Time inconsistency and reputation in monetary policy: a strategic model in continuous time

Li, Jingyuan and Tian, Guoqiang

25 July 2005

TIME INCONSISTENCY AND REPUTATION IN MONETARY POLICY: A STRATEGIC MODELLING IN CONTINUOUS TIME*

Li Jingyuan
School of Management, Huazhong University of Science and Technology, Wuhan 430074, China

Tian Guoqiang
Department of Economics, Texas A&M University, College Station, Texas 77843, USA

Abstract

This article develops a model to examine the equilibrium behavior of the time nconsistency problem in a continuous time economy with stochastic and endogenized distortion. First, the authors introduce the notion of sequentially rational equilibrium, and show that the time inconsistency problem may be solved with trigger reputation strategies for stochastic setting. The conditions for the existence of sequentially rational equilibrium are provided. Then, the concept of sequentially rational stochastically stable equilibrium is introduced. The authors compare the relative stability between the cooperative behavior and uncooperative behavior, and show that the cooperative equilibrium in this monetary policy game is a sequentially rational stochastically stable equilibrium and the uncooperative equilibrium is sequentially rational stochastically unstable equilibrium. In the long run, the zero inflation monetary policies are inherently more stable than the discretion rules, and once established, they tend to persist for longer periods of the time.

Keywords: Time inconsistency, optimal stopping, stochastically stable equilibrium

2000 MR Subject Classification: 91B64, 62P20

1 Introduction

Time inconsistency is an interesting problem in macroeconomics in general, and monetary policy in particular. Although technologies, preferences, and information are the same at different times, the policymaker's optimal policy chosen at time t1 differs from the optimal policy for t1 chosen at t0 < t1. The study of time inconsistency is important. It not only provides positive theories that help us to understand the incentives faced by policymakers and provides the

---

*Received July 25, 2005; revised December 4, 2006. Sponsored by the National Natural Science Foundationof China (70602012), Texas Advanced Research Program as well as from the Bush Program in the Economics of Public Policy, the Private Enterprise Research Center, and the Lewis Faculty Fellowship at Texas A&M University.

natural starting point for attempts to explain the actual behavior of policymakers and actual policy outcomes, but also requires one to design policy-making institutions. Such a normative task can help one understand how institutional structures affect policy outcomes.

This problem was first noted by Kydland and Prescott [4]. Several solutions were proposed to deal with this problem since then. Barro and Gordon [1] were the first to build a game model to analyze "reputation" of monetary policy.

A second solution is the basis of the incentive contracting approach to monetary policy. Persson and Tabellini [7], Walsh [12], and Svensson [10] developed models by using this approach. A third solution is built on the legislative approach. The major academic contribution in this area was by Rogoff [8].

Among these approaches, the "reputation" problem is the key. If reputation consideration discourages the monetary authorities from attempting surprise inflation, then, legal or contracting constraints on monetary authorities are unnecessary and may be harmful.

The main questions on reputation are when and how the policymaker chooses inflation optimally to minimize welfare loss, and, whether the punishment can induce the policymaker to choose zero inflation. The conclusions of Barro-Gorden models are: first, there exists a zero-inflation Nash equilibrium if the punishment for the policymaker deviating from zero-inflation is large enough. However, this equilibrium is not sequentially rational over a finite time horizon. The only sequentially rational equilibrium is achieved if the policymaker chooses discretionary inflation and the public expects it. Only over an infinite time horizon one can get a low-inflation equilibrium. Otherwise, the policymaker would be sure in the last period to produce the discretionary outcome whatever the public's expectation were and, by working backward, would be expected to do the same in the first period. Secondly, there are multiple Nash equilibria and there is no mechanism to choose between them.

This article develops a continuous time model of central bank at the spirit of Kydland and Prescott, and Barro and Gordon. The main differences between our model and previous models are the following two assumptions: (i) the natural rate of output** is a Brownian motion; (ii) the distortion of the economy is correlated to the natural rate.

The reason that we use assumption (i) is that the most recent literature shares (see Salemi [9]) the view that the natural rate changes over time and specifies the natural rate as a random walk without drift seems a plausible assumption for U.S. unemployment data.

The key aspect of this monetary time inconsistency problem is the distortion which arises from the labor-market distortions and the political pressure on the central bank. Most often, some appeal is made to the presence of labor-market distortions, for example, a wage tax. Because the larger scale of the economy implies the larger wage tax, it seems reasonable for us to assume that the distortion is an increasing function of the scale of the economy. We use a linear function to approximate this function.

In this article, we use the optimal stopping theory to study the time inconsistency problem in monetary policy with the continuous time model. By using the optimal stopping theory and

---

**The natural rate of output depends on the natural rate of unemployment. Friedman showed that monetary policy could not be used to achieve full unemployment. Unfortunately, inflation starts accelerating before full unemployment is reached. The best a nation can do is settle for the lowest level of unemployment that will not begin accelerating inflation. Friedman called this point the "natural rate of unemployment"(see Salemi [9]).

introducing the notions of sequentially rational equilibria, we give the conditions under which the time inconsistency problem may be solved with trigger reputation strategies. We provide the conditions for the existence of sequentially rational equilibrium.

We argue that the traditional concepts of equilibrium are not satisfactory as a predictor of long run behavior when the game is subjected to persistent stochastic shocks. The concept of sequentially rational stochastically stable equilibrium is introduced. Then, we compare the relative stability between the cooperative behavior and uncooperative behavior, and show that the cooperative equilibrium in this monetary policy game is a sequentially rational stochastically stable equilibrium and the uncooperative equilibrium in this monetary policy game is a sequentially rational stochastically unstable equilibrium.

The results obtained in the article imply that, in the long run, the zero inflation monetary policies are inherently more stable than the discretion rules, and once established, they tend to persist for longer periods of the time.

The article is organized as follows. Section 2 will set up the model and provides a solution for the optimal stopping problem faced by the policymaker. In Section 3, we study the equilibrium behavior. The stochastic stability of this monetary game is discussed in Section 4. Section 5 gives the conclusion.

2 Model

2.1 The Setup

We consider a continuous time game theoretical model with two players: the policymaker and the public. The policymaker's strategy space is $R^+ \times L[0,T]$, from which the policymaker is to choose an action $(\tau, \{\pi_t\}_{t\in T})$. Here, τ is the time that the policymaker changes his monetary policy from the zero-inflation rule to a discretion rule; π_t is the inflation rate chosen by the policymaker at time t; T is the lifetime of the policymaker which can be finite or infinite; and $L[0,T]$ is the class of Lebesgue integrable functions defined on $[0,T]$. The public's strategy space is $L[0,T]$, from which the public is to choose an action $(\{\pi_t^e\}_{t\in T})$. Here, π_t^e is the expected inflation rate formed by the public at time t.

Suppose that, at the beginning, the policymaker commits an inflation rate $\pi_0 = 0$, and the public believes it so that $\pi_0^e = \pi_0 = 0$. The policymaker has the right to switch from the zero-inflation to a discretion rule $\pi_t \neq 0$ at the time t between 0 and T. However, after he changes his policy, he loses his reputation.

The policymaker's loss function is described by a quadratic discounted expected loss function of the form:

$$\Lambda = E\int_0^T \mathrm{e}^{-\rho\cdot t}\Big[\frac{1}{2}\theta\Big(y_t - \bar{y}_t - k_t\Big)^2 + \frac{1}{2}\pi_t^2\Big]\mathrm{d}t, \tag{1}$$

where ρ is the discount factor with $0 < \rho < 1$, y_t is aggregate output, $\bar{y}_t$ is the economy's natural rate of output, and k_t is the distortion, which is equal $\alpha\bar{y}_t$, $\alpha > 0$. θ is a positive constant that represents the relative weight put by the policymaker on output expansions relative to inflation stabilization. Here, without loss of generality, the target inflation π is assumed to be zero. Marco-welfare function (1) has played an important role in the literature, and means that the policymaker desires to stabilize both output around $\bar{y}_t + k_t$, which exceeds the economy's equilibrium output of $\bar{y}_t$ by k_t, and inflation around zero.

Here, we assume that $\bar{y}_t = X_t$ and $\mathrm{d}X_t = \sigma \mathrm{d}B_t, X_0 = x$, where B_t is 1-dimensional Brownian motion and σ is the diffusion coefficient.

The policymaker's objective is to minimize this discounted expected loss function (1) subject to the constraint imposed by a Lucas-type aggregate supply function, the so-called Phillips curve, which describes the relationship between output and inflation in each period:

$$y_t - \bar{y}_t = a(\pi_t - \pi_t^e) + u_t, \tag{2}$$

where a is a positive constant that represents the effect of a money surprise on output, and u_t is a bounded random variable with $E[u_t] = 0$, $\mathrm{Var}[u_t] = \sigma_u^2$, $|u_t| \leq M_1$ for all t and $\mathrm{cov}(u_s, u_t) = 0$, for $t \neq s$, which represents the shock at time t. And we assume that $\bar{y}_t$ and u_t are independent. We also assume that the policymaker can observe u_t and X_t prior to setting π_t.

The public has complete information about the policymaker's objectives. It is assumed that the public forms his expectations rationally, and thus, the assumption of rational expectation implicitly defines the loss function for the public as $E[\pi_t - \pi_t^e]^2$. The public's objective is to minimize this expected inflation error. Given the public's understanding of the policymaker's decision problem, its choice of π^e is optimal.

We first examine the "one-shot" game. The single-period loss function ℓ_t for the policymaker is

$$\ell_t(\pi_t, \pi_t^e) = \frac{1}{2}\theta(y_t - \bar{y}_t - k_t)^2 + \frac{1}{2}\pi_t^2 = \frac{1}{2}\theta[a(\pi_t - \pi_t^e) - \alpha X_t + u_t]^2 + \frac{1}{2}\pi_t^2. \tag{3}$$

The equilibrium concept in this game is noncooperative Nash. Then, the policymaker minimizes ℓ_t by taking π_t^e as given, and thus, we have the best response function for the policymaker:

$$\pi_t^D = \frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha X_t - u_t). \tag{4}$$

The public is assumed to understand the incentive facing the policymaker so it uses (4) in forming its expectations about inflation so that

$$\pi_t^e = E\pi_t^D = \frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha E X_t). \tag{5}$$

Solving (5) for π_t^e, we get the unique Nash equilibrium $\pi_t^{e*} = E\pi_t^{D*} = a\theta\alpha E X_t$. Thus, as long as $EX_t \neq 0$, the policymaker has incentives to use the discretion rule although the loss at $\pi_t^e = \pi_t = 0$ is lower than at $\pi_t^{e*} = E\pi_t^{D*}$. This is the problem of time inconsistency.

A potential solution to the above time inconsistency problem is to force the policymaker to bear some consequence penalties if he deviates from his announced policy of low inflation. One of such penalties that may take is a loss of reputation. If the policymaker deviates from the low-inflation solution, credibility is lost and the public expects high inflation in the future. That is, the public expects zero-inflation as long as policymaker has fulfilled the inflation expectation in the past. However, if actual inflation exceeds what was expected, the public anticipates that the policymaker will apply discretion in the future. So, the public forms its expectation according to the trigger strategy: Observing "good" behavior induces the expectation of continual good behavior and a single observation of "bad" behavior triggers a revision of expectations.

2.2 The Optimal Stopping Problem for Policymaker

In order to solve the time inconsistency problem by using the reputation approach, we first incorporate the policymaker's loss minimization problem into a general optimal stopping time problem. During any time in $[0, T]$, the policymaker has the right to reveal his type (discretion or zero-inflation). Because he has the right but not the obligation to reveal his type, we can think it is an option for the policymaker. So, the policymaker's decision problem is to choose a best time $\tau \in [0, T]$ to exercise this option.

The policymaker considers the following optimal stopping problem: find τ^* such that

$$L^*(x) = \inf_{\tau} E^x\Big[\int_0^{\tau} f(t, X_t)\mathrm{d}t + g(\tau, X_\tau)\Big] = E^x\Big[\int_0^{\tau^*} f(t, X_t)\mathrm{d}t + g(\tau^*, X_{\tau^*})\Big], \tag{6}$$

where

$$f(s, X_t) = \frac{1}{2}\theta \mathrm{e}^{-\rho\cdot s}(\alpha X_t - u_t)^2 \tag{7}$$

is the instantaneous loss function for the policymaker when he uses the zero-inflation rate, and

$$g(s, X_\tau) = \mathrm{e}^{-\rho s} E^{X_\tau}\Big[\int_s^T \mathrm{e}^{-\rho(t-s)}\Big[\frac{\theta}{2}[a(\pi_t^D - \pi_t^e) - \alpha X_t + u_t]^2 + \frac{\pi_t^{D^2}}{2}\Big]\mathrm{d}t\Big] \tag{8}$$

is the expected loss function for policymaker, in which he begins to use the discretion rule at time s. We assume that $g(\cdot,\cdot)$ is a bounded function, i.e., $g(\cdot,\cdot) \le M$ for some constant M.

Let $\{\mathcal{F}_t\}$ be a filtration of B_t. We assume that the public's strategy π_t^e for $t > \tau$ is $\{\mathcal{F}_\tau\}$-adapted. This means that when the public forms their expectation at time t, they know the natural rate at τ.

To compute $g(\tau, X_\tau)$, substituting (4) into (8), we have

$$\begin{aligned}
g(\tau, X_\tau) &= \mathrm{e}^{-\rho\tau} E^{X_\tau}\Big[\int_\tau^T \mathrm{e}^{-\rho(t-\tau)}\Big[\frac{\theta}{2}[a(\pi_t^D - \pi_t^e) - \alpha X_t + u_t]^2 + \frac{1}{2}\pi_t^{D^2}\Big]\mathrm{d}t\Big] \\
&= \frac{1}{2}\frac{\theta}{1+a^2\theta}\mathrm{e}^{-\rho\tau} E^{X_\tau}\Big[\int_\tau^T \mathrm{e}^{-\rho(t-\tau)}\Big(\alpha X_t - u_t + a\pi_t^e\Big)^2\mathrm{d}t\Big] \\
&= \frac{\theta\alpha^2}{2(1+a^2\theta)}\mathrm{e}^{-\rho\tau}\cdot\Big\{E^{X_\tau}\Big[\int_\tau^T \mathrm{e}^{-\rho(t-\tau)} X_t^2\mathrm{d}t\Big] + 2\alpha a\pi_t^e E^{X_\tau}\Big[\int_\tau^T \mathrm{e}^{-\rho(t-\tau)} X_t\mathrm{d}t\Big] \\
&\quad + (a^2\pi_t^{e2} + \sigma_u^2) E^{X_\tau}\Big[\int_\tau^T \mathrm{e}^{-\rho(t-\tau)}\mathrm{d}t\Big]\Big\}.
\end{aligned} \tag{9}$$

We now calculate the conditional expectation for X_t^2 and X_t. Let A be the generator of Itô diffusion $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B$ (with $b(X_t) \equiv 0$). Then,

$$Af = \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Then, by Dynkin's formula (cf. Øksendal [6], p. 118), we have

$$E^{X_\tau}[X_t] = X_\tau + E^{X_\tau}\Big[\int_\tau^t AX_s\mathrm{d}s\Big] = X_\tau, \tag{10}$$

$$E^{X_\tau}[X_t^2] = X_\tau^2 + E^{X_\tau}\Big[\int_\tau^t AX_s^2\mathrm{d}s\Big] = X_\tau^2 + \sigma^2(t-\tau). \tag{11}$$

Substituting (10) and (11) into (9), we have

$$
\begin{aligned}
g(\tau, X_\tau) = \frac{1}{2}\frac{\theta}{1+a^2\theta}\Big\{\sigma^2\Big[\frac{1}{\rho^2}(\mathrm{e}^{-\rho\tau}-\mathrm{e}^{-\rho T}) - \frac{1}{\rho}(T-\tau)\mathrm{e}^{-\rho T}\Big] \\
+[(\alpha X_\tau + a\pi_\tau^e)^2+\sigma_u^2]\frac{1}{\rho}(\mathrm{e}^{-\rho\tau}-\mathrm{e}^{-\rho T})\Big\}. \qquad (12)
\end{aligned}
$$

Note that, if we define

$$
f_1(s, X_t) = -f(s, X_t), \qquad g_1(s, X_\tau) = -g(s, X_\tau) + M \geq 0,
$$

then, the loss minimization problem in (6) can be reduced to the following maximization problem: find τ^*, such that

$$
\begin{aligned}
G_0^*(x) &= \sup_{\tau\in[0,T]} E^x\Big[\int_0^\tau [-f(t, X_t)]\mathrm{d}t - g(\tau, X_\tau) + M\Big] \\
&= \sup_{\tau\in[0,T]} E^x\Big[\int_0^\tau f_1(t, X_t)\mathrm{d}t + g_1(\tau, X_\tau)\Big]. \qquad (13)
\end{aligned}
$$

In the following, we will use the optimal stopping approach to solve the optimization problem (13).

2.3 Solve the Optimal Stopping Problem

In order to solve the policymaker's optimization problem (13) by using a standard framework of the optimal stopping problem involving an integral (cf. Øksendal [6], p.213), we make the following transformations. Let

$$
W_\tau = \int_0^\tau f_1(t, X_t)\mathrm{d}t + w, \quad w \in R,
$$

and define the Itô diffusion $Z_t = Z_t^{(s,x,w)}$ in R^3 by

$$
Z_t = \begin{bmatrix} s+t \\ X_t \\ W_t \end{bmatrix},
$$

for $t \geq 0$. Then,

$$
\mathrm{d}Z_t = \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}X_t \\ \mathrm{d}W_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2}\theta\mathrm{e}^{-\rho t}(X_t-k)^2 \end{bmatrix}\mathrm{d}t + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix}\mathrm{d}B_t, \qquad Z_0 = (s, x, w).
$$

So Z_t is an Itô diffusion starting at $z := Z_0 = (s, x, w)$. Let $R^z = R^{(s,x,w)}$ denote the probability law of $\{Z_t\}$ and let $E^z = E^{(s,x,w)}$ denote the expectation with respect to R^z. In terms of Z_t the problem (13) can be written as

$$
G_0^*(x) = G^*(0, x, 0) = \sup_\tau E^{(0,x,0)}[W_\tau + g_1(\tau, X_\tau)] = \sup E^{(0,x,0)}[G(Z_\tau)],
$$

which is a special case of the problem

$$
G^*(s, x, w) = \sup_\tau E^{(s,x,w)}[W_\tau + g_1(\tau, X_\tau)] = \sup E^{(s,x,w)}[G(Z_\tau)],
$$

with

$$G(z) = G(s, x, w) := w + g_1(s, x).$$

Then, for

$$\begin{aligned}
f_1(s, x) &= -\frac{1}{2}\theta e^{-\rho \cdot s}(\alpha x - u_s)^2, \\
g_1(s, x) &= -\frac{1}{2}\frac{\theta}{1 + a^2\theta}\Big\{\sigma^2\Big[\frac{1}{\rho^2}(e^{-\rho s} - e^{-\rho T}) - \frac{1}{\rho}(T - s)e^{-\rho T}\Big] \\
&\qquad + [(\alpha x + a\pi_s^e)^2 + \sigma_u^2]\frac{1}{\rho}(e^{-\rho s} - e^{-\rho T})\Big\} + M,
\end{aligned}$$

and $G(s, x, w) = w + g_1(s, x)$, the A_Z of Z_t is given by

$$\begin{aligned}
A_Z G &= \frac{\partial G}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\theta e^{-\rho s}(x - k)^2\frac{\partial G}{\partial w} \\
&= \frac{1}{2}\frac{\theta}{1 + a^2\theta}[(\alpha x + a\pi_s^e)^2 + \sigma_u^2]e^{-\rho s} - \frac{1}{2}\theta(\alpha x - u_s)^2 e^{-\rho s} \\
&= \frac{1}{2}\frac{\theta}{1 + a^2\theta}[(\alpha x + a\pi_s^e)^2 + \sigma_u^2 - (1 + a^2\theta)(\alpha x - u_s)^2]e^{-\rho s}.
\end{aligned} \tag{14}$$

Let

$$U = \{(s, x, w) : G(s, x, w) < G^*(s, x, w)\},$$

and

$$V = \{(s, x, w) : AG(x) > 0\}.$$

Then, by (14), we have

$$\begin{aligned}
V &= \{(s, x, w) : A_Z G(s, x, w) > 0\} \\
&= R \times \{x : (\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha x - u_s)^2\} \times R.
\end{aligned} \tag{15}$$

Remark 2.1 Øksendal ([6], p.205) shows that $V \subset U$, which means that it is never optimal to stop the process before it exits from V. If we choose a suitable $\pi^e(x)$, such that $(\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha x - u_s)^2$, then, we have $U = V = R^3$. Therefore, any stopping time less than T will not be optimal for all $(s, x, w) \in V$, and thus, $\tau^* = T$ is the optimal stopping time. We will use this fact to study the time inconsistency problem of the monetary policy game in the following sections.

3 The Equilibrium Behavior of the Monetary Policy Game

In order to study the equilibrium behavior of the game, we first give the following lemma, which shows that the policymaker will keep the zero-inflation policy when the public uses trigger strategies and reputation penalties imposed by the public large enough.

Lemma 3.1 Let $\tau = \inf\{s > 0 : \pi_s \neq 0\}$. Then, for all x, any trigger strategy of the public $\{\pi_t^e(x)\}$ which has the form

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } 0 < t < \tau, \\ \pi^e(x) \in \{h : (\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2\} & \text{if } t > s \text{ and } t \geq \tau, \end{cases} \tag{16}$$

discourages the policymaker from attempting surprise inflation.

Proof For each $x \in R$, if we choose any $\pi^e \in \{h : (\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha x - u_s)^2\}$, we have

$$(\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha x - u_s)^2 \quad \text{for all } x \in R.$$

Then, V in (15) becomes $V = R^3$, and thus on any stopping time less then T is not optimal for the policymaker. Hence, $\tau^* = T$. Thus, when the public applies this trigger strategy, it is never optimal for policymaker to stop the zero-inflation policy.

Although there are (infinitely) many trigger strategies given in Lemma 1, that can discourage the policymaker from attempting surprise inflation, most of them are not optimal for the public in terms of minimizing the public's expected inflation error $(\pi_t - \pi_t^e)^2$. To rule out those non-optimal strategies, we have to impose some assumptions how the public forms an expectation and what an equilibrium solution should be used to describe the public's self-interested behavior. Different assumptions on the public's behavior may result in different optimal solutions. In the following, we introduce a concept of sequentially rational equilibrium solution.

Suppose the policymaker knows the distribution of the natural rate, X_t, exactly, that is,

$$\mathrm{d}\widetilde{P}^G = \mathrm{d}P,$$

where $\widetilde{P}^G$ is the belief of the policymaker for the movement of the shock and P is the measure of the natural rate.

We suppose that the public does not know the distribution of the natural rate, but it believes that $\widetilde{P}^P$ is absolutely continuous with respect to P, which means that if an event does not occur in probability, then the public will believe that this event will not happen.

Then, by Randon-Nikodym Theorem (Lipster & Ahiryaev [5], p.13), there exists Randon-Nikodym derivative $M(t)$ such that

$$\mathrm{d}\widetilde{P}^P = M(t)\mathrm{d}P \quad \text{(a.s.)},$$

and $M(t)$ is a martingale and bounded both from above and below (i.e., $M_1 \leq M(t) \leq M_2$ for every $0 \leq t \leq T$). This means that, whenever new information becomes available, the belief of the public is adjusted. We can interprete $M(t)$ to be the information structure of the society, which is a measurement of how the public knows the real natural rate.

We suppose that $M(t)$ is P-square-integrable and X_t is $\widetilde{P}^P$-integrable. We also suppose that $\langle X_t, M(t)\rangle = 0$ heuristically. This assumption can be interpreted as: the history of the natural rate can't help the public to predict the movement of the future natural rate in general.

We denote by $\widetilde{E}$ the expectation operator with respect to $\widetilde{P}^P$.

A strategy $(\tau, \{\pi_t, \pi_t^e\})$ is said to be a sequentially rational equilibrium strategy for the dynamic model defined above if

(i) the belief of the public for the movement of the natural rate X_t, $\widetilde{P}^P$, satisfying Bayes' rule

$$\widetilde{E}[X_t|\mathcal{F}_s] = \frac{1}{M(s)}E[X_t M(t)|\mathcal{F}_s], \tag{17}$$

for all $s < t$;

(ii) the expectation of the public is rational $\pi_t^e = E^{X_s}\pi_t^D := \widetilde{E}[\pi_t^D|\mathcal{F}_s]$ for all $s < t$;

(iii) it optimizes the objectives of the public and the policymaker.

Now, we use this type of sequentially rational equilibria to study the time inconsistency problem in monetary policy. Proposition 3.1 below shows the existence of such equilibria.

Proposition 3.1 Suppose the shocks $\{X_t\}$ satisfy the inequality

$$(\alpha x + a^2\theta\alpha X_t)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2 \text{ for all } t \in [0, T] \text{ and } x \in R. \tag{18}$$

Let $(\tau, \{\pi_s\})$ be the strategy of the policymaker, where τ is the first time that the policymaker changes its policy from zero-inflation to discretion rule, i.e., $\tau = \inf\{s > 0 : \pi_s \neq 0\}$. Let the strategy of the public $\{(\pi_t^e)\}$ be given by

$$\pi_t^e = \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } 0 < t < \tau, \\ a\theta\alpha X_\tau & \text{if } t \geq \tau. \end{cases} \tag{19}$$

Then, $(\tau^*, \{\pi_t^*, \pi_t^{e*}\})$ with $\tau^* = T$, $\pi_t^* = 0$ and $\pi_t^{e*} = 0$, for all $t \geq 0$ is a sequentially rational equilibrium strategy for the policymaker and the public.

Proof To prove $(\tau, \{\pi_t, \pi_t^e\})$ defined above results in a sequentially rational equilibrium, $\tau^* = T$, $\pi_t^* = 0$, and $\pi_t^{e*} = 0$ for all $t \geq 0$, and we need to show that (i) it satisfies Bayes' rule, (ii) the rational expectation condition holds: $\pi_t^e = E^{X_\tau}\pi_t^D := \widetilde{E}[\pi_t^D|\mathcal{F}_\tau]$, (iii) $\pi_t^e \in \{h : (\alpha x + ah)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2\}$, and (iv) $(\tau^*, \{\pi_t^*, \pi_t^{e*}\})$ optimizes the objectives of the public and the policymaker.

We first claim that the public updates its belief by Bayes' rule. Indeed, because $M(t)$ is a martingale, and for $s < t$, X_t is a $\widetilde{P}^P$-integrable random variable, then, by Lemma of Shreve & Kruzhilin ([11], p.438), the Bayes' Rule holds

$$\widetilde{E}[X_t|\mathcal{F}_s] = \frac{1}{M(s)}E[X_t M(t)|\mathcal{F}_s].$$

To show $\pi_t^e = E^{X_\tau}\pi_t^D$, first note that X_t and $M(t)$ are square-integrable martingale, using the fact that $X_t M(t) - \langle X_t, M(t)\rangle$ is a martingale (Karatzas & Shreve([3], p.31)) and the assumption $\langle X_t, M(t)\rangle = 0$, we can get that $X_t M(t)$ is a martingale by Bayes' rule

$$\widetilde{E}[X_t|\mathcal{F}_\tau] = \frac{1}{M(\tau)}E[X_t M(t)|\mathcal{F}_\tau] = \frac{1}{M(\tau)}X_\tau M(\tau) = X_\tau,$$

which means $\{X_t\}$ is also a martingale under $\widetilde{P}^P$. Because the policymaker's best response function is given by

$$\pi_t^D = \frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha x - u_s),$$

$\{X_t\}$ is a martingale under $\widetilde{P}^P$, and $\pi_t^e = a\theta\alpha X_\tau$ is a complete information at time t, we have

$$\begin{aligned} E^{X_\tau}\pi_t^D &= E^{X_\tau}\frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha x - u_s) = \frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha E^{X_\tau}X_t) \\ &= \frac{a\theta}{1 + a^2\theta}(a\pi_t^e + \alpha X_\tau). \end{aligned} \tag{20}$$

Substituting $\pi_t^e = a\theta\alpha X_\tau$ into (20), we have $E^{X_\tau}\pi_t^D = \frac{a\theta}{1+a^2\theta}(a^2\theta\alpha X_\tau + \alpha X_\tau) = a\theta\alpha X_\tau = \pi_t^e$.

Now, if condition (18) is satisfied, then we have $(\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2$ and thus, $\pi_t^e \in \{h : (\alpha x + ah)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2\}$ for all $x \in R$ with $x \neq k$. Then, by Lemma 3.1, the optimal stopping time is $\tau^* = T$. Therefore, we must have $\pi_t^* = 0$ for all $t \in [0, T]$.

Because the public only cares about his inflation prediction errors, so $\pi_t^e = a\theta\alpha X_t$ minimizes the public's expected lost when the policy change occurs at time t in this game. Hence, if both the policymaker and public believe that future shocks will grow enough to make the inequality (18) holds, the threat of the public is credible. Hence, we must have $\pi_t^{e*} = 0$ for all $t \in [0, T]$ because $\tau^* = T$. Thus, we have shown that the trigger strategies $(\tau, \{\pi_t, \pi_t^e\})$ result in a sequentially rational equilibrium, which is $\tau^* = T$, $\pi_t^* = 0$, and $\pi_t^{e*} = 0$ for all $t \geq 0$.

Thus, Proposition 3.1 implies that, as long as natural rate X_t is big enough, the public can use a trigger strategy to induce a zero-inflation sequentially rational equilibrium. Of course, the assumption that $(\alpha x + a\pi_s^e)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2$ for all $t \in [0, T]$ and $x \in R$ with $x \neq k$ seems very strong. Proposition 4.1 in the next section shows that this is a reasonable assumption. As long as this inequality holds for the initial natural rate x, both the public and the policymaker will have a strong belief that it will be true for all $t \in (0, T]$ and $x \in R$.

4 Stochastically Stable Equilibrium

In this section, we study the robustness of sequentially rational equilibrium. In order to get the sequentially rational equilibrium in Proposition 3.1, we impose the assumption that $(\alpha x + a^2\theta\alpha X_t)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2$ for all $0 \leq t \leq T$ and $x \in R$. It seems that the concept of sequentially rational equilibrium is not satisfactory as a predictor of long-run behavior when the game is subjected to persistent stochastic shocks. So, we introduce the concept of sequentially rational stochastically stable equilibrium. (In determinate dynamic systems, in order to analyze the dynamic behavior, the concepts of Lyapunov stable and asymptotically stable are always used. For stochastic evolution system, Foster and Young [2] and Young [13] first introduced the concept of stochastic stability. But the concept in their papers is different from ours.)

Definition 4.1 Let $\{S : (y, z \in R^2)\}$ be the set of sequentially rational equilibria of a dynamic game under the shock X_t, we say S is a sequentially rational stochastically stable equilibrium set if $E^x[\tau] = \infty$, where $\tau = \inf\{t : (y_t, z_t) \notin S\}$, and S is a sequentially stochastically unstable rational equilibrium set if $E^x[\tau] < \infty$.

Loosely speaking, the sequentially rational stochastically stable equilibria of a dynamic game are those equilibria such that the expected time to depart from them is infinite.

Lemma 4.1 Let $B = \{X_t : (\alpha x + a^2\theta\alpha X_t)^2 + \sigma_u^2 > (1 + a^2\theta)(\alpha|x| + M)^2 \text{ for } t \geq 0\}$, and let $\eta = \inf\{t > 0 : X_t \notin B\}$ be the first time X_t exits from B. Suppose that $x \in B$. Then, we have

$$E^x[\eta] = \infty,$$

for all $x \in R$.

Proof Solving $(\alpha x + a^2\theta\alpha X_t)^2 + \sigma_u^2 > (1 + a^2\theta)(x - u_t)^2$ for X_t, we have

$$X_t > \frac{1}{a^2\theta\alpha}[-\sigma_u^2 - \alpha x + \sqrt{1 + a^2\theta}(\alpha|x| + M)],$$

or

$$X_t < \frac{1}{a^2\theta\alpha}[(-\sigma_u^2 - \alpha x - \sqrt{1+a^2\theta}(\alpha|x|+M)].$$

Let $C = \frac{1}{a^2\theta\alpha}[u_t - \alpha x + \sqrt{1+a^2\theta}(\alpha|x|+M)]$ and $D = \frac{1}{a^2\theta\alpha}[(u_t - \alpha x - \sqrt{1+a^2\theta}(\alpha|x|+M)]$.

Because $X_0 = x \in B$ for all $x \in R$, there are two cases to be considered: 1) $x > C$ and 2) $x < D$.

Case 1) $x > C$. Let $\eta_c = \inf\{t > 0\colon\ X_t \leq C\}$ and η_n be the first exit time from the interval

$$\{X_t : C \leq X_t \leq n\},$$

for every integer n with $n > C$. We first show that $P^x(X_{\eta_n} = C) = \frac{n-x}{n-C}$ and $P^x(X_{\eta_n} = n) = \frac{x-C}{n-C}$. Consider the function $h \in C_0^2(R)$ defined by $h(x) = x$ for $C \leq x \leq n$ ($C_0^2(R)$ means the functions in $C^2(R)$ with compact support). By Dynkin's formula,

$$E^x[h(X_{\eta_n})] = h(x) + E^x\Big[\int_0^{\eta_n} Ah(X_s)\mathrm{d}s\Big] = h(x) = x, \tag{21}$$

we have

$$CP^x(X_{\eta_n} = C) + nP^x(X_{\eta_n} = n) = x.$$

Thus,

$$P^x(X_{\eta_n} = C) = \frac{n-x}{n-C},$$

and

$$P^x(X_{\eta_n} = n) = 1 - P^x(X_{\eta_n} = C) = \frac{x-C}{n-C}.$$

Now, consider $h \in C_0^2(R)$ such that $h(x) = x^2$ for $C \leq x \leq n$. Applying Dynkin's formula again, we have

$$E^x[h(X_{\eta_n})] = h(x) + E^x\Big[\int_0^{\eta_n} Ah(X_s)\mathrm{d}s\Big] = x^2 + \sigma^2 E^x[\eta_n], \tag{22}$$

and thus,

$$\sigma^2 E^x[\eta_n] = C^2 P^x(X_{\eta_n} = C) + n^2 P^x(X_{\eta_n} = n) - x^2.$$

Hence, we have

$$E^x[\eta_n] = \frac{1}{\sigma^2}\Big[C^2\frac{n-x}{n-C} + n^2\frac{x-C}{n-C} - x^2\Big].$$

Letting $n \to \infty$, we conclude that $P^x(X_{\eta_n} = n) = \frac{x-C}{n-C} \to 0$ and $\eta_c = \lim \eta_n < \infty$ a.s. Therefore, we have

$$E^x[\eta_c] = \lim_{n\to\infty} E^x[\eta_n] = \infty.$$

Case 2) $X_0 = x < D$. Define $\eta_D = \inf\{t > 0;\ X_t \geq D\}$. Let η_n be the first exit time from the interval

$$\{X_t : -n \leq X_t \leq D\},$$

for every integer n with $-n < D$. By the same method, we can prove that

$$E^x[\eta_n] = \frac{1}{\sigma^2}\Big[D^2\frac{n+x}{n+D} + n^2\frac{D-x}{n+D} - x^2\Big].$$

Letting $n \to \infty$, we conclude that $P^x(X_{\eta_n} = n) = \frac{D-x}{n+D} \to 0$ and $\eta_D = \lim \eta_n < \infty$ a.s., and thus,

$$E^x[\eta_D] = \lim_{n\to\infty} E^x[\eta_n] = \infty.$$

Thus, in either case, we have $E^x[\eta] = \infty$.

Lemma 4.1, thus, implies that, because the expected exit time from B is infinite, the policymaker will believe that the future natural rate will stay in B forever, and consequently he will likely make decisions and behave according to this belief. As a result, the sequentially rational equilibrium will likely appear in the game. So, in this sense, we can regard B as an absorbing class for X_t as long as $x \in B$.

What would happen if the initial shock x is not in B? We have the following proposition:

Lemma 4.2 Define $\tau = \inf\{t > 0 : Z_t \in B\}$. Then, for $x \notin B$, i.e., $a(1-\theta) \geq 2$, we have

$$E^x[\tau] = \frac{a(1-\theta)-2}{\sigma^2 a\theta}(k-x)^2.$$

Proof Because $x \notin B$, we have $D \leq x \leq C$. Define $\tau_C = \inf\{t > 0 : X_t \geq C\}$ and $\tau_D = \inf\{t > 0 : X_t \leq D\}$. Then, $\tau = \tau_c \wedge \tau_D := \min\{\tau_c, \tau_D\}$. We first show that $P^x(X_\tau = C) = \frac{x-D}{C-D}$ and $P^x(X_\tau = D) = \frac{C-x}{C-D}$. Consider $h \in C_0^2(R)$ such that $h(x) = x$ for $D \leq x \leq C$. By Dynkin's formula,

$$E^x[h(X_{\tau_c \wedge \tau_D})] = h(x) + E^x\Big[\int_0^{\tau_c\wedge\tau_D} Ah(X_s)\mathrm{d}s\Big] = h(x) = x, \tag{23}$$

we have

$$CP^x(X_\tau = C) + DP^x(X_\tau = D) = x.$$

Thus,

$$P^x(X_\tau = C) = \frac{x-D}{C-D},$$

and so,

$$P^x(X_\tau = D) = 1 - P^x(X_\tau = C) = \frac{C-x}{C-D}.$$

Now consider $h \in C_0^2(R)$ such that $h(x) = x^2$ for $D \leq x \leq C$. By Dynkin's formula:

$$E^x[h(X_{\tau_c \wedge \tau_D})] = h(x) + E^x\Big[\int_0^{\tau_c\wedge\tau_D} Ah(X_s)\mathrm{d}s\Big] = h(x) + \sigma^2 E^x[\tau_c \wedge \tau_D], \tag{24}$$

we have

$$\sigma^2 E^x[\tau_c \wedge \tau_D] = C^2 P^x(X_\tau = C) + D^2 P^x(X_\tau = D) - x^2,$$

and then,

$$\begin{aligned}
E^x[\tau_c \wedge \tau_D] &= \frac{1}{\sigma^2}\Big[C^2\frac{x-D}{C-D} + D^2\frac{C-x}{C-D} - x^2\Big] \\
&= \frac{1}{\sigma^2}[(C+D)x - CD - x^2] \\
&= \frac{2x}{\sigma^2 a\theta}[(1+a\theta)k - x] - \frac{1}{\sigma^2 a^2\theta^2}[(1+a\theta)k - x]^2 + \frac{1+a^2\theta}{\sigma a^2\theta^2}(x-k)^2 - \frac{1}{\sigma}x^2 \\
&= \frac{1}{\sigma^2 a^2\theta^2}\{[(1+a\theta)k - x][2xa\theta - (1+a\theta)k + x] - a^2\theta^2 x^2 + (1+a^2\theta)(k-x)^2\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sigma^2 a^2\theta^2}\{-(1+a\theta)^2k^2 - (1+2a\theta)x^2 + (1+a\theta)kx + (1+a\theta)(1+2a\theta)xk \\
&\quad -a^2\theta^2x^2 + (1+a^2\theta)(x-k)^2\} \\
&= \frac{1}{\sigma^2 a^2\theta^2}\{-(1+a\theta)^2k^2 - (1+a\theta)^2x^2 + 2(1+a\theta)^2kx + (1+a^2\theta)(x-k)^2\} \\
&= \frac{1}{\sigma^2 a^2\theta^2}\{-(1+a\theta)^2(k-x)^2 + (1+a^2\theta)(x-k)^2\} \\
&= \frac{a(1-\theta)-2}{\sigma^2 a\theta}(k-x)^2 \geq 0, \qquad (25)
\end{aligned}
$$

by noting that $a(1-\theta) \geq 2$.

Notice that, the bigger the variance of the natural rate (measured by σ), the faster the convergence rate. From Lemma 4.2, the expected time of entering B, $E^x[\tau] = E^x[\tau_c \wedge \tau_D]$ is a finite number. Suppose the public has the same belief as the policymaker. There are two cases to be considered: 1) $E^x[\tau] \geq T$. In this case, the policymaker likely believes that $X_t \notin B$ for all $t \in [0, T]$, and thus a sequentially rational stochastically stable equilibrium will not likely exist. 2) $E^x[\tau] < T$. In this case, we should not expect the zero-inflation stationary monetary policy for the time period $[0, E^x[\tau]]$ since $X_t \notin B$ for all $t \in [0, E^x[\tau]]$. However, once X_t enters B at the first time $E^x[\tau]$, we can regard X_τ as a new starting point. Then, by Lemma 4.1, both the policymaker and the public will believe that X_t will stay in B for all $t \in [E^x[\tau], T]$, and thus, we can expect to have a zero inflation stationary monetary policy on $[E^x[\tau], T]$. This implies that, although we do not have a time consistency policy on the whole time horizon $[0, T]$ when $x \notin B$, we could have a time consistency monetary policy beyond the point $E^x[\tau]$. In other words, one will have an nonstationary policy period if the initial shock $x \notin B$; however, after a certain point τ, the monetary policy may become stationary. Thus, the time inconsistency may happen at most once.

Summarizing the above discussions, we can draw the following conclusions:

(i) If the initial natural rate x is in B, one can expect all future shocks X_t are in B and thus, can expect a stationary zero-inflation outcome by the sequentially rational behavior.

(ii) If the initial natural rate x is not in B, whether or not we can expect the monetary policy to have a tendency to become stable depending on T, the lifetime of the policymaker. If the expected first entry time to B, $E^x[\tau] \geq T$, we do not expect a stationary monetary policy and thus we have the time inconsistency problem. If $E^x[\tau] < T$, we may expect a stationary monetary policy beyond the entry point $E^x[\tau]$, and monetary policy becomes stationary. Thus, the monetary policy may jump at most once.

Combine Lemma 4.1 and Lemma 4.2, we have the following proposition.

Proposition 4.1 Let $(\tau, \{\pi_s\})$ be the strategy of the policymaker, where τ is the first time that the policymaker changes his policy from zero-inflation to discretion rule, i.e., $\tau = \inf\{s > 0 : \pi_s \neq 0\}$. Let the strategy of the public $\{(\pi_t^e)\}$ be given by

$$
\pi_t^e = \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } 0 < t < \tau, \\ a\theta(k - X_\tau) & \text{if } t \geq \tau. \end{cases} \qquad (26)
$$

Then, $(\tau^*, \{\pi_t^*, \pi_t^{e*}\})$ with $\tau^* = T$, $\pi_t^* = 0$, and $\pi_t^{e*} = 0$ for all $t \geq 0$ is a sequentially rational stochastically stable equilibrium strategy for the policymaker and the public.

Then, we can see that the cooperative equilibrium in this monetary policy game is a sequentially rational stochastically stable equilibrium and the uncooperative equilibrium is a sequentially rational stochastically unstable equilibrium. In the long run, the zero inflation monetary policies are inherently more stable than the discretionary rules, and once established, they tend to persist for longer periods of time. Thus, for this continuous time dynamic stochastic game, sequentially rational stochastically stable equilibrium behavior can be predicted for any initial natural rate.

5 Conclusion

This article develops a model to examine the equilibrium behavior of monetary time inconsistency problem in a continuous time economy with stochastic natural rate and endogenized distortion. First, we introduce the notion of sequentially rational equilibrium and show that the time inconsistency problem may be solved with trigger reputation strategies in a stochastic setting. We provide the conditions for the existence of sequentially rational equilibrium. Then, the concept of sequentially rational stochastically stable equilibrium is introduced. We compare the relative stability between the so called cooperative behavior and the so-called uncooperative behavior, and show that the cooperative equilibrium in this monetary policy game is a sequentially rational stochastically stable equilibrium and the uncooperative equilibrium is sequentially rational stochastically unstable equilibrium. In the long run, the zero inflation monetary policies are inherently more stable than the discretion rules, and once established, they tend to persist for longer periods of time.

References

1 Barro R, Gordon D. Rules, discretion, and reputation in a model of monetary policy. Journal of Monetary Economics, 1983, **12**: 101–121

2 Dean P F, Young H P. Stochastic evolutionary game dynamics. Theoretical Population Biology, 1990, **38**: 219–232

3 Karatzas I. Shreve S E. Brownian Motion and Stochastic Calculus. Second Edition. Springer-Verlag, 1991

4 Kydland F, Prescott E. Rules rather than discretion: The inconsistency of optimal plan. Journal of Political Economy, 1977, **85**: 473–491

5 Liptser R S, Shiryaev A N. Statistics of Random Process: I General Theory. Second and Expanded Edition. Springer-Verlag, 2001

6 Øksendal B. Stochastic Differential Equations. 5th Edition. New York: Springer-Verlag, 1998

7 Persson T, Tabellini G. Designing institutions for monetary stability. Carnegie-Rochester Conference Series on Public Policy, 1993, **39**: 53–84

8 Rogoff K. The optimal degree of commitment to an intermediate monetary target. Quarterly Journal of Economics, 1985, **100**: 169–190

9 Salemi M K. Estimating the natural rate of unemployment and testing the natural rate hypothesis. Journal of Applied Econometrics, 1999, **14**: 1–25

10 Sevensson L. Optimal inflation targets, ‘Conservative Central Bank and Linear Inflation Contracts’. American Economic Review, 1997, **87**: 98–111

11 Shiryaev A N, Kruzhilin N. Essentials of Stochastic Finance: Facts, Models, Theory. Singapore: World Scientific, 1999

12 Walsh C. Optimal contracts for central bankers. American Economic Review, 1995, **85**: 150–167

13 Young H P. The evolution of conventions. Econometrica, 1993, **61**: 57–84