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## Two Equations on the Pareto-Efficient Sharing of Real GDP Risk

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## Two Equations on the Pareto-Efficient Sharing of Real GDP Risk

By David Eagle and Lars Christensen

This paper focuses on some theoretical findings concerning the optimal sharing of one of the most important systematic<sup>1</sup> risks related to recessions – the risk concerning the level of real GDP (RGDP) in the economy. Other recession risks, such as the systematic component of unemployment risk, are closely correlated with RGDP risk. Theoretically, in pure exchange economies with no storage or financial capital, RGDP risk is the only systematic risk. Even in our actual economies with storage and capital, RGDP risk is still the primary source of systematic risk (See Chen, Roll, and Ross, 1986). Hence, many economists such as Athanasoulis and Shiller (2001) have been concerned about RGDP risk and how that risk should be shared. Several economists (e.g., Shiller, 2002, and Koenig, 2011) have found certain results that occur for specific classes of utility functions, in particular CARA utility functions. The two equations discussed in this paper show that these results are due, not to a particular utility function per se, but rather on how one individual's relative risk aversion compares to another individual's relative risk aversion.

How economic agents share risk have many applications in economics and finance including asset pricing, insurance, and monetary economics. Futures, options, swaps and other financial derivatives have been created supposedly to facilitate the sharing of risk. Prior to the 2008 Financial Crisis, some economists (e.g., Kumar, 2007) have gone so far to say that these financial derivatives have helped complete markets in the sense of Arrow-Debreu. However, most economic agents were impacted by the recession that followed the 2008 Financial Crisis

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<sup>1</sup> We use the term “systematic risk” as used in financial economics to represent the risk that cannot be diversified away. Other terms for systematic risk are undiversifiable risk and market risk.

despite the existence of these financial derivatives. In fact, some of these financial derivatives contributed to the 2008 Financial Crisis (Financial Crisis Inquiry Commission, 2011).

By diversifying over a large base of customers, private insurance contracts handle much of the economy's idiosyncratic risk (also known as diversifiable risk, unsystematic risk or unique risk). However, for complete markets to occur in the sense of Arrow-Debreu, economic agents also need to be able to transfer systematic risks among themselves. Many would consider futures, options, swaps and other financial derivatives to be appropriate vehicles for transferring these systematic risks.

The aggregate risks related to recessions would be systematic because the whole economy is affected, which means you cannot eliminate this risk by diversifying from within that economy.<sup>2</sup> However, as Shiller (1993) and Athanasoulis and Shiller (2001) note, rather than transferring risk associated with recessions, financial derivative securities have focused on more specific risks such as equity-market risks or interest-rate risks. Most of those using these derivatives to transfer risk are professional investors, banks, farmers or other businesses. Other than unemployment insurance, which is usually handled by government agencies, very few of these derivatives are used directly by consumers to transfer their risks related to recessions.

The importance of how to share RGDP risk is now even more important given the renewed interest by monetary economists in replacing interest-rate targeting with nominal GDP targeting. As Koenig (2011) points out, an important issue concerning whether NGDP targeting is optimal or not depends on how different groups (such as borrowers and debtors) should share in RGDP risk.

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<sup>2</sup> One could reduce some RGDP risk through international diversifying as suggested by Shiller (1995). However, for close economies or for the whole world economy, no further diversifying is possible. The two equations presented in this paper are derived in the context of a closed economy.

The ultimate economic-theoretical basis for how risk should be shared is the Arrow-Debreu economy with securities that address every possible contingency, a situation economists label as “complete markets.” An Arrow-Debreu economy with state-contingent securities is Pareto efficient because it does have complete markets. Therefore, this paper uses such an economy to derive two equations that relate an individual’s coefficient of relative risk aversion to how that individual will share RGDP risk under complete markets, i.e. when the economy is Pareto efficient. The first equation is a slight extension of a formula derived by Domian and Eagle (2005). The second equation is a generalization and reinterpretation of Koenig’s (2011) formula (4). Both equations produce the same result, but have slightly different interpretations.

The purposes of the main body of this paper are (i) to present these two formulas, (ii) to discuss the interpretations of these formulas, (iii) to discuss how the formulas are similar yet different, and (iv) to apply these formulas to the issue of Nominal GDP targeting. We relegate the derivations of these formula to the appendix.

### **I. Domian and Eagle’s RGDP-sharing Formula:**

The first formula for the Pareto-efficient sharing of RGDP risk is a slight extension of the formula from Domian and Eagle (2005). This formula is:

$$\frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \frac{1}{\tilde{\alpha}_{jt}} \frac{dY_t}{Y_t} \tag{1}$$

where  $\tilde{\alpha}_{jt}$  is how individual  $j$ ’s coefficient of relative risk aversion compares to the weighted average coefficient of relative risk aversion over all consumers. The  $Y_t$  in equation (1) represents the aggregate output (RGDP) at time  $t$ , and  $\tilde{c}_{jt}(Y_t)$  is how  $j$ ’s Pareto efficient consumption depends on RGDP (Note: this is not individual  $j$ ’s consumption function; this is not the

relationship between  $j$ 's consumption and  $j$ 's income; it is the relationship between  $j$ 's Pareto-efficient consumption and *aggregate* output as measured by RGDP). The “ $\sim$ ” sign over a variable indicates that this function is a function of RGDP.

The left side of equation (1) is individual  $j$ 's percent change in consumption, whereas the term  $\frac{dY_t}{Y_t}$  is the percent change in RGDP. We derive equation (1) in the appendix. One of the purposes of this section is to help readers interpret and utilize the equation. We will do so with examples of three individuals E, A, and B respectively having average, above average, and below average risk aversion. For individual E with average relative risk aversion,  $\tilde{\alpha}_{Et} = 1$  and equation (1) states that E's Pareto-efficient consumption will change by 1% whenever RGDP changes by 1%.

Assume individual A's relative risk aversion is twice the average relative risk aversion, i.e.,  $\tilde{\alpha}_{At} = 2$ ; then equation (1) states that A's Pareto-efficient consumption will change by half a percent whenever RGDP changes by 1%. Next, assume individual B's relative risk aversion is half of the average relative risk aversion, i.e.,  $\tilde{\alpha}_{Bt} = \frac{1}{2}$ ; equation (1) states that B's Pareto-efficient consumption will change by 2% whenever RGDP changes by 1%. In essence, in an Arrow-Debreu economy, those individuals in the economy with above average relative risk aversion transfer some of their RGDP risk to those individuals with less-than-average relative risk aversion.

## **II. Koenig's RGDP-sharing Formula:**

The second formula for the Pareto-efficient sharing of RGDP risk is a generalization of Koenig's (2011) formula. This formula is:

$$\frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \tilde{\Phi}_{jt} \frac{dY_t}{Y_t} \quad (2)$$

where  $\tilde{\Phi}_{jt}$  is how  $j$ 's relative risk tolerance compares to the average relative risk tolerance. We are following the precedence in some financial-economic literature (e.g., Alexander, 2008; Duffe, 2001; Shefrin, 2001) that an individual's relative risk tolerance is defined as the reciprocal of that individual's coefficient of relative risk aversion.

Again, we will use examples involving three individuals W, X, and Z, respectively having average, above average, and below average relative risk tolerance. First, consider individual W with average relative risk tolerance, i.e.,  $\Phi_{Xt} = 1$ . Then, according to (2), W's Pareto-efficient consumption will change by 1% when RGDP changes by 1%.

Next, assume individual X has a relative risk tolerance that is twice the average relative risk tolerance. Then X's Pareto-efficient consumption will change by 2% when RGDP changes by 1%. On the other hand, if Z's relative risk tolerance is half the average relative risk tolerance, then when RGDP changes by 1%, X's Pareto-efficient consumption will change by only half a percent. In a Pareto-efficient economy like an Arrow-Debreu economy, individuals with less-than-average relative risk tolerance will transfer some of their RGDP risk to individuals with above-average relative risk tolerance.

### **III. The Similarities but Technical Differences Between the Two Equations:**

Since one's relative risk tolerance is the inverse of one's relative risk aversion, at first glance equations (1) and (2) appear very similar. While similar, there are some technical differences. Equation (1) is a little easier to interpret because people are more familiar with the term "risk aversion" than "risk tolerance." However, equation (2) is actually technically more correct.

In equation (2), the weighted average relative risk tolerance is computed in a natural way:

$$\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{1}{\tilde{\Psi}_{jt}} \quad (3)$$

where  $\tilde{\Psi}_{jt}$  is individual  $j$ 's relative risk aversion and therefore  $1/\tilde{\Psi}_{jt}$  is  $j$ 's relative risk tolerance.

The natural weights to use in computing the weighted average of everyone's relative risk tolerances are the ratios of each individual's consumption to RGDP.

However, for equation (1), the weighted average of the relative risk aversion is computed as follows:

$$\bar{\Psi}_t \equiv \sum_{j=1}^m \frac{d\tilde{c}_{jt}}{dY_t} \cdot \tilde{\Psi}_{jt} \quad (4)$$

Instead of using the natural weights of one's consumption to RGDP, this weighted average uses the weights of how one's consumption changes when RGDP changes. The weighted average in (4) is a legitimate weighted average, but it is not the natural weighted average.

Because equation (2) is based on a weighted average relative risk tolerance that uses natural weights whereas equation (1) is based on a weighted average relative risk aversion that uses unnatural weights, we conclude that equation (2) is technically superior to equation (1). On the other hand, equation (1) may be more meaningful for many who are more familiar with the term "risk aversion" than "risk tolerance."

While these two equations do differ, they do produce the same result. Therefore, it follows that the coefficient  $\tilde{\Phi}_{jt} = 1/\tilde{\alpha}_{jt}$  where  $\tilde{\Phi}_{jt}$  is how  $j$ 's relative risk tolerance compares to the weighted average relative risk tolerance and  $\tilde{\alpha}_{jt}$  is how  $j$ 's relative risk aversions compares to the weighted average relative risk aversion. However, we should keep in mind that the weighted



average risk tolerance is the natural weighted average, whereas the weighted average relative risk aversion is a little convoluted.

Even if we want to speak in terms of relative risk aversion rather than relative risk tolerance, there is a way we can use Koenig's (2011) derivation for the basis for equation (1). We can do so thinking about the average relative risk aversion as equation (3), which can be interpreted as the weighted harmonic average of the individual relative risk aversion coefficients (See Wikipedia, 2012).

#### **IV. Applying these Equation to Borrowers and Lenders and Nominal GDP Targeting:**

In this section, we illustrate an example of how one can apply equations (1) and (2). We do so by studying the impacts of how different central-bank targets impact the RGDP sharing between borrowers and lenders, or more generally the payer and receiver of any prearranged nominal amount. We will use equation (1) because of the greater understanding of the term "risk aversion," but to be more technically correct; we will interpret  $\tilde{\alpha}_j$  as being the ratio of  $j$ 's coefficient of relative risk aversion to the weighted harmonic average of the relative risk aversion coefficients as given by (3). Hence, our interpretation of (1) is now tied to Koenig's (2011) derivation of equation (2).

Assume a borrower owes a prearranged nominal  $X_t$  payment to a lender at time  $t$ . We consider four cases:

Case 1: Both the borrower and lender have average relative risk aversion

Case 2: Both the borrower and lender have the same relative risk average which may be higher or lower than average.

Case 3: The borrower has above average relative risk aversion, and the lender has below average relative risk aversion.

Case 4: The borrower has below average relative risk aversion, whereas the lender has above average relative risk aversion.

In our analysis, we assume that, other than the  $X_t$  nominal payment that the borrower must pay the lender, the net real incomes of both the borrower and the lender are proportional to RGDP. One way this could happen is if these other incomes and expenses are nominal and the central bank successfully targets NGDP. Also, realize that as long as the population does not change, then the average real income must be proportional to RGDP, because RGDP is not just real aggregate supply; it is also real aggregate income (average real income = RGDP/m where m is the number of individuals).

Let us begin with case 1, where both the borrower and the lender have average relative aversion, which mean that  $\tilde{\alpha}_j = 1$ . Equation (1) imply that the consumption of both the borrower and the lender should change by 1% when RGDP changes by 1% (i.e., their consumptions must be proportional to RGDP). When a 1% change in RGDP leads to a 1% in the borrower's and lender's other net real incomes, then their consumption will change by 1% if and only if the real value of the loan payment changes by 1%. In other words, where  $x_t \equiv X_t / P_t$  is the real value of this loan payment, Pareto efficiency requires the following condition holds when both the borrower and the lender have average relative risk aversion:

$$\frac{dx_t}{x_t} = \frac{dY_t}{Y_t} \tag{5}$$

How the real value of that loan payment changes when RGDP changes, however, depends on what the central bank targets. If the central bank successfully targets the price level or the inflation rate, the price level will not change when RGDP changes; therefore, the real value of the nominal loan payment will not change; therefore equation (1) will not be satisfied under inflation targeting or price-level targeting.

To see this in a more formal, mathematical, yet relatively simple sense, remember the equation of exchange (also called the quantity equation). which states that  $M_t V_t = N_t = P_t Y_t$  where  $M_t$  is the money supply,  $V_t$  is income velocity,  $N_t$  is nominal aggregate spending as measured by Nominal GDP (NGDP),  $P_t$  is the price level, and  $Y_t$  is RGDP. Concentrating on the  $N = PY$  part of this equation, solve for  $P_t$  to get  $P_t = N_t / Y_t$ . This tells us that there are two and only two direct determinants of the price level: (i) nominal aggregate spending as measured by NGDP and (ii) aggregate output as measured by RGDP.

Next take the logarithms of both sides of  $P_t = N_t / Y_t$  and totally differentiate to get:

$$\frac{dP_t}{P_t} = \frac{dN_t}{N_t} - \frac{dY_t}{Y_t} \quad (6)$$

Now take logarithms of both sides of the definition of the real payment,  $x_t \equiv X_t / P_t$ , and totally differentiate, treating  $X_t$  as a constant to get:

$$\frac{dx_t}{x_t} = -\frac{dP_t}{P_t} = \frac{dY_t}{Y_t} - \frac{dN_t}{N_t} \quad (7)$$

Remember that when both the borrower and lender have average relative risk aversion, equation (1) holds if and only if  $\frac{dx_t}{x_t} = \frac{dY_t}{Y_t}$ . However, under successful price-level targeting or inflation targeting, the central bank will change NGDP to offset any changes in RGDP so that the price level will not change. In other words,  $\frac{dP_t}{P_t} = \frac{dN_t}{N_t} - \frac{dY_t}{Y_t} = 0$ . However, this makes the real value of the nominal payment not change when RGDP changes.

Now consider successful NGDP targeting, which means nominal GDP will not change (i.e.,  $dN_t / N_t = 0$ ). Substituting this into (7) gives (5), which means equation (1) holds. In conclusion, successful NGDP targeting will result in the Pareto-efficient sharing of RGDP risk

between borrowers and lenders when both the borrowers and lenders have average relative risk aversion.

Now consider case 2, where both the borrower and the lender have the same level of relative risk aversion, which may be below or above average. Then equation (1) still requires that the borrower's and lender's consumption change by the same percentage. However, when the central bank targets the price level or inflation and the real value of the loan payment is unchanged. When RGDP drops by 1%, the borrower's consumption will have to drop by more than 1% because the borrower's other real net income will drop by 1%, but the real value of the borrower's payment to the lender will be unaffected. Similarly, the lender's consumption will drop less than 1%. Therefore, successful price-level targeting or inflation targeting will not lead to efficient sharing of RGDP risk even when both the borrower and the lender have the same relative risk aversion that is below or above average.

Now consider Case 3, where the borrower has above average relative risk aversion and the lender has below average relative risk aversion. Then by equation (1) a 1% change in RGDP should cause the borrower's consumption to change by less than 1%, and the lender's consumption to change by more than 1%. If the central bank successfully targeted the price level or inflation, then the borrower would be paying the lender a constant real payment to the lender. Therefore, when RGDP changes by 1%, the borrower's consumption would change by more than 1% and the lender's consumption will change by less than 1%, the opposite of equation (1) requires. Therefore, again successful price-level or inflation targeting works against the efficient sharing of RGDP risk.

Finally, consider Case 4 where the borrower with below average risk aversion and the lender who has above average risk aversion. When RGDP changes by 1%, then equation (1)

states that the borrower's consumption should change by more than 1%, whereas the lender's consumption should change by less than 1%. That is what would happen under successful price-level and inflation targeting.

In conclusion, for price-level or inflation targeting to be consistent with optimal RGDP-risk sharing, the borrowers must have less-than-average relative risk aversion and borrowers must have more than average relative risk aversion. For example, it would mean families with children that borrow money to buy a home with a mortgage have less risk aversion than an older couple who is in essence providing the funds for this mortgage. In other words, these families with children are more than happy to live more meager life styles during recessions or depressions (and more extravagant life styles during good times) in order that the older couple's consumption is less disturbed by the recession or depression.

Whether borrowers overall are more or less risk averse than the ultimate lenders is an empirical question. In the meantime, we feel the appropriate assumption when addressing what the central bank should target is to assume that both borrowers and lenders have average relative risk aversion, in which case targeting NGDP, not the price level or inflation, will lead to Pareto-efficient RGDP-risk sharing.

## V. Conclusion and Reflections

This paper presents and discusses two equations concerning the Pareto-efficient sharing of RGDP risk. The equations demonstrate how the Pareto-efficient sharing of RGDP risk depends on one's relative risk aversion, with individuals having above-average relative risk aversion transferring some of their RGDP risk to individuals having below-average relative risk aversion. While equation (1) is easier to understand because it is in terms of the well-known

concept of “risk aversion,” equation (2) which is in terms of “risk tolerances” is actually more technically correct, although if we base (1) on the weighted harmonic mean of relative risk aversion coefficients, then (1) will be as technically correct as (2), although then (1) will be based on Koenig’s (2011) derivation rather than Eagle and Domian’s (2005) derivation.

This paper applied these equations to assess different central bank targets affects the sharing of RGDP risk. In particular, this paper finds that NGDP targeting does result in Pareto-efficient sharing of RGDP risk when both the borrower and lender have average relative risk aversion.

Some economists have noticed special results (such as the Pareto-efficiency of NGDP targeting) occurring when they assume identical constant relative risk averse utility functions. However, this paper shows that those special results are actually due to how an individual’s relative risk aversion is related to average relative risk aversion. In particular, when we assume identical CRRA utility functions, we guarantee that everyone will have the same (and hence average) relative risk aversion.

The equations derived in the appendix and presented in this paper could prove useful in the development of new financial derivative markets such as in Athanasoulis, Stefano G. and Robert J. Shiller (2001), Shiller (2003), and Eagle (2005).

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## APPENDIX

Each individual  $j$  has the following utility function:

$$U_{j0}(c_{j0}) + \sum_{t=1}^T \beta^t \sum_{i=1}^{n_t} \lambda_{it} U_{jt}(c_{jit}) \quad (\text{A1})$$

where  $\beta$  is the time discount factor in common to all consumers,  $U_{j0}(\cdot)$  is individual  $j$ 's utility function at time 0,  $U_{jt}(\cdot)$  is  $j$ 's utility function at time  $t$ ,  $c_{j0}$  is individual  $j$ 's consumption at time 0,  $c_{jit}$  is  $j$ 's consumption in state  $i$  at time  $t$ ,  $T$  is the finite horizon to this economy,  $n_t$  is the number of states of nature at time  $t$ , and  $\lambda_{it}$  is the probability of state  $i$  occurring at time  $t$ . Since different consumers can have different utility functions at different times, the common time discount factor is not really a restrictive assumption.<sup>3</sup>

Consumers maximize (A1) subject to the constraints that  $c_{j0} + \sum_{t=1}^T \sum_{i=1}^{n_t} \lambda_{it} \Omega_{it} x_{jit} = y_{j0}$  and  $c_{jit} = y_{jit} + x_{jit}$  for all states  $i$  and future time periods  $t$ , where  $y_{j0}$  and  $y_{jit}$  are  $j$ 's real income (endowment) at time 0 and in state  $i$  at time 0 respectively. The symbol  $x_{jit}$  represents  $j$ 's demand for the state-contingent security that pays one consumption unit at time  $t$  if and only if state  $i$  occurs. Also,  $\Omega_{it}$  is the pricing kernel of the state-contingent security that delivers one consumption unit at time  $t$  if and only if state  $i$  occurs. Note the actual price of the state-contingent security equals  $\lambda_{it} \Omega_{it}$ . The reason economists working with state-contingent

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<sup>3</sup> Suppose for complete generality the discount factor varies by consumer, by state, and by time; that instead of  $\beta^t U_{jt}(c_{jit})$ , consumer  $j$ 's utility is  $\beta_{jit}^t \tilde{U}_{jt}(c_{jit})$  where  $\tilde{U}$  denotes the true utility function. If we set our beta equal to one and defined  $U_{jt}(c_{jit}) \equiv \beta_{jit}^t \tilde{U}_{jt}(c_{jit})$ , our formulation would take this situation into account. We chose to leave a constant time preference factor in our formulation to make it consistent with our including a time preference factor for the EOM universe.

securities break up the state-contingent security price into the probability times the pricing kernel is because the pricing kernel is unaffected by the different probabilities.

A necessary condition for this optimization problem to be satisfied for all consumers is

that for all  $i, j$ , and  $t$ ,  $U'_{j0}(c_{j0}) = \frac{\beta^t U'_{jt}(c_{jit})}{\Omega_{it}}$ , which implies

$$\frac{U'_{jt}(c_{jit})}{U'_{j0}(c_{j0})} = \frac{\Omega_{it}}{\beta^t} \text{ for all } i, j, \text{ and } t \quad (\text{A2})$$

Since  $j$ 's Pareto-efficient consumption is a function solely of aggregate supply,<sup>4</sup> define the implicit function  $\tilde{c}_{jt}(Y_t)$  to be how the Pareto-Efficient consumption by individual  $j$  at time  $t$  depends on aggregate supply.<sup>5</sup> It is extremely important to recognize  $\tilde{c}_{jt}(Y_t)$  as a reduced form; it is not the structural consumption function. To help us avoid this confusion, we refer to  $Y_t$  as aggregate supply at time  $t$ , not income.

Since equation (2) is true for all  $j$ ,

$$\frac{U'_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} = \frac{U'_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} \quad (\text{A3})$$

for  $j=2..m$ .

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<sup>4</sup> To see that  $j$ 's Pareto-efficient consumption allocation is solely a function of aggregate supply, let state 1 and state 2 be any two states where aggregate supply are the same. State 2 could still differ from state 1 because of a different distribution of endowments or different probabilities. Set  $\Omega_{2t} = \Omega_{1t}$ . If  $c_{j2t} = c_{j1t}$  for all  $j$ , then if (A3) holds for all  $j$  when  $i=1$  then it also holds for all  $j$  when  $i=2$ . Also, if  $c_{j2t} = c_{j1t}$  for all  $j$ , then if markets clear for state 1 then they clear for state 2. Therefore, if  $\tilde{c}_{jt}$  for  $j=1..m$  is the optimal consumption for one state, it is also the optimal consumption for another state with the same level of aggregate supply. Therefore, the competitive-equilibrium consumption allocation in an Arrow-Debreu economy is a function solely of aggregate supply.

<sup>5</sup> There is not just one Pareto-efficient consumption allocation, but rather a continuum of such allocations, each corresponding to a particular allocation of endowments across states. We can think about this Pareto-efficient consumption allocation as the one that corresponds to the existing allocation of endowments.

## Derivation of Generalization of Koenig's Equation of RGDP Sharing

Totally differentiating (3) gives  $\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} d\tilde{c}_{jt} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} d\tilde{c}_{1t}$ . If we divide the left and

right sides of this by the left and right sides of (A3) respectively, we get:

$$\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})} d\tilde{c}_{jt} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})} d\tilde{c}_{1t} \quad (\text{A4})$$

Next, multiply the left and right sides of (4) by  $-\tilde{c}_{jt}/\tilde{c}_{jt}$  and  $-\tilde{c}_{1t}/\tilde{c}_{1t}$  respectively and then rearrange slightly to get:

$$\frac{-\tilde{c}_{jt} U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})} \frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \frac{-\tilde{c}_{1t} U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})} \frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} \quad (\text{A5})$$

Let  $\Psi_{jt}$  be individual  $j$ 's coefficient of relative risk aversion. Therefore, since

$\tilde{\Psi}_{jt}(Y_{jt}) \equiv \frac{-\tilde{c}_{jt} U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})}$  and  $\tilde{\Psi}_{1t}(Y_{1t}) \equiv \frac{-\tilde{c}_{1t} U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})}$ , we can rewrite (A5) as:

$$\Psi_{jt} \frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \Psi_{1t} \frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} \quad (\text{A6})$$

Now solve (A6) for  $\frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}}$  to get:

$$\frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \frac{\tilde{\Psi}_{1t}}{\tilde{\Psi}_{jt}} \frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} \quad (\text{A7})$$

Now, take the equilibrium condition,  $\sum_{j=1}^m \tilde{c}_{jt} = Y_t$ , and totally differentiate to get  $\sum_{j=1}^m d\tilde{c}_{jt} = dY_t$ .

Now divide both sides by  $Y_t$  to get  $\sum_{j=1}^m \frac{d\tilde{c}_{jt}}{Y_t} = \frac{dY_t}{Y_t}$ . Finally, for each term within the summation

sign, multiply by  $\frac{\tilde{c}_{jt}}{\tilde{c}_{jt}}$  and rearrange slightly to get:

$$\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \frac{dY_t}{Y_t} \quad (\text{A8})$$

Next, substitute (A7) into (A8) to get:

$$\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{\tilde{\Psi}_{1t}}{\tilde{\Psi}_{jt}} \frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} = \frac{dY_t}{Y_t} \quad (\text{A9})$$

Solving for  $\frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}}$ , we get

$$\frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} = \frac{\frac{dY_t}{Y_t}}{\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{\tilde{\Psi}_{1t}}{\tilde{\Psi}_{jt}}} \quad (\text{A10})$$

Now, multiply both sides by  $1/\tilde{\Psi}_{1t}$  to get:

$$\frac{d\tilde{c}_{1t}}{\tilde{c}_{1t}} = \frac{1}{\tilde{\Psi}_{1t}} \frac{dY_t}{Y_t} \frac{1}{\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{1}{\tilde{\Psi}_{jt}}} \quad (\text{A11})$$

Now, substitute this into (A7) and simplify to get:

$$\frac{d\tilde{c}_{jt}}{\tilde{c}_{jt}} = \frac{1}{\tilde{\Psi}_{jt}} \frac{dY_t}{Y_t} \frac{1}{\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{1}{\tilde{\Psi}_{jt}}} \quad (\text{A12})$$

This equation is this paper's generalization of Koenig's equation (4) and is one of the two equations of RGDP risk sharing. The reciprocal of one's relative risk aversion is one's relative risk tolerance. Therefore, the term  $\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{1}{\tilde{\Psi}_{jt}}$  is the weighted average of all consumers' relative risk tolerances using the ratio of their consumption to RGDP as the weights.

To make this even simpler, define  $\tilde{\Phi}_{jt} \equiv \frac{1}{\sum_{j=1}^m \frac{\tilde{c}_{jt}}{Y_t} \frac{1}{\tilde{\Psi}_{jt}}}$ , which means how j's relative risk

tolerance compares to the average relative risk tolerance.

Using this definition of relative risk tolerance, we can rewrite (A12) as equation (2), which we present in the main body of the paper:

### **Derivation of Domian-Eagle's Equation of RGDP Sharing:**

Since equation (A3) is true for all j,

$$\frac{U'_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} = \frac{U'_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} \quad (\text{A13})$$

for  $j=2..m$ . Totally differentiating (A13) with respect to  $Y_t$  gives  $\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{j0}(c_{j0})} \frac{d\tilde{c}_{jt}}{dY_t} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1,0}(c_{1,0})} \frac{d\tilde{c}_{1t}}{dY_t}$ .

If we divide the left and right sides of this by the left and rights sides of (15) respectively, we get:

$$\frac{U''_{jt}(\tilde{c}_{jt})}{U'_{jt}(\tilde{c}_{jt})} \frac{d\tilde{c}_{jt}}{dY_t} = \frac{U''_{1t}(\tilde{c}_{1t})}{U'_{1t}(\tilde{c}_{1t})} \frac{d\tilde{c}_{1t}}{dY_t} \quad (\text{A14})$$

Define  $\tilde{a}_{jt}(Y_t) \equiv -\frac{U''_{jt}(\tilde{c}_{jt}(Y_t))}{U'_{jt}(\tilde{c}_{jt}(Y_t))}$ , which is  $j$ 's coefficient of relative risk aversion at time  $t$  as a

function aggregate supply. Multiplying both sides of (A14) by a minus sign and rearranging slightly gives:

$$\frac{\frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}} = \frac{\tilde{a}_{1t}}{\tilde{a}_{jt}} \quad (\text{A15})$$

By summing both sides of (A15) over all consumers, we get:

$$\frac{\sum_{j=1}^m \frac{d\tilde{c}_{jt}}{dY_t}}{\frac{d\tilde{c}_{1t}}{dY_t}} = \tilde{a}_{1t} \sum_{j=1}^m \frac{1}{\tilde{a}_{jt}} \quad (\text{A16})$$

By equilibrium in the market for the consumption good at time  $t$ ,  $\sum_{j=1}^m c_{jt} = Y_t$ , which also implies

that  $\sum_{j=1}^m \frac{dc_{jt}}{dY_t} = 1$ . Therefore, solving (A16) for  $\frac{d\tilde{c}_{1t}}{dY_t}$  gives  $\frac{d\tilde{c}_{1t}}{dY_t} = \frac{1/\tilde{a}_{1t}}{\sum_{j=1}^m \frac{1}{\tilde{a}_{jt}}}$ . This and (A15) imply

that the following is true for all  $j$ .

$$\frac{d\tilde{c}_{jt}}{dY_t} = \frac{1/\tilde{a}_{jt}}{\sum_{j=1}^m \frac{1}{\tilde{a}_{jt}}} \quad (\text{A17})$$

This result was first derived by Wilson (1968, see his theorem 5).

Define the function  $\tilde{\Psi}_j(Y_t)$  to be individual  $j$ 's coefficient of relative risk aversion as a function of aggregate supply.  $\tilde{\Psi}_j(Y_t) = -\tilde{c}_{jt}(Y_t)\tilde{a}_{jt}(Y_t)$ . Next, we need to determine the value of  $\bar{\Psi}_t(Y_t)$ , which is the weighted average of all individuals' coefficients of relative risk aversion using the  $\frac{d\tilde{c}_{jt}}{dY_t}$  for each  $j$  as weights. The following starts out with the definition of  $\bar{\Psi}_t(Y_t)$ , then

substitutes in the definition of  $\tilde{\Psi}_j(Y_t)$  and (A17):

$$\bar{\Psi}_t(Y_t) \equiv \sum_{j=1}^m \left( \tilde{\Psi}_j(Y_t) \cdot \frac{d\tilde{c}_{jt}}{dY_t} \right) = \sum_{j=1}^m \left( \tilde{c}_{jt}\tilde{a}_{jt} \cdot \frac{1/\tilde{a}_{jt}}{\sum_{k=1}^m \frac{1}{\tilde{a}_{kt}}} \right) = \frac{\sum_{j=1}^m \tilde{c}_{jt}}{\sum_{k=1}^m \frac{1}{\tilde{a}_{kt}}}$$

However, the sum of consumption across all consumers in this pure exchange economy equals aggregate supply for that period. Therefore,

$$\bar{\Psi}_t(Y_t) = \frac{Y_t}{\sum_{j=1}^m \tilde{a}_{jt}} \quad (\text{A18})$$

Define  $\tilde{\alpha}_j(Y_t) \equiv \frac{\tilde{\Psi}_{jt}(Y_t)}{\bar{\Psi}_t(Y_t)}$ , we can write  $\tilde{\Psi}_{jt} = \tilde{\alpha}_j \bar{\Psi}_t$  and then replace  $\tilde{\Psi}_{jt}$  with  $\tilde{c}_{jt} \tilde{a}_{jt}$  and  $\bar{\Psi}_t$

with (A18) to get  $\tilde{c}_{jt} \tilde{a}_{jt} = \tilde{\alpha}_j \frac{Y_t}{\sum_{j=1}^m \tilde{a}_{jt}}$ . Dividing both sides by  $\tilde{a}_{jt} Y_t$  gives  $\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_j \frac{1}{\sum_{j=1}^m \tilde{a}_{jt}}$ .

Using (A17), we can rewrite this as:  $\frac{\tilde{c}_{jt}}{Y_t} = \tilde{\alpha}_j \frac{d\tilde{c}_{jt}}{dY_t}$ . Dividing both sides by  $\tilde{\alpha}_j$  gives:

$$\tilde{c}'_{jt}(Y_t) = \frac{1}{\tilde{\alpha}_j} \frac{\tilde{c}_{jt}}{Y_t} \quad (\text{A19})$$

Recognize that another way to write  $\tilde{c}'_{jt}(Y_t)$  is  $\frac{dc_{jt}}{dY_t}$ , therefore another way to write (A19) is:

$$\frac{d\tilde{c}_{jt}}{dY_t} = \frac{1}{\tilde{\alpha}_j} \frac{\tilde{c}_{jt}}{Y_t} \quad (\text{A20})$$

If we multiply both sides by  $dY_t/\tilde{c}_{jt}$ , we get equation (1) that we present in the main body of the paper.