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20 August 2012

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Abstract

In this present work I shall define the basic notions of *superpositional filtrations*. Given a superposition integral I shall find a general measure theory by means of cylinder sets and then I shall define the properties of the ^{s}filtration for a general process X.

§1. We denote by $\mathcal{S}_n \equiv \mathcal{S}(\mathbb{R}^n)$ the Schwartz space endowed with a natural topology $(\mathcal{S}_\tau)$ which stems from the seminorm

$$||\varphi||_{\alpha,\beta} \equiv \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty.$$

We call this kind of space $\mathcal{S}$-space, and it is a Fréchet space [FrB]. We denote by $\mathcal{S}'_n \equiv \mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. It is the topological dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_\tau)$ [FrT]. A locally integrable function can be a tempered distribution if, for some constants a and C

$$\int_{|x|\leq G} |f(x)|dx \leq aG^C, \quad G\to\infty$$

and thus

$$\int_{\mathbb{R}^n} |f(x)\varphi(x)|dx < \infty, \quad \forall\varphi\in\mathcal{S}'.$$

Therefore we can say that $\int_{\mathbb{R}^n} f(x)\varphi(x)dx$ is a tempered distribution [FrT].
We know that the superposition integral [FrB] runs through $\mathcal{S}'$-spaces, rather given $a \in \mathcal{S}'_m$ and a summable $v \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}')$, the superposition integral of v on $\mathcal{S}'_m(\mathbb{R}^m)$ is the element of $\mathcal{S}'_n$ defined by

$$\lim_{k\to+\infty}(\alpha_k \circ \hat{v}) = \int_{\mathbb{R}^m} av = a \circ \hat{v}$$

2010 *Mathematics Subject Classification*: Primary 28C05, 28C15 ; Secondary 28A05.
Key words and phrases: Radon measures, measures on topological spaces, filters, Superposition integral.

where $\hat{v}$ is an operator generated from an $\mathcal{S}$-family v [CfS] defined on the $\mathcal{S}$-space, thus the operator is

$$\hat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi).$$

If the reader is not acquaintend with these notions it is suggested to has a glance, for example, at [CfS] , [CfD] or [FrT], [FrB], [CfF] et cetera.
In order to define a filtration on $\mathcal{S}'$-spaces, the construction of a σ-algebra is our foremost step. Given that $\mathcal{S}(\mathbb{R}^n)$ is a nuclear Fréchet space, it is known that by a cylinder set we can define a measure in $\mathcal{S}'(\mathbb{R}^n)$.
Given any fixed elements $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$, it is of utmost importance to notice that to each element $F \in \mathcal{S}'$ correspons the point $((F, \varphi_1), \cdots, (F, \varphi_n))$ in $\mathbb{R}^n$, that is to say, the elements $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$ define a mapping $F \to ((F, \varphi_1), \cdots, (F, \varphi_n))$ of $\mathcal{S}' \in \mathbb{R}^n$ [Gel]. Now we can build a partition in $\mathcal{S}'$, thus we have to decompose $\mathcal{S}'$ into cosets. It is important because, by the mapping we have seen, those functionals belonging to the same coset have the same value in $\mathbb{R}^n$. We say that two functionals F_1 and F_2 belong to the same coset iff

$$(F_1, \varphi_k) = (F_2, \varphi_k), \quad 1 \leq k \leq n. \tag{0.1}$$

Following [Gel], the decomposition of $\mathcal{S}'$ is due to the subspace $\Psi^o \subset \mathcal{S}'$. In particular, in Ψ^o, are defined those functionals carried to zero by the mapping $F \to ((F, \varphi_k), \cdots, (F, \varphi_k))$. Rather, given the (0.1), in Ψ^o it is defined the difference

$$(F_1 - F_2, \varphi_k) = 0, \quad 1 \leq k \leq n. \tag{0.2}$$

The (0.2) states that two functionals lie in the same coset iff their difference is belonged to Ψ^o (that is to say, is zero). Thus, considering the equation $(F, \varpi) = 0$, where $F \in \mathcal{S}'$ and $\varpi \in \Psi \subset \mathcal{S}$, the Ψ^o can be called the *annihilator* of the subspace Ψ [Gel]. Now we can define the quotient (factor) space $\mathcal{S}'/\Psi^o$ whose elements are cosets, and choosing a subset $A \subset \mathcal{S}'/\Psi^o$ we have a collection of $F \in \mathcal{S}'$ carried into elements of A by the mapping $\mathcal{S}' \to \mathcal{S}'/\Psi^o$, therefore we obtain the *cylinder set Z with base A and generating subspace* Ψ^o [Gel].
It is very interesting to notice that the cyilinder sets $Z_1, Z_2, \cdots$ form an *algebra*. Rather

- the complement of any cylinder set is a cylinder set;
- the intersection of two cylinder sets $Z_1 \cap Z_2$ is a cylinder set;
- the sum of any cylinder sets is a cylinder set.

§2. We have seen that we could obtain an algebra given by cylinder sets which stem from the weak-topology (or weak-$\star$topology) of our spaces. This relation gives rise a radon measure in $\mathcal{S}'$. Thus we have to define the quadruple $(\Omega, \mathscr{F}, \tau, \mu)$ where Ω is the space of tempered distribution, $\mathscr{F}$ are the Borel sets of Ω and τ the topology on Ω such that $\tau \subset \mu$, where μ is

our measure.
Given the cylinder sets Z we are interested to whose Borel bases lying in $\mathcal{S}'/\Psi^o$, thus having the same generating subspace Ψ^o, the union $\cup_{n=1}^{\infty} Z_n$ and the intersection $\cap_{n=1}^{\infty} Z_n$ of Z's cylinder sets are still cylinder sets with Borel bases. A cylinder set measure $\mu(Z)$ in $\mathcal{S}'$ is a scalar-valued function defined on the family of all cylinder sets with Borel bases endowed with these properties [Gel]:

- $0 \leq \mu(Z) \leq 1$ for all Z;
- $\mu(\mathcal{S}') = 1$, called the *normalization*;
- given the cylindrical sets Z as the union of $Z_1, Z_2, \cdots$ of pairwise disjoint cylinder sets having Borel bases and a common generating subspace Ψ^o, then

  (0.3) $$\mu(Z) = \sum_{n=1}^{\infty} \mu(Z_n)$$

- for any cylinder sets Z with Borel bases

  (0.4) $$\mu(Z) = \inf \mu(W)$$

  where W runs through all open cylinder sets containing Z.

Now consider a Borel set $A \subset \mathcal{S}'/\Psi^o$ which is Caratheodory regular

$$\gamma_\Psi(A) = \inf_W \gamma_\Psi(W),$$

it is linked to the cylinder sets Z with base A and generating subspace Ψ^o in this fashion:

(0.5) $$\gamma_\Psi(A) = \mu(Z).$$

Given the additivity of γ_Ψ on $\mathcal{S}'/\Psi^o$, if $Z_1, \cdots, Z_n$ are pairwise disjoint cylinder sets in $\mathcal{S}'$, it follows the property:

(0.6) $$\mu\left(\bigcup_{k=1}^{n} Z_k\right) = \sum_{k=1}^{n} \mu(Z_k).$$

It is worthwhile noticing that the countable additivity of μ is not always fulfilled, albeit (0.6) holds when Z_k stems from the same generating subspace Ψ^o. Alas a deep study on this fascinating field would go beyond our paper-aims.
We define the σ-algebra of sets generated by the cylinder sets as the smallest class of sets which contain the cylinder sets and is closed under the operations of complementation and countable union. Therefore the member of this σ-algebra are the Borel sets in $\mathcal{S}'$. Therefore given two transfinite numbers $\alpha > \beta$ we suppose that we have already defined the Borel sets of class β

and we want to definte those of class α. Thus, following [Gel], we call Borel sets of class α all countable unions of nonintersecting sets of class $< \alpha$ and all complements of such unions, thus

$$\mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k$$

then

$$\mu(\mathscr{B}) = \sum_{k=1}^{\infty} \mu(\mathscr{B}_k)$$

and

$$\mu(\mathcal{S}' - \mathscr{B}) = 1 - \mu(\mathscr{B}).$$

Now, consider two decompositions of the Borel set $\mathscr{B}$

$$\mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k \quad \text{and} \quad \mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}'_k$$

into pairwise disjoint Borel sets of lower classes, then it follows that we always obtain the same value of $\mu(\mathscr{B})$. Rather

$$\mathscr{B} = \bigcup_{k=1}^{\infty} Z_k = \bigcup_{k=1}^{\infty} Z'_k$$

therefore

$$\mathscr{B} = \mathcal{S}' - \bigcup_{k=1}^{\infty} Z'_k = \bigcup_{k=1}^{\infty} Z_k$$

thus the decomposition of $\mathcal{S}'$ into pairwise disjoint cylinder sets

$$\left(\bigcup_{k=1}^{\infty} Z_k\right) \cup \left(\bigcup_{k=1}^{\infty} Z'_k\right) = \mathcal{S}'$$

hence

$$\sum_{k=1}^{\infty} \mu(Z_k) = 1 - \sum_{k=1}^{\infty} \mu(Z'_k).$$

It follows that the necessary and sufficient condition so as to fulfill the countably additivity of μ on $\mathcal{S}'$ is that

$$\sum_{k=1}^{\infty} \mu(Z_k) = 1$$

for any decomposition of $\mathcal{S}' = \cup_{k=1}^{\infty} Z_k$ into pairwise cylinder sets [Gel].

§3. We have defined the σ-algebra on $\mathcal{S}'$-spaces and the way by which one are able to find their Radon measure. Now I have to define the core concept of this work: the *filtration*. Thus, given the space Ω, as defined in §2, and the σ-algebra $\mathscr{F}$, we obtain the measurable space $(\Omega, \mathscr{F})$. We can define a *filtration* as an increasing family $(\mathscr{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathscr{F}$. It is intended the sub-σ-algebra $\mathscr{F}_{t1}$ and $\mathscr{F}_{t2} \subset \mathscr{F}_{t1}$ where $t_2 < t_1$. Given this filtration we call the space $(\Omega, \mathscr{F})$ a filtered space. It is thus important to find a definition of these sub-σ-algebras, because it is of utmost importance to obtain a right definition of the *onward*-behaved algebras so as to define a stochastic process. If $\mathscr{F}$ is our σ-algebra it dovetails those $\mathscr{B}$'s we have defined. Therefore

$$\mathscr{F}_\infty = \sigma\left(\bigcup_{k\geq 0}^{\infty} \mathscr{F}_k\right) = \mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k = \bigcup_{k=1}^{\infty} Z_k.$$

Thus for the Borel set $\mathscr{B}$ we have Borel bases A_k and generating subspaces Ψ_k^o which depend on the basis A_∞ and the generating subspace Ψ_∞^o that belong to $\mathscr{F}_\infty = \mathscr{B}$. Now it is worthwile noticing that $A_k \subset A_\infty$ and $\Psi_k^o \subset \Psi_\infty^o$. Therefore we can obtain $\mathscr{F}_\infty$ by them, in few words it is made of them. We say that $A_k \subset \mathcal{S}'/\Psi_k^o$ are Borel bases and that they fulfill the property of countabily additivity, $\mathscr{F}_k = \mathscr{B}_k$. It is plain that $\mathscr{F}_k$ is embedded in A_∞ and Ψ_∞^o. Thus, given that Ψ_k^o belongs to $\mathscr{F}_k$ and that $\Psi_k^o \subset \Psi_\infty^o$, every coset of Ψ_k^o is belonged to some cosets of Ψ_∞^o. Furthermore, associating these cosets from Ψ_k^o to Ψ_∞^o, we obtain the linear mapping T_∞ of the factor space $\mathcal{S}'/\Psi_k^o$ onto the factor space $\mathcal{S}'/\Psi_\infty^o$. Now, we denote the inverse image of A_∞ under the mapping T_∞ by $T_\infty^{-1}(A_\infty)$. Given that, it is plain how $\mathscr{F}_\infty$ can be generated by the generating space Ψ_k^o and the bases $T_\infty^{-1}(A_\infty)$ and thus the family $(\mathscr{F}_k)_{k\geq 0}$ of sub-σ-algebras. It is worthwhile noticing that $\mathscr{F}_\infty = \mathscr{F}_k \cap \mathscr{F}_{k-1}$.

We have defined the mechanism of sub-σ-algebras for a filtration on an $\mathcal{S}'$-space. Therefore given a filtration $\mathscr{F}_k$ one can associate two other filtrations [CMB]:

$$\mathscr{F}_{k^-} = \bigvee_{s<k} \mathscr{F}_s \quad \text{and} \quad \mathscr{F}_{k^+} = \bigcap_{s>k} \mathscr{F}_s \tag{0.7}$$

then, $\bigvee_{k\geq 0} \mathscr{F}_{k^-} = \bigvee_{k\geq 0} \mathscr{F}_k = \bigvee_{k\geq 0} \mathscr{F}_{k^+} = \mathscr{F}_\infty$, thus we always have $\mathscr{F}_{k^-} \subset \mathscr{F}_k \subset \mathscr{F}_{k^+}$. A process X on $(\Omega, \mathscr{F})$ is called *adapted* to the filtration $(\mathscr{F}_k)$ if X_k is $\mathscr{F}_k$-measurable for each k [CMB]. If $A_k \subset \mathcal{S}'/\Psi_k^o$ is a Borel basis $\mathscr{B}_k$, it follows that a process X can be *progressively* measurable w.r.t. the filtration $\mathscr{F}_t$. In particular, given a tempered distribution a and a Borel basis A_k with generating space Ψ_k, we have

$$\{(k, a); 0 \leq k \leq t, a \in \Omega, X_k(a) \in A_k\}$$

that means

$$(k, a) \mapsto X_k(a) : ([0, t] \times \Omega, \mathscr{B}([0, t]) \otimes \mathscr{F}_t) \to \mathscr{B}[A \subset \mathcal{S}'/\Psi^o] \tag{0.8}$$

thus (0.8) is measurable for each $t \geq 0$.

§4. We have defined the basic notions of the filtration in $\mathcal{S}'$-spaces. It is a first step in order to reach the definition of the *stochastic superposition integral*, rather this kind of filtration can be employed in ^{s}martingales and thus in superposition integrals. We have defined a general ^{s}measure theory by means of cylindrical sets where the Dirac measure is a particular one. Therefore in a more general view a superposition integral can be thought of as a superposition between $a \in \Omega$ and $Z_k : \mathcal{S}' \to A \subset \mathcal{S}'/\Psi^o$

$$\int_A aZ_k$$

thus it is defined a superposition integral under cylinder sets and so we obtain a general Radon measure, hence we are able to build the measurable space $(\Omega, \mathscr{F})$ up. Furthermore, given the admissibility of sub-σ-algerbas in $\mathcal{S}'$-spaces, it is possible to define a filtration $(\mathscr{F}_k)$ so as to define a filtered space.

Given the property of continuity and boundness of the elements belonged to $\mathcal{S}'$-spaces [FrT], it is very important to define the properties of this fascinating superposition integral in order to work in a well-behaved space when we deals with probability and chiefly stochastic processes. Rather their are based upon the theory of Brownian Motion and that of the Wiener integral. These tools are very ill-behaved and become so difficult in managing with them. We do need a new well-behaved tool as strong as those that are ill-behaved, above and foremost in quantitative financial puzzles.

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