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February 2010

Online at <https://mpra.ub.uni-muenchen.de/40697/>

MPRA Paper No. 40697, posted 16 August 2012 12:27 UTC

# A Binomial Tree to Price European Options

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**Keywords:** Arbitrage, Martingale, Option, Risk-neutral, Volatility

## 1. Introduction

This short paper presents a time-changing volatility binomial tree suitable for the pricing of European options.

## 2. Binomial Tree

Time-points  $t_i$ ,  $i = 0, 1, \dots, n$ , are equidistant, and time-periods  $\Delta t = t_i - t_{i-1}$ ,  $i = 1, \dots, n$ , and time horizon  $T = n\Delta t$ , which is fixed length of time of expiration of option in years.  $t_0$  is current time-point. We also have an extra historical time-point,  $t_{-1}$ , which precedes  $t_0$ , and such that  $t_0 - t_{-1} = \Delta t$ .

The underlying security price can either rise or fall from one-point to the next,  $i = 1, \dots, n$ :

$$\begin{aligned} S_{t_i} &= S_{t_{i-1}} u_{t_i} \text{ with probability } q_i \text{ or} \\ S_{t_i} &= S_{t_{i-1}} d_{t_i} \text{ with probability } 1 - q_i, \end{aligned} \quad (1)$$

where  $u_{t_i}$  stands for *up*,  $d_{t_i}$  stands for *down*, and  $u_{t_i}$ ,  $d_{t_i}$  are variable.  $q_i$  is the risk-neutral probability of underlying security price at  $t_{i-1}$ ,  $S_{t_{i-1}}$ , rising to  $S_{t_{i-1}} u_{t_i}$  at  $t_i$ .

Further down we derive a formula for  $q_i$ .

The definition of continuously compounded return of underlying security from  $t_{i-1}$  to  $t_i$ :

$$R_{t_i} = \log S_{t_i} - \log S_{t_{i-1}}, \quad i = 0, 1, \dots, n. \quad (2)$$

We call *current return*  $R_{t_0} = \log S_{t_0} - \log S_{t_{-1}}$ , where  $S_{t_{-1}}$  is a known historical price, so *current return* is known too. Rearranging (1), and taking logarithms, and using (2) we define,  $i = 1, \dots, n$ ,

$$\begin{aligned} \log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} &= \log u_{t_i} = R_{t_i}^+ \text{ with probability } q_i \text{ or} \\ \log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} &= \log d_{t_i} = R_{t_i}^- \text{ with probability } 1 - q_i, \end{aligned} \quad (3)$$

where

$$R_{t_i}^+ = \mu\Delta t + \sigma_{t_i} \sqrt{\Delta t}, \quad (4)$$

$$R_{t_i}^- = \mu\Delta t - \sigma_{t_i} \sqrt{\Delta t}, \quad (5)$$

with  $\mu\Delta t < \sigma_{t_i} \sqrt{\Delta t}$  for large  $n$ , or equivalently small  $\Delta t$ .  $\sigma_{t_i} \sqrt{\Delta t}$  is part of a volatility process,  $\{\sigma_{t_i} \sqrt{\Delta t}\}_{i=0}^n$ , which we need to model, where  $\sigma_{t_0}$  is known current annual volatility.

### 3. Martingale Condition

Under no arbitrage, the discounted price process of the underlying security,  $\{\tilde{S}_{t_i}\}_{i=0}^n$ , must be a martingale. We now derive a formula for risk-neutral probability  $q_i$  in (1), so that  $\{\tilde{S}_{t_i}\}_{i=0}^n$  is a martingale.

Let us introduce a sample of independent Bernoulli random variables, which are independent of  $\{\tilde{S}_{t_i}\}_{i=0}^n$ :

$$\begin{aligned} Z_i &= +1 \text{ with probability } q_i \text{ or} \\ Z_i &= -1 \text{ with probability } 1 - q_i, \end{aligned}$$

where  $q_i$  is the risk-neutral probability in (1). Then (4) and (5) can be written as one equation:

$$R_{t_i} = \mu\Delta t + \sigma_{t_i} \sqrt{\Delta t} Z_i, \quad i = 1, \dots, n.$$

The martingale condition is

$$\begin{aligned} E(\tilde{S}_{t_i} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}}, \dots) &= \tilde{S}_{t_{i-1}}, & i = 1, \dots, n, \\ E(e^{-ir\Delta t} S_{t_{i-1}} e^{R_{t_i}} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}}, \dots) &= e^{-(i-1)r\Delta t} S_{t_{i-1}} \\ E(e^{\mu\Delta t + \sigma_{t_i} \sqrt{\Delta t} Z_i} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}}, \dots) &= e^{r\Delta t}, \end{aligned}$$

where  $r$  is the risk-free rate of interest, which is constant during time horizon  $T$ , and

$$\begin{aligned} e^{\mu\Delta t + \sigma_{t_i} \sqrt{\Delta t} Z_i} \mid \sigma_{t_i} &= e^{\mu\Delta t + \sigma_{t_i} \sqrt{\Delta t}} \text{ with probability } q_i \text{ or} \\ e^{\mu\Delta t + \sigma_{t_i} \sqrt{\Delta t} Z_i} \mid \sigma_{t_i} &= e^{\mu\Delta t - \sigma_{t_i} \sqrt{\Delta t}} \text{ with probability } 1 - q_i, \end{aligned}$$

so that

$$q_i e^{\mu\Delta t + \sigma_i \sqrt{\Delta t}} + (1 - q_i) e^{\mu\Delta t - \sigma_i \sqrt{\Delta t}} = e^{r\Delta t}.$$

Hence,

$$q_i = \frac{e^{r\Delta t} - e^{\mu\Delta t - \sigma_i \sqrt{\Delta t}}}{e^{\mu\Delta t + \sigma_i \sqrt{\Delta t}} - e^{\mu\Delta t - \sigma_i \sqrt{\Delta t}}}.$$

In risk-neutral pricing we set  $\mu = r$ , so that

$$q_i = \frac{1 - e^{-\sigma_i \sqrt{\Delta t}}}{e^{\sigma_i \sqrt{\Delta t}} - e^{-\sigma_i \sqrt{\Delta t}}}, \quad i = 1, \dots, n. \quad (6)$$

For large  $n$ , or equivalently small  $\Delta t$ , substituting the exponentials in (6) by their series expansions ignoring terms of order  $(\Delta t)^{3/2}$  or higher, we get

$$q_i = \frac{1}{2} - \frac{1}{4} \sigma_i \sqrt{\Delta t}, \quad i = 1, \dots, n. \quad (7)$$

So, the risk-neutral probability of  $S_{t_i}$  rising is less than for  $S_{t_{i-1}}$  falling. This is true for any  $n$ , or equivalently any  $\Delta t$ .

$$E(R_{t_i} | \sigma_{t_i}) = \mu\Delta t + (2q_i - 1)\sigma_{t_i} \sqrt{\Delta t} \quad (8)$$

$$\text{var}(R_{t_i} | \sigma_{t_i}) = 4q_i(1 - q_i)\sigma_{t_i}^2 \Delta t. \quad (9)$$

Notice that if  $q_i = 1/2$  (which it is not), then

$$E(R_{t_i} | \sigma_{t_i}) = \mu\Delta t$$

$$\text{var}(R_{t_i} | \sigma_{t_i}) = \sigma_{t_i}^2 \Delta t.$$

Setting  $q_i$  as in (6) is an artificial device which forces  $\{\tilde{S}_i\}_{i=0}^n$  to be a martingale.

#### 4. Modeling Volatility

As regards the modeling of  $\sigma_{t_i} \sqrt{\Delta t}$ , Black (1976) already noticed a negative correlation between returns and volatility, i.e. when returns are high, volatility is low, and when returns are low, volatility is high. Such negative correlation can be captured by the following equation:

$$\sigma_{t_i} \sqrt{\Delta t} = \sigma_{t_{i-1}} \sqrt{\Delta t} - \alpha(R_{t_{i-1}} - \mu\Delta t), \quad i = 1, \dots, n, \quad (10)$$

where  $0 < \alpha < 1$ . It is clear that according to (10) volatility of returns,  $\sigma_{t_i} \sqrt{\Delta t}$ , can never be negative, because, recalling (4) and (5), if  $R_{t_{i-1}} = R_{t_{i-1}}^+$ , then

$$\sigma_{t_i} \sqrt{\Delta t} = \sigma_{t_{i-1}} \sqrt{\Delta t} - \alpha (\sigma_{t_{i-1}} \sqrt{\Delta t}), \quad i = 1, \dots, n,$$

$0 < \alpha < 1$ . Alternatively, if  $R_{t_{i-1}} = R_{t_{i-1}}^-$ , then

$$\sigma_{t_i} \sqrt{\Delta t} = \sigma_{t_{i-1}} \sqrt{\Delta t} + \alpha (\sigma_{t_{i-1}} \sqrt{\Delta t}), \quad i = 1, \dots, n,$$

$0 < \alpha < 1$ .

$$\begin{aligned} \mathbb{E}(\sigma_{t_i} \sqrt{\Delta t} \mid \sigma_{t_{i-1}}) &= \sigma_{t_{i-1}} \sqrt{\Delta t} - \alpha (\mathbb{E}(R_{t_{i-1}} \mid \sigma_{t_{i-1}}) - \mu \Delta t) \\ &= \sigma_{t_{i-1}} \sqrt{\Delta t} - \alpha (2q_{i-1} - 1) \sigma_{t_{i-1}} \sqrt{\Delta t} \end{aligned}, \quad (11)$$

where, as in (6),

$$q_0 = \frac{1 - e^{-\sigma_0 \sqrt{\Delta t}}}{e^{\sigma_0 \sqrt{\Delta t}} - e^{-\sigma_0 \sqrt{\Delta t}}}, \quad (12)$$

which is known.

From (10) we see that

$$\sigma_{t_i} \sqrt{\Delta t} = \sigma_{t_0} \sqrt{\Delta t} - \alpha \sum_{j=0}^{i-1} (R_{t_j} - \mu \Delta t), \quad i = 1, \dots, n.$$

Hence,

$$\mathbb{E}(\sigma_{t_i} \sqrt{\Delta t} \mid \sigma_{t_{i-1}}, \sigma_{t_{i-2}}, \dots) = \sigma_{t_0} \sqrt{\Delta t} - \alpha \sum_{j=0}^{i-1} (2q_j - 1) \sigma_{t_j} \sqrt{\Delta t}.$$

At  $t_n$ , dropping  $\sqrt{\Delta t}$ ,

$$\mathbb{E}(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \dots) = \sigma_{t_0} - \alpha \sum_{j=0}^{n-1} (2q_j - 1) \sigma_{t_j},$$

and, using (7) for  $q_j$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \dots) = \sigma_{t_0} - \alpha \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \left( -\frac{1}{2} \sigma_{t_j} \sqrt{\Delta t} \right) \sigma_{t_j} = \infty,$$

because each term in the sum is of order  $\sqrt{\Delta t}$  (order  $1/\sqrt{n}$ ), but extra terms are added to the sum at rate  $n$ . So, the expected value of  $\sigma_{t_n}$  has no finite limit.

$$\text{var}(\sigma_{t_i} \sqrt{\Delta t} | \sigma_{t_{i-1}}) = \alpha^2 \text{var}(R_{t_{i-1}} | \sigma_{t_{i-1}}) = 4\alpha^2 q_{i-1} (1 - q_{i-1}) \sigma_{t_{i-1}}^2 \Delta t. \quad (13)$$

Looking at (13), we note that the greater the  $\alpha$ , the greater the variance of volatility of returns, and the greater the variance of volatility of returns, the greater the kurtosis of the distribution of returns.

## 5. Option Pricing Algorithm

The above tree can be implemented easily to price European call or put options. In fact, thousands of paths along the tree are simulated, and for each path the payoff of our European option is calculated, and the price of our European option equals the arithmetic mean of the thousands of discounted payoffs.

In detail, first, the (fixed) length of time  $T$  between now,  $t_0$ , and when the European option expires,  $t_n$ , needs to be determined. Second,  $n$  is chosen and fixed, so that  $T$  is split into  $n$  smaller time periods  $\Delta t = T/n = t_i - t_{i-1}$ ,  $i = 1, \dots, n$ . For example,  $\Delta t$  could be one day.

Now, step-by-step, starting at  $t_0$ , given  $\alpha$ :

1. Calculate previously defined *current return*,  $R_{t_0} = \log S_{t_0} - \log S_{t_{-1}}$ .
2. With  $R_{t_0}$  obtain  $\sigma_{t_1} \sqrt{\Delta t}$  from (10), where  $\sigma_{t_0}$  is quoted or estimated annual volatility of returns.
3. Calculate  $q_1$  from (6).
4. In order to determine whether  $S_{t_0}$  rises or falls, draw a uniformly distributed random number in  $[0,1)$ . If drawn number is in  $[0, q_1)$ , then  $S_{t_0}$  rises, and so input  $\sigma_{t_1} \sqrt{\Delta t}$  in (4) to obtain  $R_{t_1}^+$ . If drawn number is in  $[q_1, 1)$ , then  $S_{t_0}$  falls, and so input  $\sigma_{t_1} \sqrt{\Delta t}$  in (5) to obtain  $R_{t_1}^-$ .
5. With either  $R_{t_1}^+ = \log u_{t_1}$  or  $R_{t_1}^- = \log d_{t_1}$  calculate either  $S_{t_1} = S_{t_0} u_{t_1}$  or  $S_{t_1} = S_{t_0} d_{t_1}$ .
6. With  $S_{t_1}$  return to 1. shifting forward one time-point, thus calculating  $R_{t_1}$ , and repeat 1. to 5. shifting forward one time-point at each repetition until  $S_{t_n}$  is calculated.
7. With  $S_{t_n}$  calculate European call option payoff,  $X^c$ , or put payoff,  $X^p$ ,

$$X^c = \max(S_{t_n} - k, 0) \quad (14)$$

$$X^p = \max(k - S_{t_n}, 0), \quad (15)$$

where  $k$  is the strike.

8. Repeat 1. to 7. thousands of times, thus simulating thousands of paths along the tree, and attaining thousands of payoffs.

9. Calculate price of European call,  $\pi(X^c)$ , or put,  $\pi(X^p)$ ,

$$\pi(X^c) = e^{-rT} \left( \frac{1}{m} \sum_{i=1}^m X_i^c \right) \quad (16)$$

$$\pi(X^p) = e^{-rT} \left( \frac{1}{m} \sum_{i=1}^m X_i^p \right), \quad (17)$$

where  $m$  is the number of paths simulated, and payoffs calculated.  $r$  is the risk-free rate of interest, which is constant during  $T$ .

10. As a measure of the accuracy of estimate (16) or (17), calculate its standard error, given by  $s/\sqrt{m}$ , where  $s$  is the sample standard deviation of the  $m$  discounted payoffs obtained.

## 6. Advantages of the Model

- Easy to implement.
- Apart from  $\mu$  and  $\sigma_{t_0}$ , volatility process has only one parameter,  $\alpha$ .
- Relatively easy to calibrate.  $\alpha$  parameter can be calibrated by trial and error, given a sensible estimate of  $\sigma_{t_0}$ .
- Numerically stable. It can model the implied volatility (implied  $\sigma_{t_0}$ ) surface, where three option expiry dates with three months between them are considered, without the need to change the value of  $\alpha$  parameter.

## References

Black F. (1976) Studies of Stock Price Volatility Changes, *Proceedings of the 1976 Meetings of the Business and Economics Statistics Section, American Statistical Association*, 177-181.