the dynamic disequilibrium modelling approach in continuous time. In what follows variables are real, "D" denotes the operator d/dt, the hat "$\widehat{\ }$" stands for "desired", and all coefficients are written in such a way that they are supposed to be positive unless otherwise stated. Furthermore, the exchange rate (E) for the euro/dollar is defined as the number of US dollars per one euro; it follows that an increase in E means an appreciation of the euro (the euro gets stronger) vis-à-vis the US dollar. Finally, we notice that, while the rest of the world is not explicitly modelled in the theoretical framework, in the empirical estimation it will be captured by exogenous correction factors.

4.1 Investment

Saving and investment decisions are made independently by individual agents. This is equivalent to saying that families choose saving and consumption, while firms decide over investment and production. Total investment (I) is given by the sum of private and public investment. It adjusts to its partial equilibrium level with a mean time lag $(1/\alpha_1)$ due to adjustment costs:

$$\mathrm{D}I^{EU} = \alpha_1^{EU}\left(\hat{I}^{EU} - I^{EU}\right) \tag{17}$$

$$\mathrm{D}I^{US} = \alpha_1^{US}\left(\hat{I}^{US} - I^{US}\right) \tag{18}$$

The desired investment in the two countries is:

$$\hat{I}^{EU} = f_1^{EU}\left[\left(MPK^{EU} - R^{EU}\right)\right] \tag{19}$$

$$\hat{I}^{US} = f_1^{US}\left[\left(MPK^{US} - R^{US}\right)\right] \tag{20}$$

$$\text{where sgn } f_1\left[...\right] = \text{sgn}\left[...\right],\ f_1^{'} > 0$$

MPK is productivity of capital and R real interest rate. To model the investment function for $\hat{I}$, we follow Infante and Stein (1973) and Stein (1995). In a context of intertemporal optimization over infinite horizon, rational agents make their intertemporal decisions relying on a closed loop suboptimal feedback control rule (SOFC). Since agents are not endowed with perfect foresights, the optimal trajectory of the economy remains unknown. The SOFC rule based on current measurements of the marginal product of capital ensures that the economy will converge toward the unknown (and possibly changing) optimal trajectory. It predicts that the optimal rate of investment, $\hat{I}$, responds positively to the difference between the productivity of capital and the real interest rate as illustrated in equations (19) and (20).

4.2 Consumption

Social (public plus private) internal consumption (C) adjusts to its partial equilibrium level with a mean time lag $(1/\alpha_2)$:

$$\mathrm{D}C^{EU} = \alpha_2^{EU}\left(\widehat{C}^{EU} - C^{EU}\right) \tag{21}$$

$$\mathrm{D}C^{US} = \alpha_2^{US}\left(\widehat{C}^{US} - C^{US}\right) \tag{22}$$

According to Stein and Sauernheimer (1996), the appropriate optimization process entails that:

$$\hat{C}^{EU} = f_2^{EU}\left(\underset{+}{Y^{EU}}, \underset{-}{F^{EU}}\right) \tag{23}$$

$$\hat{C}^{US} = f_2^{US}\left(\underset{+}{Y^{US}}, \underset{-}{F^{US}}\right) \tag{24}$$

where Y denotes domestic output and F net foreign debt. The desired consumption function is derived as follows. Private and public agents optimize their utility under an intertemporal budget constraint. According to standard optimization theory, consumption is proportional to current capacity output. When foreign debt grows above the level considered sustainable, the government employes a restrictive fiscal policy by decreasing current expenditure. This gives that partial optimal consumption is a positive function of domestic output and a negative function of foreign debt.

4.3 Trade balance

According to the NATREX theory, the partial adjustment process also holds for the balance of trade (BT):

$$\mathrm{D}BT^{EU} = \alpha_3^{EU}\left(\widehat{BT}^{EU} - BT^{EU}\right) \tag{25}$$

$$\mathrm{D}BT^{US} = \alpha_3^{US}\left(\widehat{BT}^{US} - BT^{US}\right) \tag{26}$$

where $BT = XGS - MGS$, XGS denotes exports and MGS imports. The partial equilibrium values are:

$$\widehat{BT}^{EU} = f_3^{EU}\left(\underset{+}{Y^{US}}, \underset{-}{Y^{EU}}, \underset{-}{E}\right) \tag{27}$$

$$\widehat{BT}^{US} = f_3^{US}\left(\underset{+}{Y^{EU}}, \underset{-}{Y^{US}}, \underset{+}{E}\right) \tag{28}$$

and are obtained considering that, according to the standard assumptions in international economics, real exports are affected positively by the other country's output, whereas imports respond positively to the home country's real output. Furthermore, given our definition for the euro/dollar exchange rate (number of US dollars per one euro), an increase in E leads to an appreciation of the euro vis-à-vis the US dollar and, thus, to a decrease (increase) of the EU (US) export and an increase (decrease) of the EU (US) imports. It follows that the partial first derivatives have the sign shown below the corresponding variables in equations (27) and (28).

In a world made up of two countries (the Eurozone and the US), the following condition must hold:

$$BT^{EU} + \frac{1}{E} BT^{US} = 0 \tag{29}$$

Given (29), one between (25) and (26) is redundant. Nonetheless, in view of the empirical estimation of the model, we maintain both equations to also consider the rest of the world effect.

4.4 Real interest rates

The dynamic equations for the real interest rate (R) respectively in the Eurozone and in the United States are given by[5]:

$$\mathrm{D}R^{EU} = \alpha_4^{EU} \left(\widehat{R}^{EU} - R^{EU} \right) \tag{30}$$

$$\mathrm{D}R^{US} = \alpha_4^{US} \left(\widehat{R}^{US} - R^{US} \right) \tag{31}$$

where:

$$\widehat{R}^{EU} = R^{US} + \rho^{EU} \tag{32}$$

$$\widehat{R}^{US} = R^{EU} + \rho^{US} \tag{33}$$

and ρ is the risk premium.

The real interest rate parity (*RIP*) theory states that if investors make their decisions in real terms, then portfolio equilibrium in an open economy entails equality between expected rates of return in real terms, possibly admitting a divergence corresponding to the risk premium. In this model, investors face a long time horizon, deal with both direct and portfolio investment, and trade domestic as well as foreign assets. Agents are rational in the sense that they exploit all available information. Hence, investors keep trading, and let interest differentials adjust, until they become indifferent between domestic and foreign assets, *i.e.* the *RIP* condition with risk premium holds ((32) and (33) respectively for the Eurozone and the US). This condition, however, is not valid instantaneously, but

[5] For the long run characteristics of the system composed of (30) and (31) see the mathematical appendix A.

rather is achieved with a certain time lag $(1/\alpha_4)$, due to market imperfections and to the corresponding sluggishness in the re-equilibrating process, as described by (30) and (31).

Eaton and Gersovitz (1981) establish that, because of the moral hazard associated with sovereign risk, the risk premium of international lending varies positively with the stock of debt held by the given country. Moreover, Sachs (1984), Sachs and Cohen (1982), and Cooper and Sachs (1985) suggest that the cost of servicing the debt may be curbed by means of growth-oriented policies and policies that enhance the country's foreign exchange earning capacity. It follows that the risk premiums may be expressed as a positive function of the ratio between foreign debt and domestic output (van der Ploeg, 1996; Bhandari, Hague, and Turnovsky, 1990), that is:

$$\rho^{EU} = f_4^{EU}\left(F^{EU}\underset{+}{/}Y^{EU}\right) \tag{34}$$

$$\rho^{US} = f_4^{EU}\left(F^{US}\underset{+}{/}Y^{US}\right) \tag{35}$$

In the long run, two further conditions must hold. Indeed in a two-large economy world, a situation where one country is continuously characterized by a positive, while the other by a negative stock of foreign debt is not sustainable in the long run. Therefore, the long run equilibrium requires that:

$$F^{EU} = F^{US} = 0. \tag{36}$$

Given equations (34) and (35), it also follows that:

$$\rho^{EU} = \rho^{US} = 0 \tag{37}$$

4.5 Output

In the two economies, output adjusts with a lag $(1/\alpha_5)$ to the excess demand:

$$\mathrm{D}Y^{EU} = \alpha_5^{EU}\left(Y_D^{EU} - Y_S^{EU}\right) \tag{38}$$

$$\mathrm{D}Y^{US} = \alpha_5^{US}\left(Y_D^{US} - Y_S^{US}\right) \tag{39}$$

where Y_D and Y_S are respectively aggregate demand and aggregate supply, and by national account identity it turns out that $Y_D = C + I + (XGS - MGS)$.

In this paper we follow Federici and Gandolfo (2002) and introduce endogenous growth.

The production function is modelled in an "AK" fashion[6], that is potential output, respectively in the Eurozone and the United States, and is given by:

$$Y_S^{EU} = A^{EU} K^{EU} \quad (40)$$

$$Y_S^{US} = A^{US} K^{US} \quad (41)$$

where A is a positive constant which reflects the technological level.

We generalize this approach by assuming that $\widehat{A}$ is function of the stock of accumulated knowledge, Ω, which has a positive but decreasing effect. In addition, $A(\Omega)$ has an upper limit $\bar{A}$, since it is implausible to think that the productivity of capital can go to infinity. Therefore:

$$\widehat{A}^{EU} = A^{EU}(\Omega^{EU}) \quad (42)$$

$$\widehat{A}^{US} = A^{US}(\Omega^{US}) \quad (43)$$

$$\text{with } A' > 0,\ A'' < 0,\ A \leq \overline{A}$$

where Ω may be in turn expressed as a function of the accumulated R&D expenditure ($I_{R\&D}$):

$$\Omega^{EU} = \gamma^{EU} \int_{-\infty}^{t} I_{R\&D}^{EU}(s)\, ds \text{ or } \dot{\Omega} = \gamma^{EU} I_{R\&D}^{EU} \quad (44)$$

$$\Omega^{US} = \gamma^{US} \int_{-\infty}^{t} I_{R\&D}^{US}(s)\, ds \text{ or } \dot{\Omega} = \gamma^{US} I_{R\&D}^{US} \quad (45)$$

We assume that A adjusts with a lag $(1/\alpha_6)$ to its partial equilibrium level, hence the corresponding dynamic equations in the two economies are described by:

$$\mathrm{D} \ln A^{EU} = \alpha_6^{EU} \ln\left(\frac{\widehat{A}^{EU}}{A^{EU}}\right) \quad (46)$$

$$\mathrm{D} \ln A^{US} = \alpha_6^{US} \ln\left(\frac{\widehat{A}^{US}}{A^{US}}\right) \quad (47)$$

Total investment (I) is divided between investment in fixed capital (I_K) and investment in

[6] For obvious reasons this function has come to be known in the recent literature as the "AK" production function (Barro and Sala-i-Martin, 1995), but its use in growth theory has a long tradition: e.g. Harrod (1939) and Domar (1946); Klump and Streissler (2000) show that the von Neumann production function can be reduced to the AK type. More sophisticated forms (including other factors of production) could be considered, but on the basis of the parsimony principle we decided to start with the simplest possible form.

R&D ($I_{R\&D}$):

$$\begin{aligned} I^{EU} &= I_K^{EU} + I_{R\&D}^{EU} && (48) \\ I^{US} &= I_K^{US} + I_{R\&D}^{US} && (49) \end{aligned}$$

R&D investment enhances the marginal productivity of capital (by increasing the stock of accumulated knowledge) but with a lag (γ). The lag is crucial: more investment in R&D means less increase in the capital stock and hence smaller growth immediately; it also means higher productivity and thus higher growth later.

We need now to determine the optimal allocation of I between I_K and $I_{R\&D}$. Given that investment is private+public, we can think of the choice being determined by a maximizing policy maker, whereby the potential growth rate of output is maximized. The maximization problem is illustrated here for the Eurozone (it is analogous for the US). Since $Y = AK$, it follows that $\dot{Y}/Y = \dot{A}/A + \dot{K}/K$, hence the maximization problem is:

$$\begin{aligned} \max_{\{I_K, I_{R\&D}\}} \left[\frac{\dot{A}}{A} + \frac{\dot{K}}{K}\right] &= \max_{\{I_K, I_{R\&D}\}} \left[\frac{A'\dot{\Omega}}{A} + \frac{I_K}{K}\right] \\ &= \max_{\{I_K, I_{R\&D}\}} \left[\gamma\frac{A' I_{R\&D}}{A} + \frac{I_K}{K}\right] \\ \text{sub } I_K + I_{R\&D} - I &= 0, && (50) \end{aligned}$$

Performing the constrained optimization (see mathematical appendix B), we obtain:

$$\begin{aligned} I_{R\&D} &= \frac{1}{H} I, \text{ and} && (51) \\ I_K &= \frac{H-1}{H} I && (52) \end{aligned}$$

where $I = f_1(A - R)$, and $H = \gamma^2 K^2 \left[\left(\frac{A'}{A}\right)^2 - \frac{A''}{A}\right] + \gamma^2$ (> 0). Equations (51) and (52) enable us to define respectively investment in R&D and investment in fixed capital as (variable) fractions of total investment.

By definition, the changes in the capital stock, neglecting the depreciation, are:

$$\begin{aligned} \mathrm{D}K^{EU} &= I_K^{EU} && (53) \\ \mathrm{D}K^{US} &= I_K^{US} && (54) \end{aligned}$$

4.6 External debt constraint

Finally, in the two countries external debt constraints hold:

$$\mathrm{D}F^{EU} = -\left(XGS^{EU} - MGS^{EU}\right) + NFI^{EU} - UT^{EU} \tag{55}$$

$$\mathrm{D}F^{US} = -\left(XGS^{US} - MGS^{US}\right) + NFI^{US} - UT^{US} \tag{56}$$

where NFI is net factor income from abroad and UT stands for unilateral transfers.

Some definitional equations complete the model. (That are valid for both economies and, therefore, superscripts denoting the economy are omitted). The current account (CA) is given by:

$$CA = BT + NFI$$

where, neglecting net labour income from abroad, it turns out that $NFI = -RF$, being RF the net interest payments. The stock of foreign debt is in turn defined as:

$$F(t) = F_0 - \int_0^t CA(\tau)\, d\tau \tag{57}$$

which derives from the balance-of-payments accounting identity:

$$CA - \mathrm{D}NFA = 0, \tag{58}$$

where $\mathrm{D}NFA$ is the change in the stock of net foreign assets, NFA. For the sake of clarity, we recall that, according to the accounting principles of the balance of payments (IMF, 1993, p. 7), a decrease in foreign liabilities or an increase in foreign assets ($\mathrm{D}NFA > 0$) should be recorded as a negative figure (debit), and, conversely, an increase in foreign liabilities or a decrease in foreign assets ($\mathrm{D}NFA < 0$) should be recorded as a positive figure (credit). Since

$$F \equiv -NFA, \quad \mathrm{D}F = -\mathrm{D}NFA, \tag{59}$$

from (58) we have

$$\mathrm{D}F = -CA. \tag{60}$$

If we integrate (60) and assume that the arbitrary constant of integration is F_0, we obtain (57).

5 Derivation of the NATREX

In NATREX equilibrium, the current account is in equilibrium given output growth at capacity. This requires $CA = 0$ and $Y = Y_D = Y_S = AK$ for each economy. In addition, long-run equilibrium requires absence of any risk premium, what implies in turn $\hat{R} = R = \bar{R}$ for each economy, where $\bar{R}$ is the same for both countries and can be calculated endogenously (see mathematical appendix A). Since, on the other hand, the production function implies $MPK = A$, this shows the crucial role of A and $\bar{R}$: it is the difference between them that enhances or hinders investment according to the investment equations (19) and (20). Investment will be positive if $A > \bar{R}$. This will cause capital growth, and hence growth of output.

To derive the NATREX, we observe that current account equilibrium, $CA^{EU} = CA^{US} = 0$, implies $F^{EU} = F_0^{EU}$ ($F^{US} = F_0^{US}$). However, in the long run equilibrium the stock of net foreign assets must also be zero. Hence:

$$F^{EU} = F^{US} = 0.$$

Furthermore, in equilibrium, $BT^{EU} = \widehat{BT}^{EU}$ ($BT^{US} = \widehat{BT}^{US}$). It follows that: $CA^{EU} = \widehat{BT}^{EU} = 0$ ($CA^{EU} = \widehat{BT}^{EU} = 0$), and so E can be determined from the implicit function $\widehat{BT}^{EU} = f_3^{EU}\left(Y_S^{EU}, Y_S^{US}, E\right) = 0$ (or $\widehat{BT}^{US} = f_3^{US}\left(Y_S^{US}, Y_S^{EU}, 1/E\right) = 0$). If the appropriate invertibility conditions on the Jacobian are satisfied, we have

$$E_N = \phi\left(Y_S^{EU}, Y_S^{US}\right), \tag{61}$$

where the subscript N stands for NATREX. The same can be derived from $\widehat{BT}^{US} = f_3^{US}\left(Y_S^{US}, Y_S^{EU}, 1/E\right) = 0$,

$$E_N = \psi\left(Y_S^{US}, Y_S^{EU}\right), \tag{62}$$

where, of course, equation (61) and (62) must have the same value. It also turns out that the system is conditionally stable (see Section 6).

This concerns the long-run equilibrium growth path. In the medium run, however, the requirements of no risk premium (hence $R = \overline{R}$) and no capital flows, are a bit too stringent. A more plausible alternative is to allow for $R \neq \overline{R}$ and non-zero capital flows, while keeping the basic requirements of NATREX, namely $CA = 0$ with $Y = Y_S$. Therefore, we have $\widehat{BT}^{EU} = f_3^{EU}\left(Y_S^{EU}, Y_S^{US}, E\right) - R^{EU}F^{EU} = -\widehat{BT}^{US}$ $\left(= -\left[f_3^{US}\left(Y_S^{US}, Y_S^{EU}, 1/E\right) - R^{US}F^{US}\right]\right)$. In this formulation the NATREX turns out to be

$$E_N = \chi(Y_S^{EU}, Y_S^{US}, R^{EU}F^{EU}). \tag{63}$$

This is the *medium run* NATREX.

6 Qualitative analysis

The US and the Eurozone economies are modelled symmetrically so that the following description holds for both economies and the superscript are omitted for notational simplicity. Recapitulating the dynamic structure of the theoretical model around the growth equilibrium (where $Y = Y_D = Y_S = AK$):

$$\begin{array}{ll} \mathrm{D}I = \alpha_1(\widehat{I} - I) & \widehat{I} = f_1[A - \bar{R}] \\ \mathrm{D}C = \alpha_2(\widehat{C} - C) & \widehat{C} = c\widehat{Y} = cAK \\ \mathrm{D}BT = \alpha_3(\widehat{BT} - BT) & \widehat{BT} = 0 \\ \mathrm{D}R = \alpha_4\left(\widehat{R} - R\right) & \widehat{R} = R^* \text{ (}^*\text{ stands for the other country)} \\ \mathrm{D}A = \alpha_6(\widehat{A} - A) & \widehat{A} = \bar{A} \\ \mathrm{D}K = I_K = \kappa I & \end{array} \tag{64}$$

The equations are self-explanatory; it should only be pointed out that: in the first equations $\bar{R}$ is the long run equilibrium value of the real interest rate (see appendix A); in the fifth equations, $\widehat{A}$ (the desired long run value of A) has been set equal to its maximum value, $\bar{A}$; finally in the sixth equations, $\kappa = 1 - \dfrac{1}{H}$ where $H = \gamma^2 K^2\left[\left(\dfrac{A'}{A}\right)^2 - \dfrac{A''}{A}\right] + \gamma^2$.

System (64) is a system of 12 differential equations (6 for each economy). Fortunately its dimension can be greatly reduced by the following considerations:

1) The third equations - thanks to the fact that the desired value of BT is zero - form an independent subset that can be solved independently, and is easily seen to be stable, since the two independent equations have a characteristic root equal to $-\alpha_3 < 0$.

2) The fourth equations constitute a subsystem that can be solved independently (see mathematical appendix A) and turns out to be stable, with roots that are all real negative.

Thus we are left with the following system:

$$\begin{aligned}
&\mathrm{D}I = \alpha_1(\widehat{I} - I) && \widehat{I} = f_1[A - \bar{R}] \\
&\mathrm{D}C = \alpha_2(\widehat{C} - C) && \widehat{C} = c\widehat{Y} = cAK \\
&\mathrm{D}A = \alpha_6(\hat{A} - A) && \hat{A} = \bar{A} \\
&\mathrm{D}K = I_K = \kappa I.
\end{aligned} \tag{65}$$

This system is dichotomous, because it gives rise to two four-equation systems - one for the Eurozone and the other for the US - that can be solved independently. Consider the matrix of the linear approximation to each system:

$$\begin{bmatrix} -\alpha_1 & 0 & \alpha_1 (f_1')_0 & 0 \\ 0 & -\alpha_2 & \alpha_2 c(K)_0 & \alpha_2 c\bar{A} \\ 0 & 0 & -\alpha_6 & 0 \\ (\kappa)_0 & 0 & (\kappa_A)_0 & (\kappa_K)_0 \end{bmatrix}, \tag{66}$$

where $\kappa_A = \dfrac{1}{H^2}\dfrac{\partial H}{\partial A}$, $\kappa_K = \dfrac{1}{H^2}\dfrac{\partial H}{\partial K}$ and $(...)_0$ denotes that the variable is evaluated at the equilibrium point. Furthermore, it can be checked[7] that $\dfrac{\partial H}{\partial A} < 0, \dfrac{\partial H}{\partial K} > 0$.

The characteristic equation is

$$\begin{vmatrix} -\alpha_1 - \lambda & 0 & \alpha_1 (f_1')_0 & 0 \\ 0 & -\alpha_2 - \lambda & \alpha_2 c(K)_0 & \alpha_2 c\bar{A} \\ 0 & 0 & -\alpha_6 - \lambda & 0 \\ (\kappa)_0 & 0 & (\kappa_A)_0 & (\kappa_K)_0 - \lambda \end{vmatrix} = 0, \tag{67}$$

which gives

$$(-\alpha_2 - \lambda)(-\alpha_6 - \lambda)(-\alpha_1 - \lambda)\left[(\kappa_K)_0 - \lambda\right] = 0. \tag{68}$$

Hence the characteristic roots are $\lambda_1 = -\alpha_2$, $\lambda_2 = -\alpha_6$, $\lambda_3 = -\alpha_1$, $\lambda_4 = (\kappa_K)_0$.

In conclusion, the system under consideration has three real negative roots and one real positive root. Thus we are in the standard case of saddle-point stability. More precisely, given a first-order differential system in normal form with distinct characteristic roots, partly stable and partly unstable (a *conditionally stable* system), we can always make the

[7] $\dfrac{\partial H}{\partial A} = \gamma_5^2 K^2 \left[2\left(\dfrac{A'}{A}\right)\left(-\dfrac{A'}{A^2}\right) + \dfrac{A''}{A^2}\right] < 0;$
$\dfrac{\partial H}{\partial K} = 2\gamma_5^2 K \left[\left(\dfrac{A'}{A}\right)^2 - \dfrac{A''}{A}\right] > 0.$

system stable provided that we can choose as many initial conditions as there are unstable roots (Gandolfo, 1997, Chap. 18, Sect. 18.2.2.3, Theorem 18.3). In our model, the presence of the government in the consumption and investment equations ensures that it is possible to choose one initial condition so as to make the system stable. Stability will be monotonic, since all stable roots are real.

7 Concluding remarks

In this paper we build a two-country model for the US and the Euroarea in order to examine the determinants and dynamics of the euro/dollar equilibrium exchange rate. We adopt the NATREX approach (Stein, 1990), which is based on a specific theoretical dynamic stock-flow model to derive the equilibrium real exchange rate. The equilibrium concept ensures simultaneously internal and external equilibrium, and reflects the behavior of the fundamental variables behind investment and saving decisions in the absence of cyclical factors, speculative capital movements and movements in international reserves. This approach has already been applied to explain the medium-long term dynamics of the real exchange rate in a number of industrial countries. However, the previous literature has relied on a one-country framework where the "rest of the world" is treated as given. This work is the first to fully specify the two economies and allow for their interaction. The model presents a saddle point stability. Hence, once the initial condition is chosen, the system turns out conditionally stable. Since the government takes part in the consumption and investment decisions, the choice of the appropriate initial condition is always guaranteed. Our theoretical model offers the basis for empirical estimation of the euro/dollar equilibrium exchange rate that will be carried out in future research.

A Appendix A

In the long run, recalling that the risk premiums are zero, we have:

$$\mathrm{D}R^{EU} = \alpha_4^{EU} r \left(\widehat{R}^{EU} - R^{EU}\right) = \alpha_4^{EU}\left(R^{US} - R^{EU}\right) \tag{A.1}$$

$$\mathrm{D}R^{US} = \alpha_4^{US}\left(\widehat{R}^{US} - R^{US}\right) = \alpha_4^{US}\left(R^{EU} - R^{US}\right) \tag{A.2}$$

the characteristic equation of the system is (for simplicity in this appendix, α_4^{EU} and α_4^{US} are written omitting the "4" subscript):

$$\begin{vmatrix} -\alpha^{EU}-\lambda & \alpha^{EU} \\ \alpha^{US} & -\alpha^{US}-\lambda \end{vmatrix} = \lambda(\alpha^{EU}+\alpha^{US}+\lambda) = 0 \tag{A.3}$$

whereby $\lambda_1 = 0, \lambda_2 = -(\alpha^{EU}+\alpha^{US})$. The solution is:

$$\begin{aligned} R^{EU} &= A_1 + A_2 e^{-(\alpha^{EU}+\alpha^{US})t}, && \text{(A.4)} \\ R^{US} &= A_1 \frac{\lambda_1 - a_{11}}{a_{12}} + A_2 \frac{\lambda_2 - a_{11}}{a_{12}} e^{-(\alpha^{EU}+\alpha^{US})t} && \text{(A.5)} \end{aligned}$$

where $a_{11} = -\alpha^{EU}, a_{12} = \alpha^{EU}$. It follows that:

$$R^{US} = A_1 + \frac{\lambda_2 + \alpha^{EU}}{\alpha^{EU}} A_2 e^{-(\alpha^{EU}+\alpha^{US})t} = A_1 + A_2 \frac{-\alpha^{US}}{\alpha^{EU}} e^{-(\alpha^{EU}+\alpha^{US})t} \tag{A.6}$$

Given the initial conditions $R^{EU} = R_0^{EU}$, $R^{US} = R_0^{US}$ with $t = 0$, to obtain the arbitrary constants, A_1, A_2, we have:

$$\begin{aligned} R_0^{EU} &= A_1 + A_2, && \text{(A.7)} \\ R_0^{US} &= A_1 + \frac{-\alpha^{US}}{\alpha^{EU}} A_2, && \text{(A.8)} \end{aligned}$$

whose solution gives:

$$A_1 = \frac{\frac{-\alpha^{US}}{\alpha^{EU}} R_0^{EU} - R_0^{US}}{\frac{-\alpha^{US}}{\alpha^{EU}} - 1}, \; A_2 = \frac{R_0^{US} - R_0^{EU}}{\frac{-\alpha^{US}}{\alpha^{EU}} - 1}. \tag{A.9}$$

In the long run, the real interest rates of the two countries converge to A_1, that is:

$$A_1 = \frac{\frac{-\alpha^{US}}{\alpha^{EU}} R_0^{EU} - R_0^{US}}{\frac{-\alpha^{US}}{\alpha^{EU}} - 1} = \frac{\alpha^{US} R_0^{EU} + \alpha^{EU} R_0^{US}}{\alpha^{EU} + \alpha^{US}} = \frac{\frac{1}{\alpha^{EU}} R_0^{EU} + \frac{1}{\alpha^{US}} R_0^{US}}{\frac{1}{\alpha^{EU}} + \frac{1}{\alpha^{US}}}. \tag{A.10}$$

We observe that in the long run the real interest rates converge to the same constant value (denoted by $\overline{R}$ in the text), which is the weighted average of the initial real interest rates and the weights are given by the corresponding mean time lags.

B Appendix B

The determination of the optimal allocation of I between I_K and $I_{R\&D}$ is illustrated for the Eurozone (it is analogous for the US, thus the superscript denoting the economy is omitted).

The maximization problem is:

$$\begin{aligned}
\max_{\{I_K, I_{R\&D}\}} \left[\frac{\dot{A}}{A} + \frac{\dot{K}}{K}\right] &= \max_{\{I_K, I_{R\&D}\}} \left[\frac{A'\dot{\Omega}}{A} + \frac{I_K}{K}\right] \\
&= \max_{\{I_K, I_{R\&D}\}} \left[\gamma\frac{A' I_{R\&D}}{A} + \frac{I_K}{K}\right] \qquad \text{(A.11)} \\
\text{sub } I_K + I_{R\&D} - I &= 0.
\end{aligned}$$

From the Lagrangian

$$L = \left[\gamma\frac{A' I_{R\&D}}{A} + \frac{I_K}{K}\right] + \lambda(I - I_K - I_{R\&D}) \qquad \text{(A.12)}$$

we obtain the first-order conditions

$$\frac{\partial L}{\partial I_{R\&D}} = \gamma\frac{A'}{A} - \lambda = 0, \qquad \text{(A.13)}$$

$$\frac{\partial L}{\partial I_K} = \frac{1}{K} - \lambda = 0, \qquad \text{(A.14)}$$

whence

$$\gamma\frac{A'}{A} = \frac{1}{K}, \qquad \text{(A.15)}$$

which states that the proportional increase in the marginal productivity of capital due to R&D expenditure (which measures the marginal benefit, in terms of output growth, of a unit of expenditure devoted to R&D), should always be equal to the reciprocal of the capital stock (which measures the marginal benefit of a unit of expenditure devoted to fixed investment).

Equation (A.15) must hold at every instant of time, determining the desired magnitudes $\hat{I}_K$ and $\hat{I}_{R\&D}$. Differentiating with respect to time we have

$$\gamma\frac{(A''\dot{\Omega})A - A'(A'\dot{\Omega})}{A^2} = -\frac{\dot{K}}{K^2}, \qquad \text{(A.16)}$$

hence

$$\gamma\frac{A''\gamma I_{R\&D}A - (A')^2\gamma I_{R\&D}}{A^2} = -\frac{I_K}{K^2} = -\frac{I - I_{R\&D}}{K^2} = \frac{I_{R\&D}}{K^2} - \frac{I}{K^2}. \qquad \text{(A.17)}$$

Collecting the terms containing $I_{R\&D}$ we obtain

$$\gamma^2\left[\frac{A''}{A} - \left(\frac{A'}{A}\right)^2 - \frac{1}{\gamma^2 K^2}\right] I_{R\&D} = -\frac{I}{K^2} \qquad \text{(A.18)}$$

and multiplying through by $-K^2$ we get

$$\gamma^2 K^2\left[\left(\frac{A'}{A}\right)^2 - \frac{A''}{A} + \frac{1}{K^2}\right] I_{R\&D} = I, \qquad \text{(A.19)}$$

where the expression in square brackets is positive, since $A'' < 0$. Defining:

$$H = \gamma^2 K^2 \left[\left(\frac{A'}{A} \right)^2 - \frac{A''}{A} + \frac{1}{K^2} \right] = \gamma^2 K^2 \left[\left(\frac{A'}{A} \right)^2 - \frac{A''}{A} \right] + \gamma^2 \tag{A.20}$$

we finally have

$$I_{R\&D} = \frac{1}{H} I, \quad I_K = \frac{H-1}{H} I \tag{A.21}$$

where $I = f_1(A - R)$.

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