Preferences and Increased Risk Aversion under a General Framework of Stochastic Dominance

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Abstract

This paper analyzes increased risk aversion in the presence of two risks. Necessary and sufficient conditions for increased risk aversion across the domain of the foreground risk are found for changes in both the foreground and background risks. Preferences that satisfy the necessary and sufficient conditions are determined through a lower bound on their measure of prudence. These bounds are found through second-degree spreads of a transformation of the background risk. The necessary and sufficient conditions demonstrate that for all second degree spreads of this nature, absolute temperance plays a central role in the necessary and sufficient conditions for increased risk aversion. The approach also demonstrates that changes in risk aversion under the general framework of stochastic dominating spreads can be explained by a weighted average of terms involving absolute prudence and absolute temperance. Once a general set of necessary and sufficient conditions have been found it is shown that for preferences that are decreasing absolute risk averse in the sense of Ross, increased risk aversion due to changes in the background risk within this framework is equivalent to Ross risk vulnerability. The general conditions also find necessary and sufficient conditions for preferences to be properly risk averse toward patent increases in risk.

INDEX WORDS: Stochastic dominance, increased risk aversion, background risk, transformations, patent increase in risk, prudence, proper risk aversion, risk vulnerability

Introduction

The existence of a background risk has been seen to be of particular interest in the research on risk behavior. Ross (1981) observed that the Arrow-Pratt measure of risk aversion is inadequate in the sense that upon the introduction of a background risk a more risk averse agent may not behave in a more risk averse manner whenever a foreground risk is present. Other research has explored the effects of aversion to risk upon the introduction of a stochastically independent background risk for certain types of risk preferences as defined in Pratt and Zeckhauser (1987), Kimball (1993), and Gollier and Pratt (1996). Gollier and Pratt demonstrate how these previously defined classes of risk preferences are related by identifying sufficient conditions for any agent possessing these qualities to react to the introduction of a small, unfair, and independent background risk by becoming more risk averse to bearing a foreground risk. Their notion of preferences possessing these qualities, known as risk vulnerability, captures this common quality for all of these classes of risk preferences and is the widest set of preferences upon which the introduction of background risk will generate more risk averse behavior. However, Gollier and Pratt do not present a systematic method for obtaining the different risk measures that arise in these various cases.

The introduction of background risk has garnered a relatively greater share of the spotlight than has the more common situation of background risk already being present. Eeckhoudt et al. (1996) have derived necessary and sufficient conditions on utility for increased risk aversion under stochastic dominating shifts in the distribution of the background risk. Their method yields conditions for greater risk aversion whenever an already present background risk undergoes an unfavorable change in its distribution. They derive conditions for increased risk aversion for arbitrary first degree stochastically dominated spreads and then for arbitrary mean preserving spreads. Their approach can be extended to arbitrary second degree spreads and this is the approach taken here. In doing so, the author believes a greater understanding of increased risk aversion can be achieved under the general framework of stochastic dominating spreads by focusing on that by which first degree spreads and mean preserving spreads are related, namely second degree stochastic dominance. In doing so, an alternative set of necessary and sufficient conditions for increased risk aversion is seen to exist under the general framework of second degree stochastic dominating spreads. These conditions reveal the importance temperance and prudence both play in explaining changes in risk aversion within the framework of stochastic dominating spreads in risk. It will be shown that the measure of temperance relative to prudence helps explain any change in risk aversion under the general framework of stochastic dominance.

Once these universal conditions are known, specific transformations of the background risk will yield necessary and sufficient conditions for increased risk aversion for families of von Neumann-Morgenstern utility functions identifiable by conditions involving their measure of prudence. Once the model has been developed to account for the presence of both risks, necessary and sufficient conditions for increased risk aversion will include a condition on compensated increases in a foreground risk as well as some comparative statics related to increases in background risk. Necessary and sufficient conditions for preferences to be properly risk averse toward a patent increase in risk are identified.

Literature Review

The notion of more risk averse is equivalent to higher risk premiums. Pratt (1964) has shown the risk premium, π, to be a function of the distribution of a foreground risk, θ, and an endowment, ω. Defining F to be the cumulative distribution of this risk with compact support Θ, the risk premium solves for the equality, $\int_{\Theta} u(\omega+\theta)\,dF(\theta) = u\left(\omega + \int_{\Theta} \theta\,dF(\theta) - \pi(\omega, F)\right)$. Agent A is said to be locally more risk averse at wealth level ω than agent B when A's risk premium at ω is higher than B's, i.e. $\pi^A(\omega, F) \geq \pi^B(\omega, F)$. Pratt has also shown that an agent with a higher risk premium at a given level of wealth has a higher Arrow-Pratt measure of absolute risk aversion at this level as well, defined as $r(\omega) := \frac{-u''(\omega)}{u'(\omega)}$. Preferences are said to be decreasing absolute risk averse whenever lower levels of wealth coincide with higher measures of Arrow-Pratt risk aversion, that is $r(\omega-\delta) \geq r(\omega)$ for any level of wealth with $\delta > 0$.

The Arrow-Pratt measure of absolute risk aversion can be unreliable as an indicator that the more risk averse will behave in a more risk averse way when a second risk is introduced. An agent that faces a foreground risk may have a risk premium that changes in a manner not consistent with his Arrow-Pratt measure of risk aversion upon the introduction of a second risk. Ross (1981) has shown that despite one agent being uniformly more risk averse than another in the sense of Arrow and Pratt, i.e., $r^A(\omega) \geq a \geq r^B(\omega) \forall \omega$, it is still possible for agent A to have a lower risk premium than agent B due to an introduction of a background risk. This can occur in a lottery setting whenever the background risk is associated with a particular payoff such that the likelihood of it occurring is sufficiently small. Pratt (1990) adds clarity to this counterintuitive result by arguing that such behavior becomes more likely for any agent that is more risk averse than A due to the greater relative importance placed by these agents on changes in less desirable outcomes; while the less risk averse place greater relative importance on changes in more desirable outcomes[i].

Ross goes further and demonstrates that certain, more restricted preferences do not present such a difficulty. Given any level of wealth, ω, if there exists a scalar *a* such that $\frac{-u'''(\omega_1)}{u''(\omega_1)} \geq a \geq \frac{-u''(\omega_2)}{u'(\omega_2)}$ for any wealth levels ω_1 and ω_2 contained in a sufficiently small interval centered at ω, then the agent exhibits decreasing absolute risk aversion in the sense of Ross which implies decreasing absolute risk aversion in the Arrow-Pratt sense. Satisfaction of this local condition for a von Neumann-Morgenstern utility function assures us that the agent will have a higher risk premium whenever their Arrow-Pratt measures of risk aversion increase. An example of how such information is useful involves the relationship between risk premiums and insurance premiums. A background risk is an uninsurable risk. In the presence of a background risk, individuals at best acquire partial insurance for foreground risks. A higher risk premium following the introduction of a background risk implies a higher willingness to pay for partial insurance of a foreground risk in the presence of a background risk. Decreasing absolute risk aversion in the sense of Ross is equivalent to $p(\omega+\varepsilon) \geq \lambda(\omega) \geq r(\omega+\varepsilon)$.

Kihlstrom et al. (1981) confirmed Ross's conclusion that the Arrow-Pratt measure of risk aversion is unreliable as an indicator of risk averse behavior in general upon the introduction of background risk, even if the two risks are statistically independent. Nonetheless, they go on to demonstrate the usefulness of statistically independent risks in deriving comparative statics results and proved that under this restriction, nonincreasing absolute risk aversion is preserved for expected utility, i.e. $\frac{-u''(\omega+\theta)}{u'(\omega+\theta)} \geq \frac{-u''(\omega)}{u'(\omega)} \Rightarrow \frac{-E\left[u''(\omega+\theta+\varepsilon)|\omega,\theta\right]}{E\left[u'(\omega+\theta+\varepsilon)|\omega,\theta\right]} \geq \frac{-E\left[u''(\omega+\varepsilon)|\omega\right]}{E\left[u'(\omega+\varepsilon)|\omega\right]}$ with $\theta < 0$. They also demonstrate that when the agent is decreasing absolute risk averse in the sense of Ross, expected utility inherits this property as long as the independence condition holds between the two random variables. Under the assumption of statistically independent risks, they discover that higher risk premiums are associated with higher Arrow-Pratt measures of risk aversion following the introduction of a statistically independent background risk as long as preferences are nonincreasing absolute risk averse.

Absolute prudence has gained some prominence in the literature of the theory of risk. Kimball (1990) describes prudence as 'a propensity to forearm oneself in the face of uncertainty'. Unlike the measure of absolute risk aversion, which measures the intensity by which an individual likes or dislikes risk at a given level of wealth, prudence measures the sensitivity of a decision variable under conditions of optimality. Quite often prudence is referred to as the precautionary motive, or an individual is said to be prudent when the third derivative of the utility function is positive. It is well known that decreasing absolute risk aversion implies an agent's measure of absolute prudence is no less than his measure of absolute risk aversion. Prudence proves to play a central role in identifying preferences that experience increased risk aversion in this paper.

The behavioral condition on preferences that an unattractive lottery can never become more attractive due to the presence of another independent, unattractive lottery accurately describes properly risk averse preferences and was originally introduced by Pratt and Zeckhauser (1987). An agent who is properly risk averse will have a higher risk premium upon the introduction of an independent risk at any level of wealth, regardless of whether the wealth level is random or nonrandom. Whenever wealth is nonrandom and preferences are properly risk averse, this will be referred to as 'fixed wealth proper' risk averse preferences. Utility functions that are properly risk averse are also decreasing absolute risk averse in the Arrow and Pratt sense.

Kimball (1993) considered the set of independent loss-aggravating risks. A risk is loss-aggravating if the reduction in expected utility increases as wealth is reduced by a small amount once that risk has been introduced. That is to say, ε is loss-aggravating for a decrease in wealth of size δ > 0 if $E\left[u(\omega+\varepsilon)-u(\omega-\delta+\varepsilon)\right]\geq u(\omega)-u(\omega-\delta)$. For an infinitesimally small reduction in wealth this is equivalent in the limit to $Eu'(\omega+\varepsilon)\geq u'(\omega)$. A statistically independent background risk will be loss aggravating for preferences that are standard risk averse when an undesirable risk is already present. Kimball proves that necessary and sufficient conditions for preferences to be standard risk averse are that absolute prudence, defined as $p(\omega):=\frac{-u'''(\omega)}{u''(\omega)}$, and absolute risk aversion be decreasing in wealth. Standard risk averse preference are also properly risk averse.

Risk vulnerable preferences are the most general class of preferences that satisfy the attractive quality of more risk averse preferences behaving in a more risk averse manner when a statistically independent background risk is introduced. Gollier and Pratt (1996) define preferences as being risk vulnerable when the introduction of any unfair background risk makes the agent behave in a more risk averse manner. In the case of a small fair background risk, they provide an expression that approximates the local relative change in the risk premium due to the introduction of the risk,

$$\frac{\Pi(\omega,F)-\pi(\omega,F)}{\pi(\omega,F)}\cong\frac{-\frac{Eu''(\omega+\varepsilon)}{Eu'(\omega+\varepsilon)}-r(\omega)}{r(\omega)}\cong\frac{E\theta^2 p(\omega)\left[\frac{-u^{(4)}(\omega)}{u'''(\omega)}-r(\omega)\right]}{2+E\theta^2 p(\omega)r(\omega)}$$

where $\Pi(\omega,F)$ is the risk premium that solves for the following equality in the presence of a background risk with distribution H and compact support E;

$$\int_{\mathrm{E}}\int_{\Theta}u(\omega+\theta+\varepsilon)dF(\theta)dH(\varepsilon)=\int_{\mathrm{E}}u\left(\omega+\varepsilon+\int_{\Theta}\theta dF(\theta)-\prod(\omega,F)\right)dH(\varepsilon)$$

A necessary and sufficient condition for risk vulnerability is that for every level of wealth both absolute prudence and absolute temperance, defined as $t(\omega) := \frac{-u^{(4)}(\omega)}{u'''(\omega)}$, are no less than absolute risk aversion. Generally speaking, preferences are said to be locally risk vulnerable at ω if $p(\omega) \geq r(\omega)$ and $t(\omega) \geq r(\omega)$. They note necessary and sufficient conditions for various classes of risk preferences that must hold for local risk vulnerability and discover that properly risk averse and standard risk averse preferences are risk vulnerable.

Rothschild and Stiglitz (1970) provide a definition for an increase in risk as a mean preserving spread of the risk. A distribution F parameterized by α_1 is said to undergo a mean preserving spread indexed by parameter α_2 when the following conditions are satisfied:

$$\int_{\underline{\theta}}^{\theta}\left[F(t;\alpha_2) - F(t;\alpha_1)\right]dt \begin{cases} \geq 0, \forall \theta \in \Theta \\ = 0, \theta = \overline{\theta} \end{cases} \tag{1.1}$$

Where the 'underbar' and 'overbar' notation denotes the minimum and maximum elements of the compact support of F (Θ) respectively. This is a special case of second degree stochastic dominance, and throughout the paper this condition, commonly referred to as a Rothschild-Stiglitz increase in risk, will be expressed as the partial ordering, $F(\theta;\alpha_1) \succeq_{MP} F(\theta;\alpha_2)$. An important result of their paper is that any risk averse agent when given the choice of the lotteries indexed by α_1 and α_2 will reject the lottery indexed by α_2 whenever (1.1) holds. That is, for any concave utility function, u(θ), $\int_\Theta u(\theta)dF(\theta;\alpha_1) \geq \int_\Theta u(\theta)dF(\theta;\alpha_2)$ whenever the conditions given in (1.1) are satisfied.

Eeckhoudt et al. (1996) utilize the notion of first degree stochastically dominated and mean preserving increases in background risk to provide a framework for analyzing an agent's behavior under such changes in an already present background risk. They derive necessary and sufficient conditions for increased risk aversion within this setting that are specific to the nature of the stochastic dominating shift of the background risk that occurs. If it is a first degree stochastic dominating shift over the compact space E with cumulative distribution function H, i.e. a shift in risk satisfying the conditions,

$$\int_{\underline{\varepsilon}}^{\varepsilon} d\left[H\left(t;\beta_2\right)-H\left(t;\beta_1\right)\right]\begin{cases}\geq 0, \forall \varepsilon \in \mathrm{E} \\ = 0, \varepsilon = \bar{\varepsilon}\end{cases}$$

then decreasing absolute risk aversion in the sense of Ross is necessary and sufficient for increased risk aversion. On the other hand, if the background risk undergoes a mean preserving spread in the background risk then another boundedness condition, $t\left(\omega+\varepsilon\right)\geq\lambda\left(\omega\right)\geq r\left(\omega+\varepsilon\right)$, is necessary and sufficient for increased risk aversion. They conclude that both conditions must be satisfied if any second degree spread in background risk is to cause increased risk aversion. One result of this paper confirms their result but also proves that for any stochastic dominating spreads of a second degree nature that is not a first degree spread, their conditions are indeed sufficient but not necessary for increased risk aversion.

An alternative definition of increases in risk involves the use of utility distributions. Diamond and Stiglitz (1974) consider mean preserving spreads in the distribution of utility via change of variable techniques. They note that if marginal utility is nonzero, change of variable techniques yield a set of conditions that hold under mean preserving spreads in utility. Thus, if $\tilde{F}$ is the utility distribution function indexed by the distribution of $F\left(\theta;\alpha\right)$, and α_2 indexes a mean preserving spread in $\tilde{F}$, then $\tilde{F}\left(U;\alpha_1\right)\succeq_{MP}\tilde{F}\left(U;\alpha_2\right)$ where $U=u\left(\theta\right)$, and it follows that

$$\int_{\underline{\theta}}^{\theta} u_\theta\left(t\right)\left[\tilde{F}\left(u\left(t\right),\alpha_2\right)-\tilde{F}\left(u\left(t\right);\alpha_1\right)\right]dt=\int_{\underline{\theta}}^{\theta} u_\theta\left(t\right)\left[F\left(\theta;\alpha_2\right)-F\left(\theta;\alpha_1\right)\right]dt\begin{cases}\geq 0, \forall \theta \in \Theta \\ = 0, \theta = \bar{\theta}\end{cases}$$

A mean preserving utility spread is conveniently viewed as a compensated increase in risk. Such a spread in risk for an individual is not preferred by anyone more risk averse as indicated by a higher Arrow-Pratt measure of absolute risk aversion.

Transformations of random variables have been utilized to alter the mean and variance of risk within an optimization problem (Sandmo (1971)) and cause a Rothschild-Stiglitz increase in risk (Meyer and Ormiston (1989)). A deterministic transformation of a random variable is nondecreasing in the random variable. Such a transformation preserves the ranking of preferences in a stochastic environment. For example, if k is a function that is a deterministic transformation of the foreground risk, then

$k(\theta_1) \le k(\theta_2) \forall \theta_1 \le \theta_2$. Meyer ((1977) and (1989)) has also analyzed such transformations under a stochastic dominance framework in an effort to rank deterministic transformations of a random variable. Meyer describes stochastic dominance with respect to a function as:

$$\int_{\underline{\theta}}^{\theta} \left[F(t;\alpha_2) - F(t;\alpha_1) \right] dk(\theta) \ge \forall \theta \in \Theta \Leftrightarrow \int_{\Theta} u(\theta) d\left[F(\theta;\alpha_2) - F(\theta;\alpha_1) \right] \le 0 \forall u(\theta) \in \left(\tfrac{-k''(\theta)}{k'(\theta)}, \infty \right)$$

Hence, any agent with absolute risk aversion greater than or equal to $\frac{-k''(\theta)}{k'(\theta)}$ will choose the lottery indexed by α_1 over that indexed by α_2 when forced to choose between these two lotteries. Meyer's research has successfully generalized the notion of stochastic dominance. The results that follow differ from that of Meyer in several respects. First, all families of utility functions with prudence measures that are pointwise bounded from below by a similar looking ratio for the transformation of the background risk space will experience increased risk aversion whenever the distribution undergoes a second degree spread in the transformed background risk. Secondly, all preferences with prudence measures that exceed the aforementioned bound will experience an increase in their expected marginal utilities under the deterioration in background risk. Finally, all stochastic dominating spreads will be occurring for transformations of background risks rather than background risks directly.

Increased Aversion to Risk

Introductions of risk, often a relevant part of the definition of classes of risk preferences as in the case of Pratt and Zeckhauser (1987), Kimball (1993), and Gollier and Pratt (1996), may be considered to be nothing more than an increase in the variance of an 'improper' distribution. An improper distribution for any risk will be one in which there is no variance, i.e. all the probability mass occurs at some particular value. When this is the case, with H the distribution function for the background risk, parameter β_1 will be the parameter that represents an improper distribution for the background risk. Either a first or second degree stochastically dominated increase in background risk indexed by 'j', may occur when a background risk is already present, indexed by 'i'.

It will be convenient to work with an indirect utility function in what immediately follows. Following convention, let $v(\theta;\beta_i)=\int_{\mathrm{E}} u(\theta,\varepsilon)\,dH(\varepsilon;\beta_i)$ be the indirect utility function indexed by parameter β_i. Indirect utility is strictly monotonic in θ, indicating $v(\theta;\beta_i)=V^{\beta_i}$ can be inverted to apply the change of variable technique. For a nontrivial second degree stochastic dominating spread in indirect utility, V^{β_i}, the appropriate conditions are

$$\int_{\min\{v(\Theta)\}}^{V^{\beta_i}}\left[\tilde{F}(t;\alpha_w)-\tilde{F}(t;\alpha_h)\right]dt=\int_{\underline{\theta}}^{\theta}\left[F(s;\alpha_w)-F(s;\alpha_h)\right]v_\theta(s;\beta_i)\,ds\begin{cases}\geq 0,\ \forall\theta\in\Theta\\ >0,\ \text{some }\theta\in\Theta\end{cases}\tag{2.1}$$

Any distribution by definition is a second degree spread of itself and is sometimes referred to as a null spread. Such spreads are of no interest in this paper. Nonetheless, it is possible for null spreads to be the only types of spreads that satisfy certain conditions. For any second degree spread it will be assumed that there is some θ for which (2.1) is a strict inequality, thereby ruling out null spreads.

Keenan and Snow (2003) have shown that when background risk is initially absent, $\beta = \beta_1$, any compensated increase in risk satisfying (2.1), with equality for θ equaling the maximum value of θ, that is accompanied by the introduction of a small, fair, background risk reduces indirect utility if and only if the introduction of the background risk causes the agent to be more risk averse in the sense of Arrow and Pratt. Recognizing that a compensated increase in risk is specific to the individual's preferences, this result can be extended to include cases where the background risk is already present by making the compensated increase in risk specific to the agent's preferences given the presence of background risk. The lemma below establishes this result. The proof makes use of the fact that any monotonically increasing, concave utility function will be worse off under the distribution that is stochastically dominated than it is under the distribution that stochastically dominates, as shown by Hadar and Russell (1969).

Much of the analytical work that follows involves conditions that hold at the max or min of the compact support for the cumulative distribution functions. These values will be recognized by the 'overbar' and 'underbar' notation respectively to reduce the burden of notation.

Lemma 1*: Let E be the support of H(ε;β), Θ be the support of F(θ;α) with background risk ε, and foreground risk θ, independent random variables with c.d.f.s F and H respectively, where α_w indexes a compensated increase in risk for* V^{β_i} *as in* (2.1) *with equality when* $\theta = \bar{\theta}$. *Define* $R(\theta;\beta_i) := \frac{-v''(\theta;\beta_i)}{v'(\theta;\beta_i)}$.

For any utility function such that $u_\theta > 0$ *and* $u_{\theta\theta} \leq 0$:

$\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] \leq 0$ *for any* α_w *satisfying* (2.1) *that is not a null spread with equality holding when* $\theta = \bar{\theta}$ *if and only if* $R(\theta;\beta_j) \geq R(\theta;\beta_i)$ *for all* $\theta \in \Theta$.

Proof:

$$\begin{aligned}
&\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] \\
&= -\int_\Theta \frac{v'(\theta;\beta_j)}{v'(\theta,\beta_i)} v'(\theta;\beta_i)\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] d\theta \\
&= \int_\Theta \frac{\partial}{\partial\theta}\left(\frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)}\right) \int_{\underline{\theta}}^{\theta} v'(s;\beta_i)\left[F(s;\alpha_w) - F(s;\alpha_h)\right] ds d\theta - \frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)} \int_{\underline{\theta}}^{\theta} v'(s;\beta_i)\left[F(s;\alpha_w) - F(s;\alpha_h)\right] ds \Big|_\Theta
\end{aligned}$$

Under a mean preserving spread of indirect utility, V^{β_i}, indexed by the change in parameter from α_h to α_w the equality reduces to:

$$\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] = \int_\Theta \frac{\partial}{\partial\theta}\left(\frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)}\right) \int_{\underline{\theta}}^{\theta} v'(s;\beta_i)\left[F(s;\alpha_w) - F(s;\alpha_h)\right] ds d\theta \tag{2.2}$$

Notice that $\frac{\partial}{\partial\theta}\frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)} = \frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)}\left(R(\theta;\beta_i) - R(\theta;\beta_j)\right)$. Hence the sign of the right hand side of (2.2) will be determined by the difference in risk aversion due to a change in the parameter of the distribution involving the background risk. Thus, when $R(\theta;\beta_j) \geq R(\theta;\beta_i) \forall \theta$ it follows that

$\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] \leq 0$ for any α_w that indexes a change in the distribution of the foreground risk satisfying (2.1). This proves sufficiency.

For necessity, I will prove the contra positive. Suppose that $R(\theta;\beta_j) < R(\theta;\beta_i)$ for some $\hat{\theta} \in \Theta$.

By R continuous in θ, there exists a connected subset $A \subset \Theta$ of nonzero measure such that $\hat{\theta} \in A$ and

$R(a;\beta_j) < R(a;\beta_i)$ for any $a \in A$. Then $\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] > 0$ for the following cumulative distribution function:

$$\tilde{F}(V;\alpha_w) = \begin{cases} P, \forall V \in v(A;\beta_i) \\ \tilde{F}(V;\alpha_h), \forall V \notin v(A;\beta_i) \end{cases}$$

Where P satisfies $\int_{\inf\{v(A;\beta_i)\}}^{V} \left[P - \tilde{F}(t;\alpha_h)\right] dt \begin{cases} \geq 0 \forall V \in v(A;\beta_i) \\ = 0, V = \sup\{v(A;\beta_i)\} \end{cases}$ with a strict inequality over some subset of $v(A;\beta_i)$ and is known to exist by the integrand being continuous in P and

$$\int_{v(A;\beta_i)} \left[\tilde{F}\left(\inf\{v(A;\beta_i)\};\alpha_h\right) - \tilde{F}(t;\alpha_h)\right] dt \leq 0 \leq \int_{v(A;\beta_i)} \left[\tilde{F}\left(\sup\{v(A;\beta_i)\};\alpha_h\right) - \tilde{F}(t;\alpha_h)\right] dt$$

This distribution possesses the quality of being stochastically dominated in a second degree sense and yields the following terms after a change of variable, integrating over Θ rather than $v(\Theta;\beta_i)$.

$$\int_{\min\{v(\Theta;\beta_i)\}}^{V} \left[\tilde{F}(t;\alpha_w) - \tilde{F}(t;\alpha_h)\right] dt = \begin{cases} 0, \forall \theta \notin A \\ \int_{\inf A}^{\theta} v_\theta(s;\beta_i)\left[P - F(s;\alpha_h)\right] ds, \theta \in A \end{cases}$$

Under this particular second degree spread in risk,

$$\frac{\partial}{\partial\theta}\left(\frac{v'(\theta;\beta_j)}{v'(\theta;\beta_i)}\right) \int_{\underline{\theta}}^{\theta} v'(s;\beta_i)\left[F(s;\alpha_w) - F(s;\alpha_h)\right] ds \begin{cases} = 0, \theta \notin A \\ \geq 0, \theta \in A \end{cases}$$

with a strict equality holding over some subset of A. Therefore,

$$\int_\Theta v(\theta;\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] > 0$$

Q.E.D.

The change in parameter from β_i to β_j represents a shift in the distribution of the background risk that causes the agent to be more risk averse. This increased aversion to risk is sufficient for the compensated increase in risk for preferences V^{β_i} to be unattractive for preferences V^{β_j} indicated by the decrease in the latter's well-being. Any compensated increase in risk with background risk present that is deemed unattractive under another background risk indicates the agent simultaneously experiences greater aversion to risk for all θ and a lower level of well being under the latter background risk. That is to say, an unfavorable change in the distribution of the background risk also increases the agent's aversion to risk over the domain of the foreground risk (F). Observe at this point the distributions that both the β_i and β_j parameters index bear no specific relation to each other.

Lemma 1 involves a special case of stochastic dominance with respect to a function as described by Meyer (1977), modified only by the addition of background risk. That is, $F(\theta;\alpha_h)$ stochastically dominates $F(\theta;\alpha_w)$ with respect to $v(\theta;\beta_i)$. This is Meyer's necessary and sufficient condition for any utility function with absolute risk aversion greater than $R(\theta;\beta_i)$ that weakly prefers the distribution for the foreground risk that dominates. Lemma 1 indicates that there exists indirect utility functions that belong to this family, with the unique interpretation that all of these indirect utility functions will reject the compensated increase in risk 'constructed' under the initial distribution for the background risk indexed by β_i. If this is true for all θ then any one of these other utility functions will not prefer the compensated increase in risk for $v(\theta;\beta_i)$ under the stochastically dominated distribution indexed by α_w. Thus the family of utility functions implied by greater aversion to risk in lemma 1 includes not only indirect utility functions given by various second degree spreads of the background risk; but other indirect utility functions as well including distributions of transformations of the background risk all of whom share the common property of higher measures of absolute risk aversion for all θ. This aspect of the lemma will be useful in understanding the results acquired in this paper as we examine stochastic dominating spreads of functions of the background risk.

Transformed Background Risk and Prudence

Eeckhoudt et al. (1996) note that increased variance in the background risk can encompass more complicated changes in the distribution than the analytically common case of adding another independent risk. In the spirit of this particular observation, one goal of this paper is to treat a broad spectrum of spreads in risk which can be accomplished via stochastic dominating spreads of a transformation of the background risk domain (the domain of H). In doing so, second degree stochastic dominating spreads encompass a richer set of distributions. An example of a stochastically dominated spread of a transformation of risk has already been considered in (2.1) which involved a mean preserving spread of the transformed foreground risk given by the mapping $v^{\beta_i} : \Theta \rightarrow V^{\beta_i}$. The stochastic dominating relationship of interest in this case involved one existing over the transformed foreground risk. Generally speaking, what I refer to as a transformed risk is the transformation of the compact support for a distribution or the domain of the distribution. The transformed background risk will be given by the transformation function $\phi : \mathrm{E} \rightarrow \Phi$, where E is the domain of the background risk. The purpose of this type of transformation differs from that of deterministic transformations which seem to be most useful as a means of altering a random variable under optimal choice problems.

The transformed background risk generalizes the notion of stochastic dominating spreads in a simple manner. All spreads will be of the transformed background risk. To fix notation, denote the cumulative distribution functions for the transformed background risk as $\tilde{H}$. Let $\tilde{\varepsilon}$ be a function of the transformed background risk. Sticking with the tilde notation to emphasize what background risk is relevant, the indirect utility function is seen to be

$$\tilde{v}(\theta;\beta_i) = \int_{\Phi} u\left(\theta + \tilde{\varepsilon}(\phi)\right) d\tilde{H}(\phi,\beta_i) \tag{3.1}$$

Changes in the parameter β index changes in the distribution of ϕ. Stochastic dominance of degree one or two will be indexed by the partial ordering relation $\succeq_n$. The relation conveys the idea that one distribution stochastically dominates the other by degree 'n' where n equals 1 or 2. For a Rothschild-Stiglitz increase in risk, the partial ordering is given by $\succeq_{MP}$.

Increased risk aversion involving the transformed background risk is seen to be equivalent to the following expression after a little algebraic manipulation[1]:

$$-\tilde{v}''\left(\theta;\beta_j\right) + \tilde{v}''\left(\theta;\beta_i\right) \geq \tilde{R}\left(\theta;\beta_i\right)\left[\tilde{v}'\left(\theta;\beta_j\right) - \tilde{v}'\left(\theta;\beta_i\right)\right] \forall \theta \tag{3.2}$$

[1] Multiply $\tilde{R}\left(\theta;\beta_j\right) \geq \tilde{R}\left(\theta;\beta_i\right)$ by $\frac{\tilde{v}'\left(\theta;\beta_j\right)}{-\tilde{v}''\left(\theta;\beta_i\right)}$ and subtract 1.

To determine what conditions are associated with increased risk aversion culminating from arbitrary second degree spreads of the transformed background risk we will initially focus on the basic idea that the right hand side of (3.2) is of uniform sign. Doing so simplifies the task of signing two functions of interest that are central to the results of this analysis. Fundamentally, nonnegativity of the right hand side for an arbitrary second degree spread implies nonnegativity of the left hand side under the same conditions. Whenever the right hand side of (3.2) is positive for arbitrary second degree spreads of ϕ the sign of $\tilde{\varepsilon}'(\phi)$ offers a sensible interpretation of the function $\tilde{\varepsilon}(\phi)$ within the framework of deterministic transformations described by Meyer and Ormiston (1989). Analysis of the less restrictive condition that the right hand side of (3.2) be nonnegative is simplified by the results obtained from the more restrictive case. These results are central to all other results that follow. Given this, our first task is to find necessary and sufficient conditions for

$$\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)>(<)0\forall\beta_j\text{, where }\tilde{H}(\phi;\beta_i)\succeq_2\tilde{H}(\phi;\beta_j) \tag{3.3}$$

There are necessary and sufficient conditions for either sign in (3.3). I will proceed to informally discuss the necessary and sufficient conditions that exist for the case of positive differences. It turns out that the conditions concerning a decrease in indirect marginal utility will not be of interest in this paper due to an issue discussed in Meyer and Ormiston. If utility is affected by the risk through a function such that the function is nonincreasing in the domain of the risk, i.e. $\tilde{\varepsilon}'(\phi)\leq 0$, then the ranking of lotteries most likely will not be preserved. While conditions do exist for either sign and both will be given below, it turns out that positive differences for (3.3) is the economically relevant one.

There are several observations to be made concerning subsets of the transformed background risk for which the necessary condition for the difference in (3.3) being positive does not hold and it will be seen that it is always possible to construct a second degree spread such that the difference in (3.3) is not positive whenever one of the necessary conditions do not hold. This is a basic ‘not B implies not A’ argument often used to prove ‘A implies B’. Relationships of second degree stochastic dominance include those of a first degree nature as well as those of a mean preserving nature. Graphs of these various types of second degree spreads will be considered prior to the formal proof. The function $\tilde{\varepsilon}:\Phi \to \tilde{\varepsilon}(\Phi)$ is assumed to be twice differentiable. The necessary conditions for the difference in (3.3) to be positive are

$$\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2} - p\left(\theta + \tilde{\varepsilon}(\phi)\right) < 0 \text{ for almost all } \varphi \tag{3.4}$$

and

$$\tilde{\varepsilon}'(\phi) > 0 \text{ for almost all } \varphi \tag{3.5}$$

Considering the claim (3.5) initially, suppose $\tilde{\varepsilon}'(\phi) \le 0$ over some set denoted as $B = \bigcup_t B_t$ where $B := \left\{\phi \in \Phi : \tilde{\varepsilon}'(\phi) \le 0\right\}$ and any B_t is connected. The closure of B, denoted as $\bar{B}$ consists of all the elements of B and the limit points of B[2]. A first degree stochastic worsening in the distribution of the risk ϕ includes one in which the deterioration in the distribution involves leftward shifts of probability mass occurring only over the closure of B, i.e. $\tilde{H}\left(\phi;\beta_j\right) \ge \tilde{H}\left(\phi;\beta_i\right) \forall \phi \in \bar{B}$ and $\tilde{H}\left(\phi;\beta_j\right) = \tilde{H}\left(\phi;\beta_i\right) \forall \phi \notin \bar{B}$. Such a relationship can be generated from the distribution $\tilde{H}\left(\phi;\beta_i\right)$ by taking all of the probability density implicitly assigned by the distribution over the closure of each of the sets B_t and assigning it all to $\inf B_t$. Calling the distribution that is the result of these new assignments in probability mass $\tilde{H}\left(\phi;\beta_j\right)$, this is the distribution generated from $\tilde{H}\left(\phi;\beta_i\right)$ that causes the condition

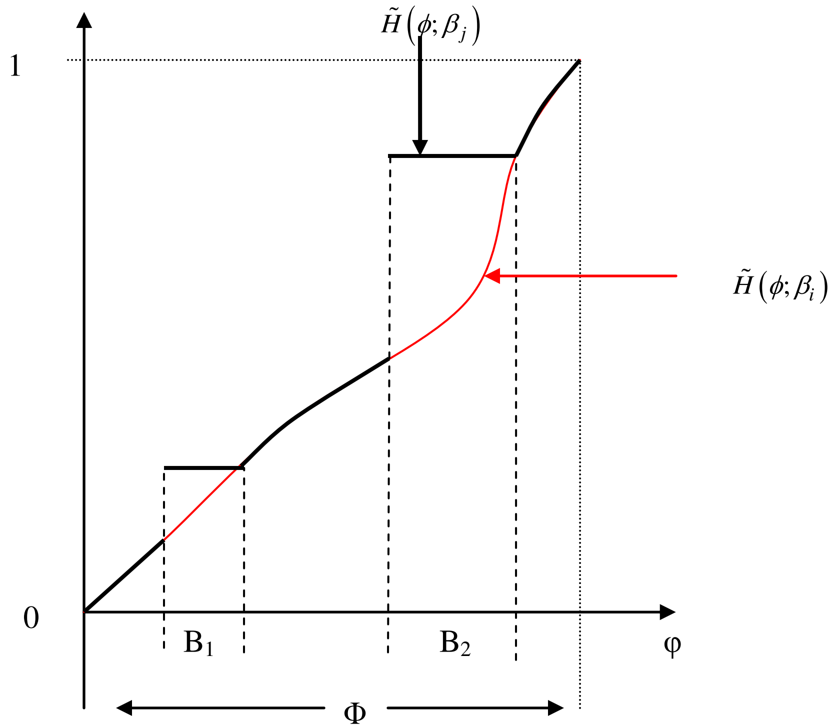
$$\tilde{v}'\left(\theta;\beta_j\right) - \tilde{v}'\left(\theta;\beta_i\right) = -\int_B u''\left(\theta + \varepsilon(\phi)\right)\varepsilon'(\phi)\left[\tilde{H}\left(\phi;\beta_j\right) - \tilde{H}\left(\phi;\beta_i\right)\right]d\phi \le 0$$

given the defined set B. If the set B is a single point, then it has no effect on the integral, therefore, only subsets of the transformed background risk that have some measurability are of issue. An example of a set B is seen in Figure 1 below.

Given the infinite number of first degree stochastic worsening shifts in the distribution, it will always be feasible to find a distribution such that the densities for both distributions differ only over

[2] More specifically, the closure of B is defined as: $\bar{B} := \left\{\phi \in \Phi : \exists b \in B : b \in \left(\phi - \delta, \phi + \delta\right) \text{ for every } \delta > 0\right\}$.

Figure 1

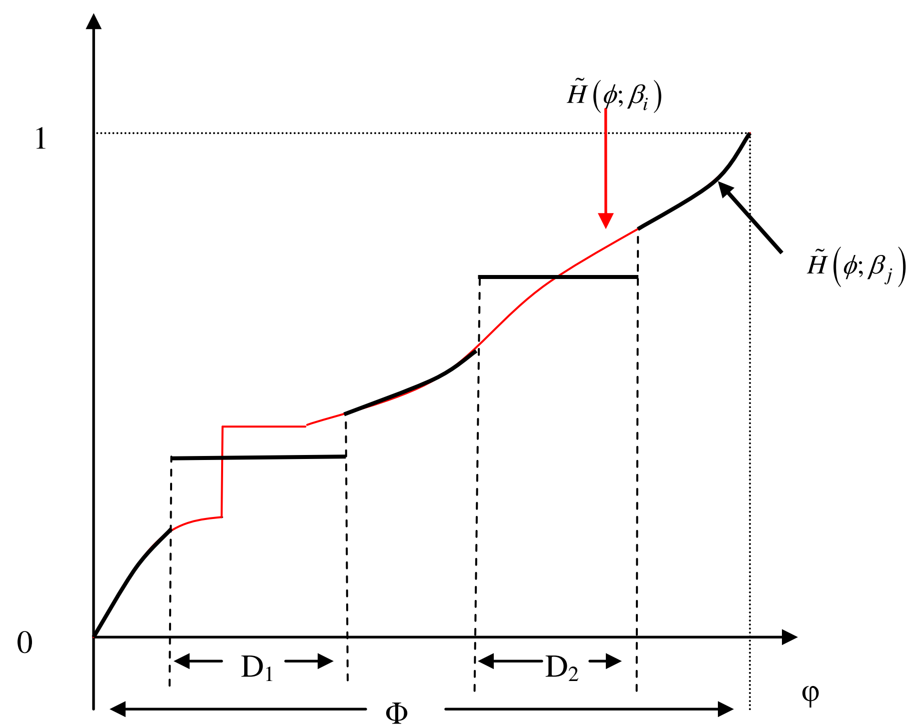


subspaces for which (3.5) is violated causing the marginal indirect utility differences to have the undesired sign. Therefore, this must be a necessary condition for a positive value to exist in (3.3).

Now suppose $\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - p(\theta + \tilde{\varepsilon}(\phi)) \geq 0$ over some set $D = \bigcup_t D_t$, where D is defined as

$D := \left\{ \phi \in \Phi : \frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - p(\theta + \tilde{\varepsilon}(\phi)) \geq 0 \right\}$ and any subset D_t is a connected set. Observe that the function $\int_{\underline{\phi}}^{\phi} \tilde{H}(s;\beta) ds$ is convex and continuous although there may be a countable number of points in which it is not differentiable. A second degree stochastically dominated spread in the distribution of ϕ includes one in which densities differ over the closure of the set D, but remain identical over the remaining space. Assume the graph of Figure 2 represents a second degree stochastic deterioration in the distribution caused by taking the probability mass assigned by $\tilde{H}(\phi;\beta_i)$ over any D_t and assigning some of it to the infimum of D_t and the rest of it to the supremum of D_t.

Figure 2



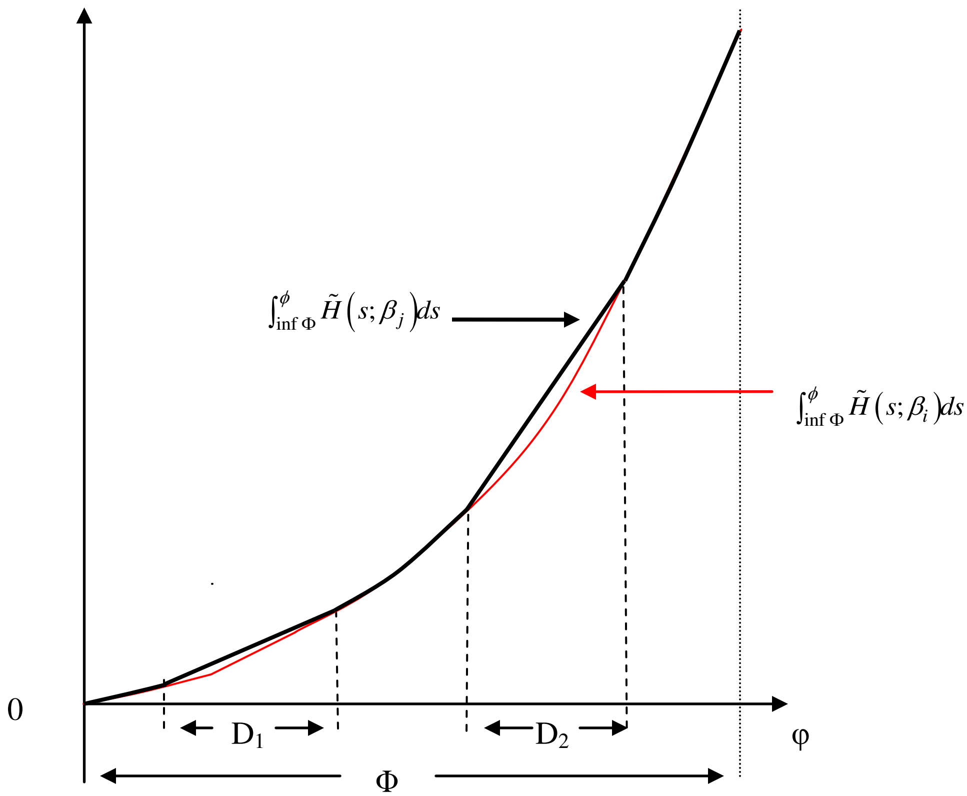
Assume that the relationship between the integrated cumulative distribution functions in Figure 2 satisfies the following conditions:

$$\int_{\inf D_i}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\begin{cases}\geq 0, \forall\phi\in D_i \text{ and strictly so for some } \phi\in D_i\\ =0, \phi=\sup D_i\end{cases}$$

It can be seen that $\tilde{H}(\phi;\beta_j)$ is a mean preserving spread of the risk given by the distribution $\tilde{H}(\phi;\beta_i)$.

There is an increase in risk that occurs due to reassigning density over the subset D such that the integrated difference of the two distributions is positive. The graph of these two integrated cumulative distribution functions may look something like Figure 3.

Figure 3



The cumulative distribution function derived from the original distribution, by redistributing the density assigned under the original distribution across any of the subspaces D_t to that subspace's infimum and supremum, possesses the following qualities:

$$\frac{\partial}{\partial\phi}\left(\int_{\underline{\phi}}^{\phi}\tilde{H}(s;\beta_j)ds\right)=\begin{cases}\tilde{H}(\phi;\beta_i),\forall\phi\in(\underline{\phi},\inf D_1)\cup(\sup D_1,\inf D_2)\cup(\sup D_2,\bar{\phi})\\ \tilde{H}(\sup D_1;\beta_i)\forall\phi\in\operatorname{int}D_1\\ \tilde{H}(\sup D_2;\beta_i)\forall\phi\in\operatorname{int}D_2\end{cases}$$

Generally speaking, the derivative of the integrated cumulative distribution function given by parameter β_j defines a cumulative distribution function for all but a countable number of points of Φ (a finite number in this example). The densities between the two distributions differ only over the subspace in which absolute prudence is sufficiently small enough to violate one of the proposed necessary conditions.

Over the subsets of the transformed background risk of which the two distributions are equal in value, i.e. $[\underline{\phi},\inf D_1)\cup[\sup D_1,\inf D_2)\cup[\sup D_2,\bar{\phi}]$, the difference between the integrated distributions seen in Figure 2 is constant. In fact, the difference in the integrated distributions over the subset excluding the set D is not only constant but is zero as well due to the differences in the integrated distributions being zero for any $\phi\in\sup D_t$. Knowing these properties, it can be seen that:

$$\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)=\int_D u''(\theta+\tilde{\varepsilon}(\phi))\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\right\}\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\,d\phi\leq 0$$

Given this, a necessary condition must be that given in (3.4) $\forall\phi\in\Phi\setminus D$ [3] and the set D has measure zero[4] whenever the difference in (3.3) is positive.

Unambigous statements concerning increases in an agent's indirect marginal utility under any mean preserving spread of an initial risk can be made about preferences based on their measure of absolute prudence. The first degree condition restricts how the function of the transformed background risk must enter the utility function for similarly unambiguous statements. Thus, satisfaction of both (3.4) and (3.5) together are necessary for unambiguous changes in the marginal indirect utility functions due to any type of second degree spread in the transformed background risk. What is immediately apparent is that the function $\tilde{\varepsilon}$ may be viewed as a deterministic transformation. In fact, once the formal proof of this is completed the function $\tilde{\varepsilon}$ will be treated as a special type of deterministic transformation.

Second degree stochastic deteriorations are obviously not all mean preserving spreads or first degree spreads of ϕ. Yet it happens to be the case that the set of necessary and sufficient conditions are given by these two specific forms of second degree spreads in risk. For a second degree spread in risk that is neither mean preserving nor a first degree spread, consider a spread in risk occurring across the space

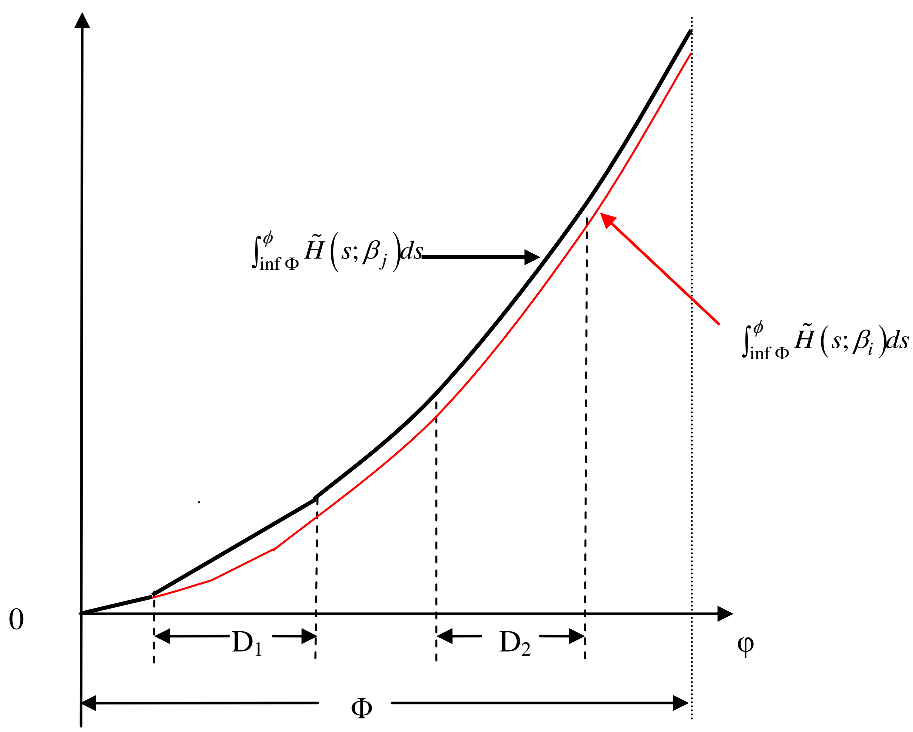
[3] Generally, the notation 'X\Y' refers to the set of all X that are not elements of Y.
[4] It is assumed that Φ is subset in a space with Lebesgue measure. $D\subset\Phi$ has Lebesgue measure zero whenever D is countable. Basic measure theory useful for economists can be found in Kirman (1981), although it is only useful if you are familiar with measure theory. For our purposes, a set D has measure zero if for every $\delta>0$ there is a closed cover of D, $\{G_1,G_2,...\}$ such that $\sum_i\mu(G_i)<\delta$. Where $\mu(G_i)$ is the length of the cover G_i, and for every

D_1 (see Figure 2). Assuming it is not a mean preserving spread, a graph of the integrated cumulative distribution functions will have slopes for both integrated cumulative distribution functions that are identical everywhere except over the space D_1. Some of the density assigned by $\tilde{H}(\phi;\beta_i)$ to D_1 has been shifted to the infimum of D_1 while the remaining portion of the density has been assigned to its supremum, generating a second degree spread in risk given by the distribution,

$$\tilde{H}(\phi;\beta_j) = \begin{cases} \tilde{H}(\phi;\beta_i), \phi \notin D_1 \\ P, \phi \in \bar{D}_1 \end{cases}$$

For all $\phi \in D_1$ the slope of $\int_{\inf D_1}^{\phi} \tilde{H}(s;\beta_j)ds$ equals P. This is seen in Figure 4 below.

Figure 4



This particular relationship indicates that the sign of the difference in the integrated cumulative distribution functions is determined entirely by the interval, $[\inf D_1, \bar{\phi}]$. Given the assumption that for any

$d \in D$ there exists a $G_k \in \{G_1, G_2, ...\}$ such that $d \in G_k$. See Spivak (1965) for an informal elementary treatment of

$\phi \notin D_1$ the difference between the distributions is zero, the difference in the integrated cumulative distribution functions is equivalent to:

$$\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)=\int_{D_1}u''(\theta+\tilde{\varepsilon}(\phi))\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\right\}\int_{\inf D_1}^{\phi}\left[P-\tilde{H}(s;\beta_i)\right]ds\,d\phi$$
$$+\int_{\sup D_1}^{\bar{\phi}}u''(\theta+\tilde{\varepsilon}(\phi))\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\right\}d\phi\int_{D_1}\left[P-\tilde{H}(\phi;\beta_i)\right]d\phi$$
$$-u''(\theta+\tilde{\varepsilon}(\bar{\phi}))\tilde{\varepsilon}'(\bar{\phi})\int_{\Phi}\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi$$

The first term on the right hand side of the equality is strictly nonpositive by assumption. The sign of the second term is unknown while the sign of the third term is positive. Recognizing the fact that $\int\left[P-\tilde{H}(\phi;\beta_i)\right]$ is continuous in P, this factor can be made arbitrarily small while preserving the second degree stochastic relationship between the two distributions. As $\int_{D_1}\left[P-\tilde{H}(s;\beta_i)\right]$ approaches zero from an initially positive value due to decreases in the value of P, the first term does not get smaller, i.e. its magnitude does not get larger. This is due to the fact that over this subset, (3.4) is assumed to not be true. In fact, a maximum value for the first term can be found. If this maximum value is negative, then the difference in (3.3) will be negative for some second degree spread in risk regardless of the sign of the derivative of the function $\tilde{\varepsilon}$. A similar shift in the density over the subspace D_2 can be performed as well, but such a pursuit will turn out to be redundant and all that is needed is to find a distribution that causes $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)\leq 0$, which has been achieved.

The preceding discussion outlining the 'not B implies not A argument' provides the proper framework to prove the necessary and sufficient conditions for the difference in (3.3) to be positive. For the difference to be negative similar arguments can be made for the conditions given in lemma 2. The proof will focus on first degree spreads and general second degree spreads in risk.

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this nature or Rudin (1976) for a more formal introductory treatment of this.

Lemma 2: *Let* $\phi:\mathrm{E}\to\Phi$ *with E compact, u' > 0, and u" < 0. Let* $\tilde{\varepsilon}\in C^2(\Phi)$ *be twice continuously differentiable and* $\tilde{H}(\phi;\beta_j)$ *be an element of the set of all distributions for the transformed background risk that satisfies*

$$\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\begin{cases}\geq 0\,\forall\phi\in\Phi\\ >0 \text{ for some } \phi\in\Phi\end{cases}$$

Then: $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)>(<)0$ *for any* β_j *if and only if,*

(i): $\tilde{\varepsilon}'(\phi)>0$ *for almost all* ϕ *(* $\tilde{\varepsilon}'(\phi)<0$ *for almost all* ϕ *)*

(ii): $p(\theta+\tilde{\varepsilon}(\phi))>\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}$ *for almost all* ϕ *(* $p(\theta+\tilde{\varepsilon}(\phi))<\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}$ *for almost all* ϕ *.)*

Proof: I will only prove the case for an increase in marginal indirect utility, seeing as the case of a decrease follows similar arguments. Necessity is proven first. Suppose for all $\phi\in B\subset\Phi$ it is true that $\tilde{\varepsilon}'(\phi)\leq 0$ with B a set of nonzero measure. If this is true, then there is a distribution indexed by β_j that has a first degree stochastic relationship with β_i such that $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)\leq 0$ where β_j parameterizes the distribution derived from $\tilde{H}(\phi;\beta_i)$ by shifting all the density over any subset of B to its infimum:

$$\tilde{H}(\phi:\beta_j)=\begin{cases}\tilde{H}(\sup B_t;\beta_i),\phi\in B=\bigcup_t B_t\\ \tilde{H}(\phi;\beta_i),\phi\notin\{B,\inf B_t\}\,\forall t\end{cases}$$

This particular distribution has the partial ordering relationship $\tilde{H}(\phi;\beta_i)\succeq_1\tilde{H}(\phi;\beta_j)$ with $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)\leq 0$.

Having established the necessary condition (i) suppose that $D:=\left\{\phi\in\Phi:\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}-p(\theta+\tilde{\varepsilon}(\phi))\geq 0\right\}$ has nonzero measure. If this is true, then let $\mathbf{P}:=\left\{P\in\mathbb{R}:\int_{\inf D_1}^{\phi}\left[P-\tilde{H}(s;\beta_i)\right]ds\geq 0\,\forall\phi\in D_1\right\}$,where D_1 is a measurable subset of D. For any $P\in\mathbf{P}$ there exists $\arg\max_{\phi}\left\{\int_{\inf D_1}^{\phi}\left[P-\tilde{H}(s;\beta_i)\right]ds:P\in\mathbf{P}\right\}$ and the argmax is increasing in P. Observe that the function

$$u''\left(\theta+\tilde{\varepsilon}(\phi)\right)\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}\int_{\inf D_1}^{\phi}\left[P-\tilde{H}\left(s;\beta_i\right)\right]ds$$

is nonincreasing in P for any $\phi\in D_1$. Consequently,

$$\int_{D_1}u''\left(\theta+\tilde{\varepsilon}(\phi)\right)\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}\int_{\inf D_1}^{\phi}\left[P-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi$$

is nonpositive and nonincreasing in P as well. Define the maximum possible value for this integral over D_1 as $M(P):=\int_{D_1}u''\left(\theta+\tilde{\varepsilon}(\phi)\right)\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}\int_{\inf D_1}^{\phi}\left[\min\{P:P\in\mathbf{P}\}-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi$.

Hence,

$$\int_{D_1}u''\left(\theta+\tilde{\varepsilon}(\phi)\right)\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}\int_{\inf D_1}^{\phi}\left[P-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi\le M(P)\le 0\,\forall P\in\mathbf{P}.$$

The following distribution is stochastically dominated in a second degree sense by $\tilde{H}(\phi;\beta_i)$:

$$\tilde{H}\left(\phi;\beta_j\right)=\begin{cases}\tilde{H}\left(\phi;\beta_i\right)\forall\phi\notin\bar{D}_1\\ P\in\operatorname{int}\{\mathbf{P}\}\forall\phi\in\bar{D}_1\end{cases}$$

It then follows that

$$\begin{aligned}\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)&=\int_{D_1}\tfrac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\inf D_1}^{\phi}\left[P-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi\\&\quad+\left[\int_{\sup D_1}^{\bar{\phi}}\tfrac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)d\theta-\tfrac{\partial}{\partial\phi}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\Big|_{\bar{\phi}}\int_{\Phi}d\phi\right]\int_{D_1}\left[P-\tilde{H}\left(\phi;\beta_i\right)\right]d\phi\\&\le M(P)+\left[\int_{\sup D_1}^{\bar{\phi}}\tfrac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)d\theta-\tfrac{\partial}{\partial\phi}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\Big|_{\bar{\phi}}\int_{\Phi}d\phi\right]\int_{D_1}\left[P-\tilde{H}\left(\phi;\beta_i\right)\right]d\phi\\&\le M(P)+\left[\left|\int_{\sup D_1}^{\bar{\phi}}\tfrac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)d\theta\right|-\tfrac{\partial}{\partial\phi}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\Big|_{\bar{\phi}}\int_{\Phi}d\phi\right]\int_{D_1}\left[P-\tilde{H}\left(\phi;\beta_i\right)\right]d\phi\end{aligned}$$

By $\int_{D_1}\left[P-\tilde{H}\left(\phi;\beta_i\right)\right]d\phi$ continuous in P there exists a $P\in\operatorname{int}\{\mathbf{P}\}$ such that

$$\frac{-M(P)}{\left|\int_{\sup D_1}^{\bar{\phi}}u''\left(\theta+\tilde{\varepsilon}(\phi)\right)\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}d\theta-u''\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)\tilde{\varepsilon}'(\bar{\phi})\int_{\Phi}d\phi\right|}\ge\int_{D_1}\left[P-\tilde{H}\left(\phi;\beta_i\right)\right]d\phi\ge 0$$

Therefore, there exists a P such that $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)\le 0$. If there exists a

$\phi < \min\left\{\arg\max_{\phi}\left\{\int_{\inf D_1}^{\phi}\left[P-\tilde{H}(s;\beta_i)\right]ds : P\in \mathbf{P}\right\}\right\}$ an element of D_1 such that $\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))>0$ then

$M(P)<0$ and by continuity in P there exists a P such that $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)<0$.

For sufficiency, suppose that $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)\le 0$ for some β_j. If this is true, then either

a. $-u''(\theta+\tilde{\varepsilon}(\phi))\tilde{\varepsilon}'(\phi)\le 0$ for some measurable subset of Φ which in turn implies B has nonzero measure. Or,

b.
$$u''(\theta+\tilde{\varepsilon}(\phi))\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\right\}\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds \le \frac{u''(\theta+\tilde{\varepsilon}(\bar{\phi}))\tilde{\varepsilon}'(\bar{\phi})}{\int_{\Phi}d\phi}\int_{\Phi}\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi$$

for some measurable subset of Φ which in turn implies that if $\tilde{\varepsilon}'(\bar{\phi})>0$ then

$\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\ge 0$ and D must have nonzero measure.

Q.E.D.

Nothing has been assumed about the nature of the transformed background risk. Also, the function $\tilde{\varepsilon}$ has been treated initially as a more abstract notion. Lemma 2 indicates that a sensible interpretation of $\tilde{\varepsilon}$ is that it is a particular type of deterministic transformation. If the function that transforms the background risk is strictly monotonically increasing and twice continuously differentiable, then it is invertible and the inverse of the function that transforms the background risk serves as a candidate for a deterministic transformation. When this is the case then $\tilde{\varepsilon}'(\phi)>0\forall\phi$, which certainly satisfies (i) of lemma 2 for second degree spreads in ϕ to be deemed undesirable for utility functions –u' when combined with (ii).

For the weaker restriction that preferences do not experience a decrease in indirect marginal utility due to arbitrary second degree spreads of the transformed background risk, the restriction placed on the deterministic transformation $\tilde{\varepsilon}$ by lemma 2 enables us to derive necessary and sufficient conditions for nonnegative indirect marginal utility differences. This is not the case for any type of deterministic transformation. For any type of deterministic transformation we have $\tilde{\varepsilon}'(\phi) \geq 0$. In the case of mean preserving spreads we are concerned with the sign of $u''(\theta+\tilde{\varepsilon}(\phi))\left[\tilde{\varepsilon}'(\phi)\right]^2\left\{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p(\theta+\tilde{\varepsilon}(\phi))\right\}$ for measurable subsets of Φ. No statements concerning necessary and sufficient conditions on preferences for nondecreasing indirect marginal utility appear to be possible for arbitrary deterministic transformations under the general framework of second degree stochastic dominance. On the other hand, if the deterministic transformation is restricted to be increasing in ϕ almost everywhere then preference restrictions are less stringent for the right hand side of (3.2) to be nonnegative. This is given by the following corollary.

Corollary to lemma 2: *Let $\phi : \mathrm{E} \to \Phi$ with E compact, u' > 0, and u'' < 0. Let $\tilde{\varepsilon} \in C^2(\Phi)$ be twice continuously differentiable such that $\tilde{\varepsilon}'(\phi) > 0$ for almost all ϕ and $\tilde{H}(\phi;\beta_j)$ be an element of the set of all distributions for the transformed background risk that satisfies*

$$\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\begin{cases}\geq 0 \,\forall \phi \in \Phi \\ > 0 \text{ for some } \phi \in \Phi\end{cases}$$

Then: $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i) \geq 0$ for any β_j if and only if

$$p(\theta+\tilde{\varepsilon}(\phi)) \geq \frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}\,\forall \phi \tag{3.6}$$

Proof: If $p(\theta+\tilde{\varepsilon}(\phi)) < \frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}$ for some ϕ then it is possible to construct a second degree spread from the initial distribution such that $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i) < 0$ for some β_j using an argument similar to that in lemma

2 with appropriate changes made for the weak inequalities. In like manner, if $\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)<0$ for some β_j, then there must exit a measurable subset of Φ such that $p(\theta+\tilde{\varepsilon}(\phi))<\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}$ for all elements of that subset whenever $\tilde{\varepsilon}'(\phi)>0$.

Q.E.D.

The corollary to lemma 2 defines sets of utility functions that will satisfy part of the sufficient conditions for a nonnegative difference in (3.3) resulting from arbitrary second degree spreads of a transformed background risk. It is worth noting that neither (3.4) nor lemma 2 make any assumptions about the third derivative of the utility function. Given condition (i) of lemma 2, the increase in marginal indirect utility occurs if and only if the measure of prudence is sufficiently high.

The transformation of the background risk remains a rather abstract notion up to this point and some examples of transformations for the background risk are instructive. Observe that for $\phi(\varepsilon)=k\varepsilon+c$, we have $\tilde{\varepsilon}(\phi)=\frac{\phi-c}{k}$ as a viable deterministic transformation. If any second degree spreads of an affine transformation of the background risk does not cause marginal indirect utility to decrease, then u''' ≥ 0 for all background risk values and the agent is prudent. At the same time, all agents that are prudent will experience an increase in their indirect marginal utility anytime an affine transformation of the background risk undergoes a second degree spread in risk.

Another example is the transformation, $\phi(\varepsilon;\alpha_h)=k\int_{\Theta}u(\theta+\varepsilon)dF(\theta,\alpha_h)+c$. This is invertible for positive marginal utility. Letting the function $\tilde{\varepsilon}$ once again be the inverse of the transformation function for the background risk, the derivatives of the inverse functions are $\tilde{\varepsilon}'(\phi(\varepsilon))=\frac{1}{\phi'(\varepsilon)}$ and $\tilde{\varepsilon}''(\phi(\varepsilon))=\frac{-\phi''(\varepsilon)}{(\phi'(\varepsilon))^3}$. Such a transformation introduces the distribution for the foreground risk into the comparative statics later on, once we consider classes of risk averse preferences. The corollary to lemma 2 indicates that whenever an agent's absolute prudence is greater than or equal to absolute risk aversion given the distribution for the foreground risk for all values of the background risk, the agent experiences an increase in his indirect marginal utility function due to arbitrary second degree spreads of $\phi(\varepsilon;\alpha_h)$. Observe that by plugging in the function for ϕ and performing some algebraic manipulations, a convenient expression is seen which is an expectation of absolute risk aversion.

$$\frac{\tilde{\varepsilon}''(\phi(\varepsilon;\alpha_h))}{\left[\tilde{\varepsilon}'(\phi(\varepsilon;\alpha_h))\right]^2} \equiv \int_\Theta \frac{-u''(\theta+\varepsilon)}{u'(\theta+\varepsilon)} d\int_{\underline{\theta}}^{\theta} \frac{u'(t+\varepsilon)dF(t;\alpha_h)}{\int_\Theta u'(\theta+\varepsilon)dF(\theta;\alpha_h)}$$

The expression $\int_{\underline{\theta}}^{\theta} \frac{u'(t+\varepsilon)dF(t;\alpha_h)}{\int_\Theta u'(\theta+\varepsilon)dF(\theta;\alpha_h)}$ is a distribution. Invoking the mean value theorem it can be seen that there exists $\theta_h \in \Theta$ such that $r(\theta_h+\varepsilon) \equiv \int_\Theta \frac{-u''(\theta+\varepsilon)}{u'(\theta+\varepsilon)} d\int_{\underline{\theta}}^{\theta} \frac{u'(t+\varepsilon)dF(t;\alpha_h)}{\int_\Theta u'(\theta+\varepsilon)dF(\theta;\alpha_h)}$. Whenever preferences possess the quality that $p(\theta+\varepsilon) \geq r(\theta_h+\varepsilon)\forall\varepsilon$ the agent will experience increased marginal indirect utility due to any second degree spread in the transformed background risk.

Viewing $\tilde{\varepsilon}(\phi)$ as a deterministic transformation such that $\tilde{\varepsilon}'(\phi) > 0$ for almost all ϕ, a necessary and sufficient condition for nondecreasing marginal indirect utility as a result of arbitrary second degree spreads of ϕ is that $p(\theta+\tilde{\varepsilon}(\phi)) \geq \frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}\forall\phi$. Armed with this, we are able to derive necessary and sufficient conditions for increased risk aversion (3.2) for all θ. Preferences satisfying the prudence relationship yield a different look at stochastic dominating spreads of a second degree nature in that an agent's greater aversion to risk can be decomposed into a weighted average of prudence at the maximal element of the background risk and an entirely new relationship involving temperance relative to prudence. This weighted average relationship sheds light on why mean preserving spreads and first degree spreads in risk effectively capture all the important comparative statics under the restriction that the agent behaves consistently under second degree spreads in the transformed background risk. This expands the analysis of stochastic dominance from simple stochastic dominating spreads in background risk to other distributions of background risk that have a stochastic dominating relationship through the transformation function $\phi(\varepsilon)$. This will be considered below.

Preferences and Increased Risk Aversion

The necessary and sufficient conditions for any second degree spread of the transformed background risk to cause marginal indirect utility to weakly increase identifies a set of preferences for which conditions for increased risk aversion due to such spreads can be described. When the transformed background risk affects utility through a deterministic transformation as in (3.5) and preferences satisfy the weaker prudence condition (3.6), increased risk aversion given by (3.2) is conveniently represented as a ratio of differences caused by the change in risk affecting both the second derivatives and first derivatives for the indirect utility functions. To accommodate for the possibility of the right hand side of (3.2) being zero, add $k\tilde{R}(\theta;\beta_i)$ to both sides of the inequality, where k > 0. This yields

$$\frac{-\left[\tilde{v}''(\theta;\beta_j)-\tilde{v}''(\theta;\beta_i)\right]+k\tilde{R}(\theta;\beta_i)}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k} \geq \tilde{R}(\theta;\beta_i)\,\forall\theta \tag{4.1}$$

For increasing indirect marginal utility, i.e. preferences that satisfy (3.4), this step is unnecessary but there is no harm in doing so. The ratio on the left hand side of (4.1) is always greater than or equal to the initial measure of risk aversion for any level of the foreground risk when the prudence condition given by (3.6) is satisfied. This interesting relationship decomposes into conditions for preferences satisfying the prudence condition that are necessary and sufficient for families of utility functions to experience increased risk aversion.

The left hand side of (4.1) can be equivalently expressed as a weighted average of terms. Define q_1 and q_2 as functions of the foreground risk, the second degree stochastic relationship of the two distributions, and k as follows:

$$q_1(\theta;\beta_i,\beta_j,k) := \frac{-\frac{\partial}{\partial\phi}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\bar{\phi}}\int_{\Phi}\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k} \tag{4.2}$$

$$q_2(\theta;\beta_i,\beta_j,k) := \frac{\int_{\Phi}\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\,d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k} \tag{4.3}$$

As defined, q_1 and q_2 are weights for any $k > 0$ such that the sum of these weights is greater than or equal to zero but less than one whenever the necessary and sufficient conditions for the corollary to lemma 2 are satisfied. On the other hand, for preferences satisfying the necessary and sufficient conditions for lemma 2 the sum of these weights will be greater than zero but less than or equal to one.

Using integration by parts twice in both the numerator and the denominator on the left hand side of (4.1), followed by a substitution of q_1, q_2, $(1-q_1-q_2)$; and subsequently multiplying the integrand that remains by $\frac{-\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)}{-\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)}$, it can be seen that the left hand side of (4.1) at a given value for the foreground risk is an average of prudence at a specific value of ϕ, an expectation, and $\tilde{R}(\theta;\beta_i)$ provided that $q_1, q_2 \in [0,1)$ and $q_1 + q_2 \le 1$. Subtracting $\tilde{R}(\theta;\beta_i)(1-q_1-q_2)$ from both sides provides an equivalent expression for (4.1). For all θ:

$$p\left(\theta+\tilde{\varepsilon}\left(\bar{\phi}\right)\right)q_1+\int_{\Phi} p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-t\left(\theta+\varepsilon(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\varepsilon(\phi)\right)}d\tilde{H}_2\left(\phi;\beta_i,\beta_j,\theta\right)q_2+\tilde{R}\left(\theta;\beta_i\right)\left(1-q_1-q_2\right)\ge\tilde{R}\left(\theta;\beta_i\right)$$ [ii]

An argument will be given following lemma 3 that $\tilde{H}_2\left(\phi;\beta_i,\beta_j,\theta\right):=\int_{\underline{\phi}}^{\phi}\frac{\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]ds}{\int_{\Phi}\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]ds\,d\phi}$ is a cumulative distribution function provided that preferences satisfy (3.6). It turns out to be the case that if the necessary and sufficient conditions of the corollary to lemma 2 are satisfied then $q_1, q_2 \in [0,1)$ and $0 \le q_1 + q_2 < 1$.

Lemma 3: *Let $\tilde{H}(\phi;\beta_i) \succeq_2 \tilde{H}(\phi;\beta_j)$ with $u' > 0$ and $u'' < 0$. Let $\phi: \mathrm{E} \to \Phi$ be the transformation for the background risk space. Let $\tilde{\varepsilon} \in C^2(\Phi)$ be a deterministic transformation such that $\tilde{\varepsilon}'(\phi) > 0$ for almost all ϕ. Then the following are equivalent.*

i. $\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)\ge 0\,\forall\beta_j$

ii. *For all β_j, there exists a $q_1, q_2 \in [0,1)$ such that $0 \le q_1 + q_2 < 1$ and for any $k > 0$*

$$
\frac{-\left[\tilde{v}''(\theta;\beta_j)-\tilde{v}''(\theta;\beta_i)\right]+k\tilde{R}(\theta;\beta_i)}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}
$$

$$
\equiv p\left(\theta+\tilde{\varepsilon}\left(\bar{\phi}\right)\right)q_1+\int_{\Phi} p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-t\left(\theta+\varepsilon(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\varepsilon(\phi)\right)}d\tilde{H}_2\left(\phi;\beta_i,\beta_j,\theta\right)q_2+\left(1-q_1-q_2\right)\tilde{R}(\theta;\beta_i) \tag{4.4}
$$

iii. $p\left(\theta+\tilde{\varepsilon}(\phi)\right)\geq\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}\forall\phi$.

Proof: (i) ⇔ (iii): This is proven by the corollary to lemma 2.

(ii) ⇔ (iii): Define $q_1\left(\theta;\beta_i,\beta_j\right):=\frac{-u''\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)\tilde{\varepsilon}'(\bar{\phi})\int_{\Phi}\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$ and

$$
q_2\left(\theta;\beta_i,\beta_j\right):=\frac{\int_{\Phi}\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\,d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}
$$

It follows that $1-q_1-q_2=\frac{k}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$.

Suppose (iii) is not true. Then following the discussion leading up to lemma 2 there must exist a measurable subset $D\subset\Phi$ and a β_j such that $\int_{\Phi}\frac{\partial^2}{\partial\phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]ds\,d\phi<0$. If $\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)+k<0$ then $q_1 < 0$ and $q_2 > 1$. Furthermore, given that $k > 0$, $q_1 + q_2 > 1$. If $\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)>0$ then $q_2 < 0$ and $q_1 > 1$ for a sufficiently small k.

Now suppose that (ii) is false for one of the following reasons:

a) $q_1 > 1$ or $q_1 < 0$: This implies that (iii) is false over a measurable subset of Φ.

b) $q_2 > 1$ or $q_2 < 0$: This implies that (iii) is false over a measurable subset of Φ.

c) $q_1 + q_2 > 1$: This implies $\frac{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}>1$ for some k which can only be true if (iii) is false over a measurable subset of Φ so that the numerator is negative for some β_j.

Q.E.D.

Given risk averse preferences, lemma 3 indicates that a necessary and sufficient condition for $\tilde{H}_2\left(\phi;\beta_i,\beta_j,\theta\right)$ to be a distribution function is that the prudence condition (3.6) is satisfied. If there is any instance of (3.6) not being satisfied, then the derivative of $\tilde{H}_2$ is negative for some ϕ. On the other hand, since $\tilde{H}_2$ is zero at $\underline{\phi}$ and one at $\overline{\phi}$, if it is not a cumulative distribution function, then it must be the case that the derivative of $\tilde{H}_2$ is negative somewhere which can only be true if (3.6) is not true. Whenever $\tilde{H}_2\left(\phi;\beta_i,\beta_j,\theta\right)$ is in fact a distribution function, it may be referred to as a 'risk adjusted' probability measure for the utility function. The distribution itself is a function of preferences in the presence of a second degree stochastic spread in risk.

The weighted average representation result of lemma 3 holds for any k > 0. If in fact the difference in the marginal indirect utilities is positive for arbitrary second degree spreads, i.e. the preference condition (3.4) applies, then the weighted average representation (4.4) exists such that $q_1 + q_2 \neq 0$. One could eliminate k in this case causing the three point weighted average representation to default to a two point version.

Machina and Pratt (1997), and Müller and Scarsini (2001) have shown that any second degree stochastically dominated distribution can be obtained from an initial distribution through a finite sequence of spreads in risk that are either first degree or mean preserving in nature[5]. The numerical value of expression (4.4) captures information about changes in the degree of concavity of the indirect utility function relative to the change in marginal indirect utility due to a second degree spread in ϕ. If the second degree spread is that of a mean preserving nature, then $p\left(\theta+\tilde{\varepsilon}\left(\overline{\phi}\right)\right)$ provides no information about the relative change in the degree of concavity of the indirect utility function. In this case all of this

[5] It should be noted that this differs from the findings of Rothschild and Stiglitz (1970) in that the latter illustrates for any increase in risk, e.g. $\eta \overset{d}{=} \varepsilon+\zeta$ where $E\left(\zeta \mid \varepsilon\right)=0 \forall \varepsilon \in \mathrm{E}$, there exist two distinct finite sequences of distributions, $\left\{G_1,G_2,...,G_n\right\},\left\{F_1,F_2,...F_n\right\}$ whereby $\lim_{n\to\infty} G_n = G, \lim_{n\to\infty} F_n = F$, such that each G_n could be obtained from F_n by a finite number of mean preserving spreads. See Leshno et al. (1997) in conjunction with Rothschild and Stiglitz for a correct proof of this important theorem.

change is captured by the relationship between prudence and temperance. If temperance is sufficiently small, then by the weighted average expression from lemma 3 it can be seen that increased risk aversion may not be realized under an arbitrary mean preserving spread in risk. Whenever the spread in risk involves any kind of decrease in the mean of the risk, a portion of the relative change in the concavity of the indirect utility function is explained by the agent's measure of prudence.

Using the corollary to lemma 2, it can be shown that for any second degree stochastically dominated spread in ϕ to cause marginal indirect utility to change in a nondecreasing manner, the following necessary condition exists:

$$t\left(\theta+\tilde{\varepsilon}(\phi)\right) \geq \frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2} \forall \phi\,^{6}. \tag{4.5}$$

Temperance is described by Kimball (1992) as a measure of the tendency to moderate the acceptance or exposure to risks and is proven to be an important measure for risk averse preferences. For example, the introduction of a statistically independent background risk might cause an individual to reduce exposure to other risks. Gollier and Pratt (1996) have shown that for an introduction of a statistically independent background risk to cause an individual's partial risk premium to increase, his measure of absolute temperance must be no less than his measure of absolute risk aversion. Thus, a negative fourth derivative of an agent's utility function is not sufficient for that agent to reduce exposure to other risks. Gollier (2001) comments that the degree to which an individual's aversion toward a given risk increases upon the introduction of a statistically independent risk is related to that agent's fourth derivative; i.e. his measure of absolute temperance[7]. It will be shown below that the importance of temperance exists in a similar manner for various types of increases in risk captured through second degree stochastically dominated spreads of the transformed background risk. For the measure of absolute temperance comparative statics exist that provide necessary and sufficient conditions, for preferences

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[6] Using the corollary to lemma 2, a necessary and sufficient condition for $-v''\left(\theta;\beta_j\right)+v''\left(\theta;\beta_i\right)\geq 0$ when $\tilde{\varepsilon}'\left(\phi\right)>0$ for almost all ϕ is (4.5).

identified by a prudence relation, to experience increased risk aversion for all θ in response to a change in the background risk. This is a result of theorem 1.

Theorem 1: *Let u' > 0, and u" < 0,* $\phi : \mathrm{E} \to \Phi$ *and* $\tilde{\varepsilon} \in C^2(\Phi)$ *be a deterministic transformation such that* $\tilde{\varepsilon}'(\phi) > 0$ *for almost all* ϕ *. Define* q_1 *and* q_2 *as functions of θ, the stochastic dominating relationship and a parameter k > 0,* $q_1(\theta;\beta_i,\beta_j,k), q_2(\theta;\beta_i,\beta_j,k)$*. Then for all* β_i*,* β_j *such that*

$\tilde{H}(\phi;\beta_i) \succeq_2 \tilde{H}(\phi;\beta_j)$ *and* $\forall\theta$: $p(\theta+\tilde{\varepsilon}(\phi)) \ge \frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} \forall\phi$ *and* $\tilde{R}(\theta;\beta_j) \ge \tilde{R}(\theta;\beta_i)$ *if and only if for each* β_i*,* β_j *there exists a* $q_1(\theta;\beta_i,\beta_j,k), q_2(\theta;\beta_i,\beta_j,k) \in [0,1)$ *such that* $0 \le q_1 + q_2 < 1$ *and*

$$p\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)q_1(\theta;\beta_i,\beta_j,k) + p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - t\left(\theta+\tilde{\varepsilon}(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - p\left(\theta+\tilde{\varepsilon}(\phi)\right)} q_2(\theta;\beta_i,\beta_j,k) \ge (q_1+q_2)(\theta;\beta_i,\beta_j,k)\, r\left(\theta+\tilde{\varepsilon}(\hat{\phi})\right) \forall\hat{\phi},\phi \quad (4.6)$$

Proof: Define the following weights: $q_1(\theta;\beta_i,\beta_j,k) := \frac{-u''(\theta+\tilde{\varepsilon}(\bar{\phi}))\tilde{\varepsilon}'(\bar{\phi})\int_\Phi\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$ and

$$q_2(\theta;\beta_i,\beta_j,k) := \frac{\int_\Phi \frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\,d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$$

Observe that if (4.6) is false then there exist sets

$L_2(\theta) := \left\{\phi \in \Phi : (q_1+q_2)\, r\left(\theta+\tilde{\varepsilon}(\phi)\right) > \min_\phi\left\{p\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)q_1 + p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - t\left(\theta+\tilde{\varepsilon}(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - p\left(\theta+\tilde{\varepsilon}(\phi)\right)} q_2\right\}\right\}$ and

$Q_2(\theta) := \left\{\phi \in \Phi : p\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)q_1 + p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - t\left(\theta+\tilde{\varepsilon}(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2} - p\left(\theta+\tilde{\varepsilon}(\phi)\right)} q_2 < \max_\phi\left\{(q_1+q_2)\, r\left(\theta+\tilde{\varepsilon}(\phi)\right)\right\}\right\}$ both of which are of nonzero measure. Choose the β_i distribution to be one which places a sufficiently large portion of its probability mass on $L_2(\theta)$ and let β_t be a distribution that places a sufficiently large portion of its distribution on $Q_2(\theta)$ such that:

[7] Page 139.

$$p\left(\theta+\tilde{\varepsilon}\left(\bar{\phi}\right)\right)q_1+\int_\Phi p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-t\left(\theta+\tilde{\varepsilon}(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)}d\tilde{H}(\phi;\beta_t)q_2<\int_\Phi r\left(\theta+\tilde{\varepsilon}(\phi)\right)d\tilde{H}(\phi;\beta_i)\left(q_1+q_2\right) \tag{4.7}$$

is true. If there does not exist a β_j such that $\tilde{H}(\phi;\beta_t)\equiv\int_{\underline{\phi}}^{\phi}\frac{\frac{\partial^2}{\partial t^2}\left[u'\left(\theta+\tilde{\varepsilon}(t)\right)\right]\int_{\underline{\phi}}^{t}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]dsdt}{\int_\Phi\frac{\partial^2}{\partial\phi^2}\left[u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right]\int_{\underline{\phi}}^{\phi}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi}$ is a distribution function, then one of the two following situations must be true:

i. There is no distribution that both satisfies $\tilde{H}(\phi;\beta_i)\succeq_2\tilde{H}\left(\phi;\beta_j\right)$ and causes (4.6) to be false.

ii. There is a distribution that satisfies $\tilde{H}(\phi;\beta_i)\succeq_2\tilde{H}\left(\phi;\beta_j\right)$ and there exists a set D(θ) of nonzero measure such that $p\left(\theta+\tilde{\varepsilon}(\phi)\right)<\frac{\tilde{\varepsilon}''(\phi)}{\tilde{\varepsilon}'(\phi)}$ for all elements of this set.

If there does exist a β_j then given the definition of q_1 and q_2 add $\left(1-q_1-q_2\right)r\left(\theta+\tilde{\varepsilon}(\phi)\right)$ to both sides of (4.6). Thereafter, perform integration by parts on (4.6). This reveals that for any k > 0

$$\tilde{R}\left(\theta;\beta_j\right)\operatorname{sgn}\left\{\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)+k\right\}<\tilde{R}\left(\theta;\beta_i\right)\operatorname{sgn}\left\{\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)+k\right\}$$

This implies either $\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)+k<0$ for some k, which requires D(θ) to have nonzero measure by lemma 3, or $\tilde{R}\left(\theta;\beta_j\right)<\tilde{R}\left(\theta;\beta_i\right)$.

If $q_i\notin[0,1)$ for i = 1 or 2, or $q_1+q_2\notin[0,1)$ then D(θ) must have nonzero measure as proven by lemma 3.

For sufficiency suppose:

i. D(θ) has nonzero measure and $\tilde{R}\left(\theta;\beta_j\right)\geq\tilde{R}\left(\theta;\beta_i\right)$. Then by lemma 3 there is a β_j such that $\tilde{H}(\phi;\beta_i)\succeq_2\tilde{H}\left(\phi;\beta_j\right)$ and $\operatorname{sgn}\left\{\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)\right\}=-1$. This implies for some k > 0, $\frac{-\left[v''\left(\theta;\beta_j\right)-v''\left(\theta;\beta_i\right)\right]+k\tilde{R}\left(\theta;\beta_i\right)}{\left[v'\left(\theta;\beta_j\right)-v'\left(\theta;\beta_i\right)+k\right]}\leq\tilde{R}\left(\theta;\beta_i\right)$. Thus, there exist sets $L_2(\theta)$ and $Q_2(\theta)$ both of which must be of nonzero measure. In addition to this, $q_1 + q_2 > 1$ for some k > 0, and neither q_1 nor q_2 are elements of [0,1) as proven by lemma 3, for some β_j.

ii. $\tilde{R}(\theta;\beta_j) < \tilde{R}(\theta;\beta_i)$ and D(θ) has zero measure. Then there exist sets $L_2(\theta)$ and $Q_2(\theta)$ both of which must be of nonzero measure.

iii. $\tilde{R}(\theta;\beta_j) < \tilde{R}(\theta;\beta_i)$ and D(θ) has nonzero measure. In this case, there exists a β_j such that $\frac{-\left[\tilde{v}''(\theta;\beta_j)-\tilde{v}''(\theta;\beta_i)\right]+k\tilde{R}(\theta;\beta_i)}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k} > \tilde{R}(\theta;\beta_i)$ but none of the desired properties for q_1 and q_2 hold given the results of lemma 3.

Q.E.D.

Given that theorem 1 concerns arbitrary second degree spreads of the transformed background risk and k is arbitrarily small, (4.6) must hold for any weighted average that is a result of second degree spreads of the risk. For any second degree spreads such that $q_1 + q_2 \neq 0$, dividing through both sides of (4.6) by $q_1 + q_2$ and defining

$$q(\theta;\beta_i,\beta_j,k) = \frac{q_1(\theta;\beta_i,\beta_j,k)}{q_1(\theta;\beta_i,\beta_j,k)+q_2(\theta;\beta_i,\beta_j,k)} \text{ whenever } q_1 + q_2 \neq 0 \qquad (4.8)$$

yields the equivalent condition:

$$p\left(\theta+\tilde{\varepsilon}(\bar{\phi})\right)q(\theta;\beta_i,\beta_j,k) + p\left(\theta+\tilde{\varepsilon}(\phi)\right)\frac{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}-t\left(\theta+\tilde{\varepsilon}(\phi)\right)}{\frac{\tilde{\varepsilon}''(\phi)}{[\tilde{\varepsilon}'(\phi)]^2}-p\left(\theta+\tilde{\varepsilon}(\phi)\right)}\left(1-q(\theta;\beta_i,\beta_j,k)\right) \geq r\left(\theta+\tilde{\varepsilon}(\hat{\phi})\right)\forall\hat{\phi},\phi \qquad (4.9)$$

This is the case of all second degree spreads for preferences that satisfy the stronger prudence condition given by (3.4). For preferences that satisfy the weaker prudence condition given by (3.6), $q_1 + q_2 = 0$ for certain mean preserving spreads. For these specific spreads we have the degenerate condition of $0 \geq 0$ which is obviously true but meaningless.

By selecting transformations for the background risk that are strictly increasing, twice continuously differentiable, and of uniform sign, choosing the inverse function for the deterministic transformation yields nice results via a change of variables that allow us to link the results of theorem 1 with lemma 1. For this type of deterministic transformation, $\tilde{\varepsilon}(\phi(\varepsilon_i)) = \varepsilon_i$ and let's establish by way of definition $\tilde{H}(\phi_i;\beta) = \tilde{H}(\phi(\varepsilon_i);\beta) \overset{def}{\equiv} H(\varepsilon_i;\beta)$ for all 'i'. The transformation function is strictly increasing so that $H(\varepsilon;\beta)$ fits the definition of a cumulative distribution function, i.e. nondecreasing in ε with a value of 0 and a value of 1 realized at the minimum and maximum of E respectively. Thus, by definition the probability density is seen to be $\tilde{H}'(\phi(\varepsilon_i);\beta)\phi'(\varepsilon_i) \overset{def}{\equiv} H'(\varepsilon_i;\beta)$; and $\tilde{R}(\theta;\beta_j) \geq \tilde{R}(\theta;\beta_i)$ if and only if $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i)$ using the corresponding distribution for the background risk that is implied by substitution. These substitutions give us the opportunity to relate increased risk aversion to changes in both the foreground risk and the background risk.

Corollary to theorem 1*: Let u' > 0, and u'' < 0,* $\phi : \mathrm{E} \to \Phi$ *such that* $\phi'(\varepsilon_i) > 0 \forall \varepsilon_i \in \mathrm{E}$*, and* $\phi \in C^2(\mathrm{E})$.

Define q_1 and q_2 as functions of θ, the stochastic dominating relationship, and a parameter k > 0.

$$q_1(\theta;\beta_i,\beta_j,k) := \frac{\frac{-u''(\theta+\bar{\varepsilon})}{\phi'(\bar{\varepsilon})}\int_{\mathrm{E}}\left[\tilde{H}(\phi(\varepsilon);\beta_j) - \tilde{H}(\phi(\varepsilon);\beta_i)\right]\phi'(\varepsilon)d\varepsilon}{\left[v'(\theta;\tilde{H},\beta_j) - v'(\theta;\tilde{H},\beta_i) + k\right]} \text{ and}$$

$$q_2(\theta;\beta_i,\beta_j,k) := \frac{-\int_{\mathrm{E}}\frac{u''(\theta+\varepsilon)}{\phi'(\varepsilon)}\left\{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - p(\theta+\varepsilon)\right\}\int_{\underline{\varepsilon}}^{\varepsilon}\left[\tilde{H}(\phi(s);\beta_j) - \tilde{H}(\phi(s);\beta_i)\right]\phi'(s)\,ds\,d\varepsilon}{\left[v'(\theta;\tilde{H},\beta_j) - v'(\theta;\tilde{H},\beta_i) + k\right]}$$

Then for all β_i, β_j such that $\tilde{H}(\phi;\beta_i) \succeq_2 \tilde{H}(\phi;\beta_j)$ *and* $\forall\theta$*:*

$p(\theta+\varepsilon) \geq \frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} \forall\varepsilon$ *and* $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i)$ *if and only if for each β_i, β_j* $\exists q_i(\theta;\beta_i,\beta_j) \in [0,1), i = 1,2$

such that $0 \leq q_1 + q_2 < 1$ *and*

$$q_1(\theta;\beta_i,\beta_j,k)\,p(\theta+\bar{\varepsilon}) + q_2(\theta;\beta_i,\beta_j,k)\,p(\theta+\varepsilon)\frac{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - p(\theta+\varepsilon)} \geq (q_1+q_2)(\theta;\beta_i,\beta_j,k)\,r(\theta+\hat{\varepsilon})\,\forall\hat{\varepsilon},\varepsilon \quad (4.10)$$

Proof: This is theorem 1 with the substitution $\phi = \phi(\varepsilon)$.

Q.E.D.

One interesting feature of (4.10) is the relatively simple relationship given by the ratio, $\frac{\frac{-\phi'''(\varepsilon)}{\phi''(\varepsilon)}-t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}-p(\theta+\varepsilon)}$. Absolute temperance measures that are greater than absolute prudence indicate that the tendency to moderate exposure to other risks dominates the precautionary motive. Hence, the weighting on $p(\theta+\varepsilon)$ found on the left hand side of (4.10) is multiplied by a factor greater than one whenever temperance exceeds prudence. This is the case for preferences that are standard risk averse. More generally, a greater tendency to moderate exposure to other risks will enhance aversion to increased risk. This behavioral quality implies that agents with higher measures of temperance are more likely to experience increased risk aversion due to an arbitrary second degree spread of ϕ. Thus, individuals with a relatively small precautionary motive for holding assets will experience increased risk aversion due to a second degree spread in ϕ provided the tendency to moderate exposure to risks is sufficiently high.

Naturally, these conditions must hold for first degree spreads as well. Thus, we have a necessary and sufficient condition for increased risk aversion under the framework of second degree stochastic spreads differing from that derived under first degree spreads by Eeckhoudt et al. For all second degree spreads that cause marginal indirect utility to increase by a positive amount 'c', the weight on $p\left(\theta+\tilde{\varepsilon}\left(\bar{\phi}\right)\right)$ will be the highest for the spread that is of a first degree nature such that it is stochastically dominated in a first degree sense by other distributions stochastically dominated in a second degree sense by the initial distribution. Stated differently, any first degree spread in risk will be a first degree spread in risk for other risks stochastically dominated by an initial distribution in a second degree sense. Relative to these other stochastically dominated distributions, the agent will place greater weight on $p\left(\theta+\tilde{\varepsilon}\left(\bar{\phi}\right)\right)$ for this first degree spread in risk relative to any of these other second degree spreads in risk that cause the same change in marginal indirect utility. This result is proven in the following lemma.

Lemma 4: *Let preferences satisfy* $p\left(\theta+\tilde{\varepsilon}\left(\phi\right)\right)\geq\frac{\tilde{\varepsilon}''(\phi)}{\left[\tilde{\varepsilon}'(\phi)\right]^{2}}\forall\phi$, *$\beta_j$ be such that* $\tilde{H}\left(\phi;\beta_i\right)\succeq_1\tilde{H}\left(\phi;\beta_j\right)$ *and*

$\tilde{v}'\left(\theta;\beta_j\right)-\tilde{v}'\left(\theta;\beta_i\right)=c>0$. *Then for any β_k satisfying* $\tilde{H}\left(\phi;\beta_i\right)\succeq_2\tilde{H}\left(\phi;\beta_k\right)$*:*

$\tilde{H}\left(\phi;\beta_k\right)\succeq_1\tilde{H}\left(\phi;\beta_j\right)\forall\beta_k \Rightarrow q_1\left(\theta;\beta_i,\beta_j,k\right)=\max_{\beta_k}\left\{q_1\left(\theta;\beta_i,\beta_k,k\right):\tilde{v}'\left(\theta;\beta_k\right)-\tilde{v}'\left(\theta;\beta_i\right)=c\right\}$ *where q_1 is defined in* (4.2).

Proof: $q_1\left(\theta;\beta_i,\beta_j,k\right)-q_1\left(\theta;\beta_i,\beta_k',k\right)<0$ for some β_k' implies $\int_\Phi\left[\tilde{H}\left(\phi;\beta_j\right)-\tilde{H}\left(\phi;\beta_k'\right)\right]d\phi<0$ which can only be true if $\tilde{H}\left(\hat{\phi};\beta_j\right)-\tilde{H}\left(\hat{\phi};\beta_k'\right)<0$ for some $\hat{\phi}\in\Phi$.

Q.E.D.

The condition that background risk is distributed through a transformation in no way affects results pertaining exclusively to arbitrary first degree spreads. Under any first degree spread, whenever $\tilde{H}\left(\phi;\beta_i\right)\succeq_1\tilde{H}\left(\phi;\beta_j\right)$ the ratio of differences in the derivatives of the indirect utility function can be expressed as an expectation of prudence via multiplying the integrand prior to the change of variables by $\frac{\frac{\partial}{\partial\phi}\left(u'\left(\theta+\tilde{\varepsilon}\left(\phi\right)\right)\right)}{\frac{\partial}{\partial\phi}\left(u'\left(\theta+\tilde{\varepsilon}\left(\phi\right)\right)\right)}$. As Eeckhoudt et al. have already proven[8], for any first degree spread of an initial background risk to cause increased risk aversion preferences must be decreasing absolute risk averse in the sense of Ross. By focusing on that which is common between all second degree spreads rather than that by which each type differs, we have arrived at conditions that allow us to make a link with lemma 1.

Increased risk aversion across the domain of the foreground risk due to a change in the background risk is necessary and sufficient for a compensated increase in risk based on an initial background risk to not be preferred due to the change. If the change in the background risk can be expressed as a second degree stochastically dominated spread of a transformation of the background risk, then preferences matching the prudence condition given by (3.6) will possess qualitative characteristics

[8] Proposition 2 on page 685.

given by the corollary to theorem 1. That is, a specific comparative relation for temperance exists for increased risk aversion. Knowing this, researchers can determine necessary and sufficient conditions for families of utilities functions that will all experience increased risk aversion across the domain of the foreground risk in a relatively simple manner.

A summary of the results concerning increased risk aversion are provided by tables in the appendix. It is assumed that the transformation function is twice continuously differentiable and $\phi'(\varepsilon) > 0 \forall \varepsilon$. Table 2 differentiates weights for first degree spreads from weights for other types of second degree spreads under the general framework of second degree stochastically dominated spreads. Simply stated, first degree spreads will assign the maximum weight on $p(\theta + \bar{\varepsilon})$ for any second degree spread in ϕ with common differences in marginal indirect utility functions.

Increased Risk Aversion and Classes of Risk Averse Preferences

The most basic transformations for the background risk are the identity function and affine transformations of the identity function. Eeckhoudt et al. have already proven the necessary and sufficient conditions for decreasing absolute risk averse preferences to experience increased risk aversion due to any first degree spreads and any mean preserving spreads. They conclude that for any second degree spread these two conditions taken together are necessary and sufficient for increased risk aversion. According to them, preferences must be decreasing absolute risk averse in the sense of Ross along with a similar condition whereby temperance bounds absolute risk aversion from above, i.e. preferences must satisfy $t(\theta+\varepsilon) \geq r(\theta+\hat{\varepsilon}) \forall \varepsilon, \hat{\varepsilon} \in \mathrm{E}$ combined with $p(\theta+\varepsilon) \geq r(\theta+\hat{\varepsilon}) \forall \varepsilon, \hat{\varepsilon} \in \mathrm{E}$ to be Ross risk vulnerable. The corollary to theorem 1 indicates for an arbitrary second degree spread regardless of its nature, preferences with convex marginal utility experience increased risk aversion across the domain of the foreground risk due to such spreads if and only if $q_1 p(\theta+\bar{\varepsilon}) + q_2 t(\theta+\varepsilon) \geq (q_1+q_2) r(\theta+\hat{\varepsilon})$ for all $\varepsilon, \hat{\varepsilon} \in \mathrm{E}$ where $q_1, q_2 \in [0,1)$ are defined by (4.2) and (4.3) with $0 < q_1 + q_2 < 1$ and this weight is determined by both preferences and the spread. Thus, it is not apparent that decreasing absolute risk aversion in the sense of Ross is necessary, under the general framework of second degree stochastic dominating spreads, for preferences to experience increased risk aversion. It is necessary if we focus our attention on arbitrary first degree spreads. If $q_1 = 0$, then the spread is mean preserving and $t(\theta+\varepsilon) \geq r(\theta+\hat{\varepsilon})$ for all $\varepsilon, \hat{\varepsilon} \in \mathrm{E}$ is the necessary and sufficient condition for all mean preserving spreads to cause risk aversion to increase across the domain of the foreground risk for all utility functions having a nonnegative third derivative.

Theorem 2: *Define q_1 and q_2 as functions of the foreground risk, the stochastic dominating relationship,and a parameter $k > 0$:*

$q_1 := \frac{\frac{-u''(\theta+\bar{\varepsilon})}{\kappa}\int_{\mathrm{E}}\left[\tilde{H}(\phi(\varepsilon);\beta_j)-\tilde{H}(\phi(\varepsilon);\beta_i)\right]d\varepsilon}{\left[v'(\theta;\tilde{H},\beta_j)-v'(\theta;\tilde{H},\beta_i)+k\right]}, q_2 := \frac{\int_{\mathrm{E}}\frac{u'''(\theta+\bar{\varepsilon})}{\kappa}\int_{\underline{\varepsilon}}^{\varepsilon}\left[\tilde{H}(\phi(s);\beta_j)-\tilde{H}(\phi(s);\beta_i)\right]ds\,d\varepsilon}{\left[v'(\theta;\tilde{H},\beta_j)-v'(\theta;\tilde{H},\beta_i)+k\right]}$. *Let $\phi(\varepsilon) = \kappa\varepsilon + c$ where $\kappa > 0$, and let β_i, β_j be such that $\tilde{H}(\phi(\varepsilon);\beta_i) \succeq_2 \tilde{H}(\phi(\varepsilon);\beta_j)$. Then for any preferences satisfying $p(\theta+\varepsilon) \geq 0 \forall \varepsilon$ the following are equivalent.*

(i): $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i) \forall \theta$

(ii): All q_1, q_2 satisfy $0 \leq q_1 + q_2 < 1$ and $q_1, q_2 \in [0,1)$. For all q_1, q_2 such that $q_1 + q_2 \neq 0$,

$qp(\theta+\bar{\varepsilon}) + (1-q)t(\theta+\varepsilon) \geq r(\theta+\hat{\varepsilon})$ for all $\varepsilon, \hat{\varepsilon} \in \mathrm{E}$, all θ, all $q := \frac{q_1}{q_1+q_2}$

(iii): $\int_{\Theta} v(\theta;\tilde{H},\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] \leq 0$ for any α_w satisfying $\int_{\Theta} v(\theta;\tilde{H},\beta_i) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] = 0$ that is not a null spread.

Proof. (i) ⇔ (ii) is proven by the corollary to theorem 1. (i)⇔ (iii) is proven by lemma 1.

Q.E.D.

Theorem 2 provides a simple proof that preferences that are decreasing absolute risk averse in the sense of Ross are also Ross risk vulnerable. This is seen by giving consideration to arbitrary second degree spreads. This is stated formally in the next theorem.

Theorem 3: *Let* $\phi(\varepsilon) = \kappa\varepsilon + c$ *where κ > 0, let* β_i, β_j *be such that* $\tilde{H}(\phi(\varepsilon);\beta_i) \succeq_2 \tilde{H}(\phi(\varepsilon);\beta_j)$ *and preferences satisfy the property of being decreasing absolute risk aversion in the sense of Ross. Then the following are equivalent.*

(i): $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i) \forall\theta$

(ii): Preferences are Ross risk vulnerable.

(iii): $\int_\Theta v(\theta;\tilde{H},\beta_j) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] \leq 0$ *for any* α_w *satisfying* $\int_\Theta v(\theta;\tilde{H},\beta_i) d\left[F(\theta;\alpha_w) - F(\theta;\alpha_h)\right] = 0$ *that is not a null spread.*

Proof: (i) ⇔ (iii) is proven by lemma 1.

(i) ⇔ (ii): If preferences are not Ross risk vulnerable, then $t(\theta+\varepsilon) < r(\theta+\hat{\varepsilon})$ for some combination of $\varepsilon, \hat{\varepsilon} \in \mathrm{E}$. Thus, for mean preserving spreads of ϕ we have $t(\theta+\varepsilon) < r(\theta+\hat{\varepsilon})$. By the corollary to theorem 1, this implies $R(\theta;\tilde{H},\beta_j) < R(\theta;\tilde{H},\beta_i)$ for some θ and some second degree spread. Conversely, suppose that $R(\theta;\tilde{H},\beta_j) < R(\theta;\tilde{H},\beta_i)$ for some θ. Given that preferences are decreasing absolute Ross risk averse, this is equivalently expressed as $\frac{-\left[v''(\theta;\tilde{H},\beta_j) - v''(\theta;\tilde{H},\beta_i)\right] + kR(\theta;\tilde{H},\beta_i)}{v'(\theta;\tilde{H},\beta_j) - v'(\theta;\tilde{H},\beta_i)} < R(\theta;\tilde{H},\beta_i)$ and it follows by lemma 3 (ignoring mean preserving spreads) that

$$p(\theta+\bar{\varepsilon})q + (1-q)\int_\mathrm{E} t(\theta+\varepsilon)\, d\tilde{H}_2(\phi(\varepsilon);\beta_i,\beta_j,\theta) < \int_\mathrm{E} \frac{-u''(\theta+\varepsilon)d\tilde{H}(\phi(\varepsilon);\beta_i)}{\int_\mathrm{E} u'(\theta+\varepsilon)d\tilde{H}(\phi(\varepsilon);\beta_i)}$$

Consequently, there must be some combination of $\varepsilon, \hat{\varepsilon} \in \mathrm{E}$ such that

$$p(\theta+\bar{\varepsilon})q + (1-q)t(\theta+\varepsilon) < r(\theta+\hat{\varepsilon}) \Rightarrow t(\theta+\varepsilon) - r(\theta+\hat{\varepsilon}) < \tfrac{q}{1-q}\left[r(\theta+\hat{\varepsilon}) - p(\theta+\bar{\varepsilon})\right] \leq 0$$

Therefore, preferences are not Ross risk vulnerable.

Q.E.D.

When preferences are decreasing absolute risk averse Pratt and Zeckhauser define properness[9] to be equivalent to $\tilde{\omega}+\tilde{\theta} \square \tilde{\omega}$ and $\tilde{\omega}+\tilde{\varepsilon} \square \tilde{\omega}$ together imply $\tilde{\omega}+\tilde{\theta}+\tilde{\varepsilon} \preceq \tilde{\omega}+\tilde{\theta}$, where the tildes represent arbitrary distributions. Using the example for the transformation $\phi(\varepsilon;\alpha_h)=\kappa\int_{\Theta} u(\theta+\varepsilon)\,dF(\theta;\alpha_h)+c$, the corollary to theorem 1 and lemma 1 taken together indicate that due to a mean preserving spread in the transformed background risk preferences will satisfy (4.10) with $q_1 = 0$ for all θ if and only if they experience increased risk aversion and $p(\theta+\varepsilon) > r(\theta_h+\varepsilon)$ for almost all ε, all θ. Using theorem 1 and lemma 1 it can be seen that when this is true both of the following conditions,

$$\int_{\Theta} v(\theta;\tilde{H},\beta_i)\,d\left[F(\theta;\alpha_w)-F(\theta;\alpha_h)\right]=0 \Rightarrow \int_{\Theta} v(\theta;\tilde{H},\beta_j)\,d\left[F(\theta;\alpha_w)-F(\theta;\alpha_h)\right]\le 0 \qquad (5.1)$$

and

$$\int_{\mathrm{E}}\int_{\Theta} u(\theta+\varepsilon)\,dF(\theta,\alpha_h)\,d\left[\tilde{H}\left(\phi(\varepsilon;\alpha_h);\beta_j\right)-\tilde{H}\left(\phi(\varepsilon;\alpha_h);\beta_i\right)\right]=0 \qquad (5.2)$$

exist simultaneously. For this family of utility functions, if the background risk deteriorates in the presence of a foreground risk such that the agent remains indifferent between the two lotteries as is the case for (5.2); and if the foreground risk deteriorates in the presence of a background risk such that the agent remains indifferent between the two lotteries as is the case for the left hand side of (5.1); then the foreground risk deteriorating in the presence of the deteriorated background risk cannot make the individual better off as seen in the right hand side of (5.1). In other words, let ξ be a random variable with a mean conditional on ε, and γ be a random variable with a mean conditional on θ. The agent's preferences concerning the lotteries given in (5.1) and (5.2) can be stated equivalently as $(\theta+\varepsilon)+\xi \square (\theta+\varepsilon)$ and $(\theta+\varepsilon)+\gamma \square (\theta+\varepsilon)$ implies $(\theta+\varepsilon)+\xi+\gamma \preceq (\theta+\varepsilon)+\gamma$, which defines properly risk averse preferences. However, the preferences are stated in terms of a specific type of risk. The left-hand side of (5.1) and (5.2) involve mean expected utility preserving noise, also referred to as patent increases in risk. According to Kimball (1993), for preferences that are decreasing absolute risk

[9] Theorem 1 (1987).

averse, a risk such as (θ + ξ) is patently more risky than θ if and only if $v\left(\theta;\tilde{H},\beta_j\right)$ is globally more risk averse than $v\left(\theta;\tilde{H},\beta_i\right)$ implies the more risk averse do not have lower risk premia in the presence of the patently greater risk. Kimball claims this is exactly the set of increases in risk that are Diamond-Stiglitz increases in risk. This leads to a theorem on increased risk aversion for preferences that are properly risk averse toward a patent increase in risk.

Theorem 4: *Let* $\phi(\varepsilon;\alpha_h)=\kappa\int_\Theta u(\theta+\varepsilon)dF(\theta;\alpha_h)+c$*, with κ > 0. Let β_i, β_j, α_h be such that* $\tilde{H}\left(\phi(\varepsilon;\alpha_h);\beta_i\right)\succeq_{MP}\tilde{H}\left(\phi(\varepsilon;\alpha_h);\beta_j\right)$*. Define q_1 and q_2 as functions of the foreground risk, the stochastic dominating relationship, and a parameter k > 0:*

$$q_1=0,q_2:=\frac{\dfrac{\int_{\mathrm{E}}\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\varepsilon)\right)\int_{\underline{\varepsilon}}^{\varepsilon}\left[\tilde{H}\left(\phi(s);\beta_j\right)-\tilde{H}\left(\phi(s);\beta_i\right)\right]\phi'(s)\,ds\,d\varepsilon}{k\int_\Theta u'(\theta+\varepsilon)dF(\theta;\alpha_h)}}{\left[v'\left(\theta;\tilde{H},\beta_j\right)-v'\left(\theta;\tilde{H},\beta_i\right)+k\right]}$$

. Then for preferences satisfying $p(\theta+\varepsilon)\geq r(\theta_h+\varepsilon)\forall\theta,\varepsilon$ *the following are equivalent.*

(i): $R\left(\theta;\tilde{H},\beta_j\right)\geq R\left(\theta;\tilde{H},\beta_i\right)\forall\theta$

(ii): $t(\theta+\varepsilon)\geq p(\theta+\varepsilon)-\frac{\left(r(\theta+\hat{\varepsilon})-p(\theta+\varepsilon)\right)\left(r(\theta_h+\varepsilon)-p(\theta+\varepsilon)\right)}{p(\theta+\varepsilon)}$ *for all* $\varepsilon,\hat{\varepsilon}\in\mathrm{E}$ *and all* θ

(iii): $\int_\Theta v\left(\theta;\tilde{H},\beta_j\right)d\left[F(\theta;\alpha_w)-F(\theta;\alpha_h)\right]\leq 0$ *for any α_w satisfying* $\int_\Theta v\left(\theta;\tilde{H},\beta_i\right)d\left[F(\theta;\alpha_w)-F(\theta;\alpha_h)\right]=0$ *that is not a null spread.*

(iv): u is properly risk averse toward patent increases in risk.

Proof. (i) ⇔ (ii) is proven by the corollary to theorem 1. Divide through (4.10) by q_2 the add and subtract $p(\theta+\varepsilon)$ in the numerator of (4.10). (i)⇔ (iii) is proven by lemma 1. (iii) ⇔ (iv) Suppose u is not properly risk averse toward patently greater risk. Then there exists an α_k and an α_x such that $\tilde{F}\left(V^{\beta_i};\alpha_k\right)\succeq_{MP}\tilde{F}\left(V^{\beta_i};\alpha_x\right)$ and for $\tilde{H}\left(\phi(\varepsilon;\alpha_k);\beta_i\right)\succeq_{MP}\tilde{H}\left(\phi(\varepsilon;\alpha_k);\beta_j\right)$, (5.2) holds for 'k' rather than 'h'. By the definition of preferences u is decreasing absolute risk averse and it follows that

$$\int_{\Theta} v\left(\theta;\tilde{H},\beta_i\right)d\left[F\left(\theta;\alpha_x\right)-F\left(\theta;\alpha_k\right)\right]=0 \nRightarrow \int_{\Theta} v\left(\theta;\tilde{H},\beta_j\right)d\left[F\left(\theta;\alpha_x\right)-F\left(\theta;\alpha_k\right)\right]\leq 0$$

Now suppose that

$$\int_{\Theta} v\left(\theta;\tilde{H},\beta_i\right)d\left[F\left(\theta;\alpha_w\right)-F\left(\theta;\alpha_h\right)\right]=0 \nRightarrow \int_{\Theta} v\left(\theta;\tilde{H},\beta_j\right)d\left[F\left(\theta;\alpha_w\right)-F\left(\theta;\alpha_h\right)\right]\leq 0$$

then u cannot be properly risk averse given that (5.2) holds.

Q.E.D.

By definition of the transformed background risk, all preferences satisfying $p\left(\theta+\varepsilon\right)\geq r\left(\theta_h+\varepsilon\right)\forall\theta,\varepsilon$ that also satisfy one of the equivalent conditions of theorem 3 are properly risk averse toward patent increases in risk. For these preferences, a mean utility preserving addition of noise in the background added to a mean utility preserving noise in the foreground will not be preferred by those that experience increased risk aversion due to the mean utility preserving addition of noise in the background[10]. Two independent and separate increases in risk that are fully compensated cannot become more attractive when they exist together. Temperance must be sufficiently large, as given in condition (ii), for the agent to experience increased risk aversion when the change in the background risk is such that the agent is indifferent between the initial state and the final state. The weight on the prudence value is zero for all of θ given the nature of the spread in risk.

Condition (ii) of theorem 3 also satisfies the necessary condition for fixed wealth properness:

$$r''\left(\omega\right)-r'\left(\omega\right)r\left(\omega\right)\equiv t\left(\omega\right)-p\left(\omega\right)+\frac{\left(r(\omega)-p(\omega)\right)^2}{p(\omega)}\geq 0\forall\omega$$ [iii]

This implies preferences that are properly risk averse toward a patent increase in risk are risk vulnerable. Thus, any preferences satisfying $p\left(\theta+\varepsilon\right)\geq r\left(\theta_h+\varepsilon\right)\forall\theta,\varepsilon$ that experiences increased risk aversion across the domain of the foreground risk due to a patent increase in risk is also risk vulnerable. This property is stronger than decreasing absolute risk averse in the sense of Ross.

Conclusion

Changes in background risk may involve more complicated shifts than second degree spreads in risk. Utilizing conditions that are known to hold for second degree stochastic dominating relationships, use of transformations of the background risk expand the applicability of techniques acquired from this field. If the change in the background risk can be expressed as a second degree stochastic dominating spread of a transformed of the background risk, then necessary and sufficient conditions exist for increased risk aversion across the domain of the foreground risk for preferences that satisfy a lower bound on the agent's prudence measure determined by the transformation of the background risk. One way of characterizing the necessary and sufficient condition is to say that temperance must be sufficiently high given the agent's absolute prudence. In general, the greater the tendency to moderate exposure to other risks the greater the likelihood that the agent will experience increased risk aversion. The importance of temperance is central to increased risk aversion under the general framework of stochastic dominating spreads. This investigation confirms Gollier and Pratt's assertions (1996) about the tempering effects of background risk.

It has also been proven that just as second degree spreads of an initial risk can be expressed as a finite sequence of mean preserving and first degree spreads, it is possible to explain the change in risk aversion as a result of a weighted combination of prudence at the maximal element of the background risk and the product of prudence and temperance relative to prudence. This sheds some light on the approach by Eeckhoudt et al. in finding unambiguous results for second degree spreads in general by considering the special cases of first degree and mean preserving spreads. The question remains as to how to resolve results that hold under arbitrary first degree spreads with the first degree spread conditions found from arbitrary second degree spreads. What can be said at this point is that for all second degree spreads of an initial risk (all ϕ related) that cause marginal indirect utility to change by an identical amount, any one of these that are first degree stochastically by the initial distribution will place greater weight on $p\left(\theta+\tilde{\varepsilon}(\phi)\right)$ than any of the other distributions that also stochastically dominate it in a first degree sense.

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[10] Pratt (1988) has more to say briefly about these types of risks in section 5.3. A patent increase in risk is also

It appears that temperance is relevant for first degree spreads under the broader scope of spreads that includes all types of second degree spreads in risk. For any second degree spread that is mean preserving, absolute temperance must be sufficiently large relative to absolute prudence for an agent to experience increased risk aversion. If the spread in risk involves any decrease in the mean of ϕ, i.e. a first degree spread in the finite sequence of constructing a second degree spread from an initial risk, then some weight is given to a prudence measure.

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equivalent to a certainty equivalent of zero for the risk increase.

Appendix

Tables

Table 1

Restrictions on preferences in addition to u' > 0 and u" < 0	***General results for the condition that any β_j satisfies $\tilde{H}(\phi;\beta_i) \succeq_2 \tilde{H}(\phi;\beta_j)$ for a given β_i (excludes null spreads), $\phi \in C^2(\mathrm{E})$ and $\phi'(\varepsilon) > 0 \forall \varepsilon$***
$p(\theta+\varepsilon) \geq \frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} \forall \varepsilon, \theta$	i. There exist $q_1, q_2 \in [0,1)$ such that $0 \leq q_1 + q_2 < 1$ where q_1 and q_2 are defined in (4.2) and (4.3) respectively ii. $\frac{-\left[v''(\theta;\tilde{H},\beta_j) - v''(\theta;\tilde{H},\beta_i)\right] + kR(\theta;\tilde{H},\beta_i)}{v'(\theta;\tilde{H},\beta_j) - v'(\theta;\tilde{H},\beta_i) + k} \equiv$ $p(\theta+\overline{\varepsilon})q_1 + \int_{\mathrm{E}} p(\theta+\varepsilon) \frac{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - p(\theta+\varepsilon)} d\tilde{H}_2(\phi(\varepsilon);\beta_i,\beta_j,\theta) q_2$ $+(1-q_1-q_2)R(\theta;\tilde{H},\beta_i)$ iii. $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i) \forall \theta$ implies $t(\theta+\varepsilon) \geq \frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} \forall \varepsilon, \theta$ iv. $R(\theta;\tilde{H},\beta_j) \geq R(\theta;\tilde{H},\beta_i) \forall \theta \Leftrightarrow$ $p(\theta+\overline{\varepsilon})\frac{q_1}{q_1+q_2} + p(\theta+\varepsilon)\frac{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)} - p(\theta+\varepsilon)}\frac{q_2}{q_1+q_2} \geq r(\theta+\hat{\varepsilon}) \forall \hat{\varepsilon}, \varepsilon$ and all $q_1 + q_2 \neq 0$

Table 2

Second degree stochastically dominated relationship	*General results for the condition that any β_j satisfies* $\tilde{H}(\phi;\beta_i)\succeq_2 \tilde{H}(\phi;\beta_j)$ *for a given β_i (excludes null spreads),* $\phi\in C^2(\mathrm{E})$ *and* $\phi'(\varepsilon)>0\,\forall\varepsilon$ *and* $p(\theta+\varepsilon)\geq\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}\forall\varepsilon,\theta$ *and concave utility*
$\tilde{H}(\phi(\varepsilon);\beta_i)\succeq_1 \tilde{H}(\phi(\varepsilon);\beta_j)$	For any β_k satisfying $\tilde{H}(\phi(\varepsilon);\beta_i)\succeq_2 \tilde{H}(\phi(\varepsilon);\beta_k)$: $\tilde{H}(\phi(\varepsilon);\beta_k)\succeq_1 \tilde{H}(\phi(\varepsilon);\beta_j)\forall\beta_k$ => $q_1(\theta;\beta_i,\beta_j,k)=\max\limits_{\beta_k}\left\{q_1(\theta;\beta_i,\beta_k,k):v'(\theta;\tilde{H},\beta_k)-v'(\theta;\tilde{H},\beta_i)=c\right\}$.
$\tilde{H}(\phi(\varepsilon);\beta_i)\succeq_2 \tilde{H}(\phi(\varepsilon);\beta_j)$ and $\tilde{H}(\phi(\varepsilon);\beta_i)\not\succeq_1 \tilde{H}(\phi(\varepsilon);\beta_j)$ $\tilde{H}(\phi(\varepsilon);\beta_i)\not\succeq_{MP} \tilde{H}(\phi(\varepsilon);\beta_j)$	$\frac{q_1}{q_1+q_2}p(\theta+\bar{\varepsilon})+\frac{q_2}{q_1+q_2}p(\theta+\varepsilon)\frac{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}-t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}-p(\theta+\varepsilon)}\geq r(\theta+\hat{\varepsilon})\forall\varepsilon,\hat{\varepsilon}$ for all q_1, q_2 such that for all β_i, β_j combinations satisfying $v'(\theta;\tilde{H},\beta_j)-v'(\theta;\tilde{H};\beta_i)=c$, $q_1\notin\max\left\{q_1:v'(\theta;\tilde{H},\beta_j)-v'(\theta;\tilde{H},\beta_i)=c,\tilde{H}(\phi(\varepsilon);\beta_i)\succeq_2 \tilde{H}(\phi(\varepsilon);\beta_j)\right\}$ and $q_1\neq 0$
$\tilde{H}(\phi(\varepsilon);\beta_i)\succeq_{MP} \tilde{H}(\phi(\varepsilon);\beta_j)$	$p(\theta+\varepsilon)\frac{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}-t(\theta+\varepsilon)}{\frac{-\phi''(\varepsilon)}{\phi'(\varepsilon)}-p(\theta+\varepsilon)}\geq r(\theta+\hat{\varepsilon})\forall\varepsilon,\hat{\varepsilon}$ for all $q_2\neq 0$

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Appendix

Notes

[i] For example, the lottery {p, ω_1; (1-p), ω_2 + {½, -θ; ½, θ}} with $\omega_2 - \theta > \omega_1$, yields a risk premium of

$\pi(\varepsilon,\omega) \approx -\frac{(1-p)}{2}\frac{u''(\omega_2)E\varepsilon^2}{pu'(\omega_1)+(1-p)u'(\omega_2)} \le -\frac{(1-p)}{2}\frac{u''(\omega_2)E\varepsilon^2}{pu'(\omega_1)}$. Thus, if an agent places greater relative importance on the less desirable outcome, ω_1, this will be reflected by a higher marginal utility measure. If 'more risk averse' is characterized by a relatively steeper utility function below ω_1 than that above ω_1, the risk premium continues to fall and the agent's aversion to the risk is mitigated by his preference for this particular heteroskedastic quality of the background risk, whereby the risk only becomes relevant with the more desirable outcome.

[ii] Using integration by parts twice, the second equality exploits the fact that $\int_a^a f(x)dx = 0$ and the third equality breaks up the numerator using $q_1 := \frac{-\frac{\partial}{\partial\phi}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\sup\Phi}\int_\Phi\left[\tilde{H}(\phi;\beta_j)-\tilde{H}(\phi;\beta_i)\right]d\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$ and

$q_2 := \frac{\int_\Phi \frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k}$:

$$\begin{aligned}
\frac{-\left[\tilde{v}''(\theta;\beta_j)-\tilde{v}''(\theta;\beta_i)\right]+k\tilde{R}(\theta;\beta_i)}{\tilde{v}'(\theta;\beta_j)-\tilde{v}'(\theta;\beta_i)+k} &= \frac{\left[\frac{\partial}{\partial\phi}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\right]_\Phi-\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi+k\tilde{R}(\theta;\beta_i)}{-\left[\frac{\partial}{\partial\phi}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds\right]_\Phi+\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi+k} \\
&= \frac{\frac{\partial}{\partial\phi}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\bar{\phi}}\int_\Phi\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]d\phi-\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi+k\tilde{R}(\theta;\beta_i)}{-\frac{\partial}{\partial\phi}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\bar{\phi}}\int_\Phi\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]d\phi+\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\tilde{\varepsilon}}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi+k} \\
&= \frac{\frac{\partial}{\partial\phi}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\bar{\phi}}}{-\frac{\partial}{\partial\phi}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\Big|_{\bar{\phi}}}q_1+\frac{-\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}{\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}q_2+(1-q_1-q_2)\tilde{R}(\theta;\beta_i)
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
\frac{-\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}{\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi} &= \int_\Phi\frac{-\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)}{\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)}\frac{\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]ds}{\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}d\phi \\
&= \int_\Phi\frac{-\frac{\partial^2}{\partial\phi^2}\left(u''(\theta+\tilde{\varepsilon}(\phi))\right)}{\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)}d\int_{\underline{\phi}}^{\phi}\frac{\frac{\partial^2}{\partial t^2}\left(u'(\theta+\tilde{\varepsilon}(t))\right)\int_{\underline{\phi}}^{t}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsdt}{\int_\Phi\frac{\partial^2}{\partial\phi^2}\left(u'(\theta+\tilde{\varepsilon}(\phi))\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}(s;\beta_j)-\tilde{H}(s;\beta_i)\right]dsd\phi}
\end{aligned}$$

The expression, $\int_{\underline{\phi}}^{\phi} \frac{\frac{\partial^2}{\partial t^2}\left(u'\left(\theta+\tilde{\varepsilon}(t)\right)\right)\int_{\underline{\phi}}^{t}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]dsdt}{\int_{\Phi}\frac{\partial^2}{\partial \phi^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)\int_{\underline{\phi}}^{\phi}\left[\tilde{H}\left(s;\beta_j\right)-\tilde{H}\left(s;\beta_i\right)\right]dsd\phi}$ is a distribution function provided that $\frac{\partial^2}{\partial t^2}\left(u'\left(\theta+\tilde{\varepsilon}(\phi)\right)\right)$ is of uniform sign for all ϕ.

iii

$$\begin{aligned}
r''(\omega)-r'(\omega)r(\omega) &= \tfrac{\partial}{\partial\omega}\left\{r(\omega)\left[r(\omega)-p(\omega)\right]\right\}-r(\omega)^2\left[r(\omega)-p(\omega)\right] \\
&= r(\omega)\left[r(\omega)-p(\omega)\right]^2+r(\omega)\left[r'(\omega)-p'(\omega)\right]-r(\omega)^2\left[r(\omega)-p(\omega)\right] \\
&= r(\omega)\left[r(\omega)-p(\omega)\right]^2+r(\omega)\left\{r(\omega)\left[r(\omega)-p(\omega)\right]-p(\omega)\left[p(\omega)-t(\omega)\right]\right\}-r(\omega)^2\left[r(\omega)-p(\omega)\right] \\
&= r(\omega)\left\{\left[r(\omega)-p(\omega)\right]^2-p(\omega)\left[p(\omega)-t(\omega)\right]\right\}\geq 0 \\
&\Rightarrow t(\omega)\geq p(\omega)-\tfrac{\left[r(\omega)-p(\omega)\right]^2}{p(\omega)}
\end{aligned}$$