Multiutility representations for incomplete difference preorders

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Abstract

A *difference preorder* is a (possibly incomplete) preorder on a space of state *changes* (rather than the states themselves); it encodes information about preference intensity, in addition to ordinal preferences. We find necessary and sufficient conditions for a difference preorder to be representable by a family of cardinal utility functions which take values in linearly ordered abelian groups. This has applications to interpersonal comparisons, social welfare, and decisions under uncertainty.

Keywords: Preference intensity; cardinal utility; linearly ordered abelian group; social welfare; uncertainty.
JEL class: D81, D60

1 Introduction

Let $\mathcal{X}$ be a set of states or alternatives available to some agent (either an individual or a group). Suppose that, for at least *some* states $x_1, x_2, y_1, y_2 \in \mathcal{X}$, it is possible to make the judgement:

"The net benefit in changing from state x_1 to state x_2 is greater than the net benefit in changing from state y_1 to y_2." (1)

For example, one interpretation of this statement would be that a $(\frac{1}{2}, \frac{1}{2})$ lottery between states y_1 and x_2 is preferable to a $(\frac{1}{2}, \frac{1}{2})$ lottery between x_1 and y_2. Another interpretation might be that, in a two-period intertemporal decision (with no discounting), the history (x_2, y_1) is preferable to the history (x_1, y_2).

Statements like (1) arise frequently in welfare economics. For example, let $\mathcal{X}$ be the set of all possible "personal states" which any person could experience at a moment in time. Suppose that an element $x \in \mathcal{X}$ encodes all the factors which could influence a person's wellbeing or happiness; this may include both psychological factors (e.g. beliefs, values, desires, personality, memories, etc.) and physical factors (e.g. health, physical location, consumption bundle, etc.). Then statement (1) becomes "The net gain in wellbeing in changing from personal state x_1 to personal state x_2 is greater than the net gain in wellbeing in changing from state y_1 to y_2." Such judgements involve interpersonal comparisons of wellbeing (because x_1 and y_1 might describe different people), so they may not always be possible. Nevertheless, we may suppose that at least *some* interpersonal comparisons are possible. For example, it seems obvious that giving a bowl of rice to a starving man will cause a greater increase in wellbeing than giving the same bowl of rice to a well-nourished man who has just eaten a feast.

Alternately, $\mathcal{X}$ could be the set of all possible states for an entire society. Suppose an element of $\mathcal{X}$ encodes enough information about the personal states of all individuals in that society to allow us to compute overall social welfare. Thus, statement (1) becomes "The net gain in social welfare in changing from social state x_1 to social state x_2 is greater than the net gain in social welfare in changing from state y_1 to y_2." This is often a difficult ethical judgement, involving tradeoffs between the interests of different individuals (and hence, interpersonal comparisons); thus, such judgements are not always possible. Nevertheless, at least *some* ethical judgements are possible. For example, most ethical systems would agree that it is better to choose a policy which slightly harms 5% of the population and greatly benefits the other 95%, rather than a policy which greatly harms 95% of the population and slightly benefits the other 5%. Interpersonal comparisons and social welfare judgements of the form (1) are considered in (Pivato, 2012b).

Another application is decision-making under uncertainty. For example, let $\mathcal{I}$ be a set of possible "states of nature", and let $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{I}}$; each element of $\mathcal{X}$ represents a "prospect", which will yield a real-valued payoff (say, of money or cardinal utility) in each state of nature. Suppose we are endowed with the pair of prospects (x_1, y_1) (either representing a $(\frac{1}{2}, \frac{1}{2})$ lottery between the two prospects, or representing a sequence of two prospects in two consecutive time periods), and we have the option to either change x_1 to x_2, or change y_1 to y_2 (but not both). Then statement (1) becomes "It would be better to change prospect x_1 to x_2, rather than change y_1 to y_2." Presumably this judgement arises from some beliefs about the likelihoods of the different elements of $\mathcal{I}$. If we had a well-defined subjective probability distribution on $\mathcal{I}$, and we were expected-utility maximizers, then statement (1) would be equivalent to "$EU(x_2) - EU(x_1) \geq EU(y_2) - EU(y_1)$" (where EU represents expected utility). However, in a situation of genuine ambiguity, we may not have such a subjective probability distribution. Nevertheless, we may still be able to estimate the *approximate* likelihoods of certain events (i.e. certain subsets of $\mathcal{I}$), and this will allow us to make judgements like (1) in at least some cases. For example, we would prefer to exchange prospects in a way which has a very high likelihood of greatly increasing our payoff, and otherwise only a small likelihood of slightly reducing it, rather than exchange prospects in a way which has a very high likelihood of greatly reducing our payoff, and otherwise only a small likelihood of slightly increasing it.

A fourth application is to multiattribute decision-making. Suppose each alternative in $\mathcal{X}$ is a bundle of many "attributes" (e.g. the consumption of different goods, perhaps at different moments in time). Clearly, statement (1) would be true if the change from x_1 to x_2 yielded a greater improvement in *every* attribute than the change from y_1 to y_2. However, in most cases, the change from x_1 to x_2 will be more beneficial for some attributes, while the change from y_1 to y_2 will be better for other attributes; in these situations, a judgement like (1) will be difficult to make. But we might still agree with (1) if, for example, the change from x_1 to x_2 involved a much larger gain, in a much larger number of attributes, than the change from y_1 to y_2.

The four previous applications all had a normative or prescriptive flavour; they asked the question, "How *should* an agent choose between different state transitions?" However, in descriptive applications, the question becomes, "How *would* the agent choose between different state transitions?" Evidently, an experimental subject's revealed preferences are *complete*, for the simple reason that the experimental protocol typically asks her to make a *choice* whenever she is confronted with a pair of alternatives. However, it is well-documented that these experimentally revealed preferences often violate transitivity (Camerer, 1995; Rabin, 1998). Presumably, these intransitivities appear in the "hard" cases, when the agent confronts a great deal of uncertainty or complexity, or a multitude of attributes, or some ethical dilemma. But the agent's revealed preferences presumably *would* be transitive if we confined our attention to the "easy" cases, where one alternative is clearly better than another. This would yield an incomplete but transitive *subrelation* of the agent's revealed preference relation. (Mandler (2005) makes a similar argument.) The model in this paper can be interpreted as a model of this transitive subrelation; the axioms presented below then become hypotheses which can be empirically tested.

We can represent judgements like (1) with a preorder $(\succeq)$ on the Cartesian product $\mathcal{X} \times \mathcal{X}$. (A ***preorder*** is a binary relation which is transitive and reflexive, but possibly incomplete.) We will write an ordered pair $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ as "$x_1 \rightsquigarrow x_2$" to emphasize that it represents a *change* from x_1 to x_2. Then statement (1) is represented by the formula "$(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$". The preorder $(\succeq)$ must satisfy three consistency conditions:

(INV) For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, if $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_2 \rightsquigarrow x_1) \preceq (y_2 \rightsquigarrow y_1)$.

(CAT) For all x_0, x_1, x_2 and $y_0, y_1, y_2 \in \mathcal{X}$, if $(x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$.

(CAT*) For all x_0, x_1, x_2 and $y_0, y_1, y_2 \in \mathcal{X}$, if $(x_0 \rightsquigarrow x_1) \succeq (y_1 \rightsquigarrow y_2)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_1)$, then $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$.

Condition (INV) ("Inversion") says that if one change is better than another, then the *reversal* of the first change is *worse* than the reversal of the second. Condition (CAT) ("Concatenation") prevents "concatenation inconsistencies", where the concatenation of two apparently superior small changes yields an inferior large change. Condition (CAT*) says that the logic of (CAT) is commutative: when aggregating the net gain of two state changes, the order doesn't matter. A preorder on $\mathcal{X} \times \mathcal{X}$ satisfying conditions (INV), (CAT), and (CAT*) will be called a ***difference preorder*** on $\mathcal{X}$.

The difference preorder $(\succeq)$ induces an (incomplete) preorder $(\underset{o}{\succeq})$ on $\mathcal{X}$, by setting $y \underset{o}{\succeq} x$ if and only if $(x \rightsquigarrow y) \succeq (x \rightsquigarrow x)$. The preorder $(\underset{o}{\succeq})$ encodes "ordinal" judgements about the relative preferability of various states in $\mathcal{X}$. However, axiom (INV) implies that $(\succeq)$ cares more about state *changes* than about the states themselves. Axioms (CAT) and (CAT*) imply that our preferences are not merely ordinal; they must have some weakly "cardinal" structure, so that the comparison of state changes can be made in a consistent fashion. However, since $(\succeq)$ is generally an *incomplete* preorder, axioms (INV), (CAT), and (CAT*) are do *not* imply the existence of a cardinal utility representation, as the following examples illustrate.

Example 1.1. (a) Let $N \geq 1$ be an integer, let $\mathcal{X} \subseteq \mathbb{R}^N$ be any subset, and for all $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathcal{X}$, define $(\mathbf{x} \rightsquigarrow \mathbf{x}') \succeq (\mathbf{y} \rightsquigarrow \mathbf{y}')$ if and only if $x'_n - x_n \geq y'_n - y_n$ for all $n \in [1 \ldots N]$. Then $(\succeq)$ is a difference preorder.

Interpretation: Suppose the N coordinates of $\mathbb{R}^N$ represent N different, incommensurable goods, such that it is impossible for us to judge the desirability of tradeoffs between one good and another. Thus, the change $(\mathbf{x} \rightsquigarrow \mathbf{x}')$ is as good as $(\mathbf{y} \rightsquigarrow \mathbf{y}')$ if and only if it yields at least as great an improvement in *every* one of the N goods.

(b) More abstractly, let $\mathcal{X}$ be any set, and let $\mathcal{V}$ be a (possibly infinite) set of real-valued functions on $\mathcal{X}$. For any $x_1, x_2, y_1, y_2 \in \mathcal{X}$, define $(x_1 \rightsquigarrow x_2) \underset{\mathcal{V}}{\succeq} (y_1 \rightsquigarrow y_2)$ if and only if $v(x_2) - v(x_1) \geq v(y_2) - v(y_1)$ for all $v \in \mathcal{V}$. Then $(\underset{\mathcal{V}}{\succeq})$ is a difference preorder.

(c) Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a real-valued "utility function", and let $\epsilon > 0$. For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, define $(x_1 \rightsquigarrow x_2) \underset{u,\epsilon}{\succeq} (y_1 \rightsquigarrow y_2)$ if and only if either $u(x_2) - u(x_1) > u(y_2) - u(y_1) + 4\epsilon$, or $x_1 = y_1$ and $x_2 = y_2$. Then $(\underset{u,\epsilon}{\succeq})$ is a difference preorder.

Interpretation: Suppose we can measure the utility of each state in $\mathcal{X}$ using u. Thus, the change $(x_1 \rightsquigarrow x_2)$ is better than $(y_1 \rightsquigarrow y_2)$ if and only if it yields a greater utility gain. However, our utility measurements are subject to an error of size at most ϵ. Thus, if $x_1 \neq y_1$ or $x_2 \neq y_2$, then we can only be *sure* that $(x_1 \rightsquigarrow x_2)$ yields a greater utility gain than $(y_1 \rightsquigarrow y_2)$ if $u(x_2) - u(x_1) > u(y_2) - u(y_1) + 4\epsilon$.

(d) Let $\mathcal{X} \subseteq \mathbb{R}^N$. Let "$>_L$" be the lexicographical order on $\mathbb{R}^N$. That is, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we have $\mathbf{x} >_L \mathbf{y}$ if there exists some $n \in [1 \ldots N]$ such that $x_1 = y_1$, $x_2 = y_2$, $\ldots$, and $x_{n-1} = y_{n-1}$, but $x_n > y_n$. Now, for all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^2, \mathbf{y}^2 \in \mathcal{X}$, define $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{L}{\succeq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if and only if $\mathbf{x}^1 - \mathbf{x}^2 >_L \mathbf{y}^1 - \mathbf{y}^2$. Then $(\underset{L}{\succeq})$ is a *complete* difference preorder on $\mathcal{X}$.

Interpretation: Each of the N coordinates of $\mathbb{R}^N$ represents a different good, over which we have cardinal preferences. However, if $n < m$, then good n is "infinitely more important" than good m, so we are willing to sacrifice an arbitrarily large amount of good m to obtain even a slight increase in good n. ◊

The collection $\mathcal{V}$ in Example 1.1(b) is an example of a *multiutility representation* for a difference preorder. Pivato (2012b) uses such multiutility representations for a difference preorder representing interpersonal comparison of wellbeing to define and axiomatically characterize a class of "quasiutilitarian" difference preorders for making social welfare comparisons. The main result of this paper provides a necessary and sufficient conditions for the existence of such a multiutility representation, via a richness condition called *solvability* and a consistency condition called *divisibility*.

Prior literature. Alt (1936, 1971) was the first to derive a cardinal utility representation from some structure of comparisons over state-transitions.[1] The literature since Alt contains cardinal utility representations for several classes of preorders on $\mathcal{X} \times \mathcal{X}$ which are very similar in philosophical content to the structures we call "difference preorders" in this paper, even if they differ in their precise axiomatizations. See in particular Suppes and Winet (1955, §5), Davidson and Marschak (1956), Scott and Suppes (1958, pp.121-122), Debreu (1958), Suppes and Zinnes (1963), Kristof (1967), Pfanzagl (1968, Ch.9), Krantz et al (1971, Theorem 4.2), Doignon and Falmagne (1974), Shapley (1975), Basu (1982), and Wakker (1988, 1989). Recently, Köbberling (2006) has proved the most general result of this type, and given an excellent survey of the earlier literature.

This paper departs from this previous literature in three ways. First, it considers *incomplete* preorders on $\mathcal{X} \times \mathcal{X}$, whereas earlier literature all assumed completeness. Second, and relatedly, this paper constructs a *multi*utility representation, whereas the earlier literature was exclusively concerned representations with a single utility function. Third, this paper allows utility functions which range over arbitrary linearly ordered abelian groups, whereas earlier literature considered only *real*-valued utility functions, which are usually obtained by imposing some sort of Archimidean or continuity condition on $(\succeq)$.

There is also an extensive literature on real-valued multiutility representations for *ordinary* preorders,[2] including Levin (1983), Sprumont (2001), Ok (2002), Mandler (2006), Knoblauch (2006), Kaminski (2007), Yılmaz (2008), and Evren and Ok (2011). However, these papers are only concerned with representing ordinal information, rather than a cardinal structure, so they use quite different methods to the aforementioned literature on difference preorders. But like that literature, these papers are all concerned with *real*-valued multiutility representations; this imposes constraints (e.g. separability) on the preorder. If we were willing to work with arbitrary linearly ordered abelian groups, these constraints would vanish. (Indeed, it is relatively easy to prove the that *any* preorder has an $\mathcal{R}$-valued multiutility representation, for some linearly ordered abelian group $\mathcal{R}$.)

The remainder of the paper is organized as follows. Section 2 sets up and states our main representation result, Theorem 2.1. Section 3 discusses the existence of "strong" utility functions for a difference preorder. Section 4 discusses the analogs of the Szpilrajn Lemma and Dushnik-Miller theorems for difference preorders, and shows by counterexample that they are not true in general. All proofs are in the appendix.

2 Model and main result

A ***linearly ordered abelian group*** is a structure $(\mathcal{R}, +, 0, >)$, where $\mathcal{R}$ is a nonempty set, "+" is an ***abelian group*** operation on $\mathcal{R}$ with identity element 0 (i.e. "+" is a binary operation on $\mathcal{R}$ which is associative, commutative, and invertible), and ">" is a ***linear order*** on $\mathcal{R}$ (i.e. a complete, antisymmetric, transitive binary relation) which is ***homogeneous***, meaning that for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$.

[1] See also (Camacho, 1980, §3) for a summary of Alt's model.

[2] A real-valued multiutility representation for a preorder $(\succeq)$ on a set $\mathcal{X}$ is a collection $\mathcal{U}$ of $\mathbb{R}$-valued utility functions on $\mathcal{X}$ such that, for all $x, y \in \mathcal{X}$, we have $x \succeq y$ if and only if $u(x) \geq u(y)$ for all $u \in \mathcal{U}$.

For example: the set $\mathbb{R}$ of real numbers is a linearly ordered abelian group (with the standard ordering and addition operator). So is any subgroup of $\mathbb{R}$ (e.g. the group $\mathbb{Q}$ of rational numbers). For any integer $N \geq 1$, the space $\mathbb{R}^N$ is a linearly ordered abelian group under vector addition and the lexicographic order "$>_L$" from Example 1.1(d). From an economic perspective, a linearly ordered abelian group is the minimum amount of mathematical structure needed to define some sort of "cardinal" utility function.

Let $\mathcal{X}$ be a set, and let $(\succeq)$ be a binary relation on $\mathcal{X} \times \mathcal{X}$. A *weak utility function* for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ (for some linearly ordered abelian group $\mathcal{R}$) such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, we have

$$\Big((x_1 \rightsquigarrow x_2) \ \succeq \ (y_1 \rightsquigarrow y_2)\Big) \quad \Longrightarrow \quad \Big(u(x_2) - u(x_1) \ \geq \ u(y_2) - u(y_1)\Big). \tag{2}$$

For example, in Example 1.1(a), any of the N coordinate projections from $\mathbb{R}^N$ to $\mathbb{R}$ is a weak utility function for $(\succeq)$. In Example 1.1(b); any element of $\mathcal{V}$ is a weak utility function for $(\underset{\mathcal{V}}{\succeq})$. In Example 1.1(d), the projection onto the *first* coordinate is a weak utility function for $(\underset{L}{\succeq})$, but the projections onto the remaining $(N-1)$ coordinates are *not* weak utility functions.

If the "$\Longrightarrow$" in statement (2) were replaced by "$\Longleftrightarrow$", then $(\succeq)$ would be a *complete* difference preorder, and u would be a cardinal utility function of the kind found in the prior literature summarized in section 1. But in general this is not the case.

There are at least three reasons for allowing utility functions to range over arbitrary linearly ordered abelian groups, rather than restricting them to the real numbers. First, at a technical level, this significantly extends the generality of our results, and simplifies the proofs. (For instance, it allows us to handle cases like Example 1.1(d).) Second, at a philosophical level, it allows for "non-Archimidean" or "lexicographical" preferences, where some desires are given infinite priority over other desires. (We do not take a descriptive or normative stance on whether agents can or should have such preferences, but nor do we wish to exclude them *a priori*.) Finally: non-real-valued utility functions sometimes arise in the setting of infinite-horizon intertemporal choice and choice under uncertainty (Pivato, 2012a).

Multiutility representations. A binary relation $(\succeq)$ on $\mathcal{X} \times \mathcal{X}$ has a *multiutility representation* if there is some collection $\mathcal{U}$ of weak utility functions for $(\succeq)$ such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

$$\Big((x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)\Big) \iff \Big(u(x_2) - u(x_1) \geq u(y_2) - u(y_1), \ \text{for all } u \in \mathcal{U}\Big). \tag{3}$$

For instance, in Example 1.1(a), the set of all N coordinate projections together yields a multiutility representation for $(\succeq)$. In Example 1.1(b), the set $\mathcal{V}$ yields a multiutility representation for $(\underset{\mathcal{V}}{\succeq})$. In Example 1.1(c), define $\mathcal{U} := \{u + \zeta; \quad \zeta : \mathcal{X} \longrightarrow [-\epsilon, \epsilon]$ any function$\}$. Then $\mathcal{U}$ yields a multiutility representation for $(\underset{u,\epsilon}{\succeq})$.

It is easy to see that any binary relation $(\succeq)$ which admits a multiutility representation (3) is a difference preorder (i.e. it satisfies (INV), (CAT), and (CAT*)). However, there exist many difference preorders *without* multiutility representations, as we shall see in sections 3 and 4 below. We will now introduce some conditions on $(\mathcal{X}, \succeq)$ which are necessary and sufficient to obtain a multiutility representation.

A *cardinal utility representation* is a multiutility representation (3) with $|\mathcal{U}| = 1$. In this case, $(\succeq)$ is necessarily a *complete* difference preorder. For instance, in Example 1.1(d), let $\mathcal{R} := \mathbb{R}^N$ with the lexicographical order $(>_L)$, and let $u : \mathcal{X} \longrightarrow \mathcal{R}$ be the identity map. Then u is a cardinal utility representation for $(\underset{L}{\succeq})$. However, it is easy to see $(\underset{L}{\succeq})$ does not admit any *real*-valued cardinal utility representation. (Indeed, $(\underset{L}{\succeq})$ does not even admit a real-valued *multi*utility representation.) The prior literature has essentially been concerned with the question: *When does a complete difference preorder have a (real-valued) cardinal utility representation?* Instead, we will be concerned with the broader question: *When does (possibly incomplete) difference preorder admit a multiutility representation?*

Solvability. Let $(\approx)$ represent the symmetric part of $(\succeq)$. (That is: $(x \rightsquigarrow x') \approx (y \rightsquigarrow y')$ if both $(x \rightsquigarrow x') \succeq (y \rightsquigarrow y')$ and $(x \rightsquigarrow x') \preceq (y \rightsquigarrow y')$.) A difference preorder $(\succeq)$ is *solvable* if, for any $x_1, x_2, y_1 \in \mathcal{X}$, there exists $y_2 \in \mathcal{X}$ such that $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$. Solvability is a "richness" or "continuity" condition which is quite common in the literature summarized in section 1. For example, if $\mathcal{X} = \mathbb{R}^N$, then the difference preorders in Example 1.1(a,d) are both solvable. However, if $\mathcal{X} \subsetneq \mathbb{R}^N$, then these difference preorders are generally not solvable. By a similar argument, the difference preorder in Example 1.1(b) is generally not solvable (unless the collection $\mathcal{V}$ has certain nice algebraic properties). Finally, Example 1.1(c) is not solvable, for the simple reason that the indifference relation $(\underset{u,\epsilon}{\approx})$ is trivial in this case —there do not exist *any* distinct $x_0, x_1, y_0, y_1 \in \mathcal{X}$ such that $(x_0 \rightsquigarrow x_1) \underset{u,\epsilon}{\approx} (y_0 \rightsquigarrow y_1)$.

Divisibility. A *standard sequence* for $(\succeq)$ is a sequence $x_0, x_1, x_2, \ldots x_N \in \mathcal{X}$ such that $(x_0 \rightsquigarrow x_1) \approx (x_1 \rightsquigarrow x_2) \approx \cdots \approx (x_{N-1} \rightsquigarrow x_N)$. A difference preorder $(\succeq)$ is *divisible* if, for any such standard sequence, we have

$$\Big((x_0 \rightsquigarrow x_N) \succeq (x_0 \rightsquigarrow x_0) \Big) \quad \Longleftrightarrow \quad \Big((x_0 \rightsquigarrow x_1) \succeq (x_0 \rightsquigarrow x_0) \Big).$$

(The direction "$\Longleftarrow$" is always true, by inductive application of axiom (CAT); the real content of divisibility is in the "$\Longrightarrow$" direction.) For example, it is easy to see that any *complete* difference preorder (e.g. Example 1.1(d)) must be divisible. Also, the difference preorder in Example 1.1(a) is divisible. To see this, let $\{\mathbf{x}^m\}_{m=1}^M$ be a standard sequence in $\mathbb{R}^N$; then for all $n \in [1 \ldots N]$, the coordinate projection $\{x_n^m\}_{m=1}^M$ is an arithmetic progression in $\mathbb{R}$, so $x_n^M - x_n^0 = M \cdot (x_n^1 - x_n^0)$. If $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^M) \succeq (\mathbf{x}^0 \rightsquigarrow \mathbf{x}^0)$, then $x_n^M \geq x_n^0$ for all $n \in [1 \ldots N]$, and thus, $x_n^1 \geq x_n^0$ for all $n \in [1 \ldots N]$, and thus $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \succeq (\mathbf{x}^0 \rightsquigarrow \mathbf{x}^0)$. By a similar argument, Example 1.1(b) is divisible. (Construct arithmetic progressions using the utility functions in $\mathcal{V}$ rather than the coordinate projections.) Indeed, the same argument leads to the following observation:

Any binary relation on $\mathcal{X} \times \mathcal{X}$ with a multiutility representation like (3) *is a divisible difference preorder.*

More surprisingly, the difference preorder in Example 1.1(c) *also* divisible, for the trivial reason that it *has* no standard sequences, because the indifference relation $(\underset{u,\epsilon}{\approx})$ is trivial. (This illustrates a weakness of the definition of divisibility: it only has traction to the extent that $(\mathcal{X}, \succeq)$ exhibits long standard sequences; we will return to this issue below.)

Embeddings. Let $\mathcal{X}'$ be another set, let $(\succeq')$ be a binary relation on $\mathcal{X}' \times \mathcal{X}'$, and let $f : \mathcal{X} \longrightarrow \mathcal{X}'$ be a function. We say that f is an *embedding* of $(\mathcal{X}, \succeq)$ into $(\mathcal{X}', \succeq')$ if f is injective, and, for all $w, x, y, z \in \mathcal{X}$, we have

$$\Big((w \rightsquigarrow x) \succeq (y \rightsquigarrow z)\Big) \iff \Big((f(w) \rightsquigarrow f(x)) \succeq' (f(y) \rightsquigarrow f(z))\Big).$$

For example, if $\mathcal{X} \subseteq \mathcal{X}'$, and $\succeq$ is the restriction of $\succeq'$ to $\mathcal{X}$, then clearly $(\mathcal{X}, \succeq)$ can be embedded in $(\mathcal{X}', \succeq')$. If $(\succeq')$ is a difference preorder, then it is easy to see that $(\succeq)$ must also be a difference preorder. If $(\succeq')$ is divisible, then $(\succeq)$ must also be divisible. We now come to our main results:

Theorem 2.1 *Let $(\succeq)$ be a binary relation on $\mathcal{X} \times \mathcal{X}$. Then $(\succeq)$ admits a multiutility representation if and only if it can be embedded in a solvable, divisible difference preorder.*

If solvability is assumed, then this characterization takes a simpler form.

Corollary 2.2 *Let $(\succeq)$ be a solvable binary relation on $\mathcal{X} \times \mathcal{X}$. Then $(\succeq)$ admits a multiutility representation if and only if it is a divisible difference preorder.*

This yields a new contribution to the aforementioned literature on cardinal utility representations.

Corollary 2.3 *Let $(\succeq)$ be a solvable binary relation on $\mathcal{X} \times \mathcal{X}$. Then $(\succeq)$ admits a* cardinal *utility representation if and only if it is a* complete *difference preorder.*

Semisolvability and induction. Divisibility is necessary to obtain a multiutility representation, but solvability is not, as we now show. A difference preorder $(\succeq)$ is ***semisolvable*** if it can be embedded in a solvable difference preorder. For example, if $(\mathcal{X}', \succeq')$ is solvable, and $\mathcal{X} \subseteq \mathcal{X}'$, and $(\succeq)$ is the restriction of $(\succeq')$ to $\mathcal{X}$, then $(\succeq)$ is semisolvable. In particular, for any subset $\mathcal{X} \subseteq \mathbb{R}^N$, the difference preorder in Example 1.1(a) is semisolvable. Indeeed, Theorem 2.1 implies that any difference preorder with a multiutility representation is semisolvable. It would be nice if Corollary 2.2 was still true with "solvable" weakened to "semisolvable". But a semisolvable system could be an extremely small subset of a solvable system —so small that the hypothesis of divisibility would have no traction at all. Thus, we must add some auxiliary condition to ensure that divisibility still has bite.

We say that $(\mathcal{X}, \succeq)$ is ***inductive*** if, for any $x_0, x_1 \in \mathcal{X}$, there exists an infinite standard sequence $\{x_0\}_{n=1}^{\infty}$ in $\mathcal{X}$ which is "generated by" $(x_0 \rightsquigarrow x_1)$, in the sense that such that $(x_n \rightsquigarrow x_{n+1}) \approx (x_0 \rightsquigarrow x_1)$ for all $n \in \mathbb{N}$. For example, any solvable system is inductive. But Example 1.1(c) is clearly not inductive. Divisibility has its strongest grip in inductive systems.

Proposition 2.4 *Suppose $(\mathcal{X}, \succeq)$ is inductive. Then $(\mathcal{X}, \succeq)$ admits a multiutility representation if and only if it is divisible and semisolvable.*

3 Strong utility functions

Let $(\succeq)$ be a difference preorder, and let $(\succ)$ denote the antisymmetric part of $(\succeq)$. Let $(\mathcal{R}, +, 0, >)$ be a linearly ordered abelian group. A ***strong utility function*** for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ which satisfies condition (2), and also such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, we have

$$\Big((x_1 \rightsquigarrow x_2) \ \succ \ (y_1 \rightsquigarrow y_2)\Big) \quad \Longrightarrow \quad \Big(u(x_2) - u(x_1) \ > \ u(y_2) - u(y_1)\Big).$$

This is not as strong as a full multiutility representation, but it is clearly more powerful than a weak utility function.[3] Strong utility functions have useful consequences in welfare economics (Pivato, 2012b). Thus, it is desirable to find sufficient conditions for their existence. A difference preorder is ***semidivisible*** if, for any standard sequence $\{x_n\}_{n=1}^{N}$, we have

$$\Big((x_0 \rightsquigarrow x_N) \approx (x_0 \rightsquigarrow x_0)\Big) \quad \Longleftrightarrow \quad \Big((x_0 \rightsquigarrow x_1) \approx (x_0 \rightsquigarrow x_0)\Big).$$

For example, any divisible difference preorder is semidivisible.

Proposition 3.1 *If $(\succeq)$ can be embedded in a solvable, semidivisible difference preorder, then $(\succeq)$ has a strong utility function.*

Combining Proposition 3.1 with Theorem 2.1, we conclude that *any difference preorder with a multiutility representation has a strong utility function.*

To see that the scope of Proposition 3.1 is strictly greater than that of Theorem 2.1, it suffices to show that not every semidivisible difference preorder is divisible. On the other hand, to see that the semidivisibility hypothesis is not entirely vacuous, it suffices to show that not every difference preorder is semidivisible. The next two examples illustrate these claims.

Example 3.2. (a) Let $\mathcal{X} = \mathbb{Z}$ (the group of integers). For any $x, x', y, y' \in \mathcal{X}$, define $(x \rightsquigarrow x') \succeq (y \rightsquigarrow y')$ if $x' - x \geq y' - y$ *and* $x' - x + y - y'$ is an even number. It is easy to check that $(\succeq)$ is a semidivisible difference preorder. But it is not divisible. For

[3] For example, any constant function is trivially a weak utility function. But it can't be a strong utility function unless $(\succeq)$ is trivial.

example, $\{0,1,2\}$ is a standard sequence (because $(0 \rightsquigarrow 1) \approx (1 \rightsquigarrow 2)$), but $(0 \rightsquigarrow 1)$ is not comparable to $(0 \rightsquigarrow 0)$, whereas $(0 \rightsquigarrow 2) \succ (0 \rightsquigarrow 0)$.

(b) Let $\mathbb{Z}_{/3} := \{0,1,2\}$, and let $+_3$ be the operation of addition mod 3 on $\mathbb{Z}_{/3}$. Let $\mathcal{X} = \mathbb{Z} \times \mathbb{Z}_{/3}$, and write a generic element as $\mathbf{x} = (x_1, x_2)$. For any $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathcal{X}$, define $\mathbf{x} \oplus \mathbf{y} := (x_1 + y_1, x_2 +_3 y_2)$; then $(\mathcal{X}, \oplus)$ is an abelian group. For any $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathcal{X}$, define $(\mathbf{x} \rightsquigarrow \mathbf{x}') \succ (\mathbf{y} \rightsquigarrow \mathbf{y}')$ if $(x_1' - x_1) > (y_1' - y_1)$; otherwise, define $(\mathbf{x} \rightsquigarrow \mathbf{x}') \approx (\mathbf{y} \rightsquigarrow \mathbf{y}')$ if and only if $(\mathbf{x}' \ominus \mathbf{x}) = (\mathbf{y}' \ominus \mathbf{y})$. It is easy to verify that this is a difference preorder. But it is not even semidivisible. For example, $\{(0,0), (0,1), (0,2), (0,0)\}$ is a standard sequence, but the transition $(0,0) \rightsquigarrow (0,1)$ is not comparable to $(0,0) \rightsquigarrow (0,0)$. $\diamondsuit$

Because they are not divisible, neither of the difference preorders in Example 3.2 admits a multiutility representation; this shows that the divisibility hypothesis of Theorem 2.1 is not vacuous. However, in Example 3.2(b), suppose we define $u : \mathcal{X} \longrightarrow \mathbb{R}$ by setting $u(x_1, x_2) := x_1$. Then it is easy to check that u is a strong utility function for $(\succeq)$. This shows that the hypothesis of Proposition 3.1 is too strong: semidivisibility is not *necessary* to obtain a strong utility function. However, some sort of hypothesis is certainly necessary; not all difference preorders admit strong utility functions, as we will show in the next section.

4 Complete extensions

If $(\succ)$ and $(\succ')$ are two binary relations on a set $\mathcal{X}$, then we say $(\succ')$ *extends* $(\succ)$ if $(x \succ y) \Longrightarrow (x \succ' y)$, for all $x, y \in \mathcal{X}$. If $(\succeq)$ and $(\succeq')$ are two difference preorders on $\mathcal{X}$, then we will say that $(\succeq')$ *strictly extends* $(\succeq)$ if $(\succeq')$ extends $(\succeq)$, *and* the antisymmetric part of $(\succeq')$ extends the antisymmetric part of $(\succeq)$.

A *partial order* is a binary relation which is transitive and antisymmetric. A *linear order* is a transitive, antisymmetric, and complete. Szpilrajn's Lemma (1930) says that every partial order on a set can be extended to a linear order. Furthermore, a result of Dushnik and Miller (1941) says that every partial order is the intersection of all its linear extensions. By analogy, we will say that a difference preorder $(\succeq)$ is *Szpilrajn* if it is strictly extended by some complete difference preorder. We will say that $(\succeq)$ is *Dushnik-Miller* if it is the intersection of all the complete difference preorders which extend it.

These properties are closely related to the existence of strong utility functions and multiutility representations. To see this, let $\mathcal{R}$ be a linearly ordered abelian group, and let $u : \mathcal{X} \longrightarrow \mathcal{R}$ be any function. We can define a complete difference preorder $(\underset{u}{\succeq})$ on $\mathcal{X}$ as follows. For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, stipulate that

$$\Big((x_1 \rightsquigarrow x_2) \underset{u}{\succeq} (y_1 \rightsquigarrow y_2)\Big) \quad \Longleftrightarrow \quad \Big(u(x_2) - u(x_1) \geq u(y_2) - u(y_1)\Big). \tag{4}$$

If $(\succeq)$ is another difference preorder on $\mathcal{X}$, then u is a (strong) utility function for $(\succeq)$ if and only if $(\underset{u}{\succeq})$ (strictly) extends $(\succeq)$. Thus, the existence of a strong utility function implies that $(\succeq)$ is Szpilrajn. Furthermore, if $(\succeq)$ has a multiutility representation (3), then $(\succeq)$ is Dushnik-Miller.

Not all difference preorders are Dushnik-Miller. For example, let $\mathcal{X} := \{x_0, x_1, x_2, y_0, y_1, y_2\}$, and define the preorder $(\succeq)$ on $\mathcal{X} \times \mathcal{X}$ as follows. Begin with the 36 "trivial" relations of the form "$(x_h \rightsquigarrow x_h) \approx (x_i \rightsquigarrow x_i) \approx (y_j \rightsquigarrow y_j) \approx (y_k \rightsquigarrow y_k)$", for any $h, i, j, k \in \{0, 1, 2\}$. To this set, add the three relations

(a) $(x_0 \rightsquigarrow x_1) \approx (x_1 \rightsquigarrow x_2)$,

(b) $(y_0 \rightsquigarrow y_1) \approx (y_1 \rightsquigarrow y_2)$, and

(c) $(x_0 \rightsquigarrow x_2) \succ (y_0 \rightsquigarrow y_2)$,

along with their "reversals" under Axiom (INV). This yields a system of 42 relations, which is closed under the application of Axioms (CAT) and (CAT*). Thus, it is a difference preorder on $\mathcal{X}$. Note that $(\succeq)$ cannot compare $(x_0 \rightsquigarrow x_1)$ with $(y_0 \rightsquigarrow y_1)$. However, if $(\underset{c}{\succeq})$ is any complete difference preorder which extends $(\succeq)$, then condition (c) implies that $(x_0 \rightsquigarrow x_2) \underset{c}{\succeq} (y_0 \rightsquigarrow y_2)$. Then conditions (a) and (b) and the contrapositive of (CAT) imply that $(x_0 \rightsquigarrow x_1) \underset{c}{\succeq} (y_0 \rightsquigarrow y_1)$. Thus, if $(\underset{DM}{\succeq})$ is the intersection of all the complete difference preorder extensions of $(\succeq)$, then we must have $(x_0 \rightsquigarrow x_1) \underset{DM}{\succeq} (y_0 \rightsquigarrow y_1)$. Thus, $(\underset{DM}{\succeq}) \neq (\succeq)$, so $(\succeq)$ is *not* Dushnik-Miller.

It follows that $(\succeq)$ does *not* have a multiutility representation. Even worse, however, is the following case.

Proposition 4.1 *For any set $\mathcal{X}$ with $|\mathcal{X}| \geq 24$, there exists a difference preorder $(\succeq)$ on $\mathcal{X}$ which is not Szpilrajn. In particular, it has no strong utility functions.*

The interpretation of these counterexamples depends upon whether we believe the incompleteness of $(\succeq)$ to be *epistemic* or *metaphysical* in origin. According to the *epistemic* account, precise comparisons between state changes are meaningful in principle; we simply lack the necessary information to make these comparisons in practice. The incomplete difference preorder $(\succeq)$ reflects our incomplete knowledge of some unknown, *complete* difference preorder $(\underset{*}{\succeq})$, which encodes the "true" ranking of state changes. Thus, $(\succeq)$ *should* be Szpilrajn in reality, so we can dismiss the pathology in Proposition 4.1 as merely showing that the axioms (INV), (CAT), and (CAT*) alone are too weak. Furthermore, if a difference preorder $(\succeq)$ is *not* Dushnik-Miller (as in the first counterexample), then it can and should be extended to its "Dushnik-Miller completion", because any extra comparisons encoded in this completion *must* be part of $(\underset{*}{\succeq})$.

According to the *metaphysical* account, however, certain comparisons are not meaningful, even in principle. Thus, there is no reason to expect $(\succeq)$ to be Szpilrajn. If $(\succeq)$ is *not* Szpilrajn, and we have good reason to regard $(\succeq)$ as our best possible model of interpersonal comparisons, then this provides evidence for the metaphysical account.

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Appendix: Proofs

We begin with two useful lemmas. The first one is a straightforward consequence of axioms (INV) and (CAT*).

Lemma A.1 *Let $(\succeq)$ be a difference preorder on $\mathcal{X}$. For all $x, y \in \mathcal{X}$, we have $(x \rightsquigarrow x) \approx (y \rightsquigarrow y)$.*

Lemma A.2 *Let $(\succeq)$ and $(\succeq')$ be difference preorders on the sets $\mathcal{X}$ and $\mathcal{X}'$, respectively. Let $f : \mathcal{X} \longrightarrow \mathcal{X}'$ be an embedding of of $(\mathcal{X}, \succeq)$ into $(\mathcal{X}', \succeq')$.*

(a) *Let $\mathcal{R}$ be a linearly ordered abelian group, and let $u : \mathcal{X}' \longrightarrow \mathcal{R}$.*

(a1) *If u is a weak utility function for $(\succeq')$, then $u \circ f : \mathcal{X} \longrightarrow \mathcal{R}$ is a weak utility function for $(\succeq)$.*

(a2) *If u is a strong utility function for $(\succeq')$, then $u \circ f : \mathcal{X} \longrightarrow \mathcal{R}$ is a strong utility function for $(\succeq)$.*

(b) *If $(\mathcal{X}', \succeq')$ has a multiutility representation, then so does $(\mathcal{X}, \succeq)$.*

Proof. (a) is obvious. To see (b), let $\mathcal{U}'$ be a set of weak utility functions for $(\mathcal{X}', \succeq')$. Define $\mathcal{U} := \{u' \circ f;\ u' \in \mathcal{U}'\}$. Then $\mathcal{U}$ is a collection of weak utility functions for $(\mathcal{X}, \succeq)$. We claim that $\mathcal{U}$ yields a multiutility representation. To see this, let $w, x, y, z \in \mathcal{X}$. Then

$$\Big((w \rightsquigarrow x) \succeq (y \rightsquigarrow z)\Big) \underset{(*)}{\iff} \Big((f(w) \rightsquigarrow f(x)) \succeq' (f(y) \rightsquigarrow f(z))\Big).$$
$$\underset{(\dagger)}{\iff} \Big(u'[f(w)] - u'[f(x)] \geq u'[f(z)] - u'[f(y)], \text{ for all } u' \in \mathcal{U}'\Big)$$
$$\iff \Big(u(w) - u(x) \geq u(z) - u(y), \text{ for all } u \in \mathcal{U}\Big),$$

as desired. Here, $(*)$ is because f is an embedding, and $(\dagger)$ is because u' is a weak utility function. □

Let $\mathcal{X}$ be a set and let $(\succeq)$ be a difference preorder on $\mathcal{X}$. For any $x \in \mathcal{X}$, define $\langle x \rangle := \{y \in \mathcal{X};\ (x \rightsquigarrow y) \approx (x \rightsquigarrow x)\}$. In other words, $\langle x \rangle$ is the set of elements in $\mathcal{X}$ which are indifferent to x, in terms of the ordinal preferences on $\mathcal{X}$ defined by $(\succeq)$. Say that $(\mathcal{X}, \succeq)$ is *perfect* if we have $|\langle x \rangle| = |\langle y \rangle|$ for all $x, y \in \mathcal{X}$. The next result says that any difference preorder can be embedded in a perfect difference preorder.

Lemma A.3 *Let $\mathcal{X}$ be a set and let $(\succeq)$ be a difference preorder on $\mathcal{X}$. Then $(\mathcal{X}, \succeq)$ can be embedded in a system $(\mathcal{X}', \succeq')$, where $(\succeq')$ is a perfect difference preorder on $\mathcal{X}'$.*

Furthermore, if $(\succeq)$ is solvable and (semi)divisible, then so is $(\succeq')$.

Proof. If $\mathcal{X}$ is finite, then let $\mathcal{W}$ be any infinite set. If $\mathcal{X}$ is infinite, then let $\mathcal{W}$ be a larger infinite set, such that $|\mathcal{W}| > |\mathcal{X}|$ (for example, we could take $|\mathcal{W}| = |2^{\mathcal{W}}|$). Either way, basic facts about (transfinite) cardinal arithmetic yield the following property:

$$\text{for any } \mathcal{Y} \subseteq \mathcal{X}, \qquad |\mathcal{Y} \times \mathcal{W}| \;=\; |\mathcal{W}|. \tag{A1}$$

Now, let $\mathcal{X}' := \mathcal{X} \times \mathcal{W}$, and define the difference preorder $(\succeq')$ on $\mathcal{X}'$ as follows. For all $(x_0, w_0), (x_1, w_1), (y_0, w_0'), (y_1, w_1')$ in $\mathcal{X}'$, stipulate that

$$\Big([(x_0, w_0) \rightsquigarrow (x_1, w_1)] \succeq' [(y_0, w_0') \rightsquigarrow (y_1, w_1')]\Big) \iff \Big((x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)\Big). \tag{A2}$$

Fix $w \in \mathcal{W}$, and define $f : \mathcal{X} \longrightarrow \mathcal{X}'$ by setting $f(x) := (x, w)$ for all $x \in \mathcal{X}$. It is easy to see that f is an embedding of $(\mathcal{X}, \succeq)$ into $(\mathcal{X}', \succeq')$.

It remains to show that $(\mathcal{X}', \succeq')$ is perfect. To see this, let $(x, w) \in \mathcal{X}'$. Then

$$\begin{aligned} \langle (x, w) \rangle &:= \{(y, w') \in \mathcal{X}' \; ; \; [(x, w) \rightsquigarrow (y, w')] \approx' [(x, w) \rightsquigarrow (x, w)]\} \\ &= \{(y, w') \in \mathcal{X}' \; ; \; (x \rightsquigarrow y) \approx (x \rightsquigarrow x)\} \\ &= \{y \in \mathcal{X} \; ; \; (x \rightsquigarrow y) \approx (x \rightsquigarrow x)\} \times \mathcal{W} \;=\; \langle x \rangle \times \mathcal{W}. \end{aligned}$$

Thus, statement (A1) implies that $|\langle (x, w) \rangle| = |\mathcal{W}|$ for all $(x, w) \in \mathcal{X}'$. Thus, $(\mathcal{X}', \succeq')$ is perfect.

Finally, suppose $(\succeq)$ is solvable and (semi)divisible. We must show that $(\succeq')$ is also solvable and (semi)divisible.

Solvable. Let (x_0, w_0), (x_1, w_1), $(y_0, w_0') \in \mathcal{X}'$; we must find $(y_1, w_1') \in \mathcal{X}'$ such that $[(x_0, w_0) \rightsquigarrow (x_1, w_1)] \approx' [(y_0, w_0') \rightsquigarrow (y_1, w_1')]$. By definition (A2), the choice of w_1' is arbitrary, and it suffices to find $y_1 \in \mathcal{X}$ such that $(x_0 \rightsquigarrow x_1) \approx (y_0 \rightsquigarrow y_1)$. But such a y_1 exists because $(\succeq)$ is solvable.

(Semi)divisible. First suppose $(\succeq)$ is divisible. Suppose $(x_0, w_0), (x_1, w_1), (x_2, w_2), \ldots, (x_N, w_N)$ is a standard sequence in $\mathcal{X}'$. Thus, $[(x_0, w_0) \rightsquigarrow (x_1, w_1)] \approx' [(x_1, w_1) \rightsquigarrow (x_2, w_2)] \approx' \cdots \approx' [(x_{N-1}, w_{N-1}) \rightsquigarrow (x_N, w_N)]$. By definition (A2), this is true if and only if $(x_0 \rightsquigarrow x_1) \approx (x_1 \rightsquigarrow x_2) \approx \cdots \approx (x_{N-1} \rightsquigarrow x_N)$. Thus, $x_0, x_1, x_2, \ldots, x_N$ is a standard sequence in $\mathcal{X}$. Thus,

$$\begin{aligned} &\Big([(x_0, w_0) \rightsquigarrow (x_N, w_N)] \succeq' [(x_0, w_0) \rightsquigarrow (x_0, w_0)]\Big) \underset{\text{(a)}}{\iff} \Big((x_0 \rightsquigarrow x_N) \succeq (x_0 \rightsquigarrow x_0)\Big) \\ &\underset{\text{(b)}}{\iff} \Big((x_0 \rightsquigarrow x_1) \succeq (x_0 \rightsquigarrow x_0)\Big) \underset{\text{(c)}}{\iff} \Big([(x_0, w_0) \rightsquigarrow (x_1, w_1)] \succeq' [(x_0, w_0) \rightsquigarrow (x_0, w_0)]\Big). \end{aligned}$$

Here, (a) and (c) are by definition (A2), and (b) is because $(\mathcal{X}, \succeq)$ is divisible by hypothesis. Thus, $(\succeq')$ is divisible.

The same argument shows that, if $(\succeq)$ semidivisible, then $(\succeq')$ is semidivisible. □

Given Lemmas A.2(b) and A.3, to prove the "$\Longleftarrow$" direction of Theorem 2.1, it suffices to prove that any perfect, solvable, divisible difference preorder has a multiutitility representation. To prove this, we need one more tool. An ***automorphism*** of $(\mathcal{X}, \succeq)$ is a bijective function $\gamma : \mathcal{X} \longrightarrow \mathcal{X}$ such that, for all $x, y \in \mathcal{X}$, we have $(x \rightsquigarrow y) \approx (\gamma(x) \rightsquigarrow \gamma(y))$. Let Γ denote the set of all automorphisms of $(\mathcal{X}, \succeq)$. It is easy to see that Γ is a group.

Lemma A.4 *Let $\succeq$ be a perfect difference preorder, and let Γ be its group of automorphisms. Then $(\succeq)$ is solvable if and only if Γ acts transitively on $\mathcal{X}$.*

Proof. "$\Longleftarrow$" Let $x_1, x_2, y_1 \in \mathcal{X}$. Since Γ acts transitively on $\mathcal{X}$, there is some $\gamma \in \Gamma$ such that $\gamma(x_1) = y_1$. Let $y_2 := \gamma(x_2)$. Then $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$, as desired.

"$\Longrightarrow$" Fix $x_0, y_0 \in \mathcal{X}$. We must construct some $\gamma \in \Gamma$ such that $\gamma(x_0) = y_0$. First note that $|\langle x_0 \rangle| = |\langle y_0 \rangle|$ (because $(\succeq)$ is perfect). So, let $\gamma : \langle x_0 \rangle \longrightarrow \langle y_0 \rangle$ be any bijection such that $\gamma(x_0) = y_0$.

Now, for every $x \in \mathcal{X}$, there is some $y \in \mathcal{X}$ such that $(x_0 \rightsquigarrow x) \approx (y_0 \rightsquigarrow y)$ (because $(\succeq)$ is solvable). Furthermore, $|\langle x \rangle| = |\langle y \rangle|$ (because $(\succeq)$ is perfect). So, let $\gamma : \langle x \rangle \longrightarrow \langle y \rangle$ be a bijection. For any $x' \in \langle x \rangle$, if $y' = \gamma(x')$, then we have

$$(x_0 \rightsquigarrow x') \underset{(a)}{\approx} (x_0 \rightsquigarrow x) \underset{(b)}{\approx} (y_0 \rightsquigarrow y) \underset{(c)}{\approx} (y_0 \rightsquigarrow y'). \tag{A3}$$

Here, (a) is by (CAT) (because $x' \in \langle x \rangle$), (b) is by construction of y, and (c) is by (CAT) (because $y' \in \langle y \rangle$).

Proceeding in this fashion, we can define the function γ on $\langle x \rangle$ for every $x \in \mathcal{X}$. Since the collection $\{\langle x \rangle;\ x \in \mathcal{X}\}$ is a partition of $\mathcal{X}$, this defines γ everywhere on $\mathcal{X}$. Since γ maps each cell of this partition bijectively to another cell of the partition, it follows that γ is a bijection from $\mathcal{X}$ to itself. From equation (A3) it follows:

$$\text{For all } x \in \mathcal{X}, \qquad (x_0 \rightsquigarrow x) \approx (y_0 \rightsquigarrow f(x)). \tag{A4}$$

It remains to show that γ is an automorphism of $(\mathcal{X}, \succeq)$. So, let $x_1, x_2 \in \mathcal{X}$. Let $y_1 := \gamma(x_1)$ and $y_2 := \gamma(x_2)$; we must show that $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$. Statement (A4) implies that $(x_0 \rightsquigarrow x_1) \approx (y_0 \rightsquigarrow y_1)$. Thus, axiom (INV) says

$$(x_1 \rightsquigarrow x_0) \approx (y_1 \rightsquigarrow y_0). \tag{A5}$$

Statement (A4) also implies that

$$(x_0 \rightsquigarrow x_2) \approx (y_0 \rightsquigarrow y_2). \tag{A6}$$

Combining statements (A5) and (A6) via (CAT), we get $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$, as desired. □

Let $(\mathcal{G}, \cdot)$ be a (possibly nonabelian) group with identity element e, and let $(>)$ be a binary relation (e.g. a partial order or preorder) on $\mathcal{G}$. We say that $(>)$ is ***homogeneous*** if, for all $f, g, h \in \mathcal{G}$, we have $(g > h) \iff (f \cdot g > f \cdot h) \iff (g \cdot f > h \cdot f)$. In particular, if $(\mathcal{R}, 0, +)$ is an abelian group, then a binary relation $(>)$ on $\mathcal{R}$ is homogeneous if, for all $r, s, t \in \mathcal{R}$, we have $(r > s) \iff (r + t > s + t)$.

An abelian group $(\mathcal{R}, +, 0)$ is ***torsion-free*** if, for any $r \in \mathcal{R}$, we have:

$$\Big(n\, r = 0 \text{ for some } n \in \mathbb{N}\Big) \iff \Big(r = 0\Big). \qquad (\text{Here, } n \cdot r := \overbrace{r + r + \cdots + r}^{n \text{ times}}.)$$

For example, it is easy to see that any linearly ordered abelian group is torsion-free. Szpilrajn (1930) proved that any partial order can be extended to a linear order. The next result is the analogous statement for homogeneous partial orders.

Homogeneous Szpilrajn Lemma. *Let $(\mathcal{R}, +, 0)$ be a torsion-free abelian group. Then any homogeneous partial order on $\mathcal{R}$ can be extended to a homogeneous linear order.*

Proof. See (Fuchs, 2011, Corollary 13, p.39). □

A homogeneous partial order $(>)$ is *isolated* if, for any $r \in \mathcal{R}$, if $n\, r > 0$ for some $n \in \mathbb{N}$, then we have $r > 0$. For example, it is easy to see that any linearly ordered abelian group is isolated. Dushnik and Miller (1941) proved that any partial order is the intersection of all the linear orders which extend it. The next result is the analogous statement for homogeneous partial orders.

Homogeneous Dushnik-Miller Theorem. *Let $(\mathcal{R}, +, 0)$ be a torsion-free abelian group. If $(>)$ is an isolated homogeneous partial order on $\mathcal{R}$, then $(>)$ is the intersection of all the homogeneous linear orders on $\mathcal{R}$ which extend it.*

Proof. See (Fuchs, 2011, Corollary 19, p.41). □

In the next proof, when invoking axioms (CAT) and (CAT*), we will sometimes write "$(x_0 \rightsquigarrow x_2) = (x_0 \rightsquigarrow x_1 \rightsquigarrow x_2)$" and "$(y_0 \rightsquigarrow y_2) = (y_0 \rightsquigarrow y_1 \rightsquigarrow y_2)$", with the implication that we are supposed to compare the transitions $(x_0 \rightsquigarrow x_1)$ and $(x_1 \rightsquigarrow x_2)$ to the transitions $(y_0 \rightsquigarrow y_1)$ and $(y_1 \rightsquigarrow y_2)$ in order to deduce a comparison between $(x_0 \rightsquigarrow x_2)$ and $(y_0 \rightsquigarrow y_2)$.

Proposition A.5 *Let $(\succeq)$ be a perfect, solvable difference preorder on a set $\mathcal{X}$.*

(a) *If $(\succeq)$ is semidivisible, then it has a strong utility function.*

(b) *If $(\succeq)$ is also divisible, then it has a multiutitility representation.*

Proof. (a) Let Γ be the automorphism group of $(\mathcal{X}, \succeq)$.

Claim 1. *Let $\gamma \in \Gamma$. Then for all $x, y \in \mathcal{X}$, we have $(x \rightsquigarrow \gamma(x)) \approx (y \rightsquigarrow \gamma(y))$.*

Proof. $(x \rightsquigarrow \gamma(x)) \;=\; (x \rightsquigarrow \gamma(y) \rightsquigarrow \gamma(x)) \underset{(*)}{\approx} (y \rightsquigarrow x \rightsquigarrow \gamma(y)) \;=\; (y \rightsquigarrow \gamma(y))$, as desired.

Here, $(*)$ is by (CAT*), using the fact that $(y \rightsquigarrow x) \approx (\gamma(y) \rightsquigarrow \gamma(x))$, because γ is an automorphism. ◇ `Claim 1`

Now define a preorder $(\succeq^{\Gamma})$ on Γ as follows: for any $\alpha, \beta \in \Gamma$, we write

$$\Big(\alpha \succeq^{\Gamma} \beta\Big) \iff \Big(\exists\, z \in \mathcal{X} \text{ with } (z \rightsquigarrow \alpha(z)) \succeq (z \rightsquigarrow \beta(z))\Big)$$

$$\underset{(a)}{\iff} \Big((y \rightsquigarrow \alpha(y)) \succeq (y \rightsquigarrow \beta(y)) \text{ for all } y \in \mathcal{X}\Big) \tag{A7}$$

$$\underset{(b)}{\iff} \Big((x \rightsquigarrow \alpha(y)) \succeq (x \rightsquigarrow \beta(y)) \text{ for all } x, y \in \mathcal{X}\Big). \tag{A8}$$

where (a) is by Claim 1, and (b) is by axiom (CAT).

Claim 2.

(a) $(\succeq^{\Gamma})$ *is a homogeneous preorder on* Γ.

(b) *For any* $\alpha, \beta \in \Gamma$, *we have* $\alpha \circ \beta \approx^{\Gamma} \beta \circ \alpha$.

Proof. (a) Let $\alpha, \beta, \gamma \in \Gamma$. Suppose $\alpha \succeq^{\Gamma} \beta$; we must show that $\gamma \circ \alpha \succeq^{\Gamma} \gamma \circ \beta$. Fix $x \in \mathcal{X}$. We have

$$\left(\alpha \succeq^{\Gamma} \beta\right) \underset{(*)}{\iff} \left((\gamma^{-1}(x) \rightsquigarrow \alpha(x)) \succeq (\gamma^{-1}(x) \rightsquigarrow \beta(x)\right)$$
$$\underset{(\dagger)}{\iff} \left((x \rightsquigarrow \gamma \circ \alpha(x)) \succeq (x \rightsquigarrow \gamma \circ \beta(x)\right) \underset{(\ddagger)}{\iff} \left(\gamma \circ \alpha \succeq^{\Gamma} \gamma \circ \beta\right).$$

Here, $(*)$ is by defining formula (A8) and $(\ddagger)$ is by defining formula (A7), while $(\dagger)$ is because γ is an automorphism of $(\succeq)$. By a similar argument, we have $\left(\alpha \succeq^{\Gamma} \beta\right) \iff \left(\alpha \circ \gamma \succeq^{\Gamma} \beta \circ \gamma\right)$. Thus, $\succeq^{\Gamma}$ is a homogeneous partial order on Γ.

(b) Let $x \in \mathcal{X}$. According to defining formula (A7), we must show that $(x \rightsquigarrow \alpha \circ \beta(x)) \approx (x \rightsquigarrow \beta \circ \alpha(x))$. Since α and β are automorphisms of $(\succeq)$ we have

$$\begin{aligned} (x \rightsquigarrow \beta(x)) &\approx (\alpha(x) \rightsquigarrow \alpha \circ \beta(x)) && \text{(A9)} \\ \text{and } (x \rightsquigarrow \alpha(x)) &\approx (\beta(x) \rightsquigarrow \beta \circ \alpha(x)). && \text{(A10)} \end{aligned}$$

Thus, we have

$$\begin{aligned} (x \rightsquigarrow \alpha \circ \beta(x)) &= (x \rightsquigarrow \alpha(x) \rightsquigarrow \alpha \circ \beta(x)) \\ &\underset{(*)}{\approx} (x \rightsquigarrow \beta(x) \rightsquigarrow \beta \circ \alpha(x)) = (x \rightsquigarrow \beta \circ \alpha(x)), \end{aligned}$$

as desired. Here, $(*)$ is by equations (A9) and (A10) and axiom (CAT*). $\diamondsuit$ `Claim 2`

Let ϵ be the identity element of Γ. Let $\mathcal{N} := \{\nu \in \Gamma;\ \nu \approx^{\Gamma} \epsilon\}$. By defining formula (A8), this means $\mathcal{N} = \{\nu \in \Gamma;\ (x \rightsquigarrow \nu(y)) \approx (x \rightsquigarrow y) \text{ for all } x, y \in \mathcal{X}\}$.

Claim 3.

(a) *For any* $\gamma \in \Gamma$, *we have* $\gamma\mathcal{N} = \mathcal{N}\gamma = \{\delta \in \Gamma;\ \delta \approx^{\Gamma} \gamma\}$.

(b) $\mathcal{N}$ *is a normal subgroup of* Γ.

(c) *The quotient group* $\Gamma/\mathcal{N}$ *is abelian.*

Proof. (a) Let $\Delta := \{\delta \in \Gamma;\ \delta \approx^{\Gamma} \gamma\}$; we must show that $\gamma\mathcal{N} = \mathcal{N}\gamma = \Delta$. To see that $\gamma\mathcal{N} \subseteq \Delta$ and $\mathcal{N}\gamma \subseteq \Delta$, let $\nu \in \mathcal{N}$. Then for any $x \in \mathcal{X}$, we have $(x \rightsquigarrow \nu \circ \gamma(x)) \approx (x \rightsquigarrow \gamma(x))$. Thus, $\nu \circ \gamma \in \Delta$, because $\nu \circ \gamma \approx^{\Gamma} \gamma$, by defining formula (A7). Meanwhile,

$$(x \rightsquigarrow \gamma \circ \nu(x)) \underset{(*)}{\approx} (\gamma^{-1}(x) \rightsquigarrow \nu(x)) \underset{(\dagger)}{\approx} (\gamma^{-1}(x) \rightsquigarrow x) \underset{(*)}{\approx} (x \rightsquigarrow \gamma(x)).$$

Thus, $\gamma \circ \nu \in \Delta$, because $\gamma \circ \nu \approx^{\Gamma} \gamma$, again by defining formula (A7). Here, both $(*)$ are because γ is an automorphism of $(\succeq)$, while $(\dagger)$ is because $\nu \in \mathcal{N}$.

These arguments hold for all $\nu \in \mathcal{N}$; it follows that $\gamma\mathcal{N} \subseteq \Delta$ and $\mathcal{N}\gamma \subseteq \Delta$.

Conversely, to see that $\gamma\mathcal{N} \supseteq \Delta$ and $\mathcal{N}\gamma \supseteq \Delta$, let $\delta \in \Delta$; thus, $\delta \overset{\Gamma}{\approx} \gamma$. Then

$$(x \rightsquigarrow \delta \circ \gamma^{-1}(x)) \underset{(*)}{\approx} (x \rightsquigarrow \gamma \circ \gamma^{-1}(x)) = (x \rightsquigarrow x),$$

and thus, if $\nu := \delta \circ \gamma^{-1}$, then $\nu \in \mathcal{N}$. Thus, $\delta = \nu \circ \gamma \in \mathcal{N}\gamma$. Here, $(*)$ is by defining formula (A8), because $\delta \overset{\Gamma}{\approx} \gamma$. Meanwhile,

$$\begin{aligned}(x \rightsquigarrow \gamma^{-1} \circ \delta(x) &\underset{(*)}{\approx} (\gamma(x) \rightsquigarrow \delta(x)) \\ &\underset{(\dagger)}{\approx} (\gamma(x) \rightsquigarrow \gamma(x)) \underset{(\ddagger)}{\approx} (x \rightsquigarrow x),\end{aligned}$$

and thus, if $\mu := \gamma^{-1} \circ \delta$, then $\mu \in \mathcal{N}$. Thus, $\delta = \gamma \circ \mu \in \gamma\mathcal{N}$. Here, $(*)$ is because γ is an automorphism, $(\dagger)$ is by defining formula (A8), because $\delta \overset{\Gamma}{\approx} \gamma$, and $(\ddagger)$ is by Lemma A.1.

These arguments hold for all $\delta \in \Delta$; it follows that $\gamma\mathcal{N} \supseteq \Delta$ and $\mathcal{N}\gamma \supseteq \Delta$.

(b) follows immediately from (a).

(c) It suffices to show that, for every $\alpha, \beta \in \Gamma$, the commutator $\alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta$ is an element of $\mathcal{N}$. Fix $z \in \mathcal{X}$. Then setting $x := \alpha \circ \beta(z)$ and $y := \beta \circ \alpha(z)$ and $\gamma := \alpha^{-1} \circ \beta^{-1}$ in Claim 1, we get

$$\Big[\alpha \circ \beta(z) \rightsquigarrow \alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta(z)\Big] \approx \Big[\beta \circ \alpha(z) \rightsquigarrow \alpha^{-1} \circ \beta^{-1} \circ \beta \circ \alpha(z)\Big]. \quad \text{(A11)}$$

Meanwhile, we have

$$(z \rightsquigarrow \alpha \circ \beta(z)) \approx (z \rightsquigarrow \beta \circ \alpha(z)), \quad \text{(A12)}$$

by Claim 2(b) and defining formula (A7). Thus,

$$\begin{aligned}(z \rightsquigarrow \alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta(z)) &= (z \rightsquigarrow \alpha \circ \beta(z) \rightsquigarrow \alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta(z)) \\ &\underset{(*)}{\approx} (z \rightsquigarrow \beta \circ \alpha(z) \rightsquigarrow \alpha^{-1} \circ \beta^{-1} \circ \beta \circ \alpha(z)) \\ &= (z \rightsquigarrow \beta \circ \alpha(z) \rightsquigarrow z) = (z \rightsquigarrow z),\end{aligned}$$

where $(*)$ is by linking formulae (A11) and (A12) via Axiom (CAT). Thus, $\alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta \in \mathcal{N}$, as desired. This holds for all $\alpha, \beta \in \Gamma$, so the commutator subgroup of Γ is contained in $\mathcal{N}$. It follows from basic group theory that $\Gamma/\mathcal{N}$ is abelian. $\diamond$ `Claim 3`

Let $\mathcal{R} := \Gamma/\mathcal{N}$. For any $\gamma \in \Gamma$, let $[\gamma]$ denote the corresponding element of $\mathcal{R}$ (that is, the coset $\gamma\mathcal{N} = \mathcal{N}\gamma$). Since $\mathcal{R}$ is abelian, we will write its operation as "+". That is, for all $\alpha, \beta \in \Gamma$, we have $[\alpha] + [\beta] := [\alpha \circ \beta]$. Let 0 denote the identity of $\mathcal{R}$. Observe that $0 = [\epsilon] = \mathcal{N} = \{\gamma \in \Gamma;\ \gamma \overset{\Gamma}{\approx} \epsilon\}$. We define a binary relation $(>)$ on $\mathcal{R}$ by setting $[\alpha] > [\beta]$ if and only if $\alpha \overset{\Gamma}{\succ} \beta$, for all $\alpha, \beta \in \Gamma$.

Claim 4. *$(>)$ is a well-defined, homogeneous partial order on $\mathcal{R}$.*

Proof. *Well-defined.* Let $\alpha, \beta \in \Gamma$. Let $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$. Thus, $\alpha' = \nu \circ \alpha$ and $\beta' = \mu \circ \beta$ for some $\nu, \mu \in \mathcal{N}$. We must show that $\alpha \succ^{\Gamma} \beta$ if and only if $\alpha' \succ^{\Gamma} \beta'$. To see this, let $x \in \mathcal{X}$. Then

$$\begin{array}{rrcccl} & (x \rightsquigarrow \alpha'(x)) & = & (x \rightsquigarrow \nu \circ \alpha(x)) & \underset{(*)}{\approx} & (x \rightsquigarrow \alpha(x)) \qquad \text{(A13)} \\ \text{and} & (x \rightsquigarrow \beta'(x)) & = & (x \rightsquigarrow \mu \circ \beta(x)) & \underset{(*)}{\approx} & (x \rightsquigarrow \beta(x)), \qquad \text{(A14)} \end{array}$$

where the $(*)$'s are because $\nu, \mu \in \mathcal{N}$. Thus,

$$\begin{array}{rcl} \left(\alpha \succ^{\Gamma} \beta\right) & \underset{(*)}{\iff} & \Big((x \rightsquigarrow \alpha(x)) \succ (x \rightsquigarrow \beta(x))\Big) \\ & \underset{(\dagger)}{\iff} & \Big((x \rightsquigarrow \alpha'(x)) \succ (x \rightsquigarrow \beta'(x))\Big) \quad \underset{(*)}{\iff} \quad \left(\alpha' \succ^{\Gamma} \beta'\right), \end{array}$$

as desired. Here, both $(*)$ are by formula (A7), while $(\dagger)$ is by equations (A13) and (A14).

Transitive. Let $\alpha, \beta, \gamma \in \Gamma$. Suppose $[\alpha] > [\beta]$ and $[\beta] > [\gamma]$. Thus, $\alpha \succ^{\Gamma} \beta$ and $\beta \succ^{\Gamma} \gamma$. Thus, $\alpha \succ^{\Gamma} \gamma$ because $(\succeq^{\Gamma})$ is transitive. Thus, $[\alpha] > [\gamma]$. *Antisymmetric* is proved similarly.

Homogeneous. Let $\alpha, \beta, \gamma \in \Gamma$. If $[\alpha] > [\beta]$, then $\alpha \succ^{\Gamma} \beta$. Thus, $\gamma \circ \alpha \succ^{\Gamma} \gamma \circ \beta$, by Claim 2(a). Thus, $[\gamma] + [\alpha] = [\gamma \circ \alpha] > [\gamma \circ \beta] = [\gamma] + [\beta]$, as desired. ◇ `Claim 4`

Fix $o \in \mathcal{X}$. For every $x \in \mathcal{X}$, Lemma A.4 yields some $\gamma_x \in \Gamma$ such that $\gamma_x(o) = x$. Define the function $u : \mathcal{X} \longrightarrow \mathcal{R}$ by setting $u(x) := [\gamma_x]$ for all $x \in \mathcal{X}$. (For example, $\gamma_o = \epsilon$, so $u(o) = [\epsilon] = 0$.)

Claim 5. For all $x, x', y, y' \in \mathcal{X}$, we have

(a) $(x \rightsquigarrow x') \succeq (y \rightsquigarrow y')$ if and only if $u(x') - u(x) \geq u(y') - u(y)$.

(b) $(x \rightsquigarrow x') \succ (y \rightsquigarrow y')$ if and only if $u(x') - u(x) > u(y') - u(y)$.

Proof. (b) follows from (a). To prove (a), First note that

$$\begin{array}{rcccl} u(x') - u(x) & = & [\gamma_{x'}] - [\gamma_x] & = & [\gamma_x^{-1} \circ \gamma_{x'}] \qquad \text{(A15)} \\ \text{and} \;\; u(y') - u(y) & = & [\gamma_{y'}] - [\gamma_y] & = & [\gamma_y^{-1} \circ \gamma_{y'}]. \qquad \text{(A16)} \end{array}$$

Thus,

$$\begin{array}{rcl} \Big(u(x') - u(x) \geq u(y') - u(y)\Big) & \underset{\text{(a)}}{\iff} & \Big([\gamma_x^{-1} \circ \gamma_{x'}] \geq [\gamma_y^{-1} \circ \gamma_{y'}]\Big) \\ & \underset{\text{(b)}}{\iff} & \Big(\gamma_x^{-1} \circ \gamma_{x'} \succeq^{\Gamma} \gamma_y^{-1} \circ \gamma_{y'}\Big) \\ & \underset{\text{(c)}}{\iff} & \Big([o \rightsquigarrow \gamma_x^{-1} \circ \gamma_{x'}(o)] \succeq [o \rightsquigarrow \gamma_y^{-1} \circ \gamma_{y'}(o)]\Big) \\ & \underset{\text{(d)}}{\iff} & \Big([\gamma_x(o) \rightsquigarrow \gamma_{x'}(o)] \succeq [\gamma_y(o) \rightsquigarrow \gamma_{y'}(o)]\Big) \\ & \underset{\text{(e)}}{\iff} & \Big((x \rightsquigarrow x') \succeq (y \rightsquigarrow y')\Big), \end{array}$$

as desired. Here (a) is by (A15) and (A16), (b) is by the definition of $(>)$, (c) is by the defining formula (A7), (d) is because $\gamma_x, \gamma_y \in \Gamma$, and (e) is by the definition of γ_x, γ_y, $\gamma_{x'}$, and $\gamma_{y'}$. $\diamond$ Claim 5

Up until now, everything in the proof has been necessary for both parts (a) and (b) of Proposition A.5. But the next claim is only required to prove part (a).

Claim 6. *If $(\succeq)$ is semidivisible, then the group $(\mathcal{R}, +, 0)$ is torsion-free.*

Proof. Let $\gamma \in \Gamma$, let $n \in \mathbb{N}$, and suppose $n \cdot [\gamma] = 0$. Note that $n \cdot [\gamma] = [\gamma^n]$; thus, we have $\gamma^n \overset{\Gamma}{\approx} \epsilon$. Thus, for any $x \in \mathcal{X}$, we have $(x \rightsquigarrow \gamma^n(x)) \approx (x \rightsquigarrow x)$ by defining formula (A7). Consider the sequence $\{x, \gamma(x), \gamma^2(x), \ldots, \gamma^n(x)\}$. This is a standard sequence: we have $(x \rightsquigarrow \gamma(x)) \approx (\gamma(x) \rightsquigarrow \gamma^2(x)) \approx \cdots \approx (\gamma^{n-1}(x) \rightsquigarrow \gamma^n(x))$, because γ is an automorphism. Thus, if $(x \rightsquigarrow \gamma^n(x)) \approx (x \rightsquigarrow x)$, then $(x \rightsquigarrow \gamma(x)) \approx (x \rightsquigarrow x)$, because $(\succeq)$ is semidivisible. Thus, $\gamma \overset{\Gamma}{\approx} \epsilon$. Thus, $[\gamma] = 0$, as desired. $\diamond$ Claim 6

Now, let $\mathcal{E}(>)$ be the set of homogeneous linear orders on $\mathcal{R}$ which extend $(>)$. Claim 6 and the the Homogeneous Szpilrajn Lemma together imply that $\mathcal{E}(>)$ is nonempty. For any order $(\gg) \in \mathcal{E}(>)$, the system $(\mathcal{R}, +, 0, \gg)$ is a linearly ordered abelian group. We can also treat $u : \mathcal{X} \longrightarrow \mathcal{R}$ as a function into $(\mathcal{R}, +, 0, \gg)$; to avoid confusion, we will denote this function by $u_{\gg}$. Finally, part (a) of the theorem follows from the next claim.

Claim 7. *For any $(\gg) \in \mathcal{E}(>)$, the function $u_{\gg}$ is a strong utility function for $(\mathcal{X}, \succeq)$.*

Proof. For all $x, x', y, y' \in \mathcal{X}$, we have

$$\begin{aligned}
\Big((x \rightsquigarrow x') \succeq (y \rightsquigarrow y')\Big) \quad &\underset{\text{(a)}}{\iff} \quad \Big(u(x') - u(x) \geq u(y') - u(y)\Big) \\
&\underset{(*)}{\Longrightarrow} \quad \Big(u_{\gg}(x') - u_{\gg}(x) \ggcurlyeq u_{\gg}(y') - u_{\gg}(y)\Big), \\
\text{and} \quad \Big((x \rightsquigarrow x') \succ (y \rightsquigarrow y')\Big) \quad &\underset{\text{(b)}}{\iff} \quad \Big(u(x') - u(x) > u(y') - u(y)\Big) \\
&\underset{(*)}{\Longrightarrow} \quad \Big(u_{\gg}(x') - u_{\gg}(x) \;\gg\; u_{\gg}(y') - u_{\gg}(y)\Big),
\end{aligned}$$

where (a) is by Claim 5(a) and (b) is by Claim 5(b), while both $(*)$ are because $(\gg) \in \mathcal{E}(>)$. $\diamond$ Claim 7

Proof of part (b). Now suppose $(\succeq)$ is divisible. We will show that the collection $\{u_{\gg};\ (\gg) \in \mathcal{E}(>)\}$ is a multiutility representation for $(\mathcal{X}, \succeq)$. For this we need the following.

Claim 8. *The order $(>)$ is isolated.*

Proof. Let $\gamma \in \Gamma$, let $n \in \mathbb{N}$, and suppose $n \cdot [\gamma] > 0$. Note that $n \cdot [\gamma] = [\gamma^n]$; thus, we have $\gamma^n \overset{\Gamma}{\succ} \epsilon$. Thus, for any $x \in \mathcal{X}$, we have $(x \rightsquigarrow \gamma^n(x)) \succ (x \rightsquigarrow x)$ by defining formula (A7). As in Claim 6, $\{x, \gamma(x), \gamma^2(x), \ldots, \gamma^n(x)\}$ is a standard sequence, because γ is an automorphism. Thus, if $(x \rightsquigarrow \gamma^n(x)) \succ (x \rightsquigarrow x)$, then $(x \rightsquigarrow \gamma(x)) \succ (x \rightsquigarrow x)$, because $(\succeq)$ is divisible. Thus, $\gamma \overset{\Gamma}{\succ} \epsilon$. Thus, $[\gamma] > 0$, as desired. $\diamond$ Claim 8

Claim 8 and the Homogeneous Dushnik-Miller Lemma together imply that $(>)$ is the intersection of all elements of $\mathcal{E}(>)$. Now, let $x, x', y, y' \in \mathcal{X}$. Then

$$\Big((x \rightsquigarrow x') \succeq (y \rightsquigarrow y')\Big) \underset{\text{(a)}}{\iff} \Big(u(x') - u(x) \geq u(y') - u(y)\Big)$$
$$\underset{\text{(b)}}{\iff} \Big(u_{\gg}(x') - u_{\gg}(x) \geqq u_{\gg}(y') - u_{\gg}(y), \text{ for all } (\gg) \in \mathcal{E}(x)\Big),$$

where (a) is by Claim 5(a), and (b) is by the Homogeneous Dushnik-Miller Theorem. □

We will prove Proposition 3.1 now, because the proof is simple and does not require any further material.

Proof of Proposition 3.1. Let $f : (\mathcal{X}, \succeq) \longrightarrow (\mathcal{X}', \succeq')$ be an embedding, where $(\mathcal{X}', \succeq')$ is solvable and semidivisible. Lemma A.3 then yields an embedding $g : \mathcal{X}' \longrightarrow \mathcal{X}''$, where $(\mathcal{X}'', \succeq'')$ is perfect, solvable, and semidivisible. Thus, $g \circ f : \mathcal{X} \longrightarrow \mathcal{X}''$ is also an embedding. Proposition A.5(a) say that $(\mathcal{X}'', \succeq'')$ has a strong utility function. Thus, Lemma A.2(a2) implies that $(\mathcal{X}, \succeq)$ also has a strong utility function. □

Now we come to the proof of our main result.

Proof of Theorem 2.1. "$\Longrightarrow$" Let $\mathcal{U}$ be a collection of utility functions yielding a multiutility representation for $(\succeq)$. Thus, each element of $\mathcal{U}$ is a function $u : \mathcal{X} \longrightarrow \mathcal{R}_u$, where $(\mathcal{R}_u, +, 0_u, >_u)$ is some linearly ordered abelian group.

Let $\mathcal{X}' := \mathcal{X} \times \prod_{u \in \mathcal{U}} \mathcal{R}_u$. Let $\mathcal{U}' := \{*\} \sqcup \mathcal{U}$. A generic element of $\mathcal{X}'$ will be written as $\mathbf{x} = (x_u)_{u \in \mathcal{U}'}$, where $x_* \in \mathcal{X}$ and $x_u \in \mathcal{R}_u$ for all $u \in \mathcal{U}$. Define the difference preorder $(\succeq')$ on $\mathcal{X}'$ as follows. For all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}'$, stipulate that

$$\Big((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \ \succeq' \ (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\Big) \iff \Big(x_u^2 - x_u^1 \ \geq_u \ y_u^2 - y_u^1, \text{ for all } u \in \mathcal{U}\Big). \quad \text{(A17)}$$

Now define $f : \mathcal{X} \longrightarrow \mathcal{X}'$ by setting $f(x) := [x, (u(x))_{u \in \mathcal{U}}]$ for all $x \in \mathcal{X}$. Then f is injective, because $f(x)_* = x$ for all $x \in \mathcal{X}$. Furthermore, f is an embedding of $(\mathcal{X}, \succeq)$ into $(\mathcal{X}', \succeq')$, as can be seen by combining statements (3) and (A17).

It is easy to check that $(\mathcal{X}', \succeq')$ is solvable (because each $\mathcal{R}_u$ is a group). It remains to show that $(\mathcal{X}', \succeq')$ is divisible. Suppose that $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N \in \mathcal{X}'$ is a $(\succeq')$-standard sequence. For any $u \in \mathcal{U}$, we have

$$\Big((\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \approx' (\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \approx' \cdots \approx' (\mathbf{x}^{N-1} \rightsquigarrow \mathbf{x}^N)\Big)$$
$$\underset{\text{(a)}}{\Longrightarrow} \Big(x_u^1 - x_u^0 = x_u^2 - x_u^1 = \cdots = x_u^N - x_u^{N-1}\Big).$$

$$\text{Thus,} \quad x_u^N - x_u^0 \ = \ x_u^N - x_u^{N-1} \ + \ x_u^{N-1} - x_u^{N-2} + \cdots + x_u^2 - x_u^1 + x_u^1 - x_u^0$$
$$= \ N \cdot \Big(x_u^1 - x_u^0\Big). \quad \text{(A18)}$$

$$\text{Thus,}\quad \Big((\mathbf{x}^0 \rightsquigarrow \mathbf{x}^N) \succeq' (\mathbf{x}^0 \rightsquigarrow \mathbf{x}^0)\Big) \underset{\text{(b)}}{\iff} \Big(x_u^N - x_u^0 \geq 0 \text{ for all } u \in \mathcal{U}\Big)$$
$$\underset{\text{(c)}}{\iff} \Big(N \cdot \Big(x_u^1 - x_u^0\Big) \geq 0 \text{ for all } u \in \mathcal{U}\Big)$$
$$\underset{\text{(d)}}{\iff} \Big(x_u^1 - x_u^0 \geq 0 \text{ for all } u \in \mathcal{U}\Big)$$
$$\underset{\text{(e)}}{\iff} \Big((\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \succeq' (\mathbf{x}^0 \rightsquigarrow \mathbf{x}^0)\Big).$$

Here, (a), (b) and (e) are by definition (A17), (c) is by equation (A18), and (d) is because, for all $u \in \mathcal{U}$, the group $\mathcal{R}_u$ is isolated, because it is linearly ordered.

"$\Longleftarrow$" Suppose there exists an embedding $f : \mathcal{X} \longrightarrow \mathcal{X}'$, where $(\mathcal{X}', \succeq')$ is solvable and divisible. Lemma A.3 then yields an embedding $g : \mathcal{X}' \longrightarrow \mathcal{X}''$, where $(\mathcal{X}'', \succeq'')$ is perfect, solvable and divisible. Thus, $g \circ f : \mathcal{X} \longrightarrow \mathcal{X}''$ is also an embedding. Proposition A.5(b) say that $(\mathcal{X}'', \succeq'')$ has a multiutility representation. Thus, Lemma A.2(b) implies that $(\mathcal{X}, \succeq)$ also has a multiutility representation. □

The proof of Proposition 2.4 requires some further lemmas.

Lemma A.6 *Let $(\succeq)$ be a difference preorder on $\mathcal{X}$. Let $x_0, x_1, \ldots, x_N, y_0, y_1, \ldots, y_N \in \mathcal{X}$.*

- **(a)** *If $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{n-1} \rightsquigarrow y_n)$ for all $n \in [1 \ldots N]$, then $(x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N)$.*
- **(b)** *Suppose $(\succeq)$ is solvable, and let $\alpha : [1 \ldots N] \longrightarrow [1 \ldots N]$ be a permutation. If $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{\alpha(n)-1} \rightsquigarrow y_{\alpha(n)})$ for all $n \in [1 \ldots N]$, then $(x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N)$.*

Proof. Part (a) is by inductive application of (CAT). Part (b) is a more complicated argument, which also involves (CAT*); see Pivato (2012c) for details. □

Let $(\succeq)$ and $(\succeq')$ be two difference preorders on the same set $\mathcal{X}$. We say $(\succeq')$ *extends* $(\succeq)$ if, for all $x_0, x_1, y_0, y_1 \in \mathcal{X}$, we have

$$\Big((x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)\Big) \Longrightarrow \Big((x_0 \rightsquigarrow x_1) \succeq' (y_0 \rightsquigarrow y_1)\Big). \tag{A19}$$

Lemma A.7 *Let $(\succeq)$ be a solvable difference preorder on $\mathcal{X}$.*

- **(a)** *There exists a solvable, divisible difference preorder $(\succeq^\dagger)$ on $\mathcal{X}$ which extends $(\succeq)$.*
- **(b)** *Let $\mathcal{Y} \subseteq \mathcal{X}$ be a subset, and suppose $(\succeq)$ is inductive and divisible when restricted to $\mathcal{Y}$. Then $(\succeq^\dagger)$ agrees with $(\succeq)$ on $\mathcal{Y}$.*

Proof. (a) For any $x_0, x_1 \in \mathcal{X}$, solvability allows us to inductively build an infinite sequence $\{x_2, x_3, x_4, \ldots\} \subseteq \mathcal{X}$ such that

$$(x_0 \rightsquigarrow x_1) \approx (x_1 \rightsquigarrow x_2) \approx (x_2 \rightsquigarrow x_3) \approx \cdots$$

Thus, for any $N \in \mathbb{N}$, the sequence $\{x_0, x_1, \ldots, x_N\}$ is a standard sequence. Observe that, if $\{x_2', x_3', x_4', \ldots\}$ is another sequence constructed by this recipe, then Lemma

A.6(a) yields $(x_0 \rightsquigarrow x_N) \approx (x_0 \rightsquigarrow x'_N)$ for all $N \in \mathbb{N}$. By a slight abuse of notation, we will use the notation "$N \cdot (x_0 \rightsquigarrow x_1)$" to mean any member of the $(\succeq)$-indifference class of $(x_0 \rightsquigarrow x_N)$.

Now we define the relation $(\succeq^\dagger)$ as follows. For any $x_0, x_1, y_0, y_1 \in \mathcal{X}$, stipulate that

$$\Big((x_0 \rightsquigarrow x_1) \succeq^\dagger (y_0 \rightsquigarrow y_1)\Big) \iff \Big(N \cdot (x_0 \rightsquigarrow x_1) \succeq N \cdot (y_0 \rightsquigarrow y_1) \text{ for some } N \in \mathbb{N}\Big). \tag{A20}$$

This relation extends $(\succeq)$ (to verify formula (A19), set $N = 1$ in (A20)). It remains to show that $(\succeq^\dagger)$ itself a solvable, divisible difference preorder on $\mathcal{X}$.

Proof of (INV). Let $x_0, x_1, y_0, y_1 \in \mathcal{X}$, and define the standard sequences $\{x_2, x_3, \ldots\}$ and $\{y_2, y_3, \ldots\}$ as above. Suppose $(x_0 \rightsquigarrow x_1) \succeq^\dagger (y_0 \rightsquigarrow y_1)$. Then $(x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N)$ for some $N \in \mathbb{N}$. Thus, (INV) implies that $(x_N \rightsquigarrow x_0) \preceq (y_N \rightsquigarrow y_0)$. It is easy to see that $(x_N \rightsquigarrow x_0) \approx N \cdot (x_1 \rightsquigarrow x_0)$ and $(y_N \rightsquigarrow y_0) \approx N \cdot (y_1 \rightsquigarrow y_0)$. Thus, we conclude that $(x_1 \rightsquigarrow x_0) \preceq' (y_1 \rightsquigarrow y_0)$, as desired.

Proof of (CAT). Let $x, x', x'', y, y', y'' \in \mathcal{X}$, and suppose that $(x \rightsquigarrow x') \succeq^\dagger (y \rightsquigarrow y')$ and $(x' \rightsquigarrow x'') \succeq^\dagger (y' \rightsquigarrow y'')$. We must show that $(x \rightsquigarrow x'') \succeq^\dagger (y \rightsquigarrow y'')$. Let $x_0 := x$, $x_1 := x'$, $y_0 := y$ and $y_1 := y'$, and let $\{x_0, x_1, x_2, x_3, \ldots\}$ and $\{y_0, y_1, y_2, y_3, \ldots\}$ be the standard sequences generated by the transitions $(x \rightsquigarrow x')$ and $(y \rightsquigarrow y')$. Thus, if $(x \rightsquigarrow x') \succeq^\dagger (y \rightsquigarrow y')$, then there exists some $N \in \mathbb{N}$ such that

$$(x_0 \rightsquigarrow x_N) \quad \succeq \quad (y_0 \rightsquigarrow y_N). \tag{A21}$$

Let $x'_0 := x'$, $x'_1 := x''$, $y'_0 := y'$ and $y'_1 := y''$, and let $\{x'_0, x'_1, x'_2, x'_3, \ldots\}$ and $\{y'_0, y'_1, y'_2, y'_3, \ldots\}$ be the standard sequences generated by the transitions $(x' \rightsquigarrow x'')$ and $(y' \rightsquigarrow y'')$. Thus, if $(x' \rightsquigarrow x'') \succeq^\dagger (y' \rightsquigarrow y'')$, then there exists some $M \in \mathbb{N}$ such that

$$(x'_0 \rightsquigarrow x'_M) \quad \succeq \quad (y'_0 \rightsquigarrow y'_M). \tag{A22}$$

Finally, let $x''_0 := x$, $x''_1 := x''$, $y''_0 := y$ and $y''_1 := y''$, and let $\{x''_0, x''_1, x''_2, x''_3, \ldots\}$ and $\{y''_0, y''_1, y''_2, y''_3, \ldots\}$ be the standard sequences generated by the transitions $(x \rightsquigarrow x'')$ and $(y \rightsquigarrow y'')$. To show that $(x \rightsquigarrow x'') \succeq^\dagger (y \rightsquigarrow y'')$, we must find some $L \in \mathbb{N}$ such that

$$(x''_0 \rightsquigarrow x''_L) \quad \succeq \quad (y''_0 \rightsquigarrow y''_L). \tag{A23}$$

For every $\ell \in \mathbb{N}$, solvability yields elements $x''_{\ell+\frac{1}{2}}$ and $y''_{\ell+\frac{1}{2}}$ in $\mathcal{X}$ such that

$$(x''_\ell \rightsquigarrow x''_{\ell+\frac{1}{2}}) \;\approx\; (x \rightsquigarrow x') \quad \text{and} \quad (y''_\ell \rightsquigarrow y''_{\ell+\frac{1}{2}}) \;\approx\; (y \rightsquigarrow y'), \tag{A24}$$

for all $\ell \in \mathbb{N}$. Since $(x''_\ell \rightsquigarrow x''_{\ell+1}) \approx (x \rightsquigarrow x'')$ and $(y''_\ell \rightsquigarrow y''_{\ell+1}) \approx (y \rightsquigarrow y'')$ by construction, it then follows from a straightforward application of (INV), (CAT) and (A24) that

$$(x''_{\ell+\frac{1}{2}} \rightsquigarrow x''_{\ell+1}) \;\approx\; (x' \rightsquigarrow x'') \quad \text{and} \quad (y''_{\ell+\frac{1}{2}} \rightsquigarrow y''_{\ell+1}) \;\approx\; (y' \rightsquigarrow y''), \tag{A25}$$

for all $\ell \in \mathbb{N}$. Now, by construction of the standard sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{x'_n\}_{n=1}^{\infty}$, and $\{y'_n\}_{n=1}^{\infty}$, the formulae (A21) and (A22) imply that

$$(x_{k\,N} \rightsquigarrow x_{(k+1)\,N}) \succeq (y_{k\,N} \rightsquigarrow y_{(k+1)\,N}) \quad \text{and} \quad (x'_{k\,M} \rightsquigarrow x'_{(k+1)\,M}) \succeq (y'_{k\,M} \rightsquigarrow y'_{(k+1)\,M}),$$

for all $k \in \mathbb{N}$. Thus, applying Lemma A.6(a), we obtain

$$(x_0 \rightsquigarrow x_{M\,N}) \succeq (y_0 \rightsquigarrow y_{M\,N}) \tag{A26}$$

$$\text{and} \quad (x'_0 \rightsquigarrow x'_{M\,N}) \succeq (y'_0 \rightsquigarrow y'_{M\,N}). \tag{A27}$$

Now, define $x_0^* := x_{M\,N}$ and $y_0^* := y_{M\,N}$. Using solvability, we can construct two more standard sequences $\{x_0^*, x_1^*, x_2^*, \ldots, x_{M\,N}^*\}$ and $\{y_0^*, y_1^*, y_2^*, \ldots, y_{M\,N}^*\}$ such that, for all $k \in [1 \ldots M\,N]$, we have

$$(x_{k-1}^* \rightsquigarrow x_k^*) \approx (x' \rightsquigarrow x'') \quad \text{and} \quad (y_{k-1}^* \rightsquigarrow y_k^*) \approx (y' \rightsquigarrow y''). \tag{A28}$$

By construction, we also have $(x'_{k-1} \rightsquigarrow x'_k) \approx (x' \rightsquigarrow x'')$ and $(y'_{k-1} \rightsquigarrow y'_k) \approx (y' \rightsquigarrow y'')$. Thus, applying Lemma A.6(a) to the relations (A28), we deduce that

$$(x_0^* \rightsquigarrow x_{M\,N}^*) \approx (x'_0 \rightsquigarrow x'_{M\,N}) \quad \text{and} \quad (y_0^* \rightsquigarrow y_{M\,N}^*) \approx (y'_0 \rightsquigarrow y'_{M\,N}). \tag{A29}$$

Recall that $x_0^* := x_{M\,N}$ and $y_0^* := x_{M\,N}$. Thus, (A27) and (A29) together yield $(x_{M\,N} \rightsquigarrow x_{M\,N}^*) \succeq (y_{M\,N} \rightsquigarrow y_{M\,N}^*)$. Combining this with (A26) via (CAT), we obtain

$$(x_0 \rightsquigarrow x_{M\,N}^*) \succeq (y_0 \rightsquigarrow y_{M\,N}^*). \tag{A30}$$

Now, by construction, $(x_k \rightsquigarrow x_{k+1}) \approx (x \rightsquigarrow x')$ and $(y_k \rightsquigarrow y_{k+1}) \approx (y \rightsquigarrow y')$ for all $k \in \mathbb{N}$. Thus, (A24) implies that

$$(x''_\ell \rightsquigarrow x''_{\ell+\frac{1}{2}}) \approx (x_k \rightsquigarrow x_{k+1}) \quad \text{and} \quad (y''_\ell \rightsquigarrow y''_{\ell+\frac{1}{2}}) \approx (y_k \rightsquigarrow y_{k+1}), \tag{A31}$$

for any $\ell, k \in \mathbb{N}$. Likewise, combining relation (A25) and (A28) yields

$$(x''_{\ell+\frac{1}{2}} \rightsquigarrow x''_{\ell+1}) \approx (x_{k-1}^* \rightsquigarrow x_k^*) \quad \text{and} \quad (y''_{\ell+\frac{1}{2}} \rightsquigarrow y''_{\ell+1}) \approx (y_{k-1}^* \rightsquigarrow y_k^*) \tag{A32}$$

for any $\ell, k \in \mathbb{N}$. Thus,

$$\begin{aligned}
(x_0 \rightsquigarrow x_{M\,N}^*) \quad &= \quad (x_0 \rightsquigarrow x_{M\,N} \rightsquigarrow x_{M\,N}^*) \\
&= \ (x_0 \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \cdots \rightsquigarrow x_{M\,N} = x_0^* \rightsquigarrow x_1^* \rightsquigarrow x_2^* \rightsquigarrow \cdots \rightsquigarrow x_{M\,N}^*) \\
&\underset{(*)}{\approx} \ (x''_0 \rightsquigarrow x''_{\frac{1}{2}} \rightsquigarrow x''_1 \rightsquigarrow x''_{1+\frac{1}{2}} \rightsquigarrow x''_2 \rightsquigarrow x''_{2+\frac{1}{2}} \rightsquigarrow \cdots \rightsquigarrow x''_{M\,N-\frac{1}{2}} \rightsquigarrow x''_{M\,N}) \\
&= \ (x''_0 \rightsquigarrow x''_{M\,N}).
\end{aligned} \tag{A33}$$

Here, $(*)$ is by applying Lemma A.6(b), and using relations (A31) and (A32). By a very similar argument, we deduce that

$$(y_0 \rightsquigarrow y_{M\,N}^*) \approx (y''_0 \rightsquigarrow y''_{M\,N}). \tag{A34}$$

Combining formulas (A30), (A33), and (A34), we finally obtain

$$(x''_0 \rightsquigarrow x''_{M\,N}) \succeq (y''_0 \rightsquigarrow y''_{M\,N}). \tag{A35}$$

Since $\{x''_n\}_{n=1}^{\infty}$ and $\{y''_n\}_{n=1}^{\infty}$ are the (infinite) standard sequences generated by the transitions $(x \rightsquigarrow x'')$ and $(y \rightsquigarrow y'')$, formula (A35) can be rewritten as $N\,M \cdot (x \rightsquigarrow x'') \succeq N\,M \cdot (y \rightsquigarrow y'')$. Thus, $(x \rightsquigarrow x'') \succeq^{\dagger} (y \rightsquigarrow y'')$, as desired.

Proof of (CAT*). Similar to the proof of (CAT).

Solvable. Let $x_0, x_1, y_0 \in \mathcal{X}$. Since $(\succeq)$ is solvable, there exists $y_1 \in \mathcal{X}$ such that $(x_0 \rightsquigarrow x_1) \approx (y_0 \rightsquigarrow y_1)$. Since $(\succeq^\dagger)$ extends $(\succeq)$, it follows that $(x_0 \rightsquigarrow x_1) \approx^\dagger (y_0 \rightsquigarrow y_1)$. Thus, $(\succeq^\dagger)$ is also solvable.

Divisible. Let $x_0, x_1 \in \mathcal{X}$, and let $\{x_n\}_{n=1}^\infty$ be an infinite $(\succeq)$-standard sequence generated by the transition $(x_0 \rightsquigarrow x_1)$. Thus, $(x_n \rightsquigarrow x_{n+1}) \approx (x_0 \rightsquigarrow x_1)$ for all $n \in \mathbb{N}$. Since $(\succeq^\dagger)$ extends $(\succeq)$, it follows that $(x_n \rightsquigarrow x_{n+1}) \approx^\dagger (x_0 \rightsquigarrow x_1)$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}_{n=1}^\infty$ is also an infinite $(\succeq^\dagger)$-standard sequence for the transition $(x_0 \rightsquigarrow x_1)$. Now, to verify that $(\succeq^\dagger)$ is divisible, suppose that $(x_0 \rightsquigarrow x_N) \succeq^\dagger (x_0 \rightsquigarrow x_0)$ for some $N \in \mathbb{N}$; we must show that $(x_0 \rightsquigarrow x_1) \succeq^\dagger (x_0 \rightsquigarrow x_0)$.

Lemma A.6(a) implies that the subsequence $\{x_0, x_N, x_{2N}, x_{3N}, \ldots\}$ is a $(\succeq)$-standard sequence for the transition $(x_0 \rightsquigarrow x_N)$. Meanwhile $\{x_0, x_0, x_0, \ldots\}$ can serve as the standard sequence for the transition $(x_0 \rightsquigarrow x_0)$. By definition (A20), if $(x_0 \rightsquigarrow x_N) \succeq^\dagger (x_0 \rightsquigarrow x_0)$, then there is some $k \in \mathbb{N}$ such that $(x_0 \rightsquigarrow x_{k\,N}) \succeq (x_0 \rightsquigarrow x_0)$. But then it follows that $(x_0 \rightsquigarrow x_1) \succeq^\dagger (x_0 \rightsquigarrow x_0)$, again by definition (A20).

Proof of part (b). Let $x_0, x_1, y_0, y_1 \in \mathcal{Y}$. Since $\mathcal{Y}$ is inductive, it contains an infinite standard sequence $\{x_n\}_{n=1}^\infty$ generated by the transition $(x_0 \rightsquigarrow x_1)$, and it also contains an infinite standard sequence $\{y_n\}_{n=1}^\infty$ generated by the transition $(y_0 \rightsquigarrow y_1)$. We have:

$$\Big((x_0 \rightsquigarrow x_1) \succeq^\dagger (y_0 \rightsquigarrow y_1)\Big) \underset{(*)}{\Longleftrightarrow} \Big((x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N) \text{ for some } N \in \mathbb{N}\Big)$$
$$\underset{(\diamond)}{\Longleftrightarrow} \Big((x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)\Big),$$

as desired. Here, $(*)$ is by defining formula (A20), and $(\diamond)$ is because $(\mathcal{Y}, \succeq)$ is divisible, and the standard sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are contained in $\mathcal{Y}$. □

Proof of Proposition 2.4. "$\Longrightarrow$" If $(\succeq)$ admits a multiutility representation, then Theorem 2.1 implies it can be embedded in a divisible, solvable system. Thus, $(\succeq)$ itself is divisible and semisolvable.

"$\Longrightarrow$" Suppose $(\mathcal{X}, \succeq)$ is semisolvable, divisible, and inductive. Then there exists an embedding $f : \mathcal{X} \longrightarrow \mathcal{X}'$, where $(\mathcal{X}', \succeq')$ is solvable. Lemma A.7(a) yields a solvable, divisible difference preorder $(\succeq^\dagger)$ on $\mathcal{X}'$ which extends $(\succeq')$.

Let $\mathcal{Y} := f(\mathcal{X}) \subseteq \mathcal{X}$. Then $(\succeq')$ is inductive and divisible when restricted to $\mathcal{Y}$, because $(\succeq)$ is inductive and divisible on $\mathcal{X}$, and f is an isomorphism from $(\mathcal{X}, \succeq)$ to $(\mathcal{Y}, \succeq')$. Thus, Lemma A.7(b) says that $(\succeq^\dagger)$ agrees with $(\succeq')$ on $\mathcal{Y}$. Thus, $f : (\mathcal{X}, \succeq) \longrightarrow (\mathcal{X}', \succeq^\dagger)$ is also an embedding. Thus, $(\mathcal{X}, \succeq)$ can be embedded in a divisible, solvable system. Thus, Theorem 2.1 says $(\mathcal{X}, \succeq)$ has a multiutility representation. □

Proof of Proposition 4.1. It suffices to show this in the case $|\mathcal{X}| = 24$. So, let $\mathcal{X} := \{x_0, x_1, \ldots, x_7, y_0, y_1, \ldots, y_7, z_0, z_1, \ldots, z_7\}$. Define the preorder $(\succeq)$ on $\mathcal{X} \times \mathcal{X}$ as follows. Begin with all $|\mathcal{X}|^2 = 576$ "trivial" relations implied by Lemma A.1. To this set, add the following relations, for all $n, m, n', m' \in [0 \ldots 7]$:

(a) $(x_n \rightsquigarrow x_m) \approx (x_{n'} \rightsquigarrow x_{m'})$ and $(y_n \rightsquigarrow y_m) \approx (y_{n'} \rightsquigarrow y_{m'})$ and $(z_n \rightsquigarrow z_m) \approx (z_{n'} \rightsquigarrow z_{m'})$ if and only if $n - m = n' - m'$.

(b) $(x_n \rightsquigarrow x_m) \succ (y_{n'} \rightsquigarrow y_{m'})$ if and only if $n - m = n' - m' > 0$ and is divisible by 3.

(c) $(y_n \rightsquigarrow y_m) \succ (z_{n'} \rightsquigarrow z_{m'})$ if and only if $n - m = n' - m' > 0$ and is divisible by 5.

(d) $(z_0 \rightsquigarrow z_7) \succ (x_0 \rightsquigarrow x_7)$.

Also add the (INV)-reversals of the sets of relations described in (b), (c) and (d) (the set (a) is already closed under (INV)). Observe that the four relation sets described in (a)-(d) are each separately closed under the application of (CAT) and (CAT*). Also, there is no way to combine a relation from one of these sets (e.g. (b)) with one from another (e.g. (c)) using (CAT) or (CAT*). Thus, the entire system is closed under (CAT) and (CAT*); thus, it is a difference preorder. We claim it is not Szpilrajn.

By contradiction, suppose that $(\underset{c}{\succeq})$ is a complete difference preorder on $\mathcal{X}$ which extends and refines $(\succeq)$. Then (a), (b) and (CAT) imply that $(x_0 \rightsquigarrow x_1) \succ (y_0 \rightsquigarrow y_1)$. Likewise, (a), (c) and (CAT) imply that $(y_0 \rightsquigarrow y_1) \succ (z_0 \rightsquigarrow z_1)$. Finally, (a), (d) and (CAT) imply that $(z_0 \rightsquigarrow z_1) \succ (x_0 \rightsquigarrow x_1)$. Thus, we have an cycle of strict preferences, yielding a contradiction. It follows that $(\succeq)$ is not Szpilrajn. Thus, $(\succeq)$ cannot have any strong utility functions. □

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