De-synchornized Clocks in Preemption Games with Risky Prospects

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Abstract

We study an optimal timing decision problem where an agent endowed with a risky investment opportunity trades the benefits of waiting for additional information against a potential loss in first-mover advantage. The players' clocks are de-synchronized in that they learn of the investment opportunity at different times. Previous literature has uncovered an inverted-U shaped relationship between a player's equilibrium expected expenditures and the measure of his competitors. This result no longer holds when the increase in the measure of players leads to a decrease in the degree of clock synchronization in the game. We show that the result reemerges if information arrives only at discrete times, and thus, a player's strategic beliefs are updated between decision times in a measurably meaningful way.

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1 Introduction

In a variety of situations that are modelled as preemption games, economic agents are heterogenous with respect to the times when they learn of an investment opportunity. For instance, some firms may become aware of a change in consumer demand earlier than others, R&D departments may make a technological breakthrough that allows them to start developing a new product at different times, and stock traders may learn of the possible existence of a financial bubble sequentially. Brunnermeier and Morgan (2010) refer to these types of situations as clock games with de-synchronized clocks. More precisely, a clock game has a set of players who, sequentially, receive a signal informing them of the opportunity to take a particular action. A player's clock starts at the instant when he receives his signal. The time interval over which all clocks start is called the awareness window, and an increase in its length corresponds to a decrease in the degree of clock synchronization in the game. Brunnermeier and Morgan (2010) study clock games with a fixed number of players and a varying degree of clock synchronization.

In many real-world examples of clock games, if the action yields a risky prospect, players have the option to wait and acquire additional payoff-relevant information before taking it.[1] In this case, the clock de-synchronization induces heterogeneity among players with respect to the amount of information that they possess at any given time. Barbos (2012) studies clock games with risky prospects and unobservable moves, and examines the strategic effect of a change in n, the measure of players in the game, when the degree of clock synchronization in the game is fixed. The resulting model is applied to investigate the relationship between competition and innovation. In many real-world situations, though, the decrease in the degree of clock synchronization is generated precisely by the increase in n, as it may take more time for a larger set of players to learn of an investment opportunity. In this paper, we consider a clock game with risky prospects, and investigate the strategic effect of an increase in n that decreases the degree of clock synchronization.

The players in our model decide on the time when to undertake a project by trading the benefits of waiting for additional information about its feasibility against a potential loss in first-mover advantage: a player's *ex-post* payoff from a feasible project is decreasing in the measure of competitors who moved before him. This results in an optimal timing decision problem in which a player compares the marginal cost of waiting for additional information (the expected loss in first-mover advantage) with its marginal benefit (the value of information), and invests as soon as the former exceeds the latter. Players are subjected to a non-negativity constraint on the *expected ex-ante* payoff that accounts for the risk and cost of investment. Barbos (2012) shows that in these games, an increase in n that does not alter the degree of clock synchronization, leads to an inverted-U relationship between a player's expected expenditures and the measure of players.

[1] For instance, a firm may acquire information about the likely profitability of the investment before undertaking it, an R&D department may perform additional tests to examine the technological feasibility of the invention before developing it into a new product, and a stock trader may examine in more detail the underlying economic activity on which the financial bubble may have been generated before altering his trading position.

The key driving force behind this result is the adverse effect of an increase in n on a player's *expected ex-post* payoff, and therefore on his marginal cost of waiting. When n is small, an increase in this measure induces then a player to undertake a riskier project by investing earlier. When n is high, and the non-negativity constraint on the expected ex-ante payoff binds, a player adjusts to a further increase in the measure of his competitors by investing with a decreasing probability while holding the risk level constant. In equilibrium, this increases the expected ex-post payoffs by reducing the measure of players who invest, and allows for non-negative ex-ante payoffs. From the viewpoint of the time a player's clock starts, an increase in n therefore leads to an increase in a player's expected expenditures when n is small, and to a decrease when n is sufficiently high.

This relationship no longer holds if the increase in n results in a decrease in the degree of clock synchronization in the game that preserves the density of clock starting times in the awareness window. In this case, a higher n does not increase a player's marginal cost of waiting by means of decreasing his expected ex-post payoff, but by increasing his belief about the event that investment is ongoing in the game at a given time. While this belief is increasing in n, it is inelastic with respect to it in a neighborhood of the equilibrium waiting time. In particular, a player knows for sure that, *at* the equilibrium waiting time, investment is ongoing in the game. Therefore, as the interval between decision times shrinks, in the limit, the marginal cost is perfectly inelastic with respect to n at the equilibrium waiting time. Since the optimal waiting time is determined by the equality between the marginal cost and the marginal benefit of waiting, the optimality condition is satisfied for the same waiting time irrespective of the value of n. Therefore, a player does *not* invest earlier in a project when the measure of his competitors increases.

In this paper we show that the result reemerges if the information about the feasibility of the project arrives only at discrete times.[2] For instance, macroeconomic data is released at various regular intervals, information about a firm's financial status comes quarterly, test results for new drugs are obtained at discrete times, etc. In these cases a player's strategic beliefs are updated in a measurably meaningful way between decision times, and thus the marginal cost of waiting is nowhere perfectly inelastic. Instead, it is increasing in n at all possible waiting times, and therefore, for low values of n, an increase in n induces a player to undertake the project earlier. For values of n for which the non-negativity constraint on the expected ex-ante payoff binds, a player reacts to a further increase in n by investing with a decreasing probability *and* by waiting longer.

Clock games were introduced in the literature by Abreu and Brunnermeier (2002, 2003) to study the persistence of mispricing in financial markets.[3] Brunnermeier and Morgan (2010) construct a finite agent analog of their model and test in an experimental setting some of its key predictions that relate the degree of clock de-synchronization in the game with the equilibrium delay. In these models, the payoff structure exhibits a mixture of preemption games and wars of attrition: the

[2] Our model is strategically isomorphic to one in which information arrives continuously but players are restricted to take actions at discrete times. Thus, our results extend also to situations captured by these types of models.

[3] See also Doblas-Madrid (2012) who consider a discrete time version of the Abreu and Brunnermeier (2002) model and assumes endogenous pricing and no behavioral types.

payoffs are increasing deterministically as a function of time up to the k^{th} player to move, and fall to a random value immediately after.[4] This payoff structure is designed so as to mimic a financial bubble where traders who ride it enjoy its benefits as long as there are still enough agents in the market, but incur a loss when a crash occurs once sufficiently many of them exited.[5]

The ex-post payoffs in our model are decreasing in the measure of players who have moved by a given time, as in pure preemption games, but players have an incentive to wait that is determined by the value of information that they can acquire. This relates out paper to the literature on timing of irreversible actions under uncertainty. Jensen (1982), Chamley and Gale (1994) or Decamps and Mariotti (2004) study models of endogenous or exogenous information acquisition in which the incentive to invest early is provided by the discounting of future payoffs rather than the competitive pressure. On the other hand, preemption games have been extensively studied in the literature, starting with the seminal papers of Reinganum (1981) and Fundenber and Tirole (1985). While the model is distinct, Weeds (2002), who also studies preemption games where the incentive to delay investment is provided by the opportunity to learn new information, is the closest paper to ours from this literature.[6] Finally, another related stream of research is the experimentation literature (see, for instance, Bolton and Harris (1999) or Cripps, Keller and Rady (2005)) that studies the trade-off between current output and information that can help increase output in the future.

2 The Model

There is a continuum set of identical and risk-neutral players who, sequentially, learn of an investment opportunity in a risky project. A mass a of players learn of the opportunity at each instant $t \in [t_0, t_0+\eta]$, with $\eta > 0$. Players do not know t_0 but have a prior distribution on it that is uniform on $\mathbb{R}$.[7] The moment when player i learns of the opportunity, i.e., when his clock starts, is denoted by t_i. Since players become aware of the project at different times, their clocks are de-synchronized. Once player i's clock starts, he may invest at any time $t_i + t$, with $t \geq 0$. There is a one-time fixed cost c of investing. A player does not observe his opponents' actions.[8]

At t_i, player i has belief p_0 that the project is feasible. Delaying action allows learning at no cost additional information about its feasibility. As in Barbos (2012), we consider that an infeasible project generates a negative signal with a Poisson rate μ, but here we assume that player i can observe such a signal only at times t_i+t, with $t \in \delta\mathbb{Z}_+ \equiv \{\delta, 2\delta, 3\delta, ...\}$, and $\delta \in \left(0, \frac{\eta}{2}\right)$. If the project

[4] Sahuguet (2006) and Park and Smith (2008) are other papers with non-nomonotonic payoff structures.

[5] Another paper that examines clock games in an experimental setting is Camerer, Kang and Ray (2010).

[6] Hoppe (2000), Lambrecht and Perraudin (2003), and Argenziano and Schmidt-Dengler (2012) are also related.

[7] The continuum set of players can be interpreted simply as the distribution of the unknown locations on the timeline of a finite number of players. The nonstandard distribution of t_0 is used to avoid boundary effects. An alternative is to discard the common prior assumption, and instead of having player i's posterior belief about t_0 at t_i be derived from a common prior about t_0, to consider this belief to be the player's prior about t_0 at that time.

[8] As Park and Smith (2008) argue "silent timing games" capture economic applications where timing decisions must be made well ahead of the time the action begins, as with high-tech market entry decisions or R&D investments.

is feasible, no negative signal is generated. Therefore, once a player receives a negative signal, he learns that the project is infeasible. On the other hand, as time passes, absent a negative signal, his belief that the project is feasible is updated favorably and the risk of investment is reduced. The signals are private information to each player.

Player i's *ex-post payoff* from investing at time $t_i + t$ in a feasible project is

$$\Pi(\overline{m}, m(t|t_i, t_0)) = A(\overline{m}) - \theta m(t|t_i, t_0), \text{ for some } \theta \in \mathbb{R}_+, \text{ and } A : \mathbb{R}_+ \to \mathbb{R}_+ \text{ with } A'(\cdot) < 0. \quad (1)$$

where $\overline{m}$ is the total measure of players that invest in the project and $m(t|t_i, t_0)$ is the measure of players that invest before player i. The specification of Π captures a congestion effect and a first-mover advantage.[9] The ex-post payoff from investing in an infeasible project is zero.[10] To isolate the effect of competitive pressure in inducing players to invest early, we assume no intertemporal discounting. The payoff of a player who does not invest is normalized at zero.

Note that the measure of players in the game is $n = a\eta$; thus, higher values of either a or η increase n. A higher a holds constant the degree of clock synchronization. A higher η lowers the degree of clock synchronization, but holds constant the density of clock starting times in the awareness window, $[t_0, t_0 + \eta]$. Our focus in this paper is on the strategic effect of an increase in η.

To simplify exposition, we make the following assumption that ensures an interior solution.

Assumption 1 *We assume $\eta \in (\eta_m, \eta_M)$, where η_m solves $\Pi(a\eta_m, a\eta_m) - c = 0$, and η_M solves $p_0 a\theta\delta\left(\delta - \frac{\delta^2}{2\eta_M}\right) = c(1 - p_0)\left(1 - e^{-\mu\delta}\right)$.*

The upper bound ensures that players do acquire some information before investing. The lower bound ensures that they do not do so indefinitely.

3 Results

In section 3.1 we introduce concepts that are used in the formal analysis of the game. In section 3.2 we present the equilibrium of the game for a fixed value of η, while in section 3.3 we present

[9] The quasilinear functional form of Π allows for a more transparent intuition of the results and a clearer exposition. With this specification, the marginal cost of waiting for one more period is essentially the expected measure of players who invest in that period, rather than the corresponding effect on the expected payoff. As in Barbos (2012), the salient results of the paper extend to more general functional forms for Π.

[10] The model can be represented as a Bayesian game in normal-form as follows. The *type* of a player i is the time t_i when his clock starts. The set of possible *sets* of types of the players is $\{[t_0, t_0 + \eta] : t_0 \in [t_i - \eta, t_i]\}$. A type profile, denoted by $T(t_0)$, is a uniform probability density function over a set of the form $[t_0, t_0 + \eta]$. The set of possible type profiles is thus $\mathcal{T} \equiv \{T(t_0) : T(t_0) \sim Uniform([t_0, t_0 + \eta]) \text{ for some } t_0 \in [t_i - \eta, t_i]\}$. A player's belief $\phi_i(T(t_0)|t_i)$ is determined by the posterior belief about t_0 as follows: $\phi_i(T(t_0)|t_i) = \frac{1}{\eta}$, if $t_0 \in [t_i - \eta, t_i]$ and 0 otherwise. In other words, player i believes that the opponents are distributed uniformly on $[t_0, t_0 + \eta]$, where t_0 is distributed uniformly on $[t_i - \eta, t_i]$. The *action space* of player i is the set of possible waiting times $\mathbb{R}_+ \cup \{\infty\}$, where $\{\infty\}$ represents the option to not invest in the project.

the main result of the paper that elicits the effect of an increase in η on the equilibrium strategies. In section 3.4 we discuss the results and provide the intuition for the contrast between the results obtained when information arrives continuously and when it arrives discretely.

3.1 Preliminaries

Note that it is never a best response for player i to invest at times $t_i + t$, with $t \notin \delta\mathbb{Z}_+$. In fact, the model is strategically isomorphic to one in which players observe the negative signals continuously, but can only take decisions in discrete time. Therefore, we restrict attention throughout to decision times $t_i + t$, with $t \in \delta\mathbb{Z}_+$. A *strategy* is then a probability distribution function $s_{(\cdot)}$ over the set of waiting times $\delta\mathbb{Z}_+ \cup \{\infty\}$. For $t \in \delta\mathbb{Z}_+$, s_t is the probability that player i invests at time $t_i + t$, conditional on no negative signal having been received up to or at that time. With a slight abuse of notation, s_∞ denotes the probability that the player does not invest even if a negative signal is never received.

We define a *simple strategy* to be one that assigns strictly positive probability to at most two finite waiting times.

Definition 2 *A simple strategy with waiting time τ is a probability distribution $s_{(\cdot)}$ over $\delta\mathbb{Z}_+ \cup \{\infty\}$ such that: (i) $s_\tau > 0$, (ii) $s_{\tau+\delta} \geq 0$, (iii) $s_t = 0$ for all $t \in \delta\mathbb{Z}_+ \backslash \{\tau, \tau + \delta\}$.*

In the rest of the paper, we use $\langle s_\tau, s_{\tau+\delta} \rangle$ to denote a generic simple strategy when $s_{\tau+\delta} \geq 0$, and use $\langle s_\tau \rangle$ to denote a simple strategy for the particular case when $s_{\tau+\delta} = 0$. We will focus on symmetric equilibria in simple strategies. Besides their relative intuitive appeal determined by the lower degree of complexity than that typically associated with strategies that involve randomization over multiple pure strategies, this class of strategies is also the smallest with the property that it contains a unique symmetric equilibrium for values of η in (η_m, η_M) except a countable subset.[11]

We will refer throughout to values computed *as of moment* t_i for player i as the *ex-ante* values. The formal analysis of the game is based on the comparison, for an arbitrary player i, of the ex-ante marginal cost and marginal benefit of waiting at $t_i + t$ for one more period δ, while keeping track of the option value of waiting. The ex-ante *marginal cost* (MC) of waiting at $t_i + t$ is the decrease in the expected payoff due to the increase in the expected measure of players that invest between $t_i + t$ and $t_i + t + \delta$. The ex-ante *marginal benefit* (MB) of waiting at $t_i + t$ is the value of the information acquired between the same times. Next, we provide precise definitions for the two concepts.

Denote by F the event that the project is feasible, by N_t the event that a negative signal is received by player i before $t_i + t$, and by F^c and N_t^c their complements. The Poisson generating

[11] Symmetric equilibria in other strategies cannot be excluded. As Barbos (2012) argues, the intuition behind the salient results of the paper would be preserved in these alternative equilibria.

process implies that, conditional on F^c, the delay of arrival of a negative signal has an exponential distribution with parameter μ. Therefore, for $t \in \delta\mathbb{Z}_+$, we have $\Pr(N_t|F^c) = 1 - e^{-\mu t}$.

The MB of waiting at $t_i + t$, computed as of moment t_i, is the expected value of the forgone costs on an infeasible project generated by the additional information, i.e.,

$$MB(t) \equiv c \cdot \Pr(N_{t+\delta} \cap N_t^c|F^c) \cdot \Pr(F^c) = c(1 - p_0)\left[1 - e^{-\mu\delta}\right] e^{-\mu t} \tag{2}$$

The equality in (2) follows because $N_t \subset N_{t+\delta}$ implies $\Pr(N_{t+\delta} \cap N_t^c|F^c) = \Pr(N_{t+\delta} \backslash N_t|F^c) = \Pr(N_{t+\delta}|F^c) - \Pr(N_t|F^c) = e^{-\mu t} - e^{-\mu(t+\delta)}$.

Next, we define formally the MC of waiting under a strategy profile in which all players adopt a simple strategy $\langle s_\tau \rangle$. Note first that for a given value of t_0, if all players adopt $\langle s_\tau \rangle$, the first player invests at $t_0 + \tau$, while the last invests at $t_0 + \tau + \eta$. Therefore, conditional on t_0, from the perspective of player i, the measure of players who have already invested by time $t_i + t$ is

$$m_{\langle s_\tau \rangle}(t|t_i, t_0) = \begin{cases} 0, & \text{when } t_i + t < t_0 + \tau \\ s_\tau a\left[(t_i + t) - (t_0 + \tau)\right], & \text{when } t_i + t \in [t_0 + \tau, t_0 + \tau + \eta] \\ s_\tau a\eta, & \text{when } t_i + t > t_0 + \tau + \eta \end{cases} \tag{3}$$

The uniform prior distribution on t_0 implies that, at t_i, player i's posterior of t_0 is uniform on $[t_i - \eta, t_i]$. Thus, the expected measure of players who have invested by time $t_i + t$ is

$$\lambda_{\langle s_\tau \rangle}(t|t_i) \equiv E_{t_0}\left[m_{\langle s_\tau \rangle}(t|t_i, t_0)\right] = \frac{1}{\eta}\int_{t_i - \eta}^{t_i} m_{\langle s_\tau \rangle}(t|t_i, t_0)dt_0 \tag{4}$$

Then, conditional on F, player i's *expected ex-post payoff* from investing at $t_i + t$ is

$$E_{t_0}\left[\Pi\left(s_\tau a\eta, m_{\langle s_\tau \rangle}(t|t_i, t_0)\right)\right] = A\left(s_\tau a\eta\right) - \theta\lambda_{\langle s_\tau \rangle}(t|t_i) \tag{5}$$

Firm i's ex-ante MC of waiting at $t_i + t$ is then the *unconditional* expected difference between the expected payoff from investing at $t_i + t$, and the expected payoff from investing at $t_i + t + \delta$

$$MC_{\langle s_\tau \rangle}(t) \equiv a\theta p_0\left[\lambda_{\langle s_\tau \rangle}(t + \delta|t_i) - \lambda_{\langle s_\tau \rangle}(t|t_i)\right] \tag{6}$$

Denote by

$$\Phi_{\langle s_\tau \rangle}(t) \equiv \begin{cases} \frac{1}{\eta}\left[\eta - \tau + \left(t + \frac{\delta}{2}\right)\right], & \text{for } t \in (\max(0, \tau - \eta), \tau) \\ \frac{1}{\eta}\left[\eta - \left(t + \frac{\delta}{2}\right) + \tau\right], & \text{for } t \in [\tau, \tau + \eta) \end{cases} \tag{7}$$

The following lemma elicits the MC of waiting for one more period δ. Its proof is in appendix A.[12]

Lemma 3 *Under a symmetric strategy profile* $\langle s_\tau \rangle$, $MC_{\langle s_\tau \rangle}(t) = p_0 a s_\tau \theta\delta\Phi_{\langle s_\tau \rangle}(t)$ *for* $t \in \delta\mathbb{Z}_+ \cap$

[12] $MC_{\langle s_\tau \rangle}(t)$ is zero for $t \in \delta\mathbb{Z}_+ \backslash (\max(0, \tau - \eta), \tau + \eta)$ since under the symmetric strategy profile $\langle s_\tau \rangle$, no player is supposed to invest before $t_i + \max(0, \tau - \eta)$ or after $t_i + \tau + \eta$.

$(\max(0, \tau - \eta), \tau + \eta)$.

To understand the lemma, consider first some $t \in [\max(0, \tau - \eta), \tau] \cap \delta\mathbb{Z}_+$. As explained above, under a strategy profile $\langle s_\tau \rangle$, from player i's viewpoint, investment has started in the game by time $t_i + t$ if and only if $t_i + t \geq t_0 + \tau$, that is, $t_0 \leq t_i - (\tau - t)$. Also, with probability one, investment has not yet ended by $t_i + t$ since $t < \tau$. Given that $t_0 \geq t_i - \eta$, it follows that player i's belief that *investment is ongoing in the game* at $t_i + t$ is $\Pr(t_0 \in [t_i - \eta, t_i - (\tau - t))) = \frac{1}{\eta}[\eta - \tau + t]$. Similarly, for $t \in [\tau, \tau + \eta]$, investing has started for sure in the game, but it has not yet ended if and only if $t_i + t \leq t_0 + \tau + \eta$, i.e., for $t_0 \in [t_i - \eta + (t - \tau), t_i)$. Therefore, player i's belief that investment is ongoing at $t_i + t$ is $\Pr(t_0 \in [t_i - \eta + (t - \tau), t_i)) = \frac{1}{\eta}[\eta - t + \tau]$. $\Phi_{\langle s_\tau \rangle}(t)$, as defined by (7), is then just the "average" belief that investment is ongoing in the game at the times in $[t, t + \delta]$.

On the other hand, from (3) and (4), by some straightforward calculations, we have that

$$\frac{1}{as_\tau}\frac{\partial}{\partial t}\lambda_{\langle s_\tau \rangle}(t|t_i) = \begin{cases} \frac{1}{\eta}\int_{t_i-\eta}^{t_i-(\tau-t)} dt_0 = \frac{1}{\eta}[\eta - \tau + t], & \text{for } t \in [\max(0, \tau - \eta), \tau] \\ \frac{1}{\eta}\int_{t_i-\eta+(t-\tau)}^{t_i} dt_0 = \frac{1}{\eta}[\eta - t + \tau], & \text{for } t \in [\tau, \tau + \eta] \end{cases} \tag{8}$$

Therefore, $\frac{1}{as_\tau}\frac{\partial}{\partial t}\lambda_{\langle s_\tau \rangle}(t|t_i)$ equals precisely the measure of the set of values of t_0 for which investment *is ongoing* at $t_i + t$. Thus, waiting for an additional infinitesimal amount of time Δt, increases $\lambda_{\langle s_\tau \rangle}(t|t_i)$ by $as_\tau \cdot \Delta t$ times the probability that investment is ongoing in the game at $t_i + t$. Lemma 3 states that the ex-ante MC of waiting for a period δ is $p_0\theta as_\tau \cdot \delta$ multiplied by $\Phi_{\langle s_\tau \rangle}(t)$, i.e., by the average belief over $[t, t + \delta]$ that investment is ongoing in the game.

The above argument underscores the key distinction between increases in n by means of increases in a or η. Note that since $\Phi_{\langle s_\tau \rangle}(t)$ is increasing in η, $MC_{\langle s_\tau \rangle}(t)$ is strictly increasing in both a and η, for all $t \in [\max(0, \tau - \eta), \tau + \eta]$. However, the two parameters increase the MC through distinct channels. A higher a increases the potential loss in ex-post payoff from being beaten to the punch by another player, conditional on the fact that investment is ongoing in the game at the time the player invests. On the other hand, a higher η increases the (average) belief that a player has on $[t, t + \delta]$ regarding the event that other players are investing. In other words, while the increase in a decreases the ex-ante expected payoff of a player by means of decreasing the ex-post payoffs, an increase in η decreases it by means of altering a player's strategic beliefs. This distinction lies at the core of the results in this paper.[13]

Next, we define formally and compute the ex-ante MC of waiting under a symmetric strategy profile $\langle s_\tau, s_{\tau+\delta} \rangle$. Note that in this case, the expected measure of players who have invested by

[13] This also underlies the role that the uncertainty about t_0 plays in the model. If t_0 is common knowledge, in a symmetric equilibrium, the MC of waiting does not increase when η increases. The MC would be either $p_0 as_\tau \theta\delta$ or zero depending on whether investment is ongoing or not. As we will see, this would imply, for instance, that players do not respond by investing earlier for low values of η.

time $t_i + t$, from the perspective of player i, is

$$\lambda_{\langle s_\tau, s_{\tau+\delta}\rangle}(t|t_i) = \lambda_{\langle s_\tau\rangle}(t|t_i) + \lambda_{\langle s_{\tau+\delta}\rangle}(t|t_i) \tag{9}$$

Essentially, one can think of the mass a of players whose clocks start at any time t_i as being split into a mass that adopts strategy $\langle s_\tau\rangle$, and a mass that adopts strategy $\langle s_{\tau+\delta}\rangle$; thus, (9) follows immediately. Then, the MC of waiting at $t_i + t$, with $t \in \delta\mathbb{Z}_+ \cap (\max(0, \tau - \eta), \tau + \eta)$ is

$$\begin{aligned} MC_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) &\equiv a\theta p_0 \left[\lambda_{\langle s_\tau, s_{\tau+\delta}\rangle}(t+\delta|t_i) - \lambda_{\langle s_\tau, s_{\tau+\delta}\rangle}(t|t_i)\right] \\ &= MC_{\langle s_\tau\rangle}(t) + MC_{\langle s_{\tau+\delta}\rangle}(t) \\ &= p_0 a\theta\delta\left[s_\tau \Phi_{\langle s_\tau\rangle}(t) + s_{\tau+\delta}\Phi_{\langle s_{\tau+\delta}\rangle}(t)\right] \end{aligned} \tag{10}$$

where the first equality follows from (6) and (9), and the second from lemma 3.

The last result of the section elicits the ex-ante expected investment expenditures of a player who adopts a simple strategy $\langle s_\tau, s_{\tau+\delta}\rangle$. We denote this amount by $\mathcal{I}_{\langle s_\tau, s_{\tau+\delta}\rangle}$. Note that the ex-ante unconditional probability of investment of a player who adopts $\langle s_\tau\rangle$ is the probability that a negative signal is not received by time τ, i.e., $\Pr(N_t^c) = p_0 + (1-p_0)e^{-\mu\tau}$. The proof of the lemma follows then immediately from the fact that $\mathcal{I}_{\langle s_\tau, s_{\tau+\delta}\rangle} = c\Pr(N_\tau^c)s_\tau + c\Pr\left(N_{\tau+\delta}^c\right)s_{\tau+\delta}$.

Lemma 4 $\mathcal{I}_{\langle s_\tau, s_{\tau+\delta}\rangle} = c\left[p_0 + (1-p_0)e^{-\mu\tau}\right]s_\tau + c\left[p_0 + (1-p_0)e^{-\mu(\tau+\delta)}\right]s_{\tau+\delta}$

3.2 The Equilibrium for a Fixed Value of η

The next proposition identifies the necessary and sufficient conditions for a symmetric equilibrium in which players adopt strategy $\langle s_\tau\rangle$ to exist for a *fixed* value of η. Its proof is in appendix B1. The equilibrium notion we employ throughout is the Bayesian Nash Equilibrium.

Proposition 5 *A symmetric equilibrium strategy $\langle s_\tau\rangle$ exists if and only if*

$$p_0\Pi\left(s_\tau a\eta, \frac{1}{2}s_\tau a\eta\right) - c\left[p_0 + (1-p_0)e^{-\mu\tau}\right] \geq 0, \text{ } (=0, \text{ if } s_\tau < 1) \tag{11}$$

$$MC_{\langle s_\tau\rangle}(\tau - \delta) \leq MB(\tau - \delta) \tag{12}$$

$$MB(\tau) \leq MC_{\langle s_\tau\rangle}(\tau) \tag{13}$$

$$\Pi(s_\tau a\eta, s_\tau a\eta) - c \leq 0, \text{ for } s_\tau \in (0,1) \tag{14}$$

To understand condition (11), note first that in the symmetric equilibrium under consideration, player i's expected ex-post payoff from a feasible project is $\Pi\left(as_\tau\eta, \frac{1}{2}as_\tau\eta\right)$. This is because the expected measure of players whose clocks started before that of player i is precisely $\frac{1}{2}a\eta$, and all players wait the same time. Second, $\Pr(N_\tau^c) = \left[p_0 + (1-p_0)e^{-\mu\tau}\right]$ is the equilibrium unconditional probability that the investment is made, so $c\Pr(N_t^c)$ is the expected ex-ante investment cost.

Therefore, condition (11) states that the *expected ex-ante payoff* from following the equilibrium strategy is non-negative. Conditions (12) and (13) state that it is enough for player i to have an incentive to wait at $t_i + \tau - \delta$, and an incentive to *not* wait at $t_i + \tau$, in order to not have an incentive to deviate from the equilibrium strategy of investing precisely at $t_i + \tau$. Condition (14) is necessary because otherwise players would wait until all uncertainty about the project is removed.

The formal proof of proposition 5 from the appendix explores the properties of a player i's expected ex-ante payoff from investing at $t_i + t$ for all $t \in \delta\mathbb{Z}_+$. Essentially, though, the proof amounts to showing a virtual single crossing property between the MC and MB curves. The MB curve is above the MC curve for $t < \tau$, and below for values of t immediately above τ.[14] While the two curves may intersect again for some higher value $t > \tau$, the shape of the ex-ante expected payoff as a function of the waiting time t and condition (14) imply that the player does not find it profitable to wait more than τ time units. Therefore, players postpone investing as long as the MB of waiting exceeds the MC, and invest as soon as the MC exceeds the MB.

Definition 6 *We say that a property holds for almost any $\eta \in (\eta_m, \eta_M)$ if it holds for all $\eta \in (\eta_m, \eta_M)$ except a countable subset.*

The next corollary states the uniqueness of the equilibrium strategy $\langle s_\tau \rangle$ except for a countable subset where some knife-edge conditions are satisfied. Its proof is in appendix B2.

Corollary 7 *For almost any $\eta \in (\eta_m, \eta_M)$, there is at most one strategy $\langle s_\tau \rangle$ satisfying (11)-(14).*

The next proposition, whose proof is in appendix B3, describes a symmetric equilibrium in which players adopt simple strategy $\langle s_\tau, s_{\tau+\delta} \rangle$, with $s_{\tau+\delta} > 0$.

Proposition 8 *A symmetric equilibrium strategy $\langle s_\tau, s_{\tau+\delta} \rangle$, with $s_{\tau+\delta} > 0$, exists if and only if*

$$p_0 \Pi\left(\left(s_\tau + s_{\tau+\delta}\right) a\eta, \lambda_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau|t_i)\right) - c\left[p_0 + (1-p_0)\, e^{-\mu\tau}\right] \geq 0, \ (= 0, \text{ if } s_\tau + s_{\tau+\delta} < 1) \quad (15)$$

$$MC_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) = MB(\tau) \quad (16)$$

$$\Pi\left(\left(s_\tau + s_{\tau+\delta}\right) a\eta, \left(s_\tau + s_{\tau+\delta}\right) a\eta\right) - c \leq 0, \text{ for } s_\tau + s_{\tau+\delta} < 1 \quad (17)$$

As in the case of proposition 5, it is necessary that players expect non-negative ex-ante payoffs, and that waiting until all uncertainty is removed is not profitable. Condition (16) requires that players are indifferent between waiting τ or $\tau + \delta$ time units. The shape of the expected ex-ante payoff as a function of the waiting time implies that these are sufficient for it to be maximized at waiting times τ and $\tau + \delta$. The next corollary, whose proof is in appendix B4, states some uniqueness properties of strategies $\langle s_\tau, s_{\tau+\delta} \rangle$ satisfying the conditions in proposition 8. Note that

[14] Note that the MB is decreasing in t, while the MC is increasing for $t < \tau$ and decreasing for $t > \tau$.

part *(i)* only claims the uniqueness of $\langle s_\tau, s_{\tau+\delta}\rangle$ for a fixed value of τ. Corollary 14 will state the uniqueness of the symmetric equilibrium in simple strategies for any τ and almost any η.

Corollary 9 *(i) For almost any $\eta \in (\eta_m, \eta_M)$ and any $\tau \in \delta\mathbb{Z}_+$, there exist at most on pair of values $(s_\tau, s_{\tau+\delta})$ satisfying (15)-(17). (ii) For almost any $\eta \in (\eta_m, \eta_M)$, there exists no strategy $\langle s_\tau, s_{\tau+\delta}\rangle$, with $s_\tau + s_{\tau+\delta} = 1$, satisfying (15)-(17).*

3.3 The Strategic Effect of a Change in the Measure of Players

The main result of the paper elicits the equilibrium strategies and expected ex-ante investment expenditures as the measure of players in the game varies. To illustrate the role of the discreteness of the information arrival, we present first the equilibrium of the game in which information arrives continuously.

The first proposition considers the case when n increases by means of an increase in a. $\langle s_\tau^a\rangle$ denotes a simple strategy in the game corresponding to a given value of a. τ^a, s_τ^a and $\mathcal{I}^a$ are the respective waiting time, probability of investment, and expected investment expenditures. The proposition, whose proof is in Barbos (2012), uncovers the inverted-U shaped relationship between the measure of players in the game and a player's equilibrium ex-ante investment expenditures.

Proposition 10 *Assume information arrives continuously. Then, there exists a unique symmetric equilibrium simple strategy $\langle s_\tau^a\rangle$, and a threshold $\widehat{a}$, such that: (i) for $a < \widehat{a}$, $\frac{d}{da}\tau^a < 0$, $s_\tau^a = 1$, and $\frac{d}{da}\mathcal{I}^a > 0$; (ii) for $a > \widehat{a}$, $\frac{d}{da}\tau^a = 0$, $\frac{d}{da}s_\tau^a < 0$, and $\frac{d}{da}\mathcal{I}^a < 0$.*

The next proposition considers the case when n increases by means of η. Its formal proof follows immediately from the characterization of the equilibrium necessary and sufficient conditions from Barbos (2012), and is thus omitted. Instead, we will present its intuition in section 3.4. $\langle s_\tau^\eta\rangle$, τ^η, s_τ^η and $\mathcal{I}^\eta$ have the obvious definitions.

Proposition 11 *Assume information arrives continuously. Then, there exists a unique symmetric equilibrium simple strategy $\langle s_\tau^\eta\rangle$, and a threshold $\widehat{\eta}$, such that: (i) for $\eta < \widehat{\eta}$, $\frac{d}{d\eta}\tau^\eta = 0$, $s_\tau^\eta = 1$, and $\frac{d}{d\eta}\mathcal{I}^\eta = 0$; (ii) for $\eta > \widehat{\eta}$, $\frac{d}{d\eta}\tau^\eta > 0$, $\frac{d}{d\eta}s_\tau^\eta < 0$, and $\frac{d}{d\eta}\mathcal{I}^\eta < 0$.*

Therefore, the inverted-U relationship between the measure of players in the game and the equilibrium ex-ante investment expenditures that emerges when the measure of players increases by means of an increase in a, does not do so when the increase is by means of η. In particular, for small values of η, players do not react by decreasing the equilibrium delay when η increases.

We now return to the case where information arrives at discrete times. If n increases by means of a, it can be shown that the equilibrium has the same properties as when information arrives continuously, only that τ^a and $\mathcal{I}^a$ are step functions when $a < \widehat{a}$. Proposition 12 presents the main result of the paper when the increase in n is by means of η. Its proof, as well as more precise statements, with the exact conditions determining the equilibrium strategies and the cutoffs, can be found in appendix C1. The corollary is proved in appendix C2.

Proposition 12 *There exists $\widehat{\eta} \in (\eta_m, \eta_M)$ and an increasing sequence $\{\eta_0 \equiv \widehat{\eta}, \eta_1, \eta_2, ...\} \subset (\eta_m, \eta_M)$, such that in equilibrium,*

1. *When $\eta \in (\eta_m, \widehat{\eta})$, players adopt $\langle s_\tau^\eta \rangle$, with $s_\tau^\eta = 1$, and τ^η a decreasing step function of η.*
2. *When $\eta \in (\widehat{\eta}, \eta_M)$, players adopt either*
   - *(a) $\langle s_\tau^\eta \rangle$, for $\eta \in \cup_{k\in\mathbb{N}} \left\{[\eta_{2k}, \eta_{2k+1})\right\}$, and $\left\langle s_\tau^\eta, s_{\tau+\delta}^\eta \right\rangle$, for $\eta \in \cup_{k\in\mathbb{N}} \left\{[\eta_{2k+1}, \eta_{2k+2})\right\}$, where $\tau^\eta = \tau^{\widehat{\eta}} + k\delta$, for $\eta \in [\eta_{2k}, \eta_{2k+2})$, or*
   - *(b) $\left\langle s_\tau^\eta, s_{\tau+\delta}^\eta \right\rangle$, for $\eta \in \cup_{k\in\mathbb{N}} \left\{[\eta_{2k}, \eta_{2k+1})\right\}$, and $\langle s_\tau^\eta \rangle$, for $\eta \in \cup_{k\in\mathbb{N}} \left\{[\eta_{2k+1}, \eta_{2k+2})\right\}$, where $\tau^\eta = \tau^{\widehat{\eta}} + (k-1)\,\delta$, for $\eta \in [\eta_{2k}, \eta_{2k+2})$.*

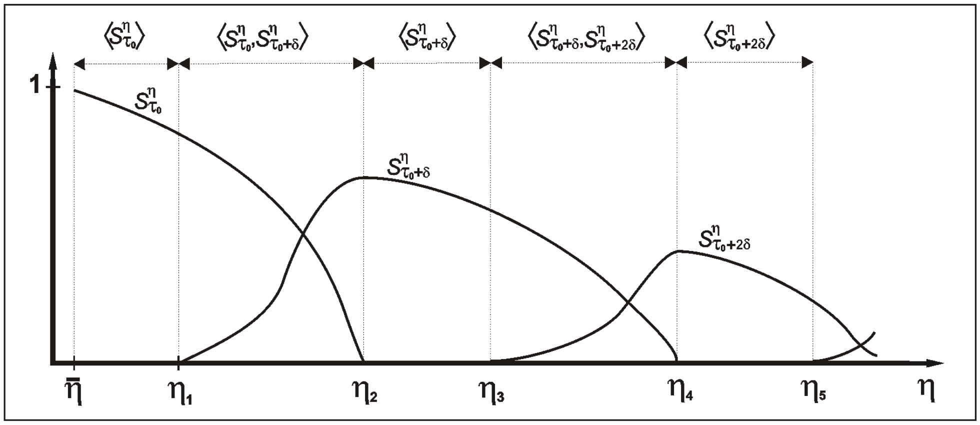
Corollary 13 *(i) For $\eta \in (\eta_m, \widehat{\eta})$, $\mathcal{I}^\eta$ is an increasing step function of η; (ii) For $\eta \in (\widehat{\eta}, \eta_M)$, $\frac{d}{d\eta}\mathcal{I}^\eta < 0$.*

For low values of η, players expect strictly positive equilibrium ex-ante payoffs and invest with probability $s_\tau^\eta = 1$ if a negative signal is not received by time τ^η. For these values, as η increases, the MC curve shifts up, exceeding the MB earlier and inducing players to weakly decrease the equilibrium delay τ^η. The *step* function τ^η emerges because the discreteness of $\delta\mathbb{Z}_+$ implies that the equilibrium strategies will prescribe investment after a given waiting time for a range of values of η. Since for $\eta \in (\eta_m, \widehat{\eta})$, players undertake riskier investments as η increases, they increase the expected ex-ante investment expenditures because, from an ex-ante point of view, the likelihood that they invest is higher.

When η is sufficiently high, i.e., at $\widehat{\eta}$, condition (11) is satisfied with equality and the equilibrium expected ex-ante payoff is zero. For $\eta \in (\widehat{\eta}, \eta_M)$, there is no equilibrium with players adopting $\langle s_\tau^\eta \rangle$, with $s_\tau^\eta = 1$. To see this, note that if such an equilibrium existed, as η increases above $\widehat{\eta}$, to continue to expect non-negative ex-ante payoffs, players would need to invest in safer projects. But if $s_\tau^\eta = 1$, the MC would continue to shift up as η increases. Thus, the trade-off between the MC and the MB of waiting would be solved earlier, inducing players to actually invest in riskier projects.

Instead, for $\eta \in (\widehat{\eta}, \eta_M)$ players decrease s_τ^η and invest later. For some values of $\eta \in (\widehat{\eta}, \eta_M)$, all players that invest, do so after waiting the same time τ^η; for the rest, there are two equilibrium waiting times. The resulting equilibrium profile of waiting times is increasing in η for $\eta > \widehat{\eta}$, while the ex-ante expected investment expenditures are decreasing.

The equilibrium strategy for $\eta \in (\widehat{\eta}, \eta_M)$ in case (a) of proposition 12.2 is presented in the following figure. $\tau_0 \equiv \tau^{\widehat{\eta}}$ denotes the equilibrium waiting time at $\widehat{\eta}$. As η increases on $[\widehat{\eta}, \eta_1]$, players adopt strategy $\langle s^{\eta}_{\tau 0}\rangle$, continuing to invest after waiting the same time as at $\eta = \widehat{\eta}$, but with a decreasing probability. On $[\eta_1, \eta_2]$, players adopt strategy $\left\langle s^{\eta}_{\tau 0}, s^{\eta}_{\tau 0+\delta}\right\rangle$; as η increases they lower $s^{\eta}_{\tau 0}$ and increase $s^{\eta}_{\tau 0+\delta}$. The upper bound of the interval, η_2, is defined by $s^{\eta_2}_{\tau 0} = 0$. On $[\eta_2, \eta_3]$, players adopt strategy $\left\langle s^{\eta}_{\tau 0+\delta}\right\rangle$, and as η increases, they decrease $s^{\eta}_{\tau 0+\delta}$ until $\eta = \eta_3$, where they start adopting strategy $\left\langle s^{\eta}_{\tau 0+\delta}, s^{\eta}_{\tau 0+2\delta}\right\rangle$.



To understand the intuition for the equilibrium strategy when $\eta \geq \widehat{\eta}$, note first that as η increases on intervals $[\widehat{\eta}, \eta_1]$, $[\eta_2, \eta_3]$, $[\eta_4, \eta_5]$, etc., s^{η}_{τ} decreases so as to keep the expected ex-ante payoff at zero, since the waiting time is constant on these intervals. We will argue next that this implies that for t in a neighborhood of τ^{η}, $MC^{\eta}_{\langle s^{\eta}_{\tau}\rangle}(t)$ *decreases* as η increases on these intervals. This is in contrast to the finding derived from lemma 3, where s^{η}_{τ} was constant as η increased.

For a fixed value of τ^{η}, let s^{η}_{τ} be the value that satisfies (11) with equality

$$p_0 \Pi\left(s^{\eta}_{\tau} a\eta, \frac{1}{2} s^{\eta}_{\tau} a\eta\right) - c\left[p_0 + (1-p_0)\, e^{-\mu\tau^{\eta}}\right] = 0 \tag{18}$$

Since Π is strictly decreasing in both arguments, $\eta \cdot s^{\eta}_{\tau}$ is constant. Now, from (7), by straightforward calculations, it follows that

$$\frac{\partial \ln \Phi_{\langle s^{\eta}_{\tau}\rangle}(t)}{\partial \ln \eta} \leq 1 \tag{19}$$

for $t \in [\tau^{\eta} - \frac{\eta}{2}, \tau^{\eta} + \frac{\eta}{2}]$. Thus, $\Phi_{\langle s^{\eta}_{\tau}\rangle}$ is *inelastic* with respect to η, for t in a neighborhood of τ^{η}. Intuitively, if all players wait the same time τ^{η} before investing, player i already assigns a high probability to the event that investment is ongoing at $t_i + t$, when t is close to τ^{η}, irrespective of the value of η. Therefore, an increase in η does not alter significantly his strategic beliefs. On the other hand, from (7), it also follows that the $MC^{\eta}_{\langle s^{\eta}_{\tau}\rangle}$ curve is *unit elastic* with respect to s^{η}_{τ}. Note now that as η increases, since $s^{\eta}_{\tau} \cdot \eta$ is constant, the absolute values of the percentage changes in s^{η}_{τ} and η must be equal. Therefore, if η increases by 1%, s^{η}_{τ} decreases by 1%, while (19) implies that $\Phi_{\langle s^{\eta}_{\tau}\rangle}$ increases by less than 1%. It follows then that, as claimed, as η increases, $MC_{\langle s^{\eta}_{\tau}\rangle}(t) = p_0 a s^{\eta}_{\tau} \theta \delta \Phi_{\langle s^{\eta}_{\tau}\rangle}(t)$ decreases for $t \in [\tau^{\eta} - \frac{\eta}{2}, \tau^{\eta} + \frac{\eta}{2}]$.

Now, as argued earlier, for $\eta < \widehat{\eta}$, as η increases, the MC shifts up and players wait less before investing. At $\eta = \widehat{\eta}$, (11) is satisfied with equality, while by proposition 5, $MC_{\langle s^{\eta}_{\tau_0} \rangle}(\tau_0 - \delta) \leq MB(\tau_0 - \delta)$ and $MB(\tau_0) \leq MC_{\langle s^{\eta}_{\tau_0} \rangle}(\tau_0)$, where, as defined previously, $\tau_0 = \tau^{\widehat{\eta}}$. As η increases above $\widehat{\eta}$, $s^{\eta}_{\tau_0}$ decreases so as to hold the expected ex-ante payoff at zero and so, the $MC_{\langle s^{\eta}_{\tau_0} \rangle}(t)$ decreases for $t \in [\tau_0 - \frac{\eta}{2}, \tau_0 + \frac{\eta}{2}]$. Since $\delta < \frac{\eta}{2}$, $MC_{\langle s^{\eta}_{\tau_0} \rangle}(\tau_0 - \delta)$ and $MC_{\langle s^{\eta}_{\tau_0} \rangle}(\tau_0)$ decrease. Let η_1 be defined by

$$MC_{\langle s^{\eta_1}_{\tau_0} \rangle}(\tau_0) = MB(\tau_0) \tag{20}$$

As η increases just above η_1 to $\eta_1 + \varepsilon$, where ε is arbitrarily small, $MC_{\left\langle s^{\eta_1+\varepsilon}_{\tau_0} \right\rangle}(\tau_0)$ falls below $MB(\tau_0)$. So $\left\langle s^{\eta_1+\varepsilon}_{\tau_0} \right\rangle$ is not an equilibrium strategy because it fails to satisfy (13); players would have an incentive to deviate and invest after waiting $\tau_0 + \delta$ time units. However, $\left\langle s^{\eta_1+\varepsilon}_{\tau_0+\delta} \right\rangle$ is also *not* an equilibrium. To understand this, note that if it was, since players would invest now in safer projects, for the zero expected ex-ante payoff condition to continue to be satisfied, the probability of investment should have an instant upward jump immediately above η_1. This would lower the expected ex-post payoff and offset the higher likelihood that the project is feasible. Thus, $s^{\eta_1+\varepsilon}_{\tau_0+\delta} > s^{\eta_1}_{\tau_0}$. Now, by inspecting (7), it follows that $\Phi_{\langle s_{\tau+\delta} \rangle}(\tau) = \Phi_{\langle s_{\tau} \rangle}(\tau)$, and thus that

$$MC_{\langle s^{\eta_1}_{\tau_0} \rangle}(\tau_0) = MC_{\left\langle s^{\eta_1}_{\tau_0+\delta} \right\rangle}(\tau_0) \tag{21}$$

Then, (20), (21), and $s^{\eta_1+\varepsilon}_{\tau_0+\delta} > s^{\eta_1}_{\tau_0}$ imply that

$$MC_{\left\langle s^{\eta_1+\varepsilon}_{\tau_0+\delta} \right\rangle}(\tau_0) > MB(\tau_0) \tag{22}$$

Thus, $\left\langle s^{\eta_1+\varepsilon}_{\tau_0+\delta} \right\rangle$ is not an equilibrium strategy because it fails (12); players would deviate from the prescribed equilibrium strategy of waiting $\tau_0 + \delta$ time units, and instead wait τ_0 time units.

The issue is resolved if the transition between the waiting times τ_0 and $\tau_0 + \delta$ is smooth, in that as η increases above η_1, players decrease the probability of waiting τ_0 time units and increase the probability of waiting $\tau_0 + \delta$ time units. Thus, as η increases on $[\eta_1, \eta_2]$, $s^{\eta}_{\tau_0}$ decreases and $s^{\eta}_{\tau_0+\delta}$ increases such that players expect zero ex-ante payoffs if investing after waiting either τ_0 or $\tau_0 + \delta$ time units. After the transition is complete at η_2, where η_2 is defined by $s^{\eta_2}_{\tau_0} = 0$, as η further increases, all players that invest, do so after waiting $\tau_0 + \delta$ time units. Then the process repeats.

The case (b) from proposition 12.2 appears because it may happen that when condition (13) binds for some value $\widehat{\eta}$ and waiting time $\tau^{\widehat{\eta}}$, as η increases slightly, if the waiting time decreases to $\tau^{\widehat{\eta}} - \delta$ and all players invest, the non-negative ex-ante payoff condition in (14) is no longer satisfied. In this case, immediately above $\widehat{\eta}$, players employ a strategy $\left\langle s^{\eta}_{\tau^{\eta}-\delta}, s^{\eta}_{\tau^{\eta}} \right\rangle$. The analysis is similar to the one from case (a).

The next corollary, whose proof is in appendix C3, states the uniqueness of the equilibrium.

Corollary 14 *For almost any $\eta \in (\eta_m, \eta_M)$, the equilibrium in proposition 12 is the unique equilibrium in simple strategies.*

3.4 Discussion

As explained in section 3.1, an increase in η leads each player to have a higher belief about the event that investment is ongoing in the game at any time, thus increasing the MC of waiting. For low levels of η, when players expect strictly positive ex-ante payoffs and thus invest with conditional probability one, the upward shift of the MC curve induces players to invest earlier. For high values of η, the non-negativity constraint on the equilibrium expected ex-ante payoff binds and players invest with a decreasing conditional probability. The effect of the belief updating on the marginal cost of waiting is of second order, and is compensated by the first order effect of the decrease in equilibrium probability of investment. On net, the MC decreases for these higher values of η, inducing players to invest later, thus further reducing investment. Therefore, for higher values of η, when η increases, players invest with a decreasing probability and wait longer.

The inverted-U shaped relationship between η and $\mathcal{I}^\eta$ does not emerge when information arrives *continuously* because the value of the MC *at* the equilibrium waiting time does not increase when η increases. To understand why, recall from the discussion motivating the results in Proposition 12 that $\Phi^\eta_{\langle s^\eta_\tau \rangle}(t)$ is inelastic with respect to η in the neighborhood around τ^η. In particular, as δ approaches 0, according to the equilibrium strategies of the other players, investment is almost surely ongoing in the game at times around the equilibrium waiting time τ^η. Thus, that belief is not altered in a measurably meaningful way between $\tau^\eta - \delta$ and $\tau^\eta + \delta$. In the limit as $\delta \to 0$, while the $MC_{\langle s^\eta_\tau \rangle}(t)$ increases in η for all $t \neq \tau^\eta$, $MC_{\langle s^\eta_\tau \rangle}(\tau^\eta)$ stays constant because at $t_i + \tau^\eta$ player i knows for sure that investment is ongoing in the game irrespective of the value of η. Thus, the MC curve crosses the MB curve at the same point as η increases for $\eta < \widehat{\eta}$, and so players do not change their waiting time. For $\eta > \widehat{\eta}$, as η increases, s^η_τ must decrease to satisfy the zero profit condition. This shifts down the MC curve everywhere and induces players to invest later.[15]

4 Conclusion

Previous literature found that an increase in the number of players in a preemption game induces them to undertake riskier actions as long as they expect non-negative payoffs from doing so. For instance, when the number of firms in an industry increases, they become more agressive in their innovative activities. This is no longer the case if the increase in the number of players is purely on an extensive margin, in that it leads to an increase in the amount of time it takes for all players to learn of an investment opportunity. For instance, if the increase in the number of firms is associated

[15] Note that when the increase in n is by means of an increase in a, the effect on the MC is of the first order and thus players do wait less for the lower values of a. For higher values, the effects of the increase in a and decrease in s^a_τ perfectly compensate each other and the equilibrium waiting time stays constant.

with a larger technological dispersion in the industry that expands the amount of time it takes for all firms to make a technological breakthrough, the positive relationship between competition and innovation no longer holds. In this paper we show that the positive relationship reemerges if players take investment decisions at discrete times either because the information arrives discretely or because they are constrained to do so.

Appendix

Appendix A. Proof of Lemma 3

First, we present a result that elicits $\lambda_{\langle s_\tau \rangle}(t|t_i)$.

Lemma 15 *Consider a strategy profile under which each player employs the strategy $\langle s_\tau \rangle$. Then, the expected measure of players who have invested before player i at moment $t_i + t$, with $t \geq 0$ is*

$$\lambda_{\langle s_\tau \rangle}(t|t_i) = \begin{cases} 0, & \text{for } t \in [0, \max(0, \tau - \eta)] \\ s_\tau a \left[\frac{(t-\tau+\eta)^2}{2\eta}\right], & \text{for } t \in (\max(0, \tau - \eta), \tau] \\ s_\tau a \left[\frac{\eta}{2} + (t - \tau) - \frac{(t-\tau)^2}{2\eta}\right], & \text{for } t \in (\tau, \tau + \eta] \\ s_\tau a \eta, & \text{for } \ t > \tau + \eta \end{cases} \tag{23}$$

Proof. This follows from (3) and (4) by direct computation. A detailed argument is presented in Barbos (2012). ■

To show the result in lemma 3, it is then sufficient then to employ (6) to compute $MC_{\langle s_\tau \rangle}(t)$. Thus, for $t \in \delta\mathbb{Z}_+ \cap [\max(0, \tau - \eta), \tau + \eta]$, we have:

$$\begin{aligned} &\lambda_{\langle s_\tau \rangle}(t + \delta|t_i) - \lambda_{\langle s_\tau \rangle}(t|t_i) = \\ &= \begin{cases} s_\tau a \left[\frac{(t+\delta-\tau+\eta)^2}{2\eta}\right] - s_\tau a \left[\frac{(t-\tau+\eta)^2}{2\eta}\right], & \text{for } t \in [\max(0, \tau - \eta), \tau - \delta] \\ s_\tau a \left[\frac{\eta}{2} + (t + \delta - \tau) - \frac{(t+\delta-\tau)^2}{2\eta}\right] - s_\tau a \left[\frac{\eta}{2} + (t - \tau) - \frac{(t-\tau)^2}{2\eta}\right], & \text{for } t \in [\tau, \tau + \eta] \end{cases} \\ &= \begin{cases} s_\tau a \left[\frac{(t+\delta-\tau+\eta)^2}{2\eta}\right] - s_\tau a \left[\frac{(t-\tau+\eta)^2}{2\eta}\right], & \text{for } t \in [\max(0, \tau - \eta), \tau] \\ s_\tau a \left[\frac{\eta}{2} + (t + \delta - \tau) - \frac{(t+\delta-\tau)^2}{2\eta}\right] - s_\tau a \left[\frac{\eta}{2} + (t - \tau) - \frac{(t-\tau)^2}{2\eta}\right], & \text{for } t \in [\tau, \tau + \eta] \end{cases} \\ &= \begin{cases} s_\tau a \frac{\delta}{\eta} \left[\eta - (\tau - t) + \frac{\delta}{2}\right], & \text{for } t \in [\max(0, \tau - \eta), \tau) \\ s_\tau a \frac{\delta}{\eta} \left[\eta - (t - \tau) - \frac{\delta}{2}\right], & \text{for } t \in [\tau, \tau + \eta] \end{cases} \end{aligned}$$

The result from the lemma 3 follows then immediately. Note that in the above, we did not compute $\lambda_{\langle s_\tau \rangle}(t + \delta|t_i) - \lambda_{\langle s_\tau \rangle}(t|t_i)$ for $t \notin \delta\mathbb{Z}_+$. Also, we did not compute the MC for $t \notin (\max(0, \tau - \eta), \tau + \eta)$ since, as it will become clear shortly, it is never a best response for a player to invest at those times under a symmetric strategy profile $\langle s_\tau \rangle$. ■

Appendix B1. Proof of Proposition 5

We will show first that the conditions in the proposition are sufficient for $\langle s_\tau \rangle$ to be a symmetric equilibrium strategy of the game. Assume all other players, but player i play strategy $\langle s_\tau \rangle$ that satisfies (11)-(14). We will show that in that case it is i's best response to play the same strategy. First, note that if $\tau > \eta$, then $\lambda_{\langle s_\tau \rangle}(t|t_i) = 0$ for $t \in [0, \tau - \eta]$ so it is not i's best response to invest before $t_i + \tau - \eta$. Also, by (14), clearly it is not his best response to invest after $t_i + \tau + \eta$. Denote by

$$\Psi_{\langle s_\tau \rangle}(t) \equiv p_0 \left[A(s_\tau a \eta) - \theta \lambda_{\langle s_\tau \rangle}(t|t_i) \right] - c \left[p_0 + (1 - p_0) e^{-\mu t} \right], \text{ for } t \geq 0. \tag{24}$$

Note that when $t \in \delta\mathbb{Z}_+$, $\Psi_{\langle s_\tau \rangle}(t)$ is player i's expected ex-ante payoff from investing at $t_i + t$. However, note that we define $\Psi_{\langle s_\tau \rangle}(t)$ for *all* values of $t \geq 0$ so as to be able to employ standard calculus methods. To prove the result it is enough to show that $\Psi_{\langle s_\tau \rangle}(\cdot)$ is maximized at $t = \tau$ in the set $[\max(0, \tau - \eta), \tau + \eta] \cap \delta\mathbb{Z}_+$. From (23) and (24), it follows that when $t < \tau$, we have $\Psi''_{\langle s_\tau \rangle}(t) = -p_0 \theta \lambda''_{\langle s_\tau \rangle}(t|t_i) - \mu^2 c (1 - p_0) e^{-\mu t} = -p_0 \theta a \frac{s_\tau}{\eta} - \mu^2 c (1 - p_0) e^{-\mu t} < 0$. On the other hand, for $t \in [\tau, \tau + \eta]$, we have $\Psi'''_{\langle s_\tau \rangle}(t) = \mu^3 c (1 - p_0) e^{-\mu t} > 0$.

Now first, the condition $p_0 \left[A(s_\tau a \eta) - \frac{1}{2} s_\tau \theta a \eta \right] - c \left[p_0 + (1 - p_0) e^{-\mu t} \right] \geq 0$ from the text of the proposition, ensures that $\Psi_{\langle s_\tau \rangle}(\tau) \geq 0$ since $\lambda_{\langle s_\tau \rangle}(\tau|t_i) = \frac{1}{2} s_\tau a \eta$. Thus, i has a non-negative expected ex-ante payoff from pursuing strategy $\langle s_\tau \rangle$. Second, from (24) it follows that

$$\begin{aligned}
\Psi_{\langle s_\tau \rangle}(\tau) &\geq \Psi_{\langle s_\tau \rangle}(\tau - \delta) \Leftrightarrow \\
&\Leftrightarrow -p_0 \theta \lambda_{\langle s_\tau \rangle}(\tau|t_i) - c(1 - p_0) e^{-\mu \tau} \geq -p_0 \theta \lambda_{\langle s_\tau \rangle}(\tau - \delta|t_i) - c(1 - p_0) e^{-\mu(\tau - \delta)} \\
&\Leftrightarrow p_0 \theta \left[\lambda_{\langle s_\tau \rangle}(\tau|t_i) - \lambda_{\langle s_\tau \rangle}(\tau - \delta|t_i) \right] \leq c(1 - p_0) \left[e^{-\mu(\tau - \delta)} - e^{-\mu \tau} \right] \\
&\Leftrightarrow MC_{\langle s_\tau \rangle}(\tau - \delta) \leq MB(\tau - \delta)
\end{aligned}$$

which is condition (12) from the text of the proposition. Therefore, since $\Psi_{\langle s_\tau \rangle}$ is concave for $t \leq \tau$, and $\Psi_{\langle s_\tau \rangle}(\tau) \geq \Psi_{\langle s_\tau \rangle}(\tau - \delta)$, it must be that it is increasing for all $t \leq \tau - \delta$. Thus, $\Psi_{\langle s_\tau \rangle}(t) \leq \Psi_{\langle s_\tau \rangle}(\tau)$ for $t \leq \tau$.

On the other hand, it is straightforward to see that

$$\Psi_{\langle s_\tau \rangle}(\tau) \geq \Psi_{\langle s_\tau \rangle}(\tau + \delta) \Leftrightarrow MB(\tau) \leq MC_{\langle s_\tau \rangle}(\tau)$$

which is condition (13) from the text of the proposition. Since $\Psi'''_{\langle s_\tau \rangle}(t) > 0$, it follows that once $\Psi_{\langle s_\tau \rangle}(\cdot)$ is convex, it will be convex for all higher values of t. Since $\Psi_{\langle s_\tau \rangle}(\tau) \geq \Psi_{\langle s_\tau \rangle}(\tau + \delta)$, $\Psi_{\langle s_\tau \rangle}$ is decreasing in between τ and $\tau + \delta$. But, $\Psi_{\langle s_\tau \rangle}$ can start increasing only after it becomes convex. So after it starts increasing, it will increase forever. Since (14), for the case when $s_\tau < 1$, and assumption 1, for the case when $s_\tau = 1$, ensure that $\Psi_{\langle s_\tau \rangle}(\tau + \eta) \leq 0$, it means that $\Psi_{\langle s_\tau \rangle}(t) \leq 0$ for $t \leq \tau + \eta$. Therefore, as desired, $\Psi_{\langle s_\tau \rangle}(\tau) \geq \Psi_{\langle s_\tau \rangle}(t)$ for all $0 \leq t \leq \tau + \eta$. This completes the proof of sufficiency in the proposition.

The necessity of (11)-(14) is straightforward. When $s_\tau < 1$, (11) is necessary to be satisfied with equality for players to be willing to mix. (13) and (12) are necessary as otherwise players would be better off waiting $\tau - \delta$ or $\tau + \delta$ time units. (14) is necessary because otherwise players deviate and invest after all uncertainty is removed. This completes the proof of proposition 5. ■

Appendix B2. Proof of Corollary 7

First, we argue that for a fixed value of s_τ there can be at most one value of τ satisfying (12) and (13). To see this, we rewrite (12) and (13) using lemma 3 as

$$p_0 a s_\tau \theta \delta \frac{1}{\eta}\left(\eta - \frac{\delta}{2}\right) \leq c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)e^{-\mu(\tau-\delta)} \tag{25}$$

$$c\left(1-p_0\right)\left[1-e^{-\mu\delta}\right]e^{-\mu\tau} \leq p_0 a s_\tau \theta \delta \frac{1}{\eta}\left(\eta - \frac{\delta}{2}\right) \tag{26}$$

which since the left hand side of (25) equals the right hand side of (26) imply

$$e^{-\mu\tau} \leq \frac{p_0 a s_\tau \theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\delta\frac{1}{\eta}\left(\eta-\frac{\delta}{2}\right) \leq e^{-\mu\tau}e^{\mu\delta} \Rightarrow$$

$$-\mu\tau \leq \ln\left\{\frac{p_0 a s_\tau \theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta-\frac{\delta^2}{2\eta}\right)\right\} \leq -\mu\tau+\mu\delta \Rightarrow$$

$$\tau \in \left[-\frac{1}{\mu}\ln\left\{\frac{p_0 a s_\tau \theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta-\frac{\delta^2}{2\eta}\right)\right\}, -\frac{1}{\mu}\ln\left\{\frac{p_0 a s_\tau \theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta-\frac{\delta^2}{2\eta}\right)\right\}+\delta\right] \tag{27}$$

Note that (27) pins down a unique value for τ for a given value of s_τ, except if

$$-\frac{1}{\mu}\ln\left\{\frac{p_0 a s_\tau \theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta-\frac{\delta^2}{2\eta}\right)\right\} \in \delta\mathbb{Z}_+ \tag{28}$$

The set of values of η for which (28) is satisfied is countable. (27) also implies that for all values of η, except a countable subset, there is at most one symmetric equilibrium in simple strategies $\langle s_\tau \rangle$ with $s_\tau = 1$.

Next, we argue that for all values of η, except a countable subset, there is a unique equilibrium in simple strategies $\langle s_\tau \rangle$. Assume by contradiction that there are two such equilibria $\langle s_\tau \rangle$ and $\langle s'_\tau \rangle$. If $\tau = \tau'$, then from (11) it follows that $s_\tau = s'_\tau$ because A is strictly decreasing in both arguments and thus the two strategies would be identical. If $\tau > \tau'$ then from (27) it must be that $s_\tau < s'_\tau$. However, from (24) it is clear that when $\tau > \tau'$ and $s_\tau < s'_\tau$, we have $\Psi_{\langle s_\tau \rangle}(\tau) > \Psi_{\langle s'_\tau \rangle}(\tau')$. This implies that $\Psi_{\langle s_\tau \rangle}(\tau) > 0$, and thus that $s_\tau = 1$. This is inconsistent with $s_\tau < s'_\tau$. Thus, indeed there is at most one symmetric equilibrium in simple strategies $\langle s_\tau \rangle$. ■

Appendix B3. Proof of Proposition 8

Showing the necessity of the conditions (15)-(17) is straightforward. We will show next their sufficiency. Assume all other players, but player i, adopt strategy $\langle s_\tau, s_{\tau+\delta}\rangle$ with $s_{\tau+\delta} > 0$. Then, as in (24), denote by

$$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) \equiv p_0 \left[A\left(\left(s_\tau + s_{\tau+\delta}\right) a\eta\right) - \theta\lambda_{\langle s_\tau, s_{\tau+\delta}\rangle}(t|t_i)\right] - c\left[p_0 + (1-p_0)\, e^{-\mu t}\right], \text{ for } t \geq 0 \quad (29)$$

and note that when $t \in \delta\mathbb{Z}_+$, $\Psi_{\langle s_\tau\rangle}(t)$ is i's expected ex-ante payoff if it invests at time $t_i + t$. We will show that $\Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) < 0$ for $t < \tau$ and $\Psi'''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) > 0$ for $t > \tau + \delta$ so that the argument from the proof of proposition 5 from appendix B1 will go through in this case as well with a slight modification. First, by (9), it follows that

$$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) = p_0 \left[A\left(\left(s_\tau + s_{\tau+\delta}\right) a\eta\right) - \theta\lambda_{\langle s_\tau\rangle}(t|t_i) - \theta\lambda_{\langle s_{\tau+\delta}\rangle}(t|t_i)\right] - c\left[p_0 + (1-p_0)\, e^{-\mu t}\right]$$

Using lemma 15, we have then that

$$\Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) = \begin{cases} -p_0\theta a\left(s_\tau + s_{\tau+\delta}\right)\frac{1}{\eta} - \mu^2 c\,(1-p_0)\, e^{-\mu t}, \text{ for } t < \tau \\ p_0\theta a s_\tau \frac{1}{\eta} - p_0\theta a s_{\tau+\delta}\frac{1}{\eta} - \mu^2 c\,(1-p_0)\, e^{-\mu t}, \text{ for } \tau < t < \tau + \delta \end{cases} \quad (30)$$

On the other hand,

$$\Psi'''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) = \mu^3 c\,(1-p_0)\, e^{-\mu t}, \text{ for all } t \geq \max(0, \tau - \eta)$$

Note that $\Psi''_{\langle s_\tau\rangle, \alpha(\tau+\delta)}(t) < 0$ for $t < \tau$ and $\Psi'''_{\langle s_\tau\rangle, \alpha(\tau+\delta)}(t) > 0$ for $t > \tau + \delta$. Moreover, it is straightforward to see that

$$\begin{aligned} \lim_{t\to(\tau+\delta)^-} \Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) &= p_0\theta a\left(s_\tau - a s_{\tau+\delta}\right)\frac{1}{\eta} - \mu^2 c\,(1-p_0)\, e^{-\mu(\tau+\delta)} \\ &< \lim_{t\to(\tau+\delta)^+} \Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) = p_0\theta a\left(s_\tau + s_{\tau+\delta}\right)\frac{1}{\eta} - \mu^2 c\,(1-p_0)\, e^{-\mu(\tau+\delta)} \end{aligned}$$

Finally, the condition from (16) in the text of the proposition 8 is equivalent to

$$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) = \Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau + \delta) \quad (31)$$

while (17) implies

$$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau + \eta + \delta) < 0$$

We will argue now that $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) \geq \Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(t)$ for all $t \in \delta\mathbb{Z}_+$. We consider two cases.

Case 1: $s_\tau \leq s_{\tau+\delta}$. In this case, from (30) it follows that $\Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) < 0$ for $t \in (\tau, \tau+\delta)$. Therefore the function $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ is concave for $t < \tau+\delta$. It is clear then that this and $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) =$

$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\delta)$ imply that $\Psi'_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) > 0$ and $\Psi'_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\delta) < 0$.[16] Since $\Psi'''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) > 0$ for $t > \tau+\delta$ and $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\eta+\delta) < 0$, an argument similar to the one from the proof of proposition 5 shows the result.

Case 2: $s_\tau > s_{\tau+\delta}$. In this case, $\Psi'''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) > 0$ for $t \geq \tau$ together with the facts that $\lim_{t\to(\tau+\delta)^-} \Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t) < \lim_{t\to(\tau+\delta)^+} \Psi''_{\langle s_\tau, s_{\tau+\delta}\rangle}(t)$ implies that once $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ is convex, it will be convex for all higher values. If $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ were increasing at $\tau+\delta$, then it should already be convex there and thus it would be increasing for all values above $\tau+\delta$. But this contradicts the fact that $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) \geq 0 > \Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\eta+\delta)$. Therefore, $\Psi'_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\delta) < 0$. Also, $\Psi'_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) > 0$ because otherwise, in order for $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ to be decreasing at $\tau+\delta$, it should have increased somewhere between τ and $\tau+\delta$, which would imply that $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ was convex at that point and therefore convex and increasing from that point to $\tau+\delta$. But this would contradict the fact that $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}$ should be decreasing at $\tau+\delta$. The rest of the argument goes as in the previous case. ■

Appendix B4. Proof of Corollary 9

We show first the uniqueness of the strategy $\langle s_\tau, s_{\tau+\delta}\rangle$ satisfying (15)-(17) for a fixed value of τ. Note first that lemma 15 implies $\lambda_{\langle s_{\tau+\delta}\rangle}(\tau|t_i) = s_{\tau+\delta} a \frac{(\eta-\delta)^2}{2\eta}$, $\lambda_{\langle s_\tau\rangle}(\tau+\delta|t_i) = s_\tau a\left(\frac{\eta}{2}+\delta-\frac{\delta^2}{2\eta}\right)$, $\lambda_{\langle s_\tau\rangle}(\tau|t_i) = s_\tau a\frac{\eta}{2}$ and $\lambda_{\langle s_{\tau+\delta}\rangle}(\tau+\delta|t_i) = s_{\tau+\delta} a\frac{\eta}{2}$. Using these in (29) it follows that

$$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) = p_0 A\left((s_\tau + s_{\tau+\delta})\, a\eta\right) - p_0\theta a\left[s_\tau\frac{\eta}{2} + s_{\tau+\delta}\frac{(\eta-\delta)^2}{2\eta}\right] - c\left[p_0 + (1-p_0)\, e^{-\mu\tau}\right] \tag{32}$$

$$\begin{aligned}\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\delta) &= \\ &= p_0 A\left((s_\tau + s_{\tau+\delta})\, a\eta\right) - p_0\theta a\left[s_\tau\left(\frac{\eta}{2}+\delta-\frac{\delta^2}{2\eta}\right) + s_{\tau+\delta}\frac{\eta}{2}\right] - c\left[p_0 + (1-p_0)\, e^{-\mu(\tau+\delta)}\right]\end{aligned} \tag{33}$$

Since $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) = \Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau+\delta)$, by subtracting the two equations we rewrite (16) as

$$p_0\theta a\,(s_\tau + s_{\tau+\delta})\left(\delta - \frac{\delta^2}{2\eta}\right) = c\,(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau} \tag{34}$$

Now, when $\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) > 0$, it must be that $s_\tau + s_{\tau+\delta} = 1$. From (34) this uniquely determines the value of τ. However, unless $-\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c(1-p_0)\left[1-e^{-\mu\delta}\right]}\left(\delta-\frac{\delta^2}{2\eta}\right)\right\} \in \delta\mathbb{Z}_+$, such an equilibrium does not exist. Therefore, a symmetric equilibrium in strategies $\langle s_\tau, s_{\tau+\delta}\rangle$ with $s_\tau + s_{\tau+\delta} = 1$ exists only for a countable set of values of η.

On the other hand, when $s_\tau + s_{\tau+\delta} < 1$, (34) gives $s_\tau + s_{\tau+\delta}$ for any given value of τ.

[16] By straightforward calculations it follows that Ψ is differentiable at both τ and $\tau+\delta$.

$\Psi_{\langle s_\tau, s_{\tau+\delta}\rangle}(\tau) = 0$, (32) can be rewritten as

$$p_0 A\left(\left(s_\tau + s_{\tau+\delta}\right) a\eta\right) - p_0\theta a\left[\left(s_\tau + s_{\tau+\delta}\right)\frac{\eta}{2} - s_{\tau+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] - c\left[p_0 + \left(1-p_0\right)e^{-\mu\tau}\right] = 0 \quad (35)$$

Since $s_\tau + s_{\tau+\delta}$ is determined by the value of τ, (35) determines $s_{\tau+\delta}$ uniquely as a function of τ, and therefore s_τ is determined as well. Therefore, for any $\tau \in \delta\mathbb{Z}_+$, there exists at most one pair of s_τ and $s_{\tau+\delta}$ satisfying (15)-(17). ■

Appendix C1. Proof of Proposition 12

We will show that for any value of $\eta \in [\eta_m, \eta_M]$, where η_m and η_M are defined in assumption 1 there exists a symmetric equilibrium in simple strategies. Note first that

$$-\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta - \frac{\delta^2}{2\eta}\right)\right\} > 0 \Leftrightarrow$$
$$p_0 a\theta\left(\delta - \frac{\delta^2}{2\eta}\right) < c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right) \Leftrightarrow \eta < \eta_M$$

It follows that there always exists at least one value of τ satisfying (27) for $s_\tau^\eta = 1$. Let $\widetilde{\tau}(\eta)$ to be the value satisfying (27) for $s_\tau^\eta = 1$ when $-\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c(1-p_0)\left(1-e^{-\mu\delta}\right)}\left(\delta - \frac{\delta^2}{2\eta}\right)\right\} \notin \delta\mathbb{Z}_+$, and let $\widetilde{\tau}(\eta)$ be the upper bound of the interval otherwise. Formally,

$$\widetilde{\tau}(\eta) \equiv \left(-\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta - \frac{\delta^2}{2\eta}\right)\right\}, -\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c\left(1-p_0\right)\left(1-e^{-\mu\delta}\right)}\left(\delta - \frac{\delta^2}{2\eta}\right)\right\} + \delta\right] \cap \delta\mathbb{Z}_+$$

The resulting $\widetilde{\tau}(\eta)$ is a decreasing *step* function of η. As η increases, the two bounds decrease, and for values of η for which $-\frac{1}{\mu}\ln\left\{\frac{p_0 a\theta}{c(1-p_0)\left(1-e^{-\mu\delta}\right)}\left(\delta - \frac{\delta^2}{2\eta}\right)\right\} \in \delta\mathbb{Z}_+$, the value of $\widetilde{\tau}(\eta)$ has a downward jump. In between these values, $\widetilde{\tau}(\eta)$ is constant.

Definition 16 *Let* $\left\langle \widetilde{s}^\eta_{\widetilde{\tau}(\eta)} \right\rangle$ *be the simple strategy with waiting time* $\widetilde{\tau}(\eta)$ *and probability of investment* $\widetilde{s}^\eta_{\widetilde{\tau}(\eta)} = 1$.

We have two cases to consider depending on the sign of $\Psi_{\left\langle \widetilde{s}^{\eta_m}_{\widetilde{\tau}(\eta_m)} \right\rangle}\left(\widetilde{\tau}(\eta_m)\right)$. If $\Psi_{\left\langle \widetilde{s}^{\eta_m}_{\widetilde{\tau}(\eta_m)} \right\rangle}\left(\widetilde{\tau}(\eta_m)\right) < 0$, η_m is too high for players to expect non-negative ex-ante payoffs if they all invest in the project. In this case, let $\widehat{\eta} \equiv \eta_m$.

If $\Psi_{\left\langle \widetilde{s}^{\eta_m}_{\widetilde{\tau}(\eta_m)} \right\rangle}\left(\widetilde{\tau}(\eta_m)\right) > 0$, let $\widehat{\eta}$ be the maximum value of η for which the following are satisfied

$$\Psi_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}\left(\widetilde{\tau}(\eta)\right) \geq 0 \quad (36)$$
$$MC_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}\left(\widetilde{\tau}(\eta) - \delta\right) \leq MB\left(\widetilde{\tau}(\eta) - \delta\right) \quad (37)$$
$$MB\left(\widetilde{\tau}(\eta)\right) \leq MC_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}\left(\widetilde{\tau}(\eta)\right) \quad (38)$$

First, note that (37) and (38) can be rewritten as in (27), with $\widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} = 1$. Thus, they will always be satisfied for some $\widetilde{\tau}(\eta)$. On the other hand, from (24) it follows that $\Psi_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta))$ is decreasing in η because $A'(\cdot) < 0$, $\lambda_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta)) = \frac{1}{2}a\eta$, and $\widetilde{\tau}(\eta)$ is weakly decreasing in η. However, since $\widetilde{\tau}(\eta)$ is not continuous, $\Psi_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta))$ is not continuous at the values of η for which $\widetilde{\tau}(\eta)$ has the downward jump. The points of discontinuity of $\widetilde{\tau}(\eta)$ occur at values of η for which $MC_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta) - \delta) = MB(\widetilde{\tau}(\eta) - \delta)$. This is intuitive since in between two points of discontinuity of $\widetilde{\tau}(\eta)$, $MC_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta) - \delta)$ and $MC_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta))$ both increase as η increases. Therefore, $\widehat{\eta}$ is defined either by

$$(i)\ \Psi_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\widetilde{\tau}(\widehat{\eta})} \right\rangle}(\widetilde{\tau}(\widehat{\eta})) = 0, \text{ or by} \tag{39}$$

$$(ii)\ MC_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\widetilde{\tau}(\widehat{\eta})} \right\rangle}(\widetilde{\tau}(\widehat{\eta}) - \delta) = MB(\widetilde{\tau}(\widehat{\eta}) - \delta),\ \Psi_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\widetilde{\tau}(\widehat{\eta})} \right\rangle}(\widetilde{\tau}(\widehat{\eta})) \geq 0, \text{ and } \lim_{\eta \to \widehat{\eta}^+} \Psi_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta)) < 0 \tag{40}$$

As we will argue shortly, case (i) will correspond to case (a) in the text of proposition 12(2) because just above $\widehat{\eta}$ players will start investing at $\widetilde{\tau}(\widehat{\eta})$ with a decreasing probability. Case (ii), which occurs when $\Psi_{\langle \widetilde{s}^{\eta}_{\tau} \rangle}(\widetilde{\tau}(\eta))$ falls below 0 at a point of discontinuity of $\widetilde{\tau}(\eta)$, will correspond to case (b). Note also that $\widehat{\eta} > \eta_m$ because $\Psi_{\left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle}(\widetilde{\tau}(\eta))$ is decreasing in η. Then for any $\eta \in [\eta_m, \widehat{\eta}]$, let $\tau^{\eta} \equiv \widetilde{\tau}(\eta)$ and $\langle s^{\eta}_{\tau} \rangle \equiv \left\langle \widetilde{s}^{\eta}_{\widetilde{\tau}(\eta)} \right\rangle$ and note that all conditions of proposition 5 are satisfied, that τ^{η} is decreasing in η, and that $s^{\eta}_{\tau} = 1$.

Lemma 17 *For any $\eta > \widehat{\eta}$ there is no symmetric equilibrium in simple strategies $\langle s^{\eta}_{\tau} \rangle$ with $s^{\eta}_{\tau} = 1$.*

Proof. Consider first case (i) from (39). Let $\tau_0 \equiv \widetilde{\tau}(\widehat{\eta})$ and denote by $\underline{\eta}_1$ the minimal value of η for which τ_0 is an equilibrium waiting time. Note that at $\underline{\eta}_1$, players have just switched from $\left\langle \widetilde{s}^{\eta}_{\tau_0+\delta} \right\rangle$ to $\langle \widetilde{s}^{\eta}_{\tau_0} \rangle$. Therefore, $MC_{\left\langle \widetilde{s}^{\underline{\eta}_1}_{\tau_0+\delta} \right\rangle}(\tau_0) = MB(\tau_0)$. Since the MC is increasing in η on $\left[\underline{\eta}_1, \widehat{\eta}\right]$, $MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+\delta} \right\rangle}(\tau_0)$ increases, exceeding $MB(\tau_0)$. Thus, $\left\langle \widetilde{s}^{\eta}_{\tau_0+\delta} \right\rangle$ is not an equilibrium strategy for $\eta \in \left[\underline{\eta}_1, \widehat{\eta}\right]$. Instead, players adopt strategy $\langle \widetilde{s}^{\eta}_{\tau_0} \rangle$.

At $\widehat{\eta}$, we have $\Psi_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0) = 0$. Above $\widehat{\eta}$, if all players were to adopt $\langle \widetilde{s}^{\eta}_{\tau_0} \rangle$, $\Psi_{\langle \widetilde{s}^{\eta}_{\tau_0} \rangle}(\tau_0) < 0$ and players would expect negative ex-ante payoffs from $\langle \widetilde{s}^{\eta}_{\tau_0} \rangle$. To avoid this, if players were to continue to invest with probability 1, they must switch immediately to waiting for more than τ_0 units of time, so as to invest in safer projects. But switching to investing at $\tau_0 + \delta$ is not a feasible equilibrium strategy because $MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+\delta} \right\rangle}(\tau_0) > MB(\tau_0)$, so condition (12) would not be satisfied. Also, since when $\widetilde{s}^{\eta}_{\tau_0+k\delta} = \widetilde{s}^{\eta}_{\tau_0+\delta} = 1$, we have that for any $k \geq 1$, $MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+k\delta} \right\rangle}(\tau_0 + k\delta - \delta) = MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+\delta} \right\rangle}(\tau_0) = p_0 a\theta \left(\delta - \frac{\delta^2}{2\eta}\right)$, and $MB(\tau_0) > MB(\tau_0 + k\delta - \delta)$, it follows that $MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+k\delta} \right\rangle}(\tau_0 + k\delta - \delta) > MB(\tau_0 + k\delta - \delta)$. Thus, no waiting time higher than τ_0 is feasible in an equilibrium with $\widetilde{s}^{\eta}_{\tau_0+k\delta} = 1$.

On the other hand, if $\widehat{\eta}$ is defined by case (ii) in (40), at $\widehat{\eta}$ we have $MC_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0 - \delta) = MB\,(\tau_0 - \delta)$. By the same argument as above, it follows then that when $\widetilde{s}^{\eta}_{\tau_0+k\delta} = 1$, we have $MC_{\left\langle \widetilde{s}^{\eta}_{\tau_0+k\delta} \right\rangle}(\tau_0 + k\delta - \delta) > MB\,(\tau_0 + k\delta - \delta)$ for all $k \geq 1$, so no equilibrium in which players invest with probability 1 exists. ■

Since by corollary 9, it is also the case that, generically, no symmetric equilibrium in simple strategies $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta} \right\rangle$ with $s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} = 1$ exists, the equilibrium probability of investment must be lower than 1 for $\eta > \widehat{\eta}$.

Definition 18 *For a fixed value of $\overline{\tau}$, let $\langle \overline{s}^{\eta}_{\overline{\tau}} \rangle$ be the simple strategy with waiting time $\overline{\tau}$ and with $\overline{s}^{\eta}_{\overline{\tau}}$ defined implicitly by the equation $\Psi_{\langle \overline{s}^{\eta}_{\overline{\tau}} \rangle}(\overline{\tau}) = 0$.*

Lemma 19 *$MC_{\langle \overline{s}^{\eta}_{\tau} \rangle}(\tau)$ and $MC_{\langle \overline{s}^{\eta}_{\tau} \rangle}(\tau - \delta)$ are decreasing in η.*

Proof. We can rewrite $\Psi_{\langle \overline{s}^{\eta}_{\overline{\tau}} \rangle}(\tau) = 0$ as $p_0 \left(A\,(a\overline{s}^{\eta}_{\overline{\tau}}\eta) - \frac{1}{2}a\theta\overline{s}^{\eta}_{\overline{\tau}}\eta \right) - c\left[p_0 + (1-p_0)\,e^{-\mu\overline{\tau}}\right] = 0$. Since $A' < 0$, the equation $p_0 \left(A\,(ax) - \frac{1}{2}a\theta x \right) - c\left[p_0 + (1-p_0)\,e^{-\mu\overline{\tau}}\right] = 0$ has a unique solution x_0 for a fixed $\overline{\tau}$. Therefore, $\overline{s}^{\eta}_{\overline{\tau}}\eta = x_0$ for all η. Now, $MC_{\langle \overline{s}^{\eta}_{\overline{\tau}} \rangle}(\tau) = MC_{\langle \overline{s}^{\eta}_{\overline{\tau}} \rangle}(\tau - \delta) = p_0 a\theta\overline{s}^{\eta}_{\overline{\tau}} \left(\delta - \frac{\delta^2}{2\eta} \right) = p_0 a\theta \frac{x_0}{\eta} \left(\delta - \frac{\delta^2}{2\eta} \right)$, which is decreasing in η for $\eta > \delta$. ■

We identify next *one* symmetric equilibrium simple strategy for each value of η with $\eta > \widehat{\eta}$. Corollary 14 will argue that generically each of these equilibria is unique in the set of simple strategies for that particular value of η.

We consider first case (i) from (39) and let again $\tau_0 \equiv \widetilde{\tau}\,(\widehat{\eta})$. Without loss of generality, we assume that $MB\,(\tau_0) < MC_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0)$. Otherwise, the analysis is the same as in case (ii) that will be discussed below. Note now that $MC_{\left\langle \widetilde{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0) = MC_{\left\langle \overline{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0)$ because at $\widehat{\eta}$, we have $\widetilde{s}^{\widehat{\eta}}_{\tau_0} = \overline{s}^{\widehat{\eta}}_{\tau_0} = 1$. Since by lemma 19, $MC_{\langle \overline{s}^{\eta}_{\tau} \rangle}(\tau)$ and $MC_{\langle \overline{s}^{\eta}_{\tau} \rangle}(\tau - \delta)$ are decreasing in η, define η_1 to be such that $MC_{\langle \overline{s}^{\eta_1}_{\tau_0} \rangle}(\tau_0) = MB(\tau_0)$. If no such value exists, then the simple strategy we will define next for $\eta \in (\widehat{\eta}, \eta_1)$ will be the equilibrium strategy for all values of $\eta > \widehat{\eta}$. Thus, for $\eta \in (\widehat{\eta}, \eta_1)$, define $\langle s^{\eta}_{\tau} \rangle$ by letting $\tau^{\eta} \equiv \tau_0$ and $s^{\eta}_{\tau} \equiv \overline{s}^{\eta}_{\tau_0}$. This strategy satisfies all conditions of proposition 5. (11) is satisfied by the definition of $\overline{s}^{\eta}_{\tau_0}$. (12) is satisfied because $MC_{\langle s^{\eta}_{\tau} \rangle}(\tau_0 - \delta) = MC_{\langle \overline{s}^{\eta}_{\tau} \rangle}(\tau_0 - \delta) < MC_{\left\langle \overline{s}^{\widehat{\eta}}_{\tau_0} \right\rangle}(\tau_0 - \delta) \leq MB\,(\tau_0 - \delta)$. (13) is satisfied because $MC_{\langle \overline{s}^{\eta}_{\tau_0} \rangle}(\tau_0)$ is decreasing in η and $\eta < \eta_1$. Finally, (14) is satisfied because $\Pi\,(s^{\eta}_{\tau}a\eta, s^{\eta}_{\tau}a\eta) = \Pi\,(a\widehat{\eta}, a\widehat{\eta}) < c$ because $\overline{s}^{\eta}_{\tau_0}\eta = \widehat{\eta}$, as shown in the proof of lemma 19.

As η increases above η_1, a profile in which all players adopt the strategy $\langle s^{\eta}_{\tau} \rangle$ that was defined on $(\widehat{\eta}, \eta_1)$ is no longer an equilibrium because $MC_{\langle \overline{s}^{\eta_1}_{\tau_0} \rangle}(\tau_0) > MB(\tau_0)$. Moreover, immediately above η_1, strategy $\left\langle \overline{s}^{\eta}_{\tau_0+\delta} \right\rangle$ also does *not* constitute an equilibrium. In other words, it is not an equilibrium for all players that invest to do so after waiting for a time $\tau_0 + \delta$. To see this, note first that $MC_{\left\langle\overline{s}^{\eta_1}_{\tau_0}\right\rangle}(\tau_0) = p_0 a\theta\overline{s}^{\eta_1}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta_1}\right)$ and $MC_{\left\langle\overline{s}^{\eta_1}_{\tau_0+\delta}\right\rangle}(\tau_0) = p_0 a\theta\overline{s}^{\eta_1}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta_1}\right)$. Now, given the definition of $\overline{s}^{\eta}_{\tau_0+\delta}$, it must be that $\overline{s}^{\eta_1}_{\tau_0+\delta} > \overline{s}^{\eta_1}_{\tau_0}$. Intuitively, if at η_1 players were to invest after waiting for $\tau_0+\delta$, since they would invest in safer projects, in order for the zero profit condition to be satisfied, the mixing probability should have an upward jump relative to $\overline{s}^{\eta_1}_{\tau_0}$. But then,

$$MC_{\left\langle\overline{s}^{\eta_1}_{\tau_0+\delta}\right\rangle}(\tau_0) = p_0 a\theta\overline{s}^{\eta_1}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta_1}\right) > p_0 a\theta\overline{s}^{\eta_1}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta_1}\right) = MC_{\left\langle\overline{s}^{\eta_1}_{\tau_0}\right\rangle}(\tau_0) = MB(\tau_0)$$

Therefore, $MC_{\left\langle\overline{s}^{\eta_1}_{\tau_0+\delta}\right\rangle}(\tau_0) > MB(\tau_0)$, which contradicts (12) from proposition 5. Let η_2 be defined by $MC_{\left\langle\overline{s}^{\eta_2}_{\tau_0+\delta}\right\rangle}(\tau_0) = MB(\tau_0)$. Since $MC_{\left\langle\overline{s}^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0)$ is decreasing in η, we have $\eta_2 > \eta_1$ and $MC_{\left\langle\overline{s}^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0) > MB(\tau_0)$ for all $\eta \in [\eta_1, \eta_2)$. Therefore, strategy $\left\langle\overline{s}^{\eta}_{\tau_0+\delta}\right\rangle$ does not constitute an equilibrium for any $\eta \in [\eta_1, \eta_2)$. By an identical argument, it follows that no other simple strategy $\left\langle\overline{s}^{\eta}_{\tau_0+k\delta}\right\rangle$ with $k \geq 1$ is an equilibrium for $\eta \in [\eta_1, \eta_2)$ since

$$MC_{\left\langle\overline{s}^{\eta}_{\tau_0+k\delta}\right\rangle}(\tau_0 + k\delta - \delta) \geq MC_{\left\langle\overline{s}^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0) > MB(\tau_0) \geq MB(\tau_0 + k\delta - \delta)$$

where the first inequality comes from the fact that $\overline{s}^{\eta}_{\tau_0+k\delta} \geq \overline{s}^{\eta}_{\tau_0+\delta}$.

Instead, for $\eta \in [\eta_1, \eta_2)$, there is an symmetric equilibrium in which players adopt strategy $\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle$, with $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta} < 1$, satisfying the conditions of proposition 8.

Lemma 20 *For $\eta \in [\eta_1, \eta_2)$ there exists a simple strategy $\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle$ satisfying (15)-(17).*

Proof. As in (34), condition (16) can be rewritten as

$$\begin{aligned} p_0\theta a\left(s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}\right)\left(\delta - \frac{\delta^2}{2\eta}\right) &= c\left(1-p_0\right)\left(1 - e^{-\mu\delta}\right)e^{-\mu\tau_0} \Rightarrow && (41) \\ s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta} &= L\frac{2\eta}{2\eta-\delta} && (42) \end{aligned}$$

where $L \equiv \frac{c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau_0}}{p_0\theta a}$. On the other hand, condition (33) with $\Psi_{\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0) = 0$ is

$$p_0 A\left(\left(s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}\right)a\eta\right) - p_0\theta a\left[\left(s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}\right)\frac{\eta}{2} + s^{\eta}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] - c\left[p_0 + \left(1-p_0\right)e^{-\mu(\tau_0+\delta)}\right] = 0 \quad (43)$$

Substituting $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}$ from (42), we obtain

$$p_0 A\left(aL\frac{2\eta^2}{2\eta-\delta}\right) - p_0\theta a\left[L\frac{\eta^2}{2\eta-\delta} + s^{\eta}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] - c\left[p_0 + \left(1-p_0\right)e^{-\mu(\tau_0+\delta)}\right] = 0$$

This equation defines $s^{\eta}_{\tau_0}$ as an implicit function of η. Using the implicit function theorem, it follows that the resulting $s^{\eta}_{\tau_0}$ is decreasing in η.

We will argue next that $s^{\eta}_{\tau_0} \in (0,1)$ for $\eta \in [\eta_1, \eta_2)$, so that $s^{\eta}_{\tau_0}$ is a well defined probability. Note that at η_1, (41) and (43) are satisfied for $s^{\eta_1}_{\tau_0} = \overline{s}^{\eta_1}_{\tau_0}$ and $s^{\eta_1}_{\tau_0+\delta} = 0$, while at η_2, they are satisfied for $s^{\eta_2}_{\tau_0} = 0$ and $s^{\eta_2}_{\tau_0+\delta} = \overline{s}^{\eta_2}_{\tau_0+\delta}$. By corollary (9), these are the unique equilibrium strategies $\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta} \right\rangle$ at those values of η. Thus, $s^{\eta}_{\tau_0}$ decreases on (η_1, η_2) from $\overline{s}^{\eta_1}_{\tau_0}$ to 0. Therefore, as claimed, $s^{\eta}_{\tau_0} \in [0,1)$.

From (42) it also follows that $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}$ is decreasing in η. Moreover, $\overline{s}^{\eta_1}_{\tau_0} > \overline{s}^{\eta_2}_{\tau_0+\delta}$. To see this, note that they solve the equations

$$\begin{aligned} MC_{\left\langle \overline{s}^{\eta_1}_{\tau_0} \right\rangle}(\tau_0) &= MB(\tau_0) \Rightarrow p_0 a\theta \overline{s}^{\eta_1}_{\tau_0} \left(\delta - \frac{\delta^2}{2\eta_1}\right) = MB(\tau_0) \\ MC_{\left\langle \overline{s}^{\eta_2}_{\tau_0+\delta} \right\rangle}(\tau_0) &= MB(\tau_0) \Rightarrow p_0 a\theta \overline{s}^{\eta_2}_{\tau_0+\delta} \left(\delta - \frac{\delta^2}{2\eta_2}\right) = MB(\tau_0) \end{aligned}$$

Therefore, $\overline{s}^{\eta_1}_{\tau_0} \left(\delta - \frac{\delta^2}{2\eta_1}\right) = \overline{s}^{\eta_2}_{\tau_0+\delta} \left(\delta - \frac{\delta^2}{2\eta_2}\right)$, which since $\eta_2 > \eta_1$ implies $\overline{s}^{\eta_1}_{\tau_0} > \overline{s}^{\eta_2}_{\tau_0+\delta}$. Since $s^{\eta_1}_{\tau_0} + s^{\eta_1}_{\tau_0+\delta} = \overline{s}^{\eta_1}_{\tau_0}$, $s^{\eta_2}_{\tau_0} + s^{\eta_2}_{\tau_0+\delta} = \overline{s}^{\eta_2}_{\tau_0+\delta}$, and $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}$ is decreasing in η, it follows that $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta} \in \left[\overline{s}^{\eta_2}_{\tau_0+\delta}, \overline{s}^{\eta_1}_{\tau_0}\right] \subset (0,1)$ for $\eta \in [\eta_1, \eta_2)$.

We will show next that $s^{\eta}_{\tau_0+\delta} \in (0,1)$. Since $s^{\eta_1}_{\tau_0+\delta} = 0$ and $s^{\eta_2}_{\tau_0+\delta} = \overline{s}^{\eta_2}_{\tau_0+\delta} \in (0,1)$, it would be enough to show that $s^{\eta}_{\tau_0+\delta}$ is increasing in η. Now, from (32) with $\Psi_{\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta} \right\rangle}(\tau_0+\delta) = 0$ and substituting $s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}$ from (42) we have

$$p_0 A\left(aL\frac{2\eta^2}{2\eta-\delta}\right) - p_0\theta a\left[L\frac{\eta^2}{2\eta-\delta} - s^{\eta}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] - c\left[p_0 + (1-p_0)\, e^{-\mu\tau_0}\right] = 0 \tag{44}$$

Because $\frac{\partial}{\partial\eta}\left[A\left(aL\frac{2\eta^2}{2\eta-\delta}\right)\right] < 0$ and the derivative of the left hand side of (44) with respect to $s^{\eta}_{\tau_0+\delta}$ is positive, using the implicit function theorem, to show that $s^{\eta}_{\tau_0+\delta}$ is increasing in η it is enough to show that

$$\frac{\partial}{\partial\eta}\left[L\frac{\eta^2}{2\eta-\delta} - s^{\eta}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] > 0 \Leftrightarrow L\frac{2\eta^2 - 2\eta\delta}{(2\eta-\delta)^2} - s^{\eta}_{\tau_0+\delta}\frac{\delta^2}{2\eta^2} > 0 \tag{45}$$

Now note from (42), $s^{\eta}_{\tau_0} > 0$ implies that

$$s^{\eta}_{\tau_0+\delta} < L\frac{2\eta}{2\eta-\delta} \tag{46}$$

Substituting (46) into (45), we have

$$L\frac{2\eta^2 - 2\eta\delta}{(2\eta-\delta)^2} - s^{\eta}_{\tau_0+\delta}\frac{\delta^2}{2\eta^2} > L\frac{2\eta^2 - 2\eta\delta}{(2\eta-\delta)^2} - L\frac{2\eta}{2\eta-\delta}\frac{\delta^2}{2\eta^2} \tag{47}$$

so using that $L > 0$ and $2\eta - \delta > 0$, it follows that it is enough to show that

$$\frac{\eta-\delta}{2\eta-\delta} - \frac{\delta^2}{2\eta^2} > 0 \Leftrightarrow H(\eta) \equiv 2\eta^3 - 2\eta^2\delta - 2\eta\delta^2 + \delta^3 > 0$$

We have $H'(\eta) \equiv 6\eta^2 - 4\eta\delta - 2\delta^2$ and $H''(\eta) \equiv 12\eta - 4\delta$. Since $H''(\eta) > 0$ for $\eta > \delta$ and $H'(\delta) = 0$, it follows that $H'(\eta) \geq 0$ for $\eta > \delta$. Also, $H(2\delta) = 5\delta^3 > 0$. Therefore, $H(\eta) > 0$ for $\eta > 2\delta$, i.e., for $\delta \in \left(0, \frac{\eta}{2}\right)$. Therefore, $s^{\eta}_{\tau_0+\delta}$ is well defined.

Finally, from (42) it also follows that $\left(s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}\right)\eta = L\frac{2\eta^2}{2\eta-\delta}$, which is increasing in η. Therefore, $\left(s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta}\right)\eta > \overline{s}^{\eta_1}_{\tau_0}\eta_1 = \widehat{\eta}$. Thus, condition (17) of the proposition 8 is also satisfied. This completes the proof of lemma 20. ■

Thus, as η increases above η_1, players gradually shift the weight of the mixing probabilities from waiting for τ_0 towards waiting for $\tau_0 + \delta$, until at η_2 where players no longer invest after waiting for τ_0. Using the same arguments as before, it can be shown that above η_2, the symmetric equilibrium strategy is $\langle s^{\eta}_{\tau}\rangle$, with $\tau^{\eta} = \tau_0 + \delta$ and $s^{\eta}_{\tau} = \overline{s}^{\eta}_{\tau_0+\delta}$. As shown in lemma 19, $MC_{\left\langle \overline{s}^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0 + \delta)$ and $MC_{\left\langle \overline{s}^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0 + 2\delta)$ decrease in η until η_3 where $MC_{\left\langle \overline{s}^{\eta_3}_{\tau_0+\delta}\right\rangle}(\tau_0 + \delta) = MB(\tau_0 + \delta)$, and the players start adopting $\left\langle s^{\eta}_{\tau_0+\delta}, s^{\eta}_{\tau_0+2\delta}\right\rangle$, with mixing probabilities defined in a similar manner as for $\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle$. Then, as η increases, the process repeats.

The argument for the case (b) of proposition 8(2) and $\tau_0 - \delta$ is sustainable in a mixed strategy equilibrium is similar to the one above. This completes the proof of proposition 12. ■

Appendix C2. Proof of Corollary 13

Part (i) of the corollary follows immediately from lemma 4 and the fact that when $\eta \in (\eta_m, \widehat{\eta})$, the equilibrium strategy $\langle s^{\eta}_{\tau}\rangle$ has $s^{\eta}_{\tau} = 1$ and τ^{η} a decreasing step function.

For part (ii), note from lemma 4 that it is sufficient to show that $\frac{\partial}{\partial\eta}s^{\eta}_{\tau} < 0$ when players adopt $\langle s^{\eta}_{\tau}\rangle$ and that $\frac{\partial}{\partial\eta}\left[s^{\eta}_{\tau} + e^{-\mu\delta}s^{\eta}_{\tau+\delta}\right] < 0$ when players adopt $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle$. The first condition follows immediately from the proof of proposition 12. From (42), we have that $s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} = L\frac{2\eta}{2\eta-\delta}$ where L does not depend on η. Clearly, $\frac{\partial}{\partial\eta}\left[s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta}\right] < 0$, and since from the proof of proposition 12 we know that $\frac{\partial}{\partial\eta}s^{\eta}_{\tau+\delta} > 0$ and $\frac{\partial}{\partial\eta}s^{\eta}_{\tau}$, we have that $\frac{\partial}{\partial\eta}s^{\eta}_{\tau} < -\frac{\partial}{\partial\eta}s^{\eta}_{\tau+\delta} < 0$. But then $\frac{\partial}{\partial\eta}s^{\eta}_{\tau} < -e^{-\mu\delta}\frac{\partial}{\partial\eta}s^{\eta}_{\tau+\delta}$, so $\frac{\partial}{\partial\eta}\left[s^{\eta}_{\tau} + e^{-\mu\delta}s^{\eta}_{\tau+\delta}\right] < 0$. This completes the proof of the corollary. ■

Appendix C3. Proof of Corollary 14

First, by corollary 7, for all values of $\eta \in (\eta_m, \widehat{\eta})$, but a countable subset, there is no other symmetric equilibrium simple strategy $\langle s^{\eta}_{\tau}\rangle$. Also, by corollary 9$(ii)$, there is no symmetric equilibrium simple strategy $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle$ with $s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} = 1$. It remains to show that there is no symmetric equilibrium simple strategy $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle$ with $s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} < 1$. Assume by contradiction that such a strategy $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle$ exists for some $\eta \in (\eta_m, \eta_M)$. In order for $s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} < 1$, it must be that players expect zero ex-ante payoffs in this equilibrium. Moreover, since players expect strictly positive ex-ante payoffs while they all invest after waiting for time $\widetilde{\tau}(\eta)$, it must be that $\tau < \widetilde{\tau}(\eta)$ because otherwise $s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}<1$ would not be possible. Now, note that the definition of $\widetilde{\tau}(\eta)$ implies

$$c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\widetilde{\tau}(\eta)}e^{\mu\delta}\geq p_0a\theta\left(\delta-\frac{\delta^2}{2\eta}\right)\geq c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\widetilde{\tau}(\eta)}$$

Now, from (34) it follows that

$$\begin{aligned} p_0a\theta\left(s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}\right)\left(\delta-\frac{\delta^2}{2\eta}\right) &= c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau} \\ &\geq c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\widetilde{\tau}(\eta)}e^{\mu\delta}\geq p_0a\theta\left(\delta-\frac{\delta^2}{2\eta}\right) \end{aligned}$$

where the first inequality follows from $\tau\leq\widetilde{\tau}(\eta)-\delta$. This contradicts $s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}<1$.

We consider next the case (a) from proposition 12.2. First, we show that the equilibrium in proposition 12.2 is unique on $(\widehat{\eta},\eta_1)$. By corollary 7 there can be no other symmetric equilibrium strategy $\langle s^{\eta}_{\tau}\rangle$. Assume by contradiction that there exists a symmetric equilibrium simple strategy $\left\langle s^{\eta}_{\tau},s^{\eta}_{\tau+\delta}\right\rangle$. Since the strategy $\langle\overline{s}^{\eta}_{\tau_0}\rangle$ is an equilibrium on $(\widehat{\eta},\eta_1)$, it follows by corollary 9(i) that it must be that $\tau\neq\tau_0$.

Assume first that $\tau<\tau_0$. Then it must be that

$$p_0a\theta\left(s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}\right)\left(\delta-\frac{\delta^2}{2\eta}\right)\geq c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau_0}e^{\mu\delta}\geq p_0a\theta\overline{s}^{\eta}_{\tau_0}\left(\delta-\frac{\delta^2}{2\eta}\right)$$

where the first inequality follows from (34) and the fact that $\tau\leq\tau_0-\delta$, while the second from $MB(\tau_0-\delta)\geq MC_{\langle\overline{s}^{\eta}_{\tau_0}\rangle}(\tau_0-\delta)$. Therefore, $s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}>\overline{s}^{\eta}_{\tau_0}$. But then from (33) we have

$$\begin{aligned} \Psi_{\left\langle s^{\eta}_{\tau},s^{\eta}_{\tau+\delta}\right\rangle}(\tau+\delta) &= p_0A\left(\left(s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}\right)a\eta\right)-p_0\theta a\left[\left(s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}\right)\frac{\eta}{2}+s^{\eta}_{\tau}\left(\delta-\frac{\delta^2}{2\eta}\right)\right]-c\left[p_0+(1-p_0)e^{-\mu(\tau+\delta)}\right] \\ &< p_0A\left(\overline{s}^{\eta}_{\tau_0}a\eta\right)-p_0\theta a\left[\overline{s}^{\eta}_{\tau_0}\frac{\eta}{2}\right]-c\left[p_0+(1-p_0)e^{-\mu\tau_0}\right]=\Psi_{\langle\overline{s}^{\eta}_{\tau_0}\rangle}(\tau_0)=0 \end{aligned}$$

where the inequality follows from $s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}>\overline{s}^{\eta}_{\tau_0}$, $A'<0$, $s^{\eta}_{\tau}>0$ and $\tau+\delta\leq\tau_0$. Therefore, $\Psi_{\left\langle s^{\eta}_{\tau},s^{\eta}_{\tau+\delta}\right\rangle}(\tau+\delta)<0$, which provides the contradiction.

Assume now that $\tau>\tau_0$. Then,

$$\begin{aligned} p_0a\theta\left(s^{\eta}_{\tau}+s^{\eta}_{\tau+\delta}\right)\left(\delta-\frac{\delta^2}{2\eta}\right) &= c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau} \\ &< c(1-p_0)\left(1-e^{-\mu\delta}\right)e^{-\mu\tau_0}\leq p_0a\theta\overline{s}^{\eta}_{\tau_0}\left(\delta-\frac{\delta^2}{2\eta}\right) \end{aligned}$$

where the equality follows from (34), and the last inequality from $MC_{\langle\overline{s}^{\eta}_{\tau_0}\rangle}(\tau_0)\geq MB(\tau_0)$. It follows thus that $s^\eta_\tau + s^\eta_{\tau+\delta} \le \overline{s}^\eta_{\tau_0}$. Now, from (32) and (29) we have

$$\begin{aligned}\Psi_{\langle s^\eta_\tau, s^\eta_{\tau+\delta}\rangle}(\tau) &= p_0 A\left(\left(s^\eta_\tau + s^\eta_{\tau+\delta}\right) a\eta\right) - p_0\theta a\left[\left(s^\eta_\tau + s^\eta_{\tau+\delta}\right)\frac{\eta}{2} - s^\eta_{\tau+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] - c\left[p_0 + (1-p_0)\, e^{-\mu\tau}\right] \\ &> p_0 A\left(\overline{s}^\eta_{\tau_0} a\eta\right) - p_0\theta a\left[\overline{s}^\eta_{\tau_0}\frac{\eta}{2}\right] - c\left[p_0 + (1-p_0)\, e^{-\mu\tau_0}\right] = \Psi_{\langle \overline{s}^\eta_{\tau_0}\rangle}(\tau_0) = 0\end{aligned}$$

because $s^\eta_\tau + s^\eta_{\tau+\delta} \le \overline{s}^\eta_{\tau_0}$, $A' < 0$, $s^\eta_{\tau+\delta} > 0$ and $\tau > \tau_0$. Therefore, $\Psi_{\langle s^\eta_\tau, s^\eta_{\tau+\delta}\rangle}(\tau) > 0$, which is a contradiction. This completes the proof of the uniqueness of the symmetric equilibrium in simple strategies on $(\widehat{\eta}, \eta_1)$.

Next, we argue that $\left\langle s^\eta_{\tau_0}, s^\eta_{\tau_0+\delta}\right\rangle$ defined in lemma 20 is the unique simple strategy symmetric equilibrium for $\eta \in (\eta_1, \eta_2)$. First, as argued above, there is no equilibrium simple strategy $\langle s^\eta_\tau\rangle$ on (η_1, η_2). Assume by contradiction that there is some other equilibrium simple strategy $\langle s^\eta_\tau, s^\eta_{\tau+\delta}\rangle$. By corollary 9, it must be that $\tau \ne \tau_0$ and that $s^\eta_\tau + s^\eta_{\tau+\delta} < 1$.

Assume first that $\tau < \tau_0$. Then, (34) applied to both $\left\langle s^\eta_{\tau_0}, s^\eta_{\tau_0+\delta}\right\rangle$ and $\langle s^\eta_\tau, s^\eta_{\tau+\delta}\rangle$ implies

$$s^\eta_{\tau_0} + s^\eta_{\tau_0+\delta} < s^\eta_\tau + s^\eta_{\tau+\delta} \tag{48}$$

On the other hand, using (33), $\Psi_{\langle s^\eta_\tau, s^\eta_{\tau+\delta}\rangle}(\tau+\delta) = 0$ and $\Psi_{\left\langle s^\eta_{\tau_0}, s^\eta_{\tau_0+\delta}\right\rangle}(\tau_0+\delta) = 0$ imply

$$\begin{aligned}p_0 A\left(\left(s^\eta_\tau + s^\eta_{\tau+\delta}\right) a\eta\right) &= p_0\theta a\left[\left(s^\eta_\tau + s^\eta_{\tau+\delta}\right)\frac{\eta}{2} + s^\eta_\tau\left(\delta - \frac{\delta^2}{2\eta}\right)\right] + c\left[p_0 + (1-p_0)\, e^{-\mu(\tau+\delta)}\right] \\ p_0 A\left(\left(s^\eta_{\tau_0} + s^\eta_{\tau_0+\delta}\right) a\eta\right) &= p_0\theta a\left[\left(s^\eta_{\tau_0} + s^\eta_{\tau_0+\delta}\right)\frac{\eta}{2} + s^\eta_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta}\right)\right] + c\left[p_0 + (1-p_0)\, e^{-\mu(\tau_0+\delta)}\right]\end{aligned}$$

Using (51) and $A' < 0$, these imply

$$p_0\theta a s^\eta_\tau\left(\delta - \frac{\delta^2}{2\eta}\right) \le p_0\theta a\left[\left(s^\eta_{\tau_0} + s^\eta_{\tau_0+\delta} - s^\eta_\tau - s^\eta_{\tau+\delta}\right)\frac{\eta}{2}\right] + c\,(1-p_0)\, e^{-\mu\delta}\left(e^{-\mu\tau_0} - e^{-\mu\tau}\right) + p_0\theta a s^\eta_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta}\right) \tag{49}$$

Now, note that, by the definition of η_1, from (41) and the fact that $s^{\eta_1}_{\tau_0+\delta} = 0$, we have

$$p_0\theta a s^{\eta_1}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta_1}\right) = c\,(1-p_0)\left(1 - e^{-\mu\delta}\right) e^{-\mu\tau_0} \tag{50}$$

Writing (49) for $\eta = \eta_1$, and using (50) to substitute $p_0\theta a s^{\eta_1}_{\tau_0}\left(\delta - \frac{\delta^2}{2\eta_1}\right)$, we have

$$\theta a s^{\eta_1}_\tau\left(\delta - \frac{\delta^2}{2\eta_1}\right) \le p_0\theta a\left[\left(s^{\eta_1}_{\tau_0} + s^{\eta_1}_{\tau_0+\delta} - s^{\eta_1}_\tau - s^{\eta_1}_{\tau+\delta}\right)\frac{\eta_1}{2}\right] + c\,(1-p_0)\left[e^{-\mu\tau_0} - e^{-\mu\delta} e^{-\mu\tau}\right]$$

But since $\tau \le \tau_0 - \delta$, we have $e^{-\mu\tau_0} - e^{-\mu\delta}e^{-\mu\tau} \le 0$, and thus $s^{\eta_1}_\tau < 0$. Since s^η_τ is decreasing in η on (η_1, η_2), it follows that $s^\eta_\tau < 0$ for all $\eta \in (\eta_1, \eta_2)$. This provides the desired contradiction.

Assume now that $\tau > \tau_0$. Then, (34) applied to $\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle$ and $\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle$ implies

$$s^{\eta}_{\tau_0} + s^{\eta}_{\tau_0+\delta} > s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} \tag{51}$$

Using (32) to write $\Psi_{\left\langle s^{\eta}_{\tau}, s^{\eta}_{\tau+\delta}\right\rangle}(\tau) = 0$ and $\Psi_{\left\langle s^{\eta}_{\tau_0}, s^{\eta}_{\tau_0+\delta}\right\rangle}(\tau_0) = 0$, and then using (51) and $A' < 0$, we have

$$p_0\theta a s^{\eta}_{\tau+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right) \leq p_0\theta a\left[\left(s^{\eta}_{\tau} + s^{\eta}_{\tau+\delta} - s^{\eta}_{\tau_0} - s^{\eta}_{\tau_0+\delta}\right)\frac{\eta}{2}\right] + c\left(1-p_0\right)\left(e^{-\mu\tau} - e^{-\mu\tau_0}\right) + p_0\theta a s^{\eta}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta}\right) \tag{52}$$

By the definition of η_2, from (41) and the fact that $s^{\eta_2}_{\tau_0} = 0$, we have

$$p_0\theta a s^{\eta_2}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta_2}\right) = c\left(1-p_0\right)\left(1 - e^{-\mu\delta}\right)e^{-\mu\tau_0} \tag{53}$$

Writing (52) for $\eta = \eta_2$, and using (53) to substitute $p_0\theta a s^{\eta_2}_{\tau_0+\delta}\left(\delta - \frac{\delta^2}{2\eta_2}\right)$, we have

$$\theta a s^{\eta_2}_{\tau+\delta}\left(\delta - \frac{\delta^2}{2\eta_2}\right) \leq p_0\theta a\left[\left(s^{\eta_2}_{\tau} + s^{\eta_2}_{\tau+\delta} - s^{\eta_2}_{\tau_0} - s^{\eta_2}_{\tau_0+\delta}\right)\frac{\eta_2}{2}\right] + c\left(1-p_0\right)\left[e^{-\mu\tau} - e^{-\mu(\tau_0+\delta)}\right]$$

From (51) and $\tau \geq \tau_0 + \delta$, this implies that $s^{\eta_2}_{\tau+\delta} < 0$. Since $s^{\eta}_{\tau+\delta}$ is increasing in η on (η_1, η_2), it follows that $s^{\eta}_{\tau+\delta} < 0$ for all $\eta \in (\eta_1, \eta_2)$. This provides the contradiction and completes the proof of the uniqueness of the equilibrium in simple strategies on (η_1, η_2).

The argument of the uniqueness of the equilibrium in simple strategies for $\eta \in (\eta_{2k}, \eta_{2k+1})$ with $k \in \mathbb{N}$ is identical with the argument presented above for $\eta \in (\widehat{\eta}, \eta_1)$, while the argument for $\eta \in (\eta_{2k+1}, \eta_{2k+2})$ is identical to the one presented for $\eta \in (\eta_1, \eta_2)$. The analysis for the case (b) of proposition 12.2 is similar. This completes the proof of corollary 14. ■

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