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# An optimal strategy for maximizing the expected real-estate selling price: accept or reject an offer?

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**Abstract.** Motivated by a real-life situation, we put forward a model and then derive an optimal strategy that maximizes the expected real-estate selling price when one of the only two remaining buyers has already made an offer but the other one is yet to make. Since the seller is not sure whether the other buyer would make a lower or higher offer, and given no recall, the seller needs a strategy to decide whether to accept or reject the first-come offer. The herein derived optimal seller's strategy, which maximizes the expected selling price, is illustrated under several scenarios, such as independent and dependent offers by the two buyers, and for several parametric price distributions.

*Keywords and phrases:* Real estate; Optimal strategy; Decision theory; Uncertainty; Two-envelope problem.

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# 1 Introduction

## 1.1 The motivating problem

The following problem has been posited by a real estate brokerage to the authors of this paper.

The seller of a house – whether on his own or with the help of a real estate agent, or perhaps both (cf., e.g., Salant 1991) – sets a list price of the house and requests the sales agent to proceed with the sale. We refer to, e.g., Zorn and Larsen (1986), Miceli (1989), Larsen and Park (1989), and Jares et al. (2000), and references therein, on how to provide incentives for real estate sales agents to act in the best interests of home sellers.

A number of buyers, some serious and others just curious, view the house, and perhaps even make exploratory offers, which the seller can use to revise his reservation price as explained by, e.g., Read (1988). After some time, the sales agent tells the seller that the matter has reached the stage when there are left only two serious buyers. Naturally, some bargaining would take place.

Denote the two buyers by the first letters of their (fictional) names,  $L$  and  $H$ , and we do not know, though perhaps attempt to guess, which of the two would be the first to make an offer. Let  $X_L$  and  $X_H$  be the (random) sale prices if the house is to be sold to  $L$  or  $H$ , respectively. After realizing one of the prices, either  $X_L$  or  $X_H$  but we do not know which of the two, the seller needs to decide whether to accept the first-come offer or reject it and then bargain with the remaining buyer.

The seller is aware of the fact that if he rejects the first-come offer, then the first-come buyer would exit the process due to reasons such as buying another house, or simply because of getting his ego hurt, as is quite often the case in such situations. Hence, there is no recall, and thus if rejected, the buyer exits the process and leaves the seller with only one buyer, whose offered price, perhaps after some bargaining, would be accepted as the final selling price.

The seller wants to have a selling strategy, which needs to be determined prior to acting on the first-come offer. The need for such a strategy arises because the seller feels, naturally, that one of the two offers is likely to be higher than the other one, but

he does not know which of them – the higher or the lower – will come first. Hence, accepting or rejecting the first-come offer is a crucial step for the seller, and the aim of the present paper is to offer an optimal strategy for the seller who wishes to maximize the expected selling price. We call the strategy *maximizing*.

Our research of the literature, especially of that concerned with strategies in the real estate business (Subsection 1.2 below), has revealed that the above formulated problem does not really fit into the models considered so far. Certainly, we have greatly benefited from the literature, but the closest solution to our problem has turned out to be related to problems, or puzzles, on the theme “which of the two numbers is larger when only one of them is shown to you?” Here we mention only two such problems: the two-envelope problem and the secretary problem, with more details and references to be provided later in this paper.

## 1.2 A glimpse of the real-estate literature

Optimal strategies for selling assets in general, and thus real estate in particular, have been actively studied in the literature (e.g., DeGroot 1970; Albright 1977; Riley and Zeckhauser 1983; Rosenfield et al. 1983). Some works assume that the seller receives a sequence of random bids arriving in a stochastic manner. Some assume that rejected offers are not lost (recall), and others that they are lost (no recall). Some assume that the distribution of offers is known, and others that it is not. It is quite often assumed that the bids are independent and identically distributed random variables. A number of authors derive stopping-type rules that lead to best strategies for selling assets. Rosenfield et al. (1983) provide a list of selling strategies within various frameworks.

Building upon, and extending, several earlier works (e.g., Stigler 1961; Nelson 1970) on the economics of information, Gastwirth (1976) has investigated the problem of consumer search for information about price and quality of goods. He explores a sequential procedure as a search strategy, which essentially suggests searching until a price below a threshold has been found. Gastwirth (1976) explores the effects of various distributions of prices (with bounded and unbounded supports) on the search length, as well as the effects of possible dependencies between the prices. Recently, Deng et al. (2013) have analyzed the reservation and asking prices, putting an emphasis –

articulated already by Stigler (1961) – on the dispersion of prices and also investigating, among other things, the influence of the dispersion on pricing strategies that maximize the return from search.

Naturally, the distribution of prices plays a pivotal role. It has been shown, for example, that the upper bound of the distribution support may coincide with the listing price, but it may also exceed it. The list price can be determined by the seller himself or with the help of a broker (cf., e.g., Salant 1991). The price can also be predicted – with some success – by inflating previous house selling prices (e.g., Brint 2009).

It should be noted that setting the ‘right’ list price is a complex task and plays a pivotal role in determining factors such as the time on the market (known as TOM) and the price of the property. These effects of the list price have been explored theoretically as well as using empirical evidence by Yavas and Yang (1995), Arnold (1999), Anglin et al. (2003), among others. In the case of a sequential search with recall, Cheng et al. (2008) have derived a closed form formula of the TOM and the price, and they have shown in particular that the two quantities follow a nonlinear positive relationship.

Sirmans et al. (1995) have examined the prices of quickly selling houses. Their model assumes that bids are independent and identically distributed random variables. Following Lippman and McCall (1976), who suggest and explore a model of job search based on wage amounts, Sirmans et al. (1995) use a stopping rule as the seller’s strategy: accept the first bid if it is larger than the reservation price and reject it otherwise. The authors analyze various quantities such as the optimal reservation price and how it is affected by holding costs and seller’s information about the distribution of offers.

Glower et al. (1998) have studied how seller’s motivation influences the selling time, list and final prices. In particular, they have investigated five factors that affect the seller’s motivation: 1) the seller has a moving plan at the date of the price listing, 2) the seller has accepted a new employment prior to the time of the listing, 3) the seller has made an offer or bought another house at the time of the listing, 4) the sale is atypical, and 5) the seller has set an incorrect price. The model of Glower et al. (1998) is without recall, and the number of received offers is not limited.

Related to the ‘motivational paradigm,’ Anglin (2004) has studied optimal strategies for households that must sell one house in order to buy another. Arnold (1999) has derived optimal asking and reservation prices. The model of Arnold (1999) assumes that the seller faces buyers arriving according to the Poisson process with some intensity, and that the asking price, which serves as a starting price in the bargaining process, affects the intensity of the arrival of potential buyers. Biswas and McHardy (2007) have studied fixed-price and asking-price strategies for selling assets in uncertain markets, as well as the determination of associated price discounts.

Naturally, earlier derived selling strategies as well as those to be explored later in this paper hinge on price distributions and other factors. A number of price distributions have been proposed in the literature. For example, Horowitz (1992) puts forward a theory of seller’s behaviour, suggests a distribution of (random) bids, derives optimal list and reservation prices, and explains why there are list prices in the housing market and why bids can sometimes be above the corresponding list prices.

Bid or price distributions can be with finite or infinite supports. In Gastwirth (1976), for example, we find uniform, triangular, and normal distributions. We can also argue in favour of the lognormal distribution, but Ohnishi et al. (2011) explain why the heavier tailed Pareto distribution might be better. In Section 3 below, we shall use some of these distributions to illustrate our proposed optimal threshold-type strategies for selling real estate.

Certainly, we have not attempted to give here a general literature overview on the topic, which is vast and spans through numerous journals on real estate, decision theory, economics, operations research, management science, and other areas. However, we hope to have provided a glimpse of those aspects that have been discussed in the literature and – in one way or another – have profoundly influenced our thinking on, and the solution of, the motivating problem formulated above.

The rest of the paper is organized as follows. In Section 2, we put forward a probabilistic model that corresponds to our motivating example, and we also formulate natural and practically sound assumptions under which we derive a formula for the expected selling price. The formula leads to a maximizing seller’s strategy in Section 3, where we explore two important cases in detail: 1) the (random) prices  $X_L$  and

$X_H$  are independent, though not necessarily identically distributed, and 2) the prices are tied by a dependency relationship. Similarly to the optimal sequential stopping strategies noted earlier, we shall see in Section 3 that in both cases 1) and 2), we also arrive at optimal threshold-type strategies: reject the first-come offer if it is below a certain threshold and accept it otherwise. Our developed theory provides a constructive definition of the threshold, which can thus be calculated or estimated in practice.

## 2 The model and the main theorem

As noted earlier, we do not know which of the two,  $L$  or  $H$ , will buy the house, as the outcome depends on factors such as who is going to offer first and at what price, and whether the seller accepts or rejects the first-come offer.

Under this uncertainty, we are interested in maximizing the expected selling price  $\mu_X = \mathbf{E}[X]$  which naturally depends on a certain seller's strategy. We want to know this strategy.

We shall next introduce some fairly natural assumptions that will facilitate the tractability of the aforementioned maximization problem.

### 2.1 Main assumptions

Let  $O_1$  be the random variable that takes on the two 'categorical' values  $L$  and  $H$ : if  $O_1 = L$ , then the first-come offer is by the buyer  $L$ , but if  $O_1 = H$ , then the first-come offer is by  $H$ .

Next, let  $R_1$  be the random variable of rejecting the first-come offer, that is,  $R_1$  takes on the 'categorical' value  $Y$  ('yes') if the first-come offer is rejected, and on the value  $N$  ('no') otherwise.

**Assumption 2.1** Whether the first-come offer  $O_1$  is made by  $L$  or  $H$  does not depend on the (random) prices  $X_L$  and  $X_H$ .

From the mathematical point of view, Assumption 2.1 means that the conditional probability  $\mathbf{P}[O_1 = L|X_L, X_H]$  is equal to the unconditional probability

$$p := \mathbf{P}[O_1 = L]$$

of the first-come offer by  $L$ . (We use ‘:=’ when we want to emphasize that the equality is by definition.) Consequently, the probability  $\mathbf{P}[O_1 = H|X_L, X_H]$  of the first-come offer by  $H$  given the prices  $X_L$  and  $X_H$  is equal to the unconditional probability  $\mathbf{P}[O_1 = H]$ ; the latter is equal to  $1 - p$ .

**Assumption 2.2** The probability of rejecting the first-come offer depends *only* on the amount that the first-come buyer offers.

Hence, for example, the probability  $\mathbf{P}[R_1 = Y|X_L = x, X_H = y, O_1 = H]$ , which can be rewritten as  $\mathbf{P}[R_1 = Y|X_L = x, X_{O_1} = y, O_1 = H]$ , is equal to

$$\mathcal{S}(y) := \mathbf{P}[R_1 = Y|X_{O_1} = y].$$

We call  $\mathcal{S}$  the seller’s strategy function, or simply the seller’s strategy. Hence,  $\mathcal{S}(y)$  is the probability of rejecting the first-come offer of size  $y$  irrespectively of whether  $L$  or  $H$  makes the offer. Analogous arguments under Assumption 2.2 imply that  $\mathbf{P}[R_1 = N|X_L = x, X_H = y, O_1 = L]$  is equal to  $1 - \mathcal{S}(x)$ .

Finally, we introduce the benchmark expected price, or BEP for short, which is the expected selling price if we were always to accept the first-come offer:

$$\text{BEP} = p \mathbf{E}[X_L] + (1 - p) \mathbf{E}[X_H].$$

Obviously, BEP is a ‘strategy-less’ quantity.

In the next subsection we shall look at the difference between  $\mu_X$  and BEP, where the seller’s strategy  $\mathcal{S}$  will play a crucial role.

## 2.2 The main theorem

Depending on the seller’s strategy  $\mathcal{S}$ , the expected selling price  $\mu_X$  might be higher or lower than the strategy-less BEP. The following theorem specifies the strategy risk parameter (SRP), which is the difference between  $\mu_X$  and BEP.

**Theorem 2.1** *The expected selling price  $\mu_X$  is the sum of the strategy-less BEP and the strategy-dependent parameter*

$$\text{SRP}(\mathcal{S}) := p \mathbf{E} \left[ \mathcal{S}(X_L) \{ \mathbf{E}[X_H|X_L] - X_L \} \right] + (1 - p) \mathbf{E} \left[ \mathcal{S}(X_H) \{ \mathbf{E}[X_L|X_H] - X_H \} \right],$$

*which we call the strategy risk parameter (SRP).*

With the help of Theorem 2.1, we can now aim at deriving a strategy  $\mathcal{S}$  that maximizes  $\text{SRP}(\mathcal{S})$  and thus, in turn,  $\mu_X$ . We shall illustrate this task in detail in Section 3 below. At the moment we note, for example, that when the prices  $X_L$  and  $X_H$  are independent, then Theorem 2.1 gives the equation

$$\begin{aligned} \mu_X = \text{BEP} + p \mathbf{E}[\mathcal{S}(X_L)] \mathbf{E}[X_H] + (1-p) \mathbf{E}[\mathcal{S}(X_H)] \mathbf{E}[X_L] \\ - p \mathbf{E}[\mathcal{S}(X_L)X_L] - (1-p) \mathbf{E}[\mathcal{S}(X_H)X_H]. \end{aligned} \quad (2.1)$$

When  $X_L$  and  $X_H$  are dependent, and assuming for the sake of concreteness that  $X_H = 2X_L$  as is the case in the two-envelope problem (details in the last paragraph of this section), we have from Theorem 2.1 that

$$\mu_X = \text{BEP}_2 + p \mathbf{E}[X_L \mathcal{S}(X_L)] - (1-p) \mathbf{E}[\mathcal{S}(2X_L)], \quad (2.2)$$

where  $\text{BEP}_2 = (2-p)\mathbf{E}[X_L]$ , which is called the benchmark base return by McDonnell and Abbott (2009). More generally, when  $X_H = \alpha X_L$  for a constant  $\alpha > 1$ , then

$$\mu_X = \text{BEP}_\alpha + (\alpha-1)p \mathbf{E}[X_L \mathcal{S}(X_L)] - (\alpha-1)(1-p) \mathbf{E}[\mathcal{S}(\alpha X_L)] \quad (2.3)$$

with the benchmark expected price  $\text{BEP}_\alpha = (\alpha - (\alpha-1)p)\mathbf{E}[X_L]$ . Obviously, equation (2.3) implies (2.2) by setting  $\alpha = 2$ .

We conclude this section by recalling that in the two-envelope problem there are two individuals: a host and a player. The host randomly chooses an amount  $X_L$  of money and places it into one envelope and also places twice the amount (i.e.,  $X_H = 2X_L$ ) into another envelope. The two envelopes are indistinguishable. The player needs to decide whether to keep the received envelope or exchange it into another one. Once a decision has been made, the game is over and the host and the player keep the money that they find in their respective envelopes.

For an optimal strategy in this game, which has greatly influenced our present research, we refer to McDonnell and Abbott (2009), and McDonnell et al. (2011). It should be noted that there are many ways in which the two-envelope problem can be formulated, and the literature on the topic is vast. The assumptions of McDonnell and Abbott (2009) may not conform with all available versions of the two-envelope problem, but the framework of the noted paper has played a pivotal role in our current research.

### 3 The maximizing strategy

Under various scenarios, in this section we demonstrate how the maximizing strategy  $\mathcal{S}_{\text{MAX}}$  can be derived and how it looks like. The following corollary to Theorem 2.1 provides an explicit form of the strategy.

**Corollary 3.1** *Assume that the random prices  $X_L$  and  $X_H$  have densities  $f_L$  and  $f_H$ , respectively. Then the maximizing strategy function  $\mathcal{S}_{\text{MAX}}(y)$  is the indicator  $\mathbf{1}_A(y)$  of the set  $A = \{x \in [0, \infty) : \mathcal{H}_{\text{MAX}}(x) > 0\}$ , where*

$$\mathcal{H}_{\text{MAX}}(x) = p \left\{ \mathbf{E}[X_H | X_L = x] - x \right\} f_L(x) + (1 - p) \left\{ \mathbf{E}[X_L | X_H = x] - x \right\} f_H(x).$$

**Proof.** By the definition of  $\text{SRP}(\mathcal{S})$  given in Theorem 2.1 and using the assumption that the random prices  $X_L$  and  $X_H$  have densities, we easily arrive at the equation

$$\text{SRP}(\mathcal{S}) = \int \mathcal{S}(x) \mathcal{H}_{\text{MAX}}(x) dx. \quad (3.1)$$

Since  $\mathcal{S}(x)$  is always in the interval  $[0, 1]$ , the maximizing strategy  $\mathcal{S}_{\text{MAX}}(x)$  must be equal to 1 when  $\mathcal{H}_{\text{MAX}}(x) > 0$  and 0 when  $\mathcal{H}_{\text{MAX}}(x) \leq 0$ . In other words,  $\mathcal{S}_{\text{MAX}}(x)$  must be the indicator function  $\mathbf{1}_A(x)$  of the set  $A$  defined in the corollary. This finishes the proof of Corollary 3.1. ■

As we shall see in the following two subsections, the maximizing strategy  $\mathcal{S}_{\text{MAX}}$  is quite often a threshold type strategy and takes on the form  $\mathcal{S}_b(z) = \mathbf{1}_{[0,b)}(z)$  where  $b$  is a ‘threshold’ that maximizes  $\text{SRP}(\mathcal{S}_b)$ . This strategy means rejecting the first-come offer of size  $z$  if  $z < b$  and accepting it if  $z \geq b$ .

#### 3.1 When the prices $X_L$ and $X_H$ are independent

Throughout this subsection we assume that the prices  $X_L$  and  $X_H$  are independent random variables. In this case, the maximizing strategy  $\mathcal{S}_{\text{MAX}}$  is specified by Corollary 3.1 with the function

$$\mathcal{H}_{\text{MAX}}(x) = p \{ \mathbf{E}[X_H] - x \} f_L(x) + (1 - p) \{ \mathbf{E}[X_L] - x \} f_H(x). \quad (3.2)$$

We need to find those  $x \geq 0$  for which  $\mathcal{H}_{\text{MAX}}(x) > 0$ . The likelihood ratio stochastic dominance (e.g., Denuit et al. 2005; Furman and Zitikis 2008) plays a natural role here.

Namely, we say that  $X_H$  is larger or, rather, not smaller than  $X_L$  in the likelihood ratio sense, denoted by  $X_H \geq_{LR} X_L$ , if and only if

$$w(x) = \frac{f_H(x)}{f_L(x)} \text{ is a non-decreasing function of } x. \quad (3.3)$$

To make the following considerations more transparent, we assume that  $X_L$  and  $X_H$  have same supports, say the interval  $(x_1, x_2)$  for some  $0 \leq x_1 < x_2 \leq +\infty$ . This means that the two densities  $f_L(x)$  and  $f_H(x)$  are (strictly) positive for all  $x \in (x_1, x_2)$  and equal to 0 outside the interval  $(x_1, x_2)$ .

Note that assumption (3.3) implies that  $\mu_H \geq \mu_L$ , where  $\mu_H = \mathbf{E}[X_H]$  and  $\mu_L = \mathbf{E}[X_L]$ . To prove this inequality, we write  $\mu_H$  as follows:

$$\mu_H = \int x \frac{f_H(x)}{f_L(x)} f_L(x) dx = \mathbf{E}[X_L w(X_L)].$$

Consequently,  $\mu_H \geq \mu_L$  if and only if  $\mathbf{Cov}[X_L, w(X_L)] \geq 0$  because  $\mathbf{E}[w(X_L)] = \int f_H(x) dx = 1$ . By Lehmann (1966), the covariance  $\mathbf{Cov}[X_L, w(X_L)]$  is non-negative because the function  $w$  is non-decreasing. For related results with possibly non-monotonic weight functions  $w$ , we refer to Egozcue et al. (2011), and references therein.

**Theorem 3.1** *Under the above assumptions on the densities  $f_L$  and  $f_H$ , and in particular assuming (3.3), we have that*

$$\mathcal{S}_{\text{MAX}}(y) = \begin{cases} \mathbf{1}_{(x_1, b)}(y) & \text{when } \mu_L < \mu_H, \\ \mathbf{1}_{(x_1, \mu)}(y) & \text{when } \mu_L = \mu_H (= \mu), \end{cases} \quad (3.4)$$

where  $b = \sup \{x > \mu_L : v(x) > w(x)\}$  with

$$v(x) = \frac{p(\mu_H - x)}{(1 - p)(x - \mu_L)}.$$

**Proof.** The assumptions on the densities  $f_L$  and  $f_H$  imply that the means  $\mu_L$  and  $\mu_H$  are in the interval  $(x_1, x_2)$ . Moreover, we already know that  $\mu_L \leq \mu_H$ . Keeping in mind that  $f_L(x)$  and  $f_H(x)$  are (strictly) positive for all  $x \in (x_1, x_2)$  and equal to 0 outside the interval  $(x_1, x_2)$ , we need to specify those  $x \in (x_1, x_2)$  for which  $\mathcal{H}_{\text{MAX}}(x) > 0$ .

To this end, we consider the cases  $\mu_L = \mu_H$  and  $\mu_L < \mu_H$  separately; and there can only be these two cases. When  $\mu_L = \mu_H$ , in which case we denote the two expectations by  $\mu$ , we have  $\mathcal{H}_{\text{MAX}}(x) > 0$  if and only if  $x < \mu$ . Hence, the maximizing strategy is  $\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(x_1, \mu)}(y)$ .

When  $\mu_L < \mu_H$ , then  $\mathcal{H}_{\text{MAX}}(x) > 0$  for all  $x \in (x_1, \mu_L]$  and  $\mathcal{H}_{\text{MAX}}(x) < 0$  for all  $x \in [\mu_H, x_2)$ . Hence, it remains to specify those  $x \in (\mu_L, \mu_H)$  for which  $\mathcal{H}_{\text{MAX}}(x) > 0$ . The latter bound is equivalent to  $v(x) > w(x)$ . The function  $v(x)$  is decreasing on the interval  $(\mu_L, \mu_H)$ : it starts with an infinite value at  $x = \mu_L$  and ends with the value 0 at  $x = \mu_H$ . Since the function  $w(x)$  is non-decreasing by assumption (3.3), we therefore must have a point  $b \in (\mu_L, \mu_H)$  such that the bound  $v(x) > w(x)$  holds for all  $x \in (\mu_L, b)$ , and the opposite bound  $v(x) \leq w(x)$  holds for all  $x \in [b, \mu_H)$ . Consequently,  $\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(x_1, b)}(y)$  with the threshold  $b$  defined in the formulation of Theorem 3.1. This concludes the proof of Theorem 3.1. ■

### 3.2 When $X_H = \alpha X_L$ for a constant $\alpha > 1$

Here we explore the case when the prices  $X_L$  and  $X_H$  are tied via the equation  $X_H = \alpha X_L$  for some constant  $\alpha > 1$ . Hence, in particular,  $f_H(x) = (1/\alpha)f_L(x/\alpha)$  and so, by equation (3.1) and some little algebra, we have that

$$\text{SRP}(\mathcal{S}) = (\alpha - 1) \int \mathcal{S}(x)x \left[ pf_L(x) - (1-p)\frac{1}{\alpha^2}f_L\left(\frac{x}{\alpha}\right) \right] dx. \quad (3.5)$$

Consequently, the maximizing strategy  $\mathcal{S}_{\text{MAX}}$  is given by Corollary 3.1 with the function

$$\mathcal{H}_{\text{MAX}}(x) = pf_L(x) - (1-p)\frac{1}{\alpha^2}f_L\left(\frac{x}{\alpha}\right), \quad (3.6)$$

which in the case  $\alpha = 2$  appears on p. 3316 of McDonnell and Abbott (2009) and also plays a pivotal role throughout the paper of McDonnell et al. (2011).

We shall next illustrate the maximizing strategy  $\mathcal{S}_{\text{MAX}}$  in the case of three parametric distributions that have been noted by several authors dealing with real estate prices (e.g., Gastwirth 1976; Ohnishi et al. 2011).

#### 3.2.1 Uniform distribution of prices

We start with the uniform on  $[A, B]$  distribution, whose density is

$$f_L(x) = \frac{\mathbf{1}_{[A, B]}(x)}{B - A}$$

for some parameters  $0 \leq A < B < +\infty$ .

**Theorem 3.2** *When  $X_L$  is uniform on  $[A, B]$ , then*

$$\mathcal{S}_{\text{MAX}}(y) = \begin{cases} \mathbf{1}_{[A, \alpha A]}(y) & \text{when } \alpha A \leq B \text{ and } p \leq 1/(1 + \alpha^2), \\ \mathbf{1}_{[A, B]}(y) & \text{otherwise.} \end{cases} \quad (3.7)$$

**Proof.** We need to specify those  $x \in [A, B]$  for which

$$p\mathbf{1}_{[A, B]}(x) - \frac{1-p}{\alpha^2}\mathbf{1}_{[\alpha A, \alpha B]}(x) > 0.$$

This is equivalent to checking the bound

$$\left(\frac{1}{p} - 1\right) \frac{1}{\alpha^2} \mathbf{1}_{[\alpha A, \alpha B]}(x) < \mathbf{1}_{[A, B]}(x).$$

By considering the cases  $\alpha A > B$  and  $\alpha A \leq B$  separately, with the latter case split into two subcases  $p \leq 1/(1 + \alpha^2)$  and  $p > 1/(1 + \alpha^2)$ , we arrive at the strategy  $\mathcal{S}_{\text{MAX}}$  given by equation (3.7). This completes the proof of Theorem 3.2. ■

### 3.2.2 Lognormal distribution of prices

Here we consider the price  $X_L$  that follows the log-normal distribution, whose density is

$$f_L(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right\} \mathbf{1}_{(0, \infty)}(x)$$

for some parameters  $\mu \in (-\infty, \infty)$  and  $\sigma > 0$ .

**Theorem 3.3** *When  $X_L$  is log-normal, then*

$$\mathcal{S}_{\text{MAX}}(y) = \mathbf{1}_{(0, b)}(y)$$

with the threshold

$$b = \sqrt{\alpha} \exp\{\mu\} \left(\frac{p\alpha}{1-p}\right)^{\sigma^2/\log \alpha}. \quad (3.8)$$

**Proof.** We check that  $\mathcal{H}_{\text{MAX}}(x)$  defined by equation (3.6) is positive if and only if

$$p \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right\} > \frac{1-p}{\alpha x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log(x/\alpha) - \mu)^2}{2\sigma^2}\right\}$$

Canceling out some terms and taking the logarithms of both sides, the above inequality becomes

$$-\frac{(\log(x) - \mu)^2}{2\sigma^2} > \log\left(\frac{1-p}{p\alpha}\right) - \frac{(\log(x/\alpha) - \mu)^2}{2\sigma^2},$$

which is equivalent to the following one:

$$\begin{aligned} x &< \exp \left\{ \mu - \frac{\sigma^2}{\log \alpha} \log \left( \frac{1-p}{p\alpha} \right) + \frac{\log \alpha}{2} \right\} \\ &= \sqrt{\alpha} \exp\{\mu\} \left( \frac{p\alpha}{1-p} \right)^{\sigma^2/\log \alpha}. \end{aligned}$$

The right-hand side is the threshold  $b$ , thus finishing the proof of Theorem 3.3. ■

### 3.2.3 Pareto distribution of prices

Our final example deals with the price  $X_L$  that follows the Pareto distribution, whose density is

$$f_L(x) = \frac{\theta}{x_0} \left( \frac{x_0}{x} \right)^{\theta+1} \mathbf{1}_{[x_0, +\infty)}(x)$$

for some parameters  $x_0 > 0$  and  $\theta > 1$ . Note that the restriction  $\theta > 1$  is necessary for the finiteness of the first moment of  $X_L$ , which we need. The preference of the Pareto distribution over the lognormal distribution when modeling house prices has been noted by Ohnishi et al. (2011).

**Theorem 3.4** *When  $X_L$  is Pareto, then*

$$\mathcal{S}_{\text{MAX}}(y) = \begin{cases} \mathbf{1}_{[x_0, \alpha x_0)}(y) & \text{when } p \leq 1/(1 + \alpha^{1-\theta}), \\ \mathbf{1}_{[x_0, +\infty)}(y) & \text{otherwise.} \end{cases}$$

**Proof.** We need to know when  $\mathcal{H}_{\text{MAX}}(x)$  defined by equation (3.6) is positive. This is equivalent to checking the inequality

$$\mathbf{1}_{[x_0, +\infty)}(x) > \frac{1-p}{p} \alpha^{\theta-1} \mathbf{1}_{[\alpha x_0, +\infty)}(x).$$

When  $x \in [x_0, \alpha x_0)$ , then the above inequality always holds, but when  $x \in [\alpha x_0, \infty)$ , then it holds if and only if  $p > 1/(1 + \alpha^{1-\theta})$ . This completes the proof of Theorem 3.4.

■

## 4 Concluding notes

The real estate business is a fascinating laboratory for testing theories and techniques of decision theory, economics, probability, psychology, sociology, and other research

areas. It also touches upon several problems, or puzzles, that have fascinated amateur and professional scientists. In this paper, whose main contribution is an optimal real-estate seller's strategy in the motivating problem, we have noted a connection between the motivating problem and the well-known two-envelope problem, in the form of McDonnell and Abbott (2009), and McDonnell et al. (2011).

Another closely related problem to developing strategies in the real estate business is the secretary problem, as noted and utilized by Mazalov and Saario (2002), who derived an optimal threshold-type strategy for setting selling prices under the assumption of the sequential arrival of buyers. Mazalov and Saario (2002) assume (for the sake of mathematical simplicity) that the prices are uniformly distributed but their ideas can be extended to other distributions as well.

One can find many fascinating connections between the real estate business and other problems or puzzles of decision theory and related areas, but this has not been the main goal of the present paper.

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