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13 August 2012

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ABSTRACT

We explore bifurcation phenomena in the open-economy New Keynesian model developed by Clarida, Gali and Gertler (2002). We find that the open economy framework can bring about more complex dynamics, along with a wider variety of qualitative behaviors and policy responses. Introducing parameters related to the open economy structure affects the values of bifurcation parameters and changes the location of bifurcation boundaries. As a result, the stratification of the confidence region, as previously seen in closed-economy New Keynesian models, remains an important research and policy risk to be considered in the context of the open-economy New Keynesian functional structures. In fact, econometrics and optimal policy design become more complex within an open economy. Dynamical inferences need to be qualified by the risk of bifurcation boundaries crossing the confidence regions. Without adequate prior econometric research, policy design needs to take into consideration that a change in monetary policy can produce an unanticipated bifurcation.

Keywords: stability; bifurcation; open economy; New Keynesian; macroeconomics; dynamic systems

JEL-codes: C52, C61, C62, E32, E37, E61, F41

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1. Introduction

Dynamical economic systems are subject to bifurcations. As Grandmont (1985) has shown, even simple dynamic economic systems may exhibit various types of dynamic behaviors within the same functional structure, with the parameter space stratified into bifurcation regions associated with different dynamical solution-path behaviors. Therefore, analyzing bifurcation boundaries is relevant to understanding the dynamic properties of an economic system. Barnett and He (1999) investigated the stability of the Bergstrom, Nowman, and Wymer (1992) continuous time macroeconometric model of the UK economy and found both transcritical and Hopf bifurcations. Barnett and He (2006) more recently detected a singularity bifurcation in the Leeper and Sims' (1994) Euler equations macroeconometric model of the U.S. economy. Barnett, Banerjee, Duzhak, and Gopalan (2011) found that including industrial organization features into a Zellner's Marshallian macroeconomic model, permitting entry and exit of firms, does not decrease the relevancy of bifurcation phenomena. Barnett and Duzhak (2008, 2010) analyzed bifurcation using a closed economy New Keynesian model, based on Walsh (2003), and found both Hopf and period doubling bifurcations within the parameter space.

Occurrence of bifurcation boundaries stratifies the parameter space. As observed by Barnett and He (1999, 2002, 2006) and Barnett and Duzhak (2008, 2010), the existence of bifurcation boundaries in the parameter space indicates the presence of different solution types corresponding to parameter values close to each other, but on different sides of the bifurcation boundary. Dynamic properties of the system can change dramatically on different sides of a bifurcation boundary. As a result, robustness of inferences about dynamical solution properties can be damaged, if parameter values are close enough to a bifurcation boundary so that the parameters' confidence regions cross the boundary.

In Barnett and Eryilmaz (2012), we previously analyzed the Gali and Monacelli (2005) model, which is an open economy New Keynesian Model, and found that introducing parameters related to the open economy structure affects the values of bifurcation parameters and changes the location of bifurcation boundaries. Thus, the stratification of the confidence region, as often seen in closed economy New Keynesian models, is an important risk to be considered in the context of open economy New Keynesian functional structures. In this study, we examine another mainstream New Keynesian model, the Clarida, Gali and Gertler (2002) model, in the open economy tradition to further explore analytically the possibility of Hopf bifurcations within open economy New Keynesian structures. Application of our theoretical results to numerical analysis with the Clarida, Gali and Gertler (2002) model would be a challenging project and is beyond the scope of this paper, but we find from theoretical analysis of the model that future research using numerical methods to locate the model's bifurcation boundaries would be justified, and we provide the theory needed to implement the numerical search and locate Hopf bifurcation boundaries.

2. Model

We investigate the possibility of bifurcations in the open-economy New Keynesian model developed by Clarida, Gali, and Gertler (2002). We thereby extend the conclusions of Barnett and Duzhak (2008, 2010) to the open economy case. Barnett and Duzhak (2008, 2010) analyze bifurcation with a closed-economy New Keynesian model and found both Hopf and period doubling bifurcations.

Clarida, Gali, and Gertler (2002) developed a two-country version of a small open economy model, which is based on Clarida, Gali, and Gertler (2001) and Gali and Monacelli (1999). Let x_t denote the output gap, π_t^h the inflation rate for domestically produced goods and services, and r_t the nominal interest rate, with E_t being the expectation operator and $\overline{r}_t$ denoting the small open economy's natural rate of interest. The lowercase letters denote the logs of the respective variables. Then, following Walsh (2003, pp. 539 - 540), the model of Clarida, Gali, and Gertler (2002) can be rewritten in the reduced form as follows:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta\left[\sigma + \eta + \left(\frac{\nu\sigma}{1+w}\right)\right]x_t , \tag{1}$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma}\right)\left(r_t - E_t \pi_{t+1}^h - \bar{r}_t\right), \tag{2}$$

$$r_t = \bar{r}_t + \phi_\pi \pi_t^h + \phi_x x_t . \tag{3}$$

The coefficients $\phi_x > 0$ and $\phi_\pi > 0$ are the policy parameters, which measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In addition, $\delta = \left[(1-\theta)(1-\beta\theta)\right]/\theta$ is a composite parameter with θ representing the probability that a firm holds its price unchanged in a given period of time, while $1-\theta$ is the probability that a firm resets its price. The parameter η denotes the wage elasticity of labor demand, and σ^{-1} denotes the elasticity of intertemporal substitution. The parameter w denotes the growth rate of nominal wages, $\rho = \beta^{-1} - 1$ is the time discount rate, and ν is the population size in the foreign country, with $1-\nu$ being the population size of the home country. Wealth effect is captured by the term $\nu\sigma$.

Equation (1) is an inflation adjustment equation for the aggregate price of domestically produced goods. Equation (2) is the dynamic IS curve, which is derived from the Euler condition of the consumers' optimization problem. The monetary policy rule (3) is a domestic-inflation-based current-looking Taylor rule, which completes the model.

Substituting (3) for $r_t - \bar{r}_t$ into the equation (2), we can reduce the system to a first order dynamic system in two equations for domestic inflation and output gap, given by:

$$\pi_t^h = \beta E_t \pi_{t+1}^h + \delta\left[\sigma + \eta + \left(\frac{\nu\sigma}{1+w}\right)\right]x_t ,$$

$$x_t = E_t x_{t+1} - \left(\frac{1+w}{\sigma}\right)\left(\phi_\pi \pi_t^h + \phi_x x_t - E_t \pi_{t+1}^h\right).$$

Clearly, $x_t = \pi_t^h = 0$ for all t constitutes a solution (equilibrium) to the system. We can write the system in the standard form $\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t$ as follows:

$$\mathbf{A}\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = \mathbf{B}\begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix}, \tag{4}$$

where $\mathbf{A} = \begin{bmatrix} 0 & \beta \\ 1 & \dfrac{1+w}{\sigma} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -\delta\left[\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right] & 1 \\ 1 + \dfrac{(1+w)\phi_x}{\sigma} & \dfrac{(1+w)\phi_\pi}{\sigma} \end{bmatrix}$.

Then, premultiplying the terms on the right hand side by the inverse of the matrix **A**, the system can be reduced to the form $E_t\mathbf{y}_{t+1} = \mathbf{C}\mathbf{y}_t$, where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$, as follows:

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = \mathbf{C}\begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix} \tag{5}$$

where $\mathbf{C} = \begin{bmatrix} 1 + \dfrac{(1+w)\phi_x}{\sigma} + \delta(1+w)\left(\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right)\dfrac{1}{\beta\sigma} & \dfrac{(1+w)\phi_\pi}{\sigma} - \dfrac{(1+w)}{\beta\sigma} \\ -\delta\left(\sigma + \eta + \left(\dfrac{\nu\sigma}{1+w}\right)\right)\dfrac{1}{\beta} & \dfrac{1}{\beta} \end{bmatrix}$.

The system (5) is in normal form, in the sense that each equation has only one unknown variable evaluated at time $t+1$. Note that there were no disturbance terms included in the model, so $\varepsilon_t = 0$. For the uniqueness and stability of the equilibrium, both eigenvalues must be outside the unit circle.

The characteristic polynomial of the coefficient matrix **C** is given by

$$p(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}) = \lambda^2 - a_1\lambda + a_0 = 0,$$

where

$$a_0 = (1+w)(\phi_x + \phi_\pi \delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi \delta(1+v+w)+1)\frac{1}{\beta},$$

and

$$a_1 = (1+w)(\delta\eta + \phi_x \beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta} + 1,$$

which yields

$$\lambda_{1,2} = \left((1+w)(\delta\eta+\phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta}+1\right)$$
$$\pm\left(\left((1+w)(\delta\eta+\phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta}+1\right)^2 - 4\left((1+w)(\phi_x+\phi_\pi\delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi\delta(1+v+w)+1)\frac{1}{\beta}\right)\right)^{\frac{1}{2}}.$$

To examine the nature of the eigenvalues we need to check the sign of the discriminant $\Delta \equiv a_1^2 - 4a_0$, as shown in Gandolfo (1996). If the discriminant of the quadratic equation is strictly negative, so that

$$\Delta \equiv a_1^2 - 4a_0 = \left((1+w)(\delta\eta+\phi_x\beta)\frac{1}{\beta\sigma} + (1+\delta(1+v+w))\frac{1}{\beta}+1\right)^2 - 4\left((1+w)(\phi_x+\phi_\pi\delta\eta)\frac{1}{\beta\sigma} + (\phi_\pi\delta(1+v+w)+1)\frac{1}{\beta}\right) < 0,$$

then the roots of the coefficient matrix **C** will be complex conjugate numbers in the form $\lambda_{1,2} = a \pm ib$, with $a,b \in \mathbb{R}$, where $i = +\sqrt{-1}$ is the imaginary unit.

Regarding the system (5), it is algebraically cumbursome to identify the sign of the discriminant. Therefore, we assume that the eigenvalues of the system (5) are complex conjugates, $\lambda_{1,2} = a \pm ib$,

where

$$a = \frac{a_1}{2} = \left((1+w)(\delta\eta+\phi_x\beta)\frac{1}{2\beta\sigma} + (1+\delta(1+v+w))\frac{1}{2\beta} + \frac{1}{2}\right) \quad (6)$$

and

$$b=\frac{\sqrt{-\Delta}}{2}=\frac{1}{2}\sqrt{4\left(\left(1+w\right)\left(\phi_x+\phi_\pi\delta\eta\right)\frac{1}{\beta\sigma}+\left(\phi_\pi\delta\left(1+v+w\right)+1\right)\frac{1}{\beta}\right)-\left(\left(1+w\right)\left(\delta\eta+\phi_x\beta\right)\frac{1}{\beta\sigma}+\left(1+\delta\left(1+v+w\right)\right)\frac{1}{\beta}+1\right)^2}. \tag{7}$$

3. Bifurcation Analysis

To determine whether a Hopf bifurcation exists in the Clarida, Gali, and Gertler (2002) model, we use the methodology suggested by Gandolfo (1996) and Barnett and Duzhak (2008, 2010). We first evaluate the Jacobian of the system at the equilibrium point $\pi_t^h = x_t = 0$ for all $t = 1,2,...$, and then check whether the conditions of the Hopf Bifurcation Theorem are satisfied. For two dimensional systems, we apply the existence part of the Hopf Bifurcation Theorem given in Gandolfo (1996, p 492).

Theorem 1: *Consider the class of two-dimensional first-order difference equation systems produced by the map* $\mathbf{y} \to \mathbf{f}(\mathbf{y},\phi)$, $\mathbf{y} \in \mathbb{R}^2$, *with vector of parameters,* $\boldsymbol{\phi} \in \mathbb{R}^N$. *Assume for each* $\boldsymbol{\phi}$, *there exists a local fixed point,* $\mathbf{y}^* = \mathbf{y}^*(\boldsymbol{\phi})$, *in the relevant interval at which the eigenvalues of the Jacobian matrix, evaluated at* $(\mathbf{y}^*(\boldsymbol{\phi}),\boldsymbol{\phi})$, *are complex conjugates,* $\lambda_{1,2} = a \pm ib$, *and satisfy the following properties:*

(i) $|\lambda_1| = |\lambda_2| = +\sqrt{a^2+b^2} = 1$, *with* $\lambda_i \neq 1$ *for* $i = 1,2$,

where $|\lambda_i|$ *is the modulus of the eigenvalue* λ_i. *Also assume there exists* j = *1, 2, ..., N such that*

(ii) $\left.\frac{\partial|\lambda_i(\boldsymbol{\phi})|}{\partial\phi_j}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \neq 0$ *for i = 1,2.*

Then, there exists a Hopf bifurcation at the equilibrium point $(\mathbf{y}^*(\boldsymbol{\phi}^*),\boldsymbol{\phi}^*)$.

With the assumption of a pair of complex conjugate eigenvalues, we may expect to see a Hopf bifurcation, if the transversality conditions are satisfied. Using Theorem 1, the conditions for the existence of a Hopf bifurcation are stated in the following proposition.

Proposition 1: *Let* Δ *be the discriminant of the characteristic equation. Then the system (5) undergoes a Hopf bifurcation, if and only if* $\Delta < 0$ *and*

$$\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right). \tag{8}$$

Proof: Suppose the system (5) goes through a Hopf bifurcation at (y^*, ϕ_x^*), where $y^* = (x^*, \pi^*)$. Then, we need to show that $\Delta < 0$ and $\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right)$. The existence of a Hopf bifurcation requires a pair of complex conjugate eigenvalues on the unit circle. For the eigenvalues to be complex conjugate, the discriminant must be strictly negative, so that $\Delta < 0$.

For the second part of the theorem, note that the existence of a Hopf bifurcation requires $\operatorname{mod}(\lambda_1) = \operatorname{mod}(\lambda_2) = +\sqrt{a^2 + b^2} = 1$ by the first condition of Theorem 1. Rewriting the condition explicitly by substituting (6) and (7) into it, taking the square of both sides, and solving for ϕ_x, we obtain (8). Therefore, the first condition of Theorem (1) holds, only if

$$\phi_x = \frac{\beta\sigma - 1}{1+w} - \phi_\pi \left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta \right).$$

In the theorem's converse direction, suppose $\Delta < 0$ and

$\phi_x = \frac{\beta\sigma - 1}{1+w} - \phi_\pi\left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta\right)$. Substituting for ϕ_x^* into $\sqrt{a^2+b^2}$ yields

$\mathrm{mod}(\lambda_1) = \mathrm{mod}(\lambda_2) = 1$, which is the first condition in Theorem 1. In order to show that the critical value of the parameter ϕ_x is a Hopf bifurcation parameter, we check Theorem 1's second condition, which yields

$$\left.\frac{d\left|\lambda_i(\phi_x)\right|}{d\phi_x}\right|_{\phi_x=\phi_x^*} = \left.\frac{d}{d\phi_x}\left(\sqrt{a^2+b^2}\right)\right|_{\phi_x=\phi_x^*} = \frac{1+w}{2\beta\sigma} \neq 0 \text{ for } i = 1,2.$$

Thus, both conditions of Theorem 1 are satisfied and we have

$$\phi_x^* = \frac{\beta\sigma - 1}{1+w} - \phi_\pi\left(\frac{\delta\sigma(1+v+w)}{1+w} + \delta\eta\right).$$

■

Proposition 1 shows formally that taking the parameter ϕ_x free to vary and keeping the other parameters constant at plausible settings, the model of Clarida, Gali, and Gertler (2002) can be expected to undergo a Hopf bifurcation at ϕ_x^*.

Note that, the model of Clarida, Gali, and Gertler (2002) differs in several aspects from the Gali and Monacelli (2005) model, which we used in another study. Additional paramaters exist in the former model. In that model, the parameters w, v, and δ play an important role in determining the critical value of the bifurcation parameter, as we have shown. The degree to which the two models differ depends upon the parameter settings. But it is clear that numerical implementation of our theory to locating Hopf bifurcation boundaris in the Clarida, Gali, and Gertler (2002) model would be a challenging project, which we now advocate.

4. Conclusions

Bifurcation analysis has been widely used to examine and classify the dynamic behavior of a variety of economic models in economic literature. In this paper, we derive the analytical conditions for Hopf bifurcation in the open economy New Keynesian model developed by Clarida, Gali, and Gertler (2002). Using the Hopf Bifurcation Theorem, we establish the conditions for Hopf bifurcation of the model. On theoretical grounds, we show that by varying the parameter ϕ_x, while keeping the other parameters constant, the model of Clarida, Gali, and Gertler (2002) is vulnerable to Hopf bifurcation at ϕ_x^*. We also show that the structural parameters, w, v, and δ, play a significant role in determining the critical value of the bifurcation parameter, ϕ_x. Our theoretical results need to be confirmed by subsequent numerical analysis to locate the Hopf bifurcation boundary and map its shape. But that numerical analysis is beyond the scope of this paper limited to determining the relevant theory.

A primary objective of the subsequent numerical analysis should be to determine whether the Hopf bifurcation boundary crosses relevant confidence regions of the model's parameters. If so, a serious robustness problem would exist in dynamical inferences using the model. But even if the bifurcation boundary does not cross the confidence region, policy can move the location of the bifurcation boundary by changing the values of policy parameters. Within this model, the central bank should react cautiously to changes in the rate of domestic inflation and the output gap and should particularly take into consideration the following structural parameters: price rigidity, θ, wage inflation , w, and the wealth effect , $v\sigma$, to avoid inducing instability from a possible Hopf bifurcation.

Our theoretical results are consistent with prior results with other New Keynesian models in Barnett and Duzhak (2008, 2010) and Barnett and Eryilmaz (2012). Those results, which have been confirmed by numerical analysis, reinforce our conclusion that our theoretical results should be used in numerical analysis of bifurcation boundary locations in the New Keynesian Clarida, Gali, and Gertler (2002) open-economy model.

REFERENCES:

Barnett W.A. and E.A. Duzhak (2008). Non-robust dynamic inferences from macroeconometric models: Bifurcation of confidence regions. Physica A, Vol 387, No 15, June , 3817-3825.

Barnett W.A. and E.A. Duzhak (2010). Empirical assessment of bifurcation regions within New Keynesian models. Economic Theory, Vol. 45, 99–128.

Barnett, W. A. and U. Eryilmaz (2012), An analytical and numerical search for bifurcations in open economy New Keynesian models, University of Kansas working paper, Lawrence, Kansas.

Barnett, W.A. and Y. He (1999). Stability analysis of continuous time macroeconometric systems. Studies in Nonlinear Dynamics and Econometrics, 3(4), 169-188.

Barnett, W.A. and Y. He,(2002). Stabilization policy as bifurcation selection: Would stabilization policy work if the economy really were unstable? Macroeconomic Dynamics, 6, 713–747.

Barnett, W.A. and Y. He (2006). Singularity bifurcations. Journal of Macroeconomics, Vol 28, Issue 1, March , 5-22.

Barnett, W. A., S. Banerjee, E. Duzhak, and R. Gopalan (2011). Bifurcation analysis of Zellner's Marshallian macro model, Journal of Economic Dynamics and Control 9(35), 1577-1585.

Bergstrom, A.R., Nowman, K.B., and C.R. Wymer (1992). Gaussian estimation of a second order continuous time macroeconometric model of the United Kingdom. Economic Modelling, 9, 313-352.

Clarida, R., Gali, J., and M. Gertler (2001). Optimal monetary policy in open versus closed economies. American Economic Review, Vol. 91 (2), May, 248-252.

Clarida, R., Gali, J., and M. Gertler (2002). A simple framework for international monetary policy analysis. Journal of Monetary Economics, Elsevier, Vol. 49(5), July, 879-904.

Gali, J. and T. Monacelli (1999). Optimal monetary policy and exchange rate volatility in a small open economy. Boston College Working Papers in Economics, 438, Boston College Department of Economics.

Gali, J. and T. Monacelli (2005). Monetary policy and exchange rate volatility in a small open economy. Review of Economic Studies, Vol. 72, No 3, July.

Gandolfo, G. (1996). Economic Dynamics. 3rd Edition, New York and Heidelburg, Springer-Verlag. Gandolfo (1996).

Grandmont, J.M. (1985). On endogenous competitive business. Econometrica, Vol.53, 995-1045.

Leeper, E. and C. Sims (1994). Toward a modern macro model usable for policy analysis. NBER Macroeconomics Annual, 81-117.

Walsh, E. C. (2003). Monetary Theory and Policy, Cambridge, MA, MIT Press, Second Edition.