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Abstract

We consider the indifference valuation of an uncertain monetary payoff from the perspective of an uncertainty averse decision maker. We study how the indifference valuation depends on the decision maker's attitudes toward uncertainty. We obtain a characterization of comparative uncertainty aversion and various characterizations of increasing, decreasing, and constant uncertainty aversion.

Keywords: Indifference Pricing, Uncertainty Aversion, Risk Measures, Quasiconvexity, Cash-Subadditivity.

JEL Classifications: C02, D01, D52, D81, G22.

1 Introduction

The indifference prices are the boundaries delimiting the prices of a contract that would be agreed to by an individual who prefers more money to less money and who endeavors to maximize the relative desirability of her monetary endowment. The technique of indifference pricing was introduced by Bernoulli (1738) contextually with the prediction that an individual evaluates the relative desirability of alternative monetary prospects by their expected utility. The consistency of the paradigm of expected utility maximization and, accordingly, of the resulting indifference prices, with criteria of logic and rationality, was established by von Neumann and Morgenstern (1953) in a framework where the probabilities of future events are objectively determined, and extended by Savage (1972) to a framework where the probabilities of future events are subjectively determined (see also Ramsey (1931) and de Finetti (1964)). The indifference prices defined by the expected utility representation of preferences were further studied by Pratt (1964) in relation to the characterization of an individual's attitudes toward risk. Pratt (1964) found that an individual is more risk averse than another if and only if the maximum price that she would offer to avoid a risky monetary prospect is larger than for the other, and that an individual is increasingly risk averse if and only if the maximum price that she would offer to avoid a risky monetary prospect is larger the larger her constant initial wealth. Pratt (1964) further observed that an individual is increasingly risk averse if and only if the degree of relative convexity of her utility function (de Finetti (1952)) or Arrow-Pratt coefficient of absolute risk aversion (Arrow (1970) and Pratt (1964)) is an increasing function, and that this technical condition allows to immediately verify whether an expected utility representation of preferences exhibits increasing, decreasing, or constant risk aversion.

In this paper we study the indifference prices defined by the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a) and their relationship with an individual's attitudes toward uncertainty. We find that an individual is more uncertainty averse than another if and only if the maximum price that she would offer to avoid an uncertain monetary prospect is larger than for the other, and that an individual is increasingly uncertainty averse if and only if the maximum price that she would offer to avoid an uncertain monetary prospect is larger the larger her constant initial wealth. We further develop the characterization of increasing, decreasing, and constant uncertainty aversion and we provide a technical condition that allows to easily verify whether an uncertainty averse representation of preferences exhibits increasing, decreasing, or constant uncertainty aversion. We find that the variational representation of preferences of Maccheroni et al. (2006) is decreasingly uncertainty averse and that the multiple priors representation of preferences of Gilboa and Schmeidler (1989) is constantly uncertainty averse.

The rest of the paper is organized as follows. In Section 2 we introduce the notation, the background, and the assumptions. In Section 3 we study the properties of the indifference prices of an uncertainty averse decision maker. In Section 4 we study the characterization of comparative uncertainty aversion (Subsection 4.1) and of increasing, decreasing, and constant uncertainty aversion (Subsection 4.2) in terms of the indifference prices. In Section 5 we provide some further characterizations of increasing, decreasing, and constant uncertainty aversion in terms of cash-subadditivity of an indifference price (Subsection 5.1) and of inequalities between the indifference prices (Subsection 5.2). In Section 6 we study the uncertainty averse representations of preferences that are increasingly, decreasingly, or constantly uncertainty averse. In Section 7 we conclude summarizing the results and providing some possible directions for future research.

2 Framework

The pair (S, Σ) denotes a measurable space where S is a set of future states of nature and Σ is a σ-algebra of subsets of S representing future events. The set of all the available monetary payoffs is indicated by $\mathcal{X} := B(S, \Sigma)$ and it corresponds to the set of all bounded, real-valued, Σ-measurable functions X on S. The set $\mathcal{X}$ includes a subset of constant functions

$X(s) = x \in \mathbb{R} \, \forall s \in S$ which is identified with $\mathbb{R}$. Every equality or inequality involving elements of $\mathcal{X}$ is intended as holding for all $s \in S$. The set of all probabilistic models is indicated by $\Delta := \mathrm{ba}_1^+(S, \Sigma)$ and it corresponds to the set of all bounded, real-valued, finitely additive set functions P on Σ which are positive and normalized.
The *uncertainty averse* representation of preferences $U^{u,G} : \mathcal{X} \to \mathbb{R}$ of Cerreia Vioglio et al. (2011a) is given by,

$$U^{u,G}(X) := \inf_{P \in \Delta} G(E_P[u(X)], P) \tag{1}$$

for all $X \in \mathcal{X}$. The function $u : \mathbb{R} \to \mathbb{R}$ is a utility function reflecting the decision maker's attitudes toward risk. The function $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ is an *uncertainty index* reflecting the decision maker's attitudes toward uncertainty. We assume that the uncertainty index G is strictly increasing in the first component, quasiconvex, normalized, that is such that $\inf_{P \in \Delta} G(x, P) = x$ for all $x \in \mathbb{R}$, lower semi-continuous, and extended-valued uniformly equicontinuous on $\mathbb{R}$ with respect to $P \in \Delta$ (see Section 2.2 in Cerreia Vioglio et al. (2011a)). We further assume that the utility function u is strictly increasing and concave. Note that, as u is concave and finite on all of $\mathbb{R}$, it is necessarily continuous (see Rockafellar (1970, Corollary 10.1.1)).

Remark 1. Note that, under the above assumption on u, the functional $U^{u,G}$ defined by Equation (1) represents the preferences of a decision-maker that is both *risk averse* and *uncertainty averse*. Moreover, $U^{u,G}$ is strictly increasing, quasiconcave, and continuous.

Remark 2. Note that under some technical conditions the infimum in Equation (1) can be equivalently taken over the subset $\Delta^\sigma := \mathrm{ca}_1^+(S, \Sigma)$ of countably additive elements in Δ. See Theorem 7 in Cerreia Vioglio et al. (2011a).

An example of uncertainty averse representation of preferences is the *variational* representation of preferences $U^{u,c} : \mathcal{X} \to \mathbb{R}$ of Maccheroni et al. (2006) which is given by,

$$U^{u,c}(X) = \inf_{P \in \Delta} \Big(\mathbb{E}_P[u(X)] + c(P) \Big)$$

for all $X \in \mathcal{X}$ and for some convex and lower semi-continuous *ambiguity index* $c : \Delta \to (-\infty, +\infty]$ such that $\inf_{P \in \Delta} c(P) = 0$. The variational representation of preferences is obtained as a particular case of Equation (1) when the uncertainty index G satisfies,

$$G(x, P) = x + c(P) \tag{2}$$

for all $(x, P) \in \mathbb{R} \times \Delta$.
A further example of uncertainty averse representation of preferences is the *multiple priors* representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ of Gilboa and Schmeidler (1989) which is given by,

$$U^{u,\mathcal{P}}(X) = \inf_{P \in \mathcal{P}} \mathbb{E}_P[u(X)]$$

for all $X \in \mathcal{X}$ and for some non-empty, closed, and convex *set of priors* $\mathcal{P} \subset \Delta$. The multiple priors representation of preferences is obtained as a particular case of Equation (1) when the uncertainty index G satisfies,

$$G(x, P) = x + \delta(P \,|\, \mathcal{P}) \tag{3}$$

for all $(x, P) \in \mathbb{R} \times \Delta$ and where $\delta(.\,|\,\mathcal{P}) \to [0, +\infty]$ is defined by,

$$\delta(P \,|\, \mathcal{P}) = \begin{cases} 0 & \text{if } P \in \mathcal{P} \\ +\infty & \text{otherwise} \end{cases}$$

for all $P \in \Delta$.

Remark 3. Note that the representations of preferences of Cerreia Vioglio et al. (2011a), Maccheroni et al. (2006), and Gilboa and Schmeidler (1989) were originally obtained in a generalized Anscombe and Aumann (1963) framework in which the objects of choice are *uncertain acts*, that is Σ-measurable functions on S taking values in a convex subset of a general vector space. As in this paper we are specifically concerned with the uncertain acts whose consequences are money payments, we consider the restriction of the original preference functional to the subset of *monetary acts*, that is to the Σ-measurable functions on S taking values in the set of all Dirac measures δ_x on $\mathbb{R}$, and by virtue of the correspondence $X(s) \mapsto \delta_{X(s)}$ we identify monetary acts with monetary payoffs in $\mathcal{X}$ (see Föllmer and Schied (2004, Section 2.5)).

3 Indifference Pricing with Uncertainty Averse Preferences

In this section we study the indifference prices defined by the uncertainty averse representation of preferences described in Section 2. The properties of the indifference prices will be further investigated in Section 4 and in Section 5 in relation to the characterization of comparative uncertainty aversion and of increasing, decreasing, and constant uncertainty aversion.

3.1 Indifference Buyer's Price

The indifference buyer's price, which in this paper is considered from an actuarial perspective, is defined as a functional $\pi_{w_0}^{u,G}$ yielding the maximum price that a decision maker with uncertainty averse preferences $U^{u,G}$ and with constant initial wealth w_0 would offer to avoid an uncertain monetary prospect in $\mathcal{X}$ (e.g. to receive insurance).

Definition 1. A functional $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be an *indifference buyer's price* if it satisfies,

$$u(w_0 - \pi_{w_0}^{u,G}(X)) = U^{u,G}(w_0 + X)$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$.

Proposition 1 asserts that an indifference buyer's price is monotone decreasing, quasiconvex, and normalized. As a consequence of these properties, an indifference buyer's price is a *quasiconvex risk measure*. Quasiconvex risk measures were introduced in the financial mathematics literature by Cerreia Vioglio et al. (2011b) and further developed by Drapeau and Kupper (2010).

Proposition 1. *An indifference buyer's price $\pi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ satisfies the following properties for all $X, Y \in \mathcal{X}$.*

(i) Decreasing monotonicity: If $X \geq Y$, then $\pi_{w_0}^{u,G}(X) \leq \pi_{w_0}^{u,G}(Y)$.

(ii) Quasiconvexity: $\pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\}$ for all $\lambda \in [0,1]$.

(iii) Normalization: $\pi_{w_0}^{u,G}(-m) = m$ for all $m \in \mathbb{R}$.

Proof. (i) Let $X, Y \in \mathcal{X}$. If $X \geq Y$, then by Definition 1 and by the increasing monotonicity of $U^{u,G}$,

$$u(w_0 - \pi_{w_0}^{u,G}(X)) = U^{u,G}(w_0 + X) \geq U^{u,G}(w_0 + Y) = u(w_0 - \pi_{w_0}^{u,G}(Y))$$

and the increasing monotonicity of u yields $\pi_{w_0}^{u,G}(X) \leq \pi_{w_0}^{u,G}(Y)$. Thus, $\pi_{w_0}^{u,G}$ is monotone decreasing.
(ii) If $\lambda \in [0,1]$, then by Definition 1, by the quasiconcavity of $U^{u,G}$ and by the increasing monotonicity of u,

$$\begin{aligned}
&u(w_0 - \pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y)) \\
&= U^{u,G}(w_0 + \lambda X + (1-\lambda)Y) \\
&= U^{u,G}(\lambda(w_0+X) + (1-\lambda)(w_0+Y)) \\
&\geq \min\{U^{u,G}(w_0+X), U^{u,G}(w_0+Y)\} \\
&= \min\{u(w_0 - \pi_{w_0}^{u,G}(X)), u(w_0 - \pi_{w_0}^{u,G}(Y))\} \\
&= u(w_0 - \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\})
\end{aligned}$$

and the increasing monotonicity of u yields,

$$\pi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \max\{\pi_{w_0}^{u,G}(X), \pi_{w_0}^{u,G}(Y)\}$$

for all $\lambda \in [0,1]$. Thus, $\pi_{w_0}^{u,G}$ is quasiconvex.
(iii) If $m \in \mathbb{R}$, then by Definition 1,

$$u(w_0 - \pi_{w_0}^{u,G}(m)) = u(w_0 + m)$$

and the strict monotonicity of u yields $\pi_{w_0}^{u,G}(m) = -m$. Thus, $\pi_{w_0}^{u,G}$ is normalized. □

For further characterizations of the indifference buyer's price in terms of its acceptance family and of its maximal risk function we refer to Giammarino (2011).

3.2 Indifference Seller's Price

The indifference seller's price, which in this paper is considered from an actuarial perspective, is defined as a functional $\phi_{w_0}^{u,G}$ yielding the minimum price that a decision maker with uncertainty averse preferences $U^{u,G}$ and with constant initial wealth w_0 would demand to accept an uncertain monetary prospect in $\mathcal{X}$ (e.g. to provide insurance).

Definition 2. A functional $\phi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be an *indifference seller's price* if it satisfies,

$$u(w_0) = U^{u,G}(w_0 + X + \phi_{w_0}^{u,G}(X))$$

for all $X \in \mathcal{X}$ and $w_0 \in \mathbb{R}$.

Proposition 2 asserts that an indifference seller's price is monotone decreasing, convex, cash-additive and normalized. As a result of these properties, an indifference seller's price is a *cash-additive convex risk measure*. Cash-additive convex risk measures were introduced by Deprez and Gerber (1985) in the actuarial mathematics literature and by Frittelli and Rosazza Gianin (2002) and Föllmer and Schied (2002) in the financial mathematics literature.

Proposition 2. *An indifference seller's price $\phi_{w_0}^{u,G} : \mathcal{X} \to \mathbb{R}$ satisfies the following properties for all $X, Y \in \mathcal{X}$.*

(i) Decreasing monotonicity: If $X \geq Y$, then $\phi_{w_0}^{u,G}(X) \leq \phi_{w_0}^{u,G}(Y)$.

(ii) Convexity: $\phi_{w_0}^{u,G}(\lambda X + (1-\lambda)Y) \leq \lambda\phi_{w_0}^{u,G}(X) + (1-\lambda)\phi_{w_0}^{u,G}(Y)$ for all $\lambda \in [0,1]$.

(iii) Cash-additivity: $\phi_{w_0}^{u,G}(X + m) = \phi_{w_0}^{u,G}(X) - m$ for all $m \in \mathbb{R}$.

(iv) Normalization: $\phi_{w_0}^{u,G}(0) = 0$.

Proof. (i) Let $X, Y \in \mathcal{X}$. If $X \geq Y$, then by Definition 2 and by the increasing monotonicity of $U^{u,G}$,

$$\begin{aligned} u(w_0) = U^{u,G}(w_0 + &Y + \phi_{w_0}^{u,G}(Y)) \\ &= U^{u,G}(w_0 + X + \phi_{w_0}^{u,G}(X)) \\ &\qquad \geq U^{u,G}(w_0 + Y + \phi_{w_0}^{u,G}(X)) \end{aligned}$$

and the increasing monotonicity of $U^{u,G}$ yields $\phi_{w_0}^{u,G}(X) \leq \phi_{w_0}^{u,G}(Y)$. Thus, $\phi_{w_0}^{u,G}$ is monotone decreasing.

(ii) If $m \in \mathbb{R}$, then by Definition 2,

$$\begin{aligned}u(w_0) = U^{u,G}(&w_0{+}X + \phi_{w_0}^{u,G}(X))\\ &= U^{u,G}(w_0 + X + m + \phi_{w_0}^{u,G}(X + m))\\ &= u(w_0)\end{aligned}$$

and the strict monotonicity of $U^{u,G}$ yields,

$$\phi_{w_0}^{u,G}(X) = m + \phi_{w_0}^{u,G}(X + m)$$

Thus, $\phi_{w_0}^{u,G}$ is cash-additive.
(iii) If $\lambda \in [0, 1]$, then by the quasiconcavity of $U^{u,G}$ and by Definition 2,

$$\begin{aligned}&U^{u,G}(w_0 + \lambda X + (1 - \lambda)Y + \lambda\phi_{w_0}^{u,G}(X) + (1 - \lambda)\phi_{w_0}^{u,G}(Y))\\ &= U^{u,G}(\lambda(w_0 + X + \phi_{w_0}^{u,G}(X)) + (1 - \lambda)(w_0 + Y + \phi_{w_0}^{u,G}(Y)))\\ &\geq \min\{U^{u,G}(w_0 + X + \phi_{w_0}^{u,G}(X)), U^{u,G}(w_0 + Y + \phi_{w_0}^{u,G}(Y))\}\\ &=U^{u,G}(w_0 + \lambda X + (1 - \lambda)Y + \phi_{w_0}^{u,G}(\lambda X + (1 - \lambda)Y))\\ &= u(w_0)\end{aligned}$$

and the increasing monotonicity of $U^{u,G}$ yields,

$$\phi_{w_0}^{u,G}(\lambda X + (1 - \lambda)Y) \leq \lambda\phi_{w_0}^{u,G}(X) + (1 - \lambda)\phi_{w_0}^{u,G}(Y)$$

Thus, $\phi_{w_0}^{u,G}$ is convex.
(iv) As u is strictly increasing,

$$u(w_0 + \phi_{w_0}^{u,G}(0)) = u(w_0)$$

if and only if $\phi_{w_0}^{u,G}(0) = 0$. Thus, $\phi_{w_0}^{u,G}$ is normalized. □

For further characterizations of the indifference seller's price in terms of its acceptance set and of its minimal penalty function we refer to Giammarino (2011).

4 Characterizations of Uncertainty Attitudes

In this section we study the characterization of a decision maker's uncertainty attitudes in terms of the indifference prices introduced in Section 3. Further results on the characterization of uncertainty attitudes will be presented in Section 5 and in Section 6.

Remark 4. Note that the characterizations of the *uncertainty attitudes* (only) are obtained as special cases of the characterizations of the *risk and uncertainty attitudes* (jointly) under suitable normalization conditions on the risk attitudes (i.e. on the utility functions).

4.1 Comparative Uncertainty Aversion

The notion of comparative risk and uncertainty aversion presented in Definition 3 is consistent with the definition of comparative risk aversion of Yaari (1969) and with the definition of comparative uncertainty aversion of Ghirardato and Marinacci (2001). The intuition underlying the notion of comparative risk and uncertainty aversion presented in Definition 3 is that if a decision maker U^{u_1,G_1} with constant initial wealth w_0 prefers a constant monetary payoff x to an uncertain monetary payoff X, then a more risk and uncertainty averse decision maker U^{u_2,G_2} with the same constant initial wealth w_0 will do the same.

Definition 3. A decision maker $U^{u_1,G_1} : \mathcal{X} \to \mathbb{R}$ is said to be *less risk and uncertainty averse* than another $U^{u_2,G_2} : \mathcal{X} \to \mathbb{R}$ if,

$$u_1(w_0 + x) \geq U^{u_1,G_1}(w_0 + X) \;\Rightarrow\; u_2(w_0 + x) \geq U^{u_2,G_2}(w_0 + X) \tag{4}$$

for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and $w_0 \in \mathbb{R}$.

Note that U^{u_1,G_1} is said to be more risk and uncertainty averse than U^{u_2,G_2} when the implication in Equation (4) holds true in the opposite direction and that U^{u_1,G_1} is said to be as risk and uncertainty averse as U^{u_2,G_2} when the implication in Equation (4) holds true in both directions.

Remark 5. Note that the definition of comparative uncertainty aversion is obtained as a particular case of Definition 3 requiring that $u_1 \approx u_2$[1]. This normalization condition ensures that both decision makers U^{u_1,G_1} and U^{u_2,G_2} display the *same risk attitudes* u_1 and u_2 and that, as a consequence, their choices are compared only in terms of their *different uncertainty attitudes* G_1 and G_2 (see Cerreia Vioglio et al. (2011a, Section 3.3)).

Theorem 1 asserts that a decision maker is less risk and uncertainty averse than another if and only if the maximum price that she would offer to avoid an uncertain monetary prospect (e.g. to receive insurance) or, equivalently, the minimum price that she would demand to accept and uncertain monetary prospect (e.g. to provide insurance), is smaller than for the the other at every level of constant initial wealth.

Theorem 1. *The following statements are equivalent.*

(i) U^{u_1,G_1} *is less risk and uncertainty averse than* U^{u_2,G_2}*.*

(ii) $\pi^{u_1,G_1}_{w_0} \leq \pi^{u_2,G_2}_{w_0}$ *for all* $w_0 \in \mathbb{R}$*.*

(iii) $\phi^{u_1,G_1}_{w_0} \leq \phi^{u_2,G_2}_{w_0}$ *for all* $w_0 \in \mathbb{R}$*.*

[1]That is,$u_1 = \alpha u_2 + \beta$ for some $\alpha \in (0, +\infty)$ and $\beta \in \mathbb{R}$.

Proof. Let $X \in \mathcal{X}$ and $x \in \mathbb{R}$. $(i) \Leftrightarrow (ii)$ By Definition 3 and Definition 1, U^{u_1,G_1} is less risk and uncertainty averse than U^{u_2,G_2} if and only if,

$$u_1(w_0 + x) \geq u_1(w_0 - \pi_{w_0}^{u_1,G_1}(X)) \;\Rightarrow\; u_2(w_0 + x) \geq u_2(w_0 - \pi_{w_0}^{u_2,G_2}(X))$$

that is, since u_1 and u_2 are strictly increasing, if and only if,

$$\pi_{w_0}^{u_1,G_1}(X) \geq -x \;\Rightarrow\; \pi_{w_0}^{u_2,G_2}(X) \geq -x$$

Thus, U^{u_1,G_1} is less risk and uncertainty averse than U^{u_2,G_2} if and only if,

$$\pi_{w_0}^{u_1,G_1}(X) \leq \pi_{w_0}^{u_2,G_2}(X).$$

$(i) \Leftrightarrow (iii)$ By Definition 3 and Definition 2, U^{u_1,G_1} is less risk and uncertainty averse than U^{u_2,G_2} if and only if,

$$U^{u_1,G_1}(w_0 + x + X + \phi_{w_0+x}^{u_1,G_1}(X)) \geq U^{u_1,G_1}(w_0 + X) \;\Rightarrow$$
$$U^{u_2,G_2}(w_0 + x + X + \phi_{w_0+x}^{u_2,G_2}(X)) \geq U^{u_2,G_2}(w_0 + X)$$

that is, since U^{u_1,G_1} and U^{u_2,G_2} are strictly increasing, if and only if,

$$\phi_{w_0+x}^{u_1,G_1}(X) \geq -x \;\Rightarrow\; \phi_{w_0+x}^{u_2,G_2}(X) \geq -x$$

Thus, U^{u_1,G_1} is less risk and uncertainty averse than U^{u_2,G_2} if and only if,

$$\phi_{w_0+x}^{u_1,G_1}(X) \leq \phi_{w_0+x}^{u_2,G_2}(X).$$

□

Remark 6. Note that Theorem 1 implies that U^{u_1,G_1} is as risk and uncertainty averse as U^{u_2,G_2} if and only if $\pi_{w_0}^{u_1,G_1} = \pi_{w_0}^{u_2,G_2}$ and $\phi_{w_0}^{u_1,G_1} = \phi_{w_0}^{u_2,G_2}$ for all $w_0 \in \mathbb{R}$.

4.2 Increasing, Decreasing, and Constant Uncertainty Aversion

The notion of increasing risk and uncertainty aversion presented in Definition 4 is consistent with the definition of increasing risk aversion in Kreps (1988, Chapter 6, page 75). The intuition underlying the notion of increasing uncertainty aversion presented in Definition 4 is that if an increasingly risk and uncertainty averse decision maker $U^{u,G}$ prefers a constant monetary payoff x to an uncertain monetary payoff X when her constant initial wealth is w_1, then when her constant initial wealth is increased to w_2 she will do the same.

Definition 4. A decision maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is said to be *increasingly risk and uncertainty* *averse* if,

$$u(w_1 + x) \geq U^{u,G}(w_1 + X) \;\Rightarrow\; u(w_2 + x) \geq U^{u,G}(w_2 + X) \tag{5}$$

for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and $w_1, w_2 \in \mathbb{R}$ such that $w_1 \leq w_2$.

Note that $U^{u,G}$ is said to be decreasingly risk and uncertainty averse if the implication in Equation (5) holds true in the opposite direction and that $U^{u,G}$ is said to be constantly risk and uncertainty averse if the implication in Equation (5) holds true in both directions.

Remark 7. Note that the definition of increasing uncertainty aversion is obtained as a particular case of Definition 4 requiring the utility function u to be constantly absolute risk averse (CARA)[2]. This normalization condition ensures that the decision maker displays the *same risk aversion* at different levels of constant initial wealth w_1 and w_2 and that, as a consequences, her choices are compared only in terms of her *different uncertainty aversion*.

Theorem 2 asserts that a decision maker is increasingly risk and uncertainty averse if and only if the maximum price that she would offer to avoid an uncertain monetary prospect (e.g. to receive insurance) or, equivalently, the minimum price that she would demand to accept an uncertain monetary prospect (e.g. to provide insurance), is larger the larger her constant initial wealth.

Theorem 2. *The following statements are equivalent.*

(i) $U^{u,G}$ *is increasingly risk and uncertainty averse.*

(ii) $\pi^{u,G}_{w_1} \leq \pi^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

(iii) $\phi^{u,G}_{w_1} \leq \phi^{u,G}_{w_2}$ *for all* $w_1, w_2 \in \mathbb{R}$ *such that* $w_1 \leq w_2$*.*

Proof. Follows from the same arguments applied in the proof of Theorem 1. □

Remark 8. Note that Theorem 2 implies that $U^{u,G}$ is constantly risk and uncertainty averse if and only if $\pi^{u,G}_{w_1} = \pi^{u,G}_{w_2}$ and $\phi^{u,G}_{w_1} = \phi^{u,G}_{w_2}$ for all $w_1, w_2 \in \mathbb{R}$.

5 Further Characterizations of Uncertainty Attitudes

This section illustrates some further characterizations of increasing, decreasing, and constant risk and uncertainty aversion which do not rely on the dependence of the indifference prices on the decision maker's constant initial wealth w_0. The characterization results presented in this section rely instead on the observation that a decision maker's increasing, decreasing, or constant risk and uncertainty aversion describes how her choice between an uncertain monetary payoff X and a constant monetary payoff x is altered if a positive constant amount of money m is added to both alternatives.

[2]A utility function $u : \mathbb{R} \to \mathbb{R}$ is CARA if it is either linear $u(x) = \alpha x + \beta$ or exponential $u(x) = -\alpha e^{-\theta x}$ with $\alpha, \theta > 0$ and $\beta \in \mathbb{R}$.

5.1 Cash-Subadditivity, Cash-Superadditivity, and Cash-Additivity

Increasing, decreasing, and constant risk and uncertainty aversion are equivalently characterized by the additive properties that the indifference buyer's price satisfies with respect to the constant monetary payoffs m for any given w_0. Theorem 3 asserts that a decision maker is increasingly risk and uncertainty averse if and only if the indifference buyer's price is a *cash-subadditive* quasiconvex risk measure. The property of cash-subadditivity was introduced in the mathematical finance literature on risk measures by El Karoui and Ravanelli (2009) to model the impact of default risk and interest rate ambiguity on the overall risk of a financial or insurance contract. The property of cash-subadditivity is a weakening of the property of cash-additivity considered by Deprez and Gerber (1985), Frittelli and Rosazza Gianin (2002), and Föllmer and Schied (2002).

Theorem 3. *A decision maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,*

$$\pi_{w_0}^{u,G}(X + m) \geq \pi_{w_0}^{u,G}(X) - m$$

for all $m \in [0, +\infty)$ and $X \in \mathcal{X}$.

Proof. Let $X \in \mathcal{X}$, $x \in \mathbb{R}$ and $m \in [0, +\infty)$. By Definition 4, $U^{u,G}$ is increasingly risk and uncertainty averse if and only if,

$$u(w_0 + x) \geq U^{u,G}(w_0 + X) \;\Rightarrow\; u(w_0 + x + m) \geq U^{u,G}(w_0 + X + m)$$

It follows from Definition 1 that $U^{u,G}$ is increasingly risk and uncertainty averse if and only if,

$$u(w_0 + x) \geq u(w_0 - \pi_{w_0}^{u,G}(X)) \;\Rightarrow\; u(w_0 + m + x) \geq u(w_0 - \pi_{w_0}^{u,G}(X + m))$$

or, equivalently, as u is strictly increasing, if and only if,

$$\pi_{w_0}^{u,G}(X) \geq -x \;\Rightarrow\; \pi_{w_0}^{u,G}(X + m) + m \geq -x$$

Thus, $U^{u,G}$ is increasingly risk and uncertainty averse if and only if,

$$\pi_{w_0}^{u,G}(X + m) \geq \pi_{w_0}^{u,G}(X) - m.$$

□

Remark 9. Note that Theorem 3 implies that $U^{u,G}$ is constantly risk and uncertainty averse if and only if $\pi_{w_0}^{u,G}(X + m) = \pi_{w_0}^{u,G}(X) - m$ for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$. As a quasiconvex risk measure which is cash-additive is necessarily convex (see Cerreia Vioglio et al. (2011b)), the indifference buyer's price of a constantly risk and uncertainty averse decision maker is a cash-additive convex risk measure.

5.2 Inequalities

As a consequence of Theorem 3, increasing, decreasing, and constant risk and uncertainty aversion are equivalently characterized by the inequalities that the indifference prices fulfill for every w_0.

Theorem 4. *A decision maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly risk and uncertainty averse if and only if,*

$$\pi_{w_0}^{u,G}(X) \leq \phi_{w_0}^{u,G}(X)$$

for all $X \in \mathcal{X}$ such that $\phi_{w_0}^{u,G}(X) \in [0, +\infty)$.

Proof. Let $X \in \mathcal{X}$. By Definition 1 and by Definition 2,

$$u(w_0 - \pi_{w_0}^{u,G}(X + \phi_{w_0}^{u,G}(X))) = U^{u,G}(w_0 + X + \phi_{w_0}^{u,G}(X)) = u(w_0)$$

and the strict monotonicity of u yields,

$$0 = \pi_{w_0}^{u,G}(X + \phi_{w_0}^{u,G}(X))$$

Thus, by Theorem 3, $U^{u,G}$ is increasingly risk and uncertainty averse if and only if,

$$0 = \pi_{w_0}^{u,G}(X + \phi_{w_0}^{u,G}(X)) \geq \pi_{w_0}^{u,G}(X) - \phi_{w_0}^{u,G}(X)$$

for all $X \in \mathcal{X}$ such that $\phi_{w_0}^{u,G}(X) \in [0, +\infty)$. $\square$

Remark 10. Note that Theorem 4 implies that $U^{u,G}$ is constantly risk and uncertainty averse if and only if $\pi_{w_0}^{u,G}(X) = \phi_{w_0}^{u,G}(X)$ for all $X \in \mathcal{X}$.

6 Increasingly, Decreasingly, and Constantly Uncertainty Averse Preferences

In this section we study the characterization of uncertainty attitudes in terms of the uncertainty index G appearing in the uncertainty averse representation of preferences $U^{u,G}$ in Equation (1). As remarked in Section 4, in order to characterize a decision maker's uncertainty attitudes (as reflected by G), it is necessary to impose some suitable normalization conditions on her risk attitudes (as reflected by u).

The uncertainty averse representations of preferences that are minimally and maximally uncertainty averse consistently with the notion of comparative uncertainty aversion in Definition 3 and Remark 5 were studied by Cerreia Vioglio et al. (2011a). Cerreia Vioglio et al. (2011a, Proposition 6) found that a decision maker U^{u_1,G_1} is more uncertainty averse than another U^{u_2,G_2} for any choice of u_1 and u_2 such that $u_1 \approx u_2$ if and only if $G_1 \leq G_2$.

In this section we study the uncertainty averse representations of preferences that are increasingly, decreasingly, and constantly uncertainty averse consistently with the notion of increasing, decreasing, and constant uncertainty aversion in Definition 4 and Remark 7. We find that a decision maker $U^{u,G}$ is increasingly uncertainty averse for any choice of u in the class of CARA utility functions if and only if G is *star-shaped*[3] and cash-subadditive.
The property of star-shapedness was introduced in the mathematical finance literature by Cerreia Vioglio et al. (2011b) to model the impact of liquidity risk on the minimal reserve amount that must be added to an uncertain monetary payoff such that it becomes acceptable to a decision maker. The property of star-shapedness is a weakening of the property of *positive homogeneity*[4] considered by Artzner et al. (1997, 1999).

Theorem 5. *A decision maker $U^{u,G} : \mathcal{X} \to \mathbb{R}$ is increasingly uncertainty averse for any choice of $u : \mathbb{R} \to \mathbb{R}$ in the class of CARA utility functions if and only if,*

$$G(\lambda x + m, P) \leq \lambda G(x, P) + m$$

for all $\lambda \in (0, 1]$, $m \in [0, +\infty)$, and $(x, P) \in \mathbb{R} \times \Delta$.

Proof. Let $X \in \mathcal{X}$ and $(x, P) \in \mathbb{R} \times \Delta$. Without loss of generality, set $w_0 = 0$. The indifference buyer's price $\pi_0^{\mathcal{L},G}$ defined in terms of the linear utility function $\mathcal{L}(y) = y$ for all $y \in \mathbb{R}$ is given by,

$$\pi_0^{\mathcal{L},G}(X) = \sup_{P \in \Delta} -G(E_P[X], P)$$

and by Drapeau and Kupper (2010, Proposition 2.11) it is cash-subadditive if and only if $G(x + m, P) \leq G(x, P) + m$ for all $m \in [0, +\infty)$.
The indifference buyer's price $\pi_0^{\mathcal{E},G}$ defined in terms of the exponential utility function $\mathcal{E}(y) = -e^{-y}$ for all $y \in \mathbb{R}$ is given by,

$$\pi_{w_0}^{\mathcal{E},G}(X) = \ln\Big(\sup_{P \in \Delta} -G(E_P[-e^{-X}], P)\Big)$$

and it is cash-subadditive if and only if $G(\lambda x, P) \leq \lambda G(x, P)$ for all $\lambda \in (0, 1]$.
It follows that the indifference buyer's price $\pi_{w_0}^{u,G}$ is cash-subadditive for any choice of u in the class of CARA utility functions if and only if $G(\lambda x + m, P) \leq \lambda G(x, P) + m$ for all $\lambda \in (0, 1]$ and $m \in [0, +\infty)$. Thus, the statement follows from Theorem 3 and Remark 7. $\square$

Remark 11. Note that Theorem 5 implies that $U^{u,G}$ is constantly uncertainty averse for any choice of u in the class of CARA utility functions if and only if $G(\lambda x + m, P) = \lambda G(x, P) + m$ for all $\lambda \in (0, +\infty)$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$. Thus, the uncertainty index of a constantly uncertainty averse decision maker is cash-additive and positively homogeneous.

[3]A function $h : \mathbb{R} \to \mathbb{R}$ is said to be star-shaped if $h(\lambda x) \leq \lambda h(x)$ for all $\lambda \in (0, 1]$ and $x \in \mathbb{R}$.
[4]A function $h : \mathbb{R} \to \mathbb{R}$ is said to be positively homogeneous if $h(\lambda x) = \lambda h(x)$ for all $\lambda \in (0, +\infty)$ and $x \in \mathbb{R}$.

Example 1. By Theorem 5, the variational representation of preferences $U^{u,c} : \mathcal{X} \to \mathbb{R}$ is decreasingly uncertainty averse. In fact, as $c(P) \geq 0$ for all $P \in \Delta$, the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (2) satisfies,

$$\begin{aligned} G(\lambda x + m, P) &= \lambda x + m + c(P) \\ &\geq \lambda x + m + \lambda c(P) \\ &= \lambda G(x, P) + m \end{aligned}$$

for all $\lambda \in (0, 1]$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$.

Example 2. By Theorem 5, the multiple priors representation of preferences $U^{u,\mathcal{P}} : \mathcal{X} \to \mathbb{R}$ is constantly uncertainty averse. In fact, as for every $P \in \Delta$ either $\delta(P \,|\, \mathcal{P}) = 0$ or $\delta(P \,|\, \mathcal{P}) = +\infty$, the uncertainty index $G : \mathbb{R} \times \Delta \to (-\infty, +\infty]$ in Equation (3) satisfies,

$$\begin{aligned} G(\lambda x + m, P) &= \lambda x + m + \delta(P \,|\, \mathcal{P}) \\ &= \lambda x + m + \lambda \delta(P \,|\, \mathcal{P}) \\ &= \lambda G(x, P) + m \end{aligned}$$

for all $\lambda \in (0, +\infty)$, $m \in \mathbb{R}$, and $(x, P) \in \mathbb{R} \times \Delta$.

7 Conclusion

In this paper the indifference buyer's price and the indifference seller's price are defined in terms of the uncertainty averse representation of preferences of Cerreia Vioglio et al. (2011a). The indifference buyer's price is a quasiconvex risk measure and the indifference seller's price is a cash-additive convex risk measure. A decision maker is more uncertainty averse than another if and only if her indifference prices are pointwise larger than the other's. A decision maker is increasingly (respectively, decreasingly, constantly) uncertainty averse if and only if her indifference prices are increasing (respectively, decreasing, constant) functions of her constant initial wealth. Future research might investigate the extension of the analysis of this paper to a dynamic setting along the lines of Frittelli and Maggis (2011a,b).

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