Existence of Equilibria in Discontinuous Games

Nessah, Rabia and Tian, Guoqiang

November 2008

Existence of Equilibrium in Discontinuous Games

Rabia Nessah*
CNRS-LEM (UMR 8179)
IESEG School of Management
3 rue de la Digue F-59000 Lille
France

Guoqiang Tian†
Department of Economics
Texas A&M University
College Station, Texas 77843
USA

This version: March, 2010

Abstract

This paper investigates the existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous games. We introduce a new notion of weak continuity, called *weak transfer quasi-continuity*, which is weaker than most known weak notions of continuity, including diagonal transfer continuity in Baye *et al* (1993) and better-reply security in Reny (1999), and holds in a large class of discontinuous games. We show that it, together with strong diagonal transfer quasiconcavity introduced in the paper, is enough to guarantee the existence of Nash equilibria in compact and convex normal form games. We provide sufficient conditions for weak transfer quasi-continuity by introducing notions of weak transfer continuity, weak transfer upper continuity, and weak transfer lower continuity. Moreover, an analogous analysis is applied to show the existence of dominant strategy and mixed strategy Nash equilibria in discontinuous games.

Keywords: Discontinuous games, weak transfer quasi-continuity, pure strategy, mixed strategy, dominant strategy, Nash equilibrium, existence.

*E-mail address: r.nessah@ieseg.fr

†We thank Paulo Barelli for comments and suggestions. Financial support from the National Natural Science Foundation of China (NSFC-70773073) and the Program to Enhance Scholarly and Creative Activities at Texas A&M University as well as from Cheung Kong Scholars Program at the Ministry of Education of China is gratefully acknowledged. E-mail address: gtian@tamu.edu

1 Introduction

The concept of Nash equilibrium in Nash (1950, 1951) is probably the most important solution in game theory. It is immune from unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that other players do not deviate from theirs. Nash (1951) proved that a finite game has a Nash equilibrium in mixed strategies. Debreu (1952) then showed that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are nonempty, convex and compact, and (2) players have continuous and quasiconcave payoff functions. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Milgrom and Weber (1982), Dasgupta and Maskin (1986), and Jackson (2005), etc., and payoffs are discontinuous and/or non-quasiconcave.

Economists then seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some forms of transitivity/monotonicity of payoffs (cf. McManus (1964), Roberts and Sonnenschein (1976), Nishimura and Friedman (1981), Topkis (1979), Vives (1990), and Milgrom and Roberts (1990)), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin (1986), Simon (1987), Simon and Zame (1990), Tian (1992a, 1992b, 1992c, 1994), Reny (1999), Bagh and Jofre (2006), Reny (2009), while others seek to weaken both quasiconcavity and continuity (cf. Baye, Tian, and Zhou (1993), and Barelli and Soza (2009), McLennan, Monteiro, and Tourky (2009)).

This paper investigates the existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous games. We introduce a new notion of very weak continuity, called *weak transfer quasi-continuity*, which is weaker than most known weak notions of continuity, including diagonal transfer continuity in Baye *et al* (1993) and better-reply security in Reny (1999), and holds in a large class of discontinuous games. Roughly speaking, a game is weakly transfer quasi-continuous if for every nonequilibrium strategy x^*, there exists a neighborhood and a securing strategy profile such that for every strategy profile in the neighborhood, there is player i who is strictly better off by using his securing strategy.

Weak transfer quasi-continuity is indeed very weak. Recently, Tian (2009) extends the notion of weak transfer quasi-continuity to its transitive closure, called *recursive weak transfer quasi-continuity*. It is shown that recursive weak transfer quasi-continuity is necessary and sufficient for the existence of Nash equilibria in general games with any number of players that may be finite, infinite, or even uncountable; arbitrary strategy spaces that may be discrete, continuum, non-compact or non-convex; payoffs that may be discontinuous or do not have any form of quasi-concavity. Weak transfer quasi-continuity holds in many economic games. Besides those known sufficient conditions such as diagonal transfer continuity and better-reply security, we give four additional sets of sufficient conditions, each of which implies weak transfer quasi-continuity: (1) transfer continuity, (2) weak transfer continuity, (3) weak transfer upper continuity and payoff security,[1] and (4) upper semicontinuity and weak transfer lower continuity. These conditions are satisfied in many economic games and are often simple to check.

We also introduce the notions of strong diagonal transfer quasiconcavity and weak diagonal transfer quasiconcavity, which are stronger and weaker than diagonal transfer quasiconcavity introduced in Baye *et al* (1993), respectively. Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity, weak diagonal transfer quasiconcavity are all very weak notions of quasiconcavity. A game is (strongly/weakly) diagonal transfer quasiconcave provided it has a pure strategy Nash equilibrium.

We show that weak transfer quasi-continuity, together with strong diagonal transfer quasiconcavity, guarantees the existence of pure strategy Nash equilibrium in convex and compact games. We then show that shows that weak transfer continuity, together with quasiconcavity (or weak diagonal transfer quasiconcavity), guarantees the existence of pure strategy Nash equilibrium in bounded, compact and convex games. Moreover, by introducing the notion of weak dominant transfer upper continuity, an analogous analysis is applied to show the existence of dominant strategy and mixed strategy Nash equilibria in discontinuous games.

The remainder of the paper is organized as follows. Section 2 describes the notation, and provides a number of preliminary definitions. Section 3 introduces the notions of weak transfer continuity/quasi-continuity and weak/strong diagonal transfer quasiconcavity, and provides the main results on the existence of pure strategy Nash equilibrium. Examples and applications illustrating the theorems are also given. Section 4 considers the existence of dominant strategy equilibrium by introducing a similar condition, weak dominant transfer continuity. Section 5 considers the existence of mixed strategy Nash equilibrium by applying the main results obtained in Section 3 on the existence of pure strategy Nash equilibrium. It is shown there that the mixed strategy theorems of Nash (1950), Glicksberg (1951), Dasgupta and Maskin (1986), Robson (1994), Simon (1987) and Reny (1999) imply our main results presented in Section 5. Concluding remarks are offered in Section 6. The proofs of the theorems and propositions are presented in Appendix.

2 Preliminaries

Consider the following noncooperative game in a normal form:

$$G = (X_i,\ u_i)_{i\in I} \tag{2.1}$$

where $I = \{1, ..., n\}$ is a finite set of players, X_i is player i's strategy space that is a nonempty subset of a topological space E_i, and u_i is player i's payoff function from the set of strategy

[1] It is worth pointing out that, while reciprocal upper semicontinuity combined with payoff security implies better-reply security, here weak transfer upper semicontinuity combined with payoff security implies weak transfer continuity.

profiles $X = \prod_{i \in I} X_i$ to $\mathbb{R}$. For each player $i \in I$, denote by $-i$ all players rather than player i. Also denote by $X_{-i} = \prod_{j \neq i} X_j$ the set of strategies of the players in coalition $-i$.

We say that a game $G = (X_i, \ u_i)_{i \in I}$ is compact, convex, bounded, and semi-continuous, respectively if, for all $i \in I$, X_i is compact and convex, and u_i is bounded and semi-continuous on X, respectively. We say that a game $G = (X_i, \ u_i)_{i \in I}$ is quasiconcave if, for every $i \in I$, X_i is convex and the function u_i is quasiconcave in x_i.

We say that a strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium* of a game G if,

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*) \ \forall i \in I, \ \forall y_i \in X_i.$$

We say that a strategy profile $x^* \in X$ is a *pure dominant strategy equilibrium* of a game G if,

$$\forall (y_i, y_{-i}) \in X, \ u_i(y_i, y_{-i}) \leq u_i(x^*_i, y_{-i}) \ \forall i \in I.$$

Define a function $U : X \times X \to \mathbb{R}$ by

$$U(x, y) = \sum_{i=1}^{n} u_i(y_i, x_{-i}), \ \ \forall (x, y) \in X \times X. \tag{2.2}$$

Baye *et al* (1993) study the existence of pure strategy Nash equilibria in games with possibly discontinuous and nonquasiconcave payoffs by introducing the concepts of diagonal transfer continuity and diagonal transfer quasiconcavity of U.

DEFINITION 2.1 A game $G = (X_i, \ u_i)_{i \in I}$ is diagonally transfer continuous if whenever x is not an equilibrium, there exist a strategy profile $y \in X$ and a neighborhood $\mathcal{V}(x) \subset X$ of x such that $U(z, y) > U(z, z)$ for all $z \in \mathcal{V}(x)$.

REMARK 2.1 The point y in the above definition can be termed as a securing profile of strategies since whenever a strategy profile x is not an equilibrium, it secures a strictly higher utility for all strategy profiles in some neighborhood of x. It is clear that continuity of U implies diagonal transfer continuity of U.

DEFINITION 2.2 A game $G = (X_i, \ u_i)_{i \in I}$ is diagonally transfer quasiconcave if, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$.

Theorem 1 in Baye *et al* (1993) shows that, a game that is compact, convex, diagonally transfer continuous, and diagonally transfer quasiconcave must possess a pure strategy Nash equilibrium.

Reny (1999) studies the existence of pure strategy Nash equilibria in discontinuous games by introducing the concepts of payoff security and better-reply security.

The graph of the game is $\Gamma = \{(x,u) \in X \times \mathbb{R}^n : u_i(x) = u_i,\ \forall i \in I\}$. The closure of Γ in $X \times \mathbb{R}^n$ is denoted by cl Γ. The frontier of Γ, which is the set of points that are in cl Γ but not in the interior of Γ, is denoted by Fr(Γ).

DEFINITION 2.3 A game $G = (X_i,\ u_i)_{i \in I}$ is payoff secure if for every $x \in X$, every $\epsilon > 0$, and every player i, there exists $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, y_{-i}) \geq u_i(x) - \epsilon$ for all y_{-i} in some open neighborhood of x_{-i}.

DEFINITION 2.4 A game $G = (X_i,\ u_i)_{i \in I}$ is better-reply secure if whenever $(x^*, u^*) \in$ cl Γ and x^* is not an equilibrium, there is a player i and a strategy $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, y_{-i}) > u_i^*$ for all y_{-i} in some open neighborhood of x_{-i}.

DEFINITION 2.5 A game $G = (X_i,\ u_i)_{i \in I}$ is reciprocally upper semicontinuous if, whenever $(x, u) \in$ cl Γ and $u_i(x) \leq u_i$ for every player i, $u_i(x) = u_i$ for every player i.

The following notions are introduced by Bagh and Jofre (2006) and Morgan and Scalzo (2007), respectively.

DEFINITION 2.6 A game $G = (X_i,\ u_i)_{i \in I}$ is weakly reciprocal upper semicontinuous, if for any $(x, u) \in$ Fr(Γ), there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$.

DEFINITION 2.7 Let Z be a topological space and f be an extended real-valued function defined on Z. f is upper pseudocontinuous at z_0 if for all $z \in Z$ such that $f(z_0) < f(z)$, we have $\limsup_{y \to z_0} f(y) < f(z)$. f is said to be lower pseudocontinuous at z_0 if $-f$ is upper pseudocontinuous at z_0. f is said to be pseudocontinuous if it is both upper and lower pseudocontinuous.

Theorem 3.1 in Reny (1999) shows that a game $G = (X_i,\ u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, bounded, quasiconcave and better-reply secure. Reny (1999) and Bagh and Jofre (2006) provided sufficient conditions for a game to be better-reply secure. Reny (1999) showed that a game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and reciprocally upper semicontinuous. Bagh and Jofre (2006) further showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and weakly reciprocal upper semicontinuous. Morgan and Scalzo (2007) showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if u_i is pseudocontinuous, $\forall i \in I$.

REMARK 2.2 Since payoff security requires taking an open neighborhood in the upper contour set of a given level of payoff, it is a weak notion of lower semicontinuity. Also, since better-reply security requires the limit payoff resulting from strategies to approach a nonequilibrium point, it is a weak notion of continuity (which displays a certain form of both lower semicontinuity and upper semicontinuity). In addition, both notions use the same idea of transferring nonequilibrium strategy profile to a securing strategy profile that results in a strictly better-off payoff, and thus they actually fall in the forms of transfer continuity.

3 Existence of Nash Equilibria

In this section we investigate the existence of pure strategy Nash equilibria in discontinuous games. We then show how our main existence results are applied to some important economic games.

3.1 Nash Equilibria in Discontinuous Games

We start by introducing some weak forms of continuities.

DEFINITION 3.1 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer quasi-continuous* if whenever $x \in X$ is not an equilibrium, there exist a strategy profile $y \in X$ and a neighborhood $\mathcal{V}(x)$ of x so that for every $x' \in \mathcal{V}(x)$, there exists a player i such as $u_i(y_i, x'_{-i}) > u_i(x')$.

Roughly speaking, weak transfer quasi-continuity means whenever a strategy profile is not an equilibrium, there exist some of its neighborhood and a securing strategy profile such that for every strategy profile in the neighborhood, some player will be strictly better off by using his securing strategy. Note that the notion of weak transfer quasi-continuity only requires for every strategy profile in the neighborhood, there is some player, but not all players, such that is upset by the securing strategy profile.

The following notions of transfer continuity are clearly sufficient conditions for weak transfer quasi-continuity.

DEFINITION 3.2 A game $G = (X_i, u_i)_{i \in I}$ is said to be *transfer continuous* if for all player i, u_i is transfer continuous in x with respect to X_i, *i.e.*, if $u_i(z_i, x_{-i}) > u_i(x)$ for $z_i \in X_i$ and $x \in X$, then there is some neighborhood $\mathcal{V}(x)$ of x and $y_i \in X_i$ such that $u_i(y_i, x'_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

DEFINITION 3.3 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exist player i, $y_i \in X_i$ and a neighborhood $\mathcal{V}(x)$ of x such that $u_i(y_i, x'_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

Note that transfer continuity clearly implies weak transfer continuity which in turn implies weak transfer quasi-continuity, but the reverse may not be true. We will give such examples in the next subsection. However, for one-player game, transfer continuity, weak transfer continuity, weak transfer quasi-continuity, and diagonal transfer continuity all become the same.

PROPOSITION 3.1 *If a game $G = (X_i, u_i)_{i \in I}$ is weakly transfer continuous, diagonally transfer continuous or better reply secure, then it is weakly transfer quasi-continuous.*

REMARK 3.1 Define a correspondence $F : X \to 2^X$ by $F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x),\ \ \forall i \in I\}$. Then F is transfer closed-valued if and only if the game is weakly transfer quasi-continuous.[2]

We now introduce the notion of strong diagonal transfer quasiconcavity.

DEFINITION 3.4 A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly diagonal transfer quasiconcave* if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, there exists $y^h \in \{y^{k^1}, ..., y^{k^s}\}$ so that

$$u_i(y_i^{k^h}, x_{-i}) \leq u_i(x) \quad \forall i \in I. \tag{3.1}$$

Strong diagonal transfer quasiconcavity roughly says that given any finite subset $Y^m = \{y^1, ..., y^m\}$ of deviation profiles, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\}$ of candidate profiles such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, its convex combinations are not upset by all deviation profiles in $X^{\tilde{m}}$ for all players simultaneously. Note that a game is diagonally transfer quasiconcave if it is strongly diagonal transfer quasiconcave. Indeed, summing up (3.1), we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$.

REMARK 3.2 It is clear that F is transfer FS-convex if and only if the game is strongly diagonal transfer quasiconcave.[3]

We then have the following result.

THEOREM 3.1 *If a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, weakly transfer quasi-continuous, and strongly diagonal transfer quasiconcave, then it possesses a pure strategy Nash equilibrium.*

While weak transfer quasi-continuity in Theorem 3.1 is weaker than the better-reply security and diagonal transfer continuity, it requires that the game be strongly diagonal transfer quasiconcave. Can strong diagonal transfer quasiconcavity in the theorem be replaced by conventional quasiconcavity? Unfortunately, the answer is no. Recently, Reny (2009) shows this by giving a counterexample (Example 3.1 in his paper) where a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, bounded, quasiconcave and weakly transfer quasi-continuous, but it may not possess a pure strategy Nash equilibrium. However, for each player $i \in I$, defining a function $\varphi_i : X_i \times X \to \mathbb{R}$ by

$$\varphi_i(x_i, y) = \sup_{\mathcal{V} \in \Omega(y)} \inf_{z \in \mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)]$$

[2]A correspondence $H : X \to 2^X$ is transfer closed-valued on X if for every $y \in X$, $x \notin H(y)$ implies that there exists a point $y' \in X$ such that $x \notin clH(y)$.

[3]A correspondence $H : X \to 2^X$ is transfer FS-convex if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for each $J \subset \{1, ..., m\}$, we have $co\{x^j, j \in J\} \subset \bigcup_{j \in J} H(y^j)$.

where $\Omega(y)$ is the set of all neighborhoods of y, and then applying the weak transfer quasi-continuity to φ_i[4], Reny (2009) shows that the game G possesses a Nash equilibrium if it is bounded, convex, compact, quasiconcave, and has the lower single-deviation property.

Nevertheless, if we strengthen weak transfer quasi-continuity to weak transfer continuity, we can replace the strong diagonal transfer quasiconcavity by quasiconcavity, and obtain the following theorem.

THEOREM 3.2 *If a game $G = (X_i, u_i)_{i\in I}$ is convex, compact, bounded, quasiconcave, and weakly transfer continuous, then it possesses a pure strategy Nash equilibrium.*

It may be remarked that this theorem neither implies nor is implied by Baye *et al* (1993) and Reny (1999). With strong diagonal transfer quasiconcavity, the proof of Theorem 3.1 is much simpler than that of Theorem 3.2. Strong diagonal transfer quasiconcavity can be further weakened if one is willing to impose the boundedness of payoffs and weak transfer continuity. Indeed, we can do so by introducing weak diagonal transfer quasiconcavity.

Let $m \in \mathbb{N}^*$[5] and the following special simplex: [6]

$$\Delta(n,m) = \{\lambda = (\lambda_{i,j})_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \in \mathcal{M}_{\mathbb{R}}(n,m) : \lambda_{i,j} \geq 0 \text{ and } \sum_{i,j} \lambda_{i,j} = 1\}.$$

DEFINITION 3.5 A game $G = (X_i, u_i)_{i\in I}$ is said to be *weakly diagonal transfer quasiconcave* if for any finite subset $\{y^1, \dots, y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, \dots, x^m\} \subset X$ such that for each $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h,\ h = 1, \dots, m\}$, we have

$$\min_{(i,j)\in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0, \text{ with } J = \{(i,j) :\ \lambda_{i,j} > 0\}. \tag{3.2}$$

REMARK 3.3 Definition 3.5 is equivalent to the following definition: A game $G = (X_i, u_i)_{i\in I}$ is weakly diagonal transfer quasiconcave if and only if for any finite subset $\{y^1, \dots, y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, \dots, x^m\} \subset X$ such that for each $\lambda \in \Delta(n,m)$, there exists a player $i \in I$ such that

$$\min_{j\in J(i)} u_i(y_i^j, \tilde{x}_{-i}) \leq u_i(\tilde{x}),$$

where $J(i) = \{j = 1, \dots, m :\ \lambda_{i,j} > 0\}$ and $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j$.

Weak diagonal transfer quasiconcavity roughly says that given any finite subset $Y^m = \{y^1, \dots, y^m\}$ of deviation profiles, there exists a corresponding finite subset $X^m = \{x^1, \dots, x^m\}$ of candidate profiles such that for any subset $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, there exists

[4]Reny (2009) calls this condition *lower single-deviation property.*

[5]$\mathbb{N}^*$ is the set of strictly positive integer numbers.

[6]$\mathcal{M}_{\mathbb{R}}(n,m)$ is the matrix space with n lines, m columns and scalars in $\mathbb{R}$.

some player i so that its convex combinations are not upset by those deviation profiles in $X^{\tilde{m}}$ that have nonzero weights.

Weak diagonal transfer quasiconcavity is also weaker than diagonal transfer quasiconcavity as shown in the following proposition.

PROPOSITION 3.2 *If the aggregate function defined by* (2.2) *is diagonally transfer quasiconcave, then the game G is weakly diagonal transfer quasiconcave.*

THEOREM 3.3 *If a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, bounded, weakly transfer continuous, and weakly diagonal transfer quasiconcave, then it possesses a pure strategy Nash equilibrium.*

REMARK 3.4 Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity as well as weak diagonal transfer quasiconcavity are all very weak notions of quasiconcavity, and in fact, similar to the proof of Theorem 3.1 in Baye *et al* (1993), a game must be (strongly/weakly) diagonal transfer quasiconcave as long as it possesses a pure strategy Nash equilibrium.[7]

3.2 Discussion and Examples

Various notions of continuity presented in our results, such as transfer continuity, weak transfer continuity, weak transfer quasi-continuity, diagonal transfer continuity, better-reply security, etc., are quite weak, which hold in a large class of discontinuous games. In this subsection we illustrate the relationships of these weak notions of continuity and show the usefulness of our main results with examples.

It is clear that a game G is weakly transfer continuous if it is transfer continuous. However, the following example shows the reverse may not be true.

EXAMPLE 3.1 Consider a two-player game with $X_1 = X_2 = [0, 1]$ and

$$u_1(x_1, x_2) = \begin{cases} 2 + x_1 + x_2, & \text{if } x_1 = x_2, \\ x_1 + x_2, & \text{otherwise,} \end{cases} \qquad \text{and } u_2(x) = x_1 + x_2.$$

This game is weakly transfer continuous. Indeed, since the unique Nash equilibrium is given by $x_1 = x_2 = 1$, any nonequilibrium strategy profile (x_1, x_2) contains a component that is not equal to one.

If $x_2 < 1$, let $y_2 = 1$. Then, for any neighborhood $\mathcal{V}(x)$ of x where $\mathcal{V}(x) \subset [0, 1] \times [0, 1)$ such that for all $z \in \mathcal{V}(x)$, we have $u_2(z_1, y_2) = 1 + z_1 > u_2(z_1, z_2) = z_1 + z_2$.

[7] Strong diagonal transfer quasiconcavity, diagonal transfer quasiconcavity, and weak diagonal transfer quasiconcavity all become the same for one-player games.

If $x_2 = 1$, then $x_1 < 1$. Letting $y_1 = 1$, then for any neighborhood $\mathcal{V}(x)$ of x such that $\mathcal{V}(x) \subset [0,1) \times [0,1]$ and for all $z \in \mathcal{V}(x)$, $z_1 < z_2$, we have $u_1(y_1, z_2) = 3 + z_2$ if $z_2 = 1$ and $u_1(y_1, z_2) = 1 + z_2$ otherwise. Thus $u_1(y_1, z_2) > z_1 + z_2 = u_1(z_1, z_2)$ for all z_2.

Hence, the game is weakly transfer continuous. However, it is not transfer continuous. To see this, consider the nonequilibrium $x = (1, 0)$. Then, for any $y_1 \in [0, 1]$ and any neighborhood $\mathcal{V}(x) \subset X$ of x, choosing $z \in \mathcal{V}(x)$ with $z_1 = 1$ and $1 \neq z_2 \neq y_1$, we then have $u_1(y_1, z_2) = y_1 + z_2 \leq 1 + z_2 = u_1(1, z_2)$. Thus, u_1 is not transfer continuous.

Also, weak transfer quasi-continuity is strictly weaker than weak transfer continuity and better-reply security. To see this, consider the following example.

EXAMPLE 3.2 Consider the two-player game with the following payoff functions defined on $X = [0, 1] \times [0, 1]$.

$$u_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \in (0,1) \\ 1 & \text{if } x_1 = 0 \text{ and } x_2 \in \mathbb{Q} \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 \notin \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \text{, and } u_2(x_1, x_2) = x_1 - x_2$$

where $\mathbb{Q} = \{x \in [0, 1] : x \text{ is a rational number}\}$.

The payoff function of player 1 is taken from Barelli and Soza (2009). This game is neither weakly transfer continuous, better-reply secure, nor diagonally transfer continuous, but is weakly transfer quasi-continuous.

To show the game is not weakly transfer continuous, consider the nonequilibrium $x = (1, 0)$. Then, for any $y_1 \in [0, 1]$ and any neighborhood $\mathcal{V}(x) \subset X$ of x, choosing $z \in \mathcal{V}(x)$ with $z_1 = 1$ and $z_2 \notin \mathbb{Q}$, we have $u_1(y_1, z_2) \leq u_1(z_1, z_2) = 1$. Also, for any $y_2 \in [0, 1]$ and any neighborhood $\mathcal{V}(x) \subset X$ of x, choosing $z \in \mathcal{V}(x)$ with $z_2 = 0$, we have $u_2(z_1, y_2) = z_1 - y_2 \leq z_1 = u_2(z_1, z_2)$. So it is not weakly transfer continuous.

To show the game is not better-reply secure either, consider $x = (1, 0)$ and $u = (0, 1)$. Clearly (x, u) is in the closure of the graph of its vector function, and x is not a Nash equilibrium. We show that player 1 cannot obtain a payoff strictly above $u_1 = 0$. Indeed, for all $y_1 \in [0, 1]$ and any neighborhood $\mathcal{V}(x_2) \subset [0, 1]$ of x_2, choosing $z_2 \in \mathcal{V}(x_2) \setminus \mathbb{Q}$ if $y_1 = 0$, or $z_2 = 0$ otherwise, we then have $u_1(y_1, z_2) = 0 \leq u_1 = 0$. Player 2 cannot obtain a payoff strictly above $u_2 = 1$ either. To see this, for all $y_2 \in [0, 1]$ and any neighborhood $\mathcal{V}(x_1) \subset [0, 1]$ of x_1, we have $u_2(z_1, y_2) = z_1 - y_2 \leq z_1 \leq 1 = u_2$ for $z_1 \in \mathcal{V}(x_1)$. Thus, this game is not better-reply secure, so Theorem 3.1 of Reny (1999) cannot be applied.

Now we show the game is not diagonally transfer continuous. Let $x = (1, 0)$ and $y = (0, 0)$. Then, $U(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2) = u_1(0, 0) + u_2(1, 0) = 2 > U(x, x) = u_1(1, 0) + u_2(1, 0) = 1$. However, for all $y' \in [0, 1] \times [0, 1]$ and any neighborhood $\mathcal{V}(x) \subset X$ of x, choosing $z \in \mathcal{V}(x)$ with $z_1 = 1$ and $z_2 \notin \mathbb{Q}$ if $y_1' < 1$, or $z_2 = 0$ otherwise, we then have: (1) If $y_1' < 1$, then $z_2 \notin \mathbb{Q}$. Thus, $u_1(y_1^{'}, z_2) = 0$, $u_1(z_1, z_2) = 1$, $u_2(z_1, y_2^{'}) = 1 - y_2^{'}$ and $u_2(z_1, z_2) = 1 - z_2$. Therefore $U(z, y^{'}) = 1 - y_2^{'} \leq 2 - z_2 = U(z, z)$. (2) If $y_1' = 1$, then $z_2 = 0$. Thus, $u_1(y_1^{'}, z_2) = 0$, $u_1(z_1, z_2) \leq 1$, $u_2(z_1, y_2^{'}) = 1 - y_2^{'}$ and $u_2(z_1, z_2) = 1$. Therefore $U(z, y^{'}) = 1 - y_2^{'} \leq 1 + u_1(z_1, z_2) = U(z, z)$. Thus, this game is not diagonally transfer continuous, so Theorem 1 of Baye *et al* (1993) cannot be applied.

However, it is weakly transfer quasi-continuous. Indeed, let (x_1, x_2) be a nonequilibrium strategy profile with at least one non-zero coordinate. There are two cases to be considered.

1. $x_2 > 0$. Letting $y = (y_1, 0)$ and taking a neighborhood $\mathcal{V}(x) \subset [0, 1] \times (0, 1]$ of x, then for each $z \in \mathcal{V}(x)$ and player $i = 2$, we have $u_2(z_1, z_2) = z_1 - z_2 < z_1 = u_2(z_1, y_2)$.

2. $x_2 = 0$ and $x_1 > 0$. Letting $y = (0, 0)$ and taking a neighborhood $\mathcal{V}(x) \subset (0, 1] \times [0, 1]$ of x, then for each $z \in \mathcal{V}(x)$, we have $u_1(z_1, z_2) = 0 < 1 = u_1(y_1, z_2)$ for player 1 when $z_2 \in \mathbb{Q}$ and $u_2(z_1, z_2) = z_1 - z_2 < z_1 = u_2(z_1, y_2)$ for player 2 when $z_2 \notin \mathbb{Q}$.

Since the game is also convex, compact, and strongly diagonal transfer quasiconcave, by Theorem 3.1, the game considered possesses a Nash equilibrium.

Although diagonal transfer continuity, better-reply security, weak transfer (quasi-)continuity are all transfer types of continuities that are satisfied by many discontinuous economic games, a main difference among them is that, while better-reply security takes an open neighborhood of strategy profiles only for opponents' strategies rather than those of deviation player i, diagonal transfer continuity and weak transfer (quasi-)continuity take open neighborhoods of the strategy profile x for all players to the aggregate payoff function U and individual payoffs u_i, respectively.

Also, although weak transfer quasi-continuity is implied by better-reply security in Reny (1999) or diagonal transfer continuity in Baye *et al* (1993), weak transfer continuity neither implies nor is implied by better-reply security in Reny (1999) or diagonal transfer continuity in Baye *et al* (1993). The following examples show this.

EXAMPLE 3.3 Consider the two-player game with the following payoff functions defined on $[0, 1] \times [0, 1]$ studied by Carmona (2008).

$$u_i(x_1, x_2) = \begin{cases} \varphi_i(x_1, x_2), & \text{if } x_1 = x_2, \\ \psi_i(x_1, x_2), & \text{otherwise,} \end{cases}$$

where $\varphi_i, \psi_i : [0, 1]^2 \to \mathbb{R}$ are continuous functions. In addition, assume that G is bounded and quasiconcave and satisfies the following conditions:

- (i) For each $i \in I$, $\epsilon > 0$ and $y \in [0, 1]$, there exist $\bar{x} \in [0, 1]$ and a neighborhood $\mathcal{V}(y) \subset [0, 1]$ of y with $\bar{x} \notin \mathcal{V}(y)$ such that $\psi_i(\bar{x}, z) \geq \varphi_i(y, y) - \epsilon$ for each $z \in \mathcal{V}(y)$.

- (ii) If for each $x, y \in [0,1]$ with $x \neq y$ and for some i, $\psi_i(x,y) > \varphi_i(y,y)$, then there exist a player j, $\bar{x} \neq y \in [0,1]$ and a neighborhood $\mathcal{V}(y,y) \subset [0,1]^2$ of (y,y) such that $\psi_j(\bar{x}, y) > u_j(z)$ for each $z \in \mathcal{V}(y,y)$.

Carmona (2008) shows that the functions φ_i and ψ_i can be chosen so as to violate diagonal transfer continuity and/or better-reply security.

However, under conditions (i)-(ii), we can show that the game is weakly transfer continuous so that it has a Nash equilibrium by Theorem 3.2. Indeed, suppose x is not a Nash equilibrium. Then there exist a player i and a strategy $y_i \in [0,1]$ such that $u_i(y_i, x_{-i}) > u_i(x)$.

1. If $x_1 = x_2 = x$, then $\psi_i(y_i, x) > \varphi_i(x,x)$. By condition (ii), there exist a player j, $\bar{x}_j \neq x \in [0,1]$, and a neighborhood $\mathcal{V}(x_1,x_1) \subset [0,1]^2$ of (x_1,x_2) such that $\psi_j(\bar{x}_j, x) > u_j(z)$ for each $z \in \mathcal{V}(x_1,y_2)$. Let $\epsilon > 0$ such that $\psi_j(\bar{x}_j, x) - \epsilon > \sup\ u_j(z)$. Since $\bar{x}_j \neq x$ and the function $\psi_j(\bar{x}_j, .)$ is continuous, then there exists a neighborhood $\mathcal{V}(x) \subset [0,1]$ such that $\bar{x}_j \notin \mathcal{V}(x)$ and $\psi_j(\bar{x}_j, x) - \epsilon \leq \psi_j(\bar{x}_j, z_{-j})$ for all $z_{-j} \in \mathcal{V}(x)$. Thus, there exist a player j, a neighborhood $\mathcal{V}(x_1,x_1) \subset [0,1]^2$, and a strategy $\bar{x}_j \in [0,1]$ with $\bar{x}_j \neq z_{-j}$ such that $u_j(\bar{x}_j, z_{-j}) > u_j(z)$ for all $z \in \mathcal{V}(x_1,x_1)$.

2. If $x_1 \neq x_2$, then $u_i(y_i, x_{-i}) - \epsilon > \psi_i(x_1,x_2)$ for some $\epsilon > 0$. If $y_i \neq x_{-i}$, then by continuity of ψ_i, there exists a $\mathcal{V}(x_1,x_2)$ such that for all $z \in \mathcal{V}(x_1,x_1)$, $z_1 \neq z_2$ and $u_i(y_i, z_{-i}) > u_i(z)$.

   If $y_i = x_{-i}$, then $\varphi_i(y_i,y_i) - \epsilon > \psi_i(x_1,x_2)$. By condition (i), there exist $\bar{x}_i \in [0,1]$ and a neighborhood $\mathcal{V}(y_i) \subset [0,1]$ of y_i with $\bar{x}_i \notin \mathcal{V}(y_i)$ such that $\psi_i(\bar{x}_i, z) \geq \varphi_i(y_i,y_i) - \frac{\epsilon}{2}$ for each $z \in \mathcal{V}(y_i)$. Since the function $\psi_j(.,.)$ is continuous, then there exists a neighborhood $\mathcal{V}(x_1,x_2) \subset [0,1]^2$ such that for all $z \in \mathcal{V}(x_1,x_2)$, $z_1 \neq z_2$ and $\psi_i(x_1,x_2) + \frac{\epsilon}{2} \geq \psi_i(z_j, z_{-j})$. Thus, for each $z \in \mathcal{V}(x_1,x_2)$, we have $\psi_i(z_j,z_{-j}) \leq \psi_i(x_1,x_2) + \frac{\epsilon}{2} < \varphi_i(y_i,y_i) - \frac{\epsilon}{2} \leq \psi_i(\bar{x}_i, z_{-i}) = u_i(\bar{x}_i, z_{-i})$, *i.e.*, $u_i(\bar{x}_i, z_{-i}) > u_i(z)$.

3.3 Sufficient Conditions for Weak Transfer (Quasi-)Continuity

In this subsection we provide some new sufficient conditions for weak transfer (quasi-)continuity. While it is simple to verify weak transfer continuity, it is sometimes even simpler to verify other conditions leading to it and consequently weak transfer quasi-continuity. In addition to the fact that diagonal transfer continuity, better-reply security, transfer continuity, and weak transfer continuity all imply weak transfer quasi-continuity, weak transfer upper continuity and weak transfer lower continuity introduced below, when combined respectively with payoff security and upper semicontinuity, they also imply weak transfer continuity, and consequently weak transfer quasi-continuity.

DEFINITION 3.6 A game $G = (X_i,\ u_i)_{i\in I}$ is said to be *weakly transfer upper continuous* if whenever $x \in X$ is not an equilibrium, there exist player i, $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(x)$ of x such that $u_i(\hat{x}_i, x_{-i}) > u_i(x')$ for all $x' \in \mathcal{V}(x)$.

REMARK 3.5 If a game G is upper semicontinuous, then G is weakly transfer upper continuous. Indeed, suppose x is not a Nash equilibrium, then there exist a player i and a strategy y_i such that $u_i(y_i, x_{-i}) > u_i(x)$. Choose $\epsilon > 0$ such that $u_i(y_i, x_{-i}) - \epsilon > u_i(x)$. Since G is upper semicontinuous, then there exists a neighborhood $\mathcal{V}(x)$ of x such that $u_i(y_i, x_{-i}) - \epsilon > u_i(x) \geq u_i(x') - \epsilon$, for each $x' \in \mathcal{V}(x)$.

DEFINITION 3.7 A game $G = (X_i,\ u_i)_{i\in I}$ is said to be *weakly transfer lower continuous* if x is not a Nash equilibrium, which implies that there exist a player i, $y_i \in X_i$, and a neighborhood $\mathcal{V}(x_{-i})$ of x_{-i} such that $u_i(y_i, x'_{-i}) > u_i(x)$ for all $x'_{-i} \in \mathcal{V}(x_{-i})$.

REMARK 3.6 If a game G is payoff secure, then G is weakly transfer lower continuous. To see this, suppose $x \in X$ and x is not a Nash equilibrium, then there exists a player i that has a strategy $\hat{x}_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i(x)$. Choose $\epsilon > 0$ such that $u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$. Since G is payoff secure, then there exist a strategy y_i and a neighborhood $\mathcal{V}(x_{-i})$ of x_{-i} such that $u_i(y_i, x'_{-i}) \geq u_i(\hat{x}_i, x_{-i}) - \epsilon > u_i(x)$, for each $x'_{-i} \in \mathcal{V}(x_{-i})$.

We then have the following propositions that provide sufficient conditions for weak transfer (quasi-)continuity.

PROPOSITION 3.3 *If a game $G = (X_i,\ u_i)_{i\in I}$ is weakly transfer upper continuous and payoff secure, then it is weakly transfer continuous.*

PROPOSITION 3.4 *If a game $G = (X_i,\ u_i)_{i\in I}$ is weakly transfer lower continuous and upper semicontinuous, then it is weakly transfer continuous.*

Propositions 3.3-3.4, together with Theorem 3.2 or Theorem 3.3, immediately yield the following useful results.

COROLLARY 3.1 *If a game $G = (X_i, u_i)_{i\in I}$ is convex, compact, bounded, weakly transfer upper continuous, payoff secure, and quasiconcave or weakly diagonal transfer quasiconcave, then it possesses a pure strategy Nash equilibrium.*

COROLLARY 3.2 *If a game $G = (X_i, u_i)_{i\in I}$ is convex, compact, bounded, weakly transfer lower continuous, upper semicontinuous, and quasiconcave or weakly diagonal transfer quasiconcave, then it possesses a pure strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known noisy game.

EXAMPLE 3.4 Consider the two-player, nonzero sum, noisy game with the following payoff functions defined from $[0, 1] \times [0.1]$.

$$f_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_{-i}), & \text{if } x_i > x_{-i}, \end{cases}$$

where $l_i(.)$, $m_i(.)$ and $\phi_i(.)$ are upper semicontinuous over $[0, 1]$, $l_i(.)$ is strictly nondecreasing on $[0, 1]$ and satisfies the following additional conditions:

a) $\forall x \in [0, 1]$, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}(x)$ of x such that $\phi_i(x) \geq \max(l_i(z), m_i(z)) - \epsilon$, for every $z \in \mathcal{V}(x)$.

b) if $m_i(x) > \phi_i(x)$ with $x < 1$, then there exists a neighborhood $\mathcal{V}(x) \subset [0, 1)$ of x such that $m_i(z) > \phi_i(x)$, for every $z \in \mathcal{V}(x)$.

c) if $\phi_i(x) > m_i(x)$ with $x < 1$, then there exists a neighborhood $\mathcal{V}(x) \subset [0, 1)$ of x such that $\phi_i(z) > m_i(x)$, for every $z \in \mathcal{V}(x)$.

It is clear that this game G is compact and convex. Suppose that G is quasiconcave. The condition a) and the upper semicontinuity of $l_i(.)$, $m_i(.)$ and $\phi_i(.)$ over $[0, 1]$ imply that the noisy game is upper semicontinuous. The conditions b) and c) imply that the game is weakly transfer lower continuous. Then, the game considered is weakly transfer continuous by Proposition 3.4.[8], and thus it has a Nash equilibrium by Theorem 3.2.

REMARK 3.7 All the definitions of weak transfer quasi-continuity, weak transfer continuity, weak transfer upper continuity, weak transfer lower continuity and upper semicontinuity can be easily extended to the quasi-symmetric games and to get the existence results on symmetric Nash equilibrium.

3.4 Applications

In this subsection we show how our main existence results are applied to some important economic games. We provide two applications: one is in the shared resource games that is intensively studied by Rothstein (2007), and the other is in the classic Bertrand price competition games.

3.4.1 The Shared Resource Games

Rothstein (2007) studies a class of shared resource games with discontinuous payoffs, which includes a wide class of games such as the canonical game of fiscal competition for mobile capital. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps

[8]As Reny (1999) showed, if $\phi_i(x) \in co\{l_i(x), m_i(x)\}$ and $l_i(x)$ is nondecreasing, then the game is quasiconcave.

at certain joint strategies. Each player's payoff depends on her opponents' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains.

Formally, for such a game $G = (X_i, u_i)_{i \in I}$, each player i has a convex and compact strategy space $X_i \subset \mathbb{R}^l$ and a payoff function u_i that associates the *sharing rule* defined by $S_i : X \to [0, \overline{s}]$ with $\overline{s} \in (0, +\infty)$. That is to say, each player has a payoff function $u_i : X \to \mathbb{R}$ with the form $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$ where $F_i : X_i \times [0, \overline{s}] \to \mathbb{R}$ and u_i is bounded.[9].

Define $D_i \subset X$ to the set of joint strategies at which S_i is discontinuous and let the set $\Delta = \bigcup_{i \in I} D_i$ then be all of the joint strategies at which one or more of the sharing rules are discontinuous. The set $X \backslash \Delta$ is all of the joint strategies at which all of the sharing rules are continuous.

Rothstein (2007) shows a shared resource game possesses a pure strategy Nash equilibrium if the following conditions are satisfied:

(1) X is compact and convex;

(2) u_i is continuous on X and quasiconcave in x_i,

(3) S_i satisfies:

(3.i) For all $x \in X \setminus \Delta$, $\sum_{i=1}^n S_i(x) = \bar{s}$;

(3.ii) There exists $\underline{s} \in [0, \bar{s}]$ such that for all $x \in \Delta$, $\sum_{i=1}^n S_i(x) = \underline{s}$;

(3.iii) For all i, all $(x_i, x_{-i}) \in D_i$ and every neighborhood of x_i , there exists $x'_i \in X_i$ such that $(x'_i, x_{-i}) \in X \setminus D_i$;

(3.iv) For all i, there exists a constant $\tilde{s}$ satisfying $\bar{s} \geq \tilde{s} > \bar{s}/n$ such that for all $(x_i, x_{-i}) \in \Delta$ and all $(x_i, x_{-i}) \in X \setminus D_i$, $S_i(x'_i, x_{-i} \geq s_i \geq S_i(x_i, x_{-i})$.

(4) For all i, F_i satisfies:

(4.i) F_i is continuous;

(4.ii) For all $x_i \in X_i$, $F_i(x_i, \cdot)$ is nondecreasing in s_i;

(4.iii) Given any $s_i > \bar{s}/n$, $\max_{x_i \in X_i} F_i(x'_i, s_i) > \max_{x_i \in X_i} F_i(x_i, \bar{s}/n)$.

In the following, we will give an existence result with much simpler conditions:

Assumption 1: For each $i \in I$, X_i is convex and compact, and $u_i(., x_{-i})$ is bounded and quasiconcave for each $x_{-i} \in X_{-i}$.

Assumption 2: If $(y_i, x_{-i}) \in D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exist some player $j \in I$ and y'_j such that $(y'_j, x_{-j}) \in X \backslash D_j$ and $F_j(y'_j, S_j(y'_j, x_{-j})) > F_j(x_j, S_j(x))$.

[9]For more details on this model, see Rothstein (2007).

Assumption 3: If $(y_i, x_{-i}) \in X \backslash D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exist a player $j \in I$, a deviation strategy profile y' and a neighborhood $\mathcal{V}(x)$ of x such that for each $z \in \mathcal{V}(x)$, we have $F_j(y'_j, S_j(y'_j, z_{-j})) > F_j(z_j, S_j(z))$.

Assumption 1 is standard. A well-known sufficient condition for a composed function $u_i = F_i[x_i, S_i(x_i, x_{-i})]$ to be quasiconcave is that F_i is quasiconcave and nondecreasing in s_i, and S_i is concave. Assumption 2 means that if x is not an equilibrium and can be improved at a discontinuous strategy profile (y_i, x_{-i}) when player i uses the deviation strategy y_i, then there exists a player j such that it must also be improved by a continuous strategy profile (y'_i, x_{-i}) when player j uses the deviation strategy y'_i. Assumption 3 means that if a strategy profile x is not an equilibrium and can be improved by a continuous strategy profile (y_i, x_{-i}) when player i uses a deviation strategy y_i, then there exist a securing strategy profile y' and a neighborhood of x such that all points in the neighborhood cannot be equilibria. Note that, if F_i is continuous, then Assumption 3 is satisfied.

We then have the following result.

PROPOSITION 3.5 *Each shared resource game possesses a pure strategy Nash equilibrium if it satisfies Assumptions 1-3.*

3.4.2 The Bertrand Price Competition Games

Bertrand competition is a normal form game in which each of $n \geq 2$ firms, $i = 1, 2, ..., n$, simultaneously sets a price $p_i \in P_i = [0, \overline{p}]$. Under the assumption of profit maximization, the payoff to each firm i is

$$\pi_i(p_i, p_{-i}) = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i})),$$

where p_{-i} denotes the vector of prices charged by all firms other than i, $D_i(p_i, p_{-i})$ represents the total demand for firm i's product at prices (p_i, p_{-i}), and $C_i(D_i(p_i, p_{-i}))$ is firm i's total cost of producing the output $D_i(p_i, p_{-i})$. A Bertrand equilibrium is a Nash equilibrium of this game.

Let $A_i \subset X = \prod_{i \in I} P_i$ be the set of joint strategies at which π_i is discontinuous, $\Delta = \bigcup_{i \in I} A_i$ be the set of all of the joint strategies at which one or more of the payoffs are discontinuous, and $X \backslash \Delta$ be the set of all joint strategies at which all of the payoffs are continuous.

We make the following assumptions:

Assumption 1: For each $i \in I$, $\pi_i(., p_{-i})$ is quasiconcave for each $p_{-i} \in X_{-i}$.

Assumption 2: If $(q_i, p_{-i}) \in A_i$ and $\pi_i(q_i, p_{-i}) > \pi_i(p_i, p_{-i})$ for $i \in I$, then there exist a firm $j \in I$, and q'_j such that $(q'_j, p_{-j}) \in X \backslash A_j$ and $\pi_j(q'_j, p_{-j}) > \pi_j(p_j, p_{-j})$.

We then have the following result.

PROPOSITION 3.6 *Each Bertrand price competition game has a pure strategy Nash equilibrium if it satisfies (A1)-(A2).*

PROOF. It is similar to the proof of Proposition 3.5. ■

EXAMPLE 3.5 Consider a two-player Bertrand price competition game on the square $[0, a] \times [0, a]$, with $a > 0$. Assume that the demand function is discontinuous and is defined by

$$D_i(p_i, p_{-i}) = \begin{cases} \alpha_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma_i f(p_i) & \text{if } p_i > p_{-i} \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $\alpha_i, \beta_i > 0, \gamma_i \geq 0$ and $\alpha_i > \beta_i$. Suppose that the total cost of production is zero for each firm. Then, the payoff of each firm i becomes

$$\pi_i(p_i, p_{-i}) = \begin{cases} \alpha_i p_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta_i p_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma_i p_i f(p_i) & \text{if } p_i > p_{-i} \end{cases} .$$

We show that Assumption 2 is satisfied. To see this, note that $A_1 = A_2 = \{(p_1, p_2) : p_1 = p_2 \in [0, a]\}$. Suppose $(q_i, q_i) \in A_i$ and $\pi_i(q_i, q_i) > \pi_i(p_i, q_i)$ for some $p_i \in [0, a]$, we then must have $q_i = 0$. Thus

$$\beta_i q_i f(q_i) > \pi_i(p_i, q_i), \tag{3.3}$$

and therefore $f(q_i) > 0$. Since the function $xf(x)$ is continuous, then for $\epsilon = q_i f(q_i)/\theta > 0$ with $\theta = \frac{\alpha_i}{\alpha_i - \beta_i}$, there exists $\delta > 0$ such as for all x with $q_i - \delta < x < q_i + \delta$, $q_i f(q_i) - \epsilon \leq xf(x) \leq q_i f(q_i) + \epsilon$. Thus, there exists $q_i^{'} \in [0, a]$ such that

$$0 < q_i^{'} < q_i \text{ and } \alpha_i q_i^{'} f(q_i^{'}) \geq \beta_i q_i f(q_i). \tag{3.4}$$

(3.3) and (3.4) imply that there exists $q_i^{'} \in [0, a]$ such that $(q_i^{'}, p_{-i}) \in X \backslash A_i$ and $\pi_i(q_i^{'}, p_{-i}) > \pi_i(p_i, p_{-i})$. Then, by Proposition 3.6, the game has a pure strategy Nash equilibrium if it is quasiconcave.

4 Existence of Dominant Strategy Equilibria

In this section we investigate the existence of dominant strategy equilibria in discontinuous. We start by reviewing some of the basic definitions and results introduced and obtained in Baye *et al* (1993).

DEFINITION 4.1 A game $G = (X_i, u_i)_{i \in I}$ is transfer upper semicontinuous if for each $i \in I$, $x_i \in X_i$ and $y \in X$, $u_i(y) > u_i(x_i, y_{-i})$ implies that there exist a point $y' \in X$ and a neighborhood $\mathcal{V}(x_i)$ of x_i such that $u_i(y') > u_i(x_i', y_{-i}')$, for all $x_i' \in \mathcal{V}(x_i)$.

DEFINITION 4.2 A game $G = (X_i, u_i)_{i\in I}$ is uniformly transfer quasiconcave on X if, for each $i \in I$ and any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x_i^1, ..., x_i^m\} \subset X_i$ such that for any subset $\{y_i^{k^1}, y_i^{k^2}, ..., y_i^{k^s}\}$, $1 \leq s \leq m$, and any $x_i \in co\{x_i^{k^1}, x_i^{k^2}, ..., x_i^{k^s}\}$, we have $\min_{1\leq l\leq s}\{u_i(y^{k^l}) - u_i(x_i, y_{-i}^{k^l})\} \leq 0$.

Baye *et al* (1993) showed that a game $G = (X_i,\ u_i)_{i\in I}$ that is convex, compact and transfer upper continuous must possess a dominant strategy equilibrium if and only if it is uniformly transfer quasiconcave.

In the following, we provide a new result on the existence of dominant strategy equilibria in discontinuous games. We start by introducing the notion of weak dominant transfer upper continuity.

DEFINITION 4.3 A game $G = (X_i, u_i)_{i\in I}$ is said to be *weakly dominant transfer upper continuous* if whenever $\overline{x} \in X$ is not a dominant strategy equilibrium, there exist a player i, a strategy $y \in X$ and a neighborhood $\mathcal{V}(\overline{x}_i)$ of $\overline{x}_i$ such that $u_i(y) > u_i(z_i, y_{-i})$, for each $z_i \in \mathcal{V}(\overline{x}_i)$.

A game is weakly dominant transfer upper continuous if for every nondominant strategy equilibrium x^*, some player i has a strategy y_i that dominates all other strategy z_i in a neighborhood of x_i^* when other players play y_{-i}.

An even weaker form of dominant transfer continuity is presented below.

DEFINITION 4.4 A game $G = (X_i, u_i)_{i\in I}$ is said to be *weakly dominant transfer upper quasi-continuous* if $\overline{x}$ is not a dominant strategy equilibrium, then there exist a strategy $y \in X$ and a neighborhood $\mathcal{V}(\overline{x})$ of $\overline{x}$ so that for each $z \in \mathcal{V}(\overline{x})$ there exists a player $i \in I$ such as $u_i(y) > u_i(z_i, y_{-i})$.

A game is weakly dominant transfer upper quasi-continuous if for every nondominant strategy equilibrium x, there is a neighborhood $\mathcal{V}(x)$ of x that does not contain a dominant strategy equilibrium.

REMARK 4.1 It is clear that if the game G is weakly dominant transfer upper continuous or transfer upper semicontinuous (See Definition 4.1), then it is weakly dominant transfer upper quasi-continuous.

DEFINITION 4.5 A game $G = (X_i, u_i)_{i\in I}$ is said to be *strongly uniformly transfer quasi-concave* if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, there exists $y^h \in \{y^{k^1}, ..., y^{k^s}\}$ so that

$$u_i(y^{k^h}) \leq u_i(x_i, y_{-i}^{k^h}) \quad \forall i \in I. \tag{4.1}$$

Strong uniform transfer quasiconcavity roughly says that given any finite subset $Y^m = \{y^1, ..., y^m\}$ of deviation profiles, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\}$ of candidate profiles such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, its convex combinations are not dominated simultaneously by all deviations in $X^{\tilde{m}}$ for all players. We will see from Theorem 4.1 below that strong uniform transfer quasiconcavity is necessary for the existence of a dominant strategy equilibrium of a game when it is weakly dominant transfer upper quasi-continuous. It is clear that a game is uniformly transfer quasiconcave if it is strongly uniformly transfer quasiconcave. Indeed, by (4.1), we have $\min_{1\leq l\leq s}\{u_i(y^{k^l}) - u_i(x_i, y^{k^l}_{-i})\} \leq 0$.

REMARK 4.2 Define a correspondence $F : X \to 2^X$ by $F(y) = \{x \in X : u_i(y) \leq u_i(x_i, y_{-i}),\ \forall i \in I\}$. Then it is transfer FS-convex if and only if the game is strongly uniformly transfer quasiconcave.

The following theorem characterizes the existence of dominant strategy equilibrium if the game is weakly dominant transfer upper quasi-continuous and the strategy spaces of players are convex.

THEOREM 4.1 *If a game $G = (X_i, u_i)_{i\in I}$ is convex, compact, weakly dominant transfer upper quasi-continuous, and strongly uniformly transfer quasiconcave, then it possesses a dominant strategy equilibrium.*

Let $m \in \mathbb{N}^*$ and the following special simplex:

$$\Delta(n, m) = \{\lambda = (\lambda_{i,j})_{\substack{i=1,...,n\\ j=1,...,m}} \in \mathcal{M}_{\mathbb{R}}(n, m) : \lambda_{i,j} \geq 0 \text{ and } \sum_{i,j}\lambda_{i,j} = 1\}.$$

DEFINITION 4.6 A game $G = (X_i, u_i)_{i\in I}$ is said to be *weakly uniformly transfer quasiconcave* if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for each $\tilde{x} = \sum_{i,j}\lambda_{i,j}x^j \in \text{co}\{x^h,\ h = 1, ..., m\}$, we have

$$\min_{(i,j)\in J}[u_i(y^j) - u_i(\tilde{x}_i, y^j_{-i})] \leq 0, \tag{4.2}$$

where $J = \{(i, j) :\ \lambda_{i,j} > 0\}$.

REMARK 4.3 Definition 4.6 is equivalent to the following definition: A game $G = (X_i, u_i)_{i\in I}$ is weakly transfer quasiconcave if and only if for any finite subset $\{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for each $\lambda \in \Delta(n, m)$, there exists a player $i \in I$ such that

$$\min_{j\in J(i)}[u_i(y^j) - u_i(\tilde{x}_i, y^j_{-i})] \leq 0,$$

where $J(i) = \{j = 1, ..., m :\ \lambda_{i,j} > 0\}$ and $\tilde{x} = \sum_{i,j}\lambda_{i,j}x^j$.

Weak uniform transfer quasiconcavity roughly says that given any finite subset $Y^m = \{y^1, ..., y^m\}$ of deviation profiles, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\}$ of candidate profiles such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subset X^m$, $1 \leq s \leq m$, there exists some player i so that its convex combinations are not dominated by all deviations in $X^{\tilde{m}}$ that have nonzero weights. We will see from Theorem 4.2 below that weak uniform transfer quasiconcavity is necessary for the existence of a dominant strategy equilibrium of a game when it is weakly dominant transfer upper continuous. If a game G is strongly uniformly transfer quasiconcave, then it is weakly uniformly transfer quasiconcave.

The following theorem characterizes the existence of dominant strategy equilibrium if a game is weakly dominant transfer upper continuous and the strategy spaces of players are convex.

THEOREM 4.2 *If a game $G = (X_i, u_i)_{i\in I}$ is compact, bounded, convex, weakly uniformly transfer quasiconcave, and weakly dominant transfer upper continuous, then it has a dominant strategy equilibrium.*

The following proposition provides sufficient conditions for a game to be weakly dominant transfer upper continuous.

PROPOSITION 4.1 *Any of the following conditions implies that the game $G = (X_i, u_i)_{i\in I}$ is weakly dominant transfer upper continuous.*

(a) u_i is continuous in x_i.

(b) u_i is upper semi-continuous in x_i.

(c) u_i is transfer upper continuous in x_i.

5 Nash Equilibria in Mixed Strategies

In this section, we consider the existence of mixed strategy Nash equilibrium by applying the pure strategy existence results derived in the previous sections. Assume that each X_i is a compact *Hausdorff* space. Let u_i be bounded and measurable for all $i \in I$ and M_i be the regular, countably additive probability measures on the *Borel* subsets of X_i, then M_i is compact in the weak* topology. Let us consider U_i to be the extension of u_i to $M = \prod_{i\in I} M_i$ by defining $U_i(\mu) = \int_X u_i(x)d\mu(x)$ for all $\mu \in M$ with $d\mu(x) = d\mu_1(x_1) \times d\mu_2(x_2) \times ... \times d\mu_n(x_n)$, and let $\overline{G} = (M_i, U_i)_{i\in I}$ denote the mixed extension of G.

DEFINITION 5.1 A mixed strategy Nash equilibrium of the game G is an n-tuple of probability measures $(\mu_1^*, ..., \mu_n^*) \in M$ such that for all $i \in I$

$$U_i(\mu^*) = \int_X u_i(x)d\mu^*(x) \geq \max_{\mu_i\in M_i} \int_X u_i(x)d\mu_1^*(x_1) \times ... \times d\mu_i(x_i) \times .... \times d\mu_n^*(x_n).$$

The definitions of weak transfer continuity, weak transfer upper continuity, weak transfer lower continuity, upper semicontinuity, payoff security, etc. given in Subsection 3.1 apply in obvious ways to the mixed extension $\overline{G}$ by replacing X_i with M_i in each definition. However, it may be noted that weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of $\overline{G}$ neither implies nor is implied by weak transfer continuity (resp., weak transfer upper continuity, weak transfer lower continuity, payoff security) of G.

LEMMA 5.1 *If G is upper semicontinuous, then the mixed extension of G is also upper semicontinuous.*

PROOF. See the proof of Proposition 5.1 in Reny (1999) page 1052. ■

Nash (1950) and Glicksberg (1952) show that a game that is compact, Hausdorff and continuous possesses mixed strategy Nash equilibrium. Robson (1954) proves that in a compact game with metric strategy spaces, if each player's payoff is upper semicontinuous in all players' strategies, and continuous in other players' strategies, then the game possesses a mixed strategy Nash equilibrium.

The following theorem strictly generalizes the mixed strategy Nash equilibrium existence results of Nash (1950), Glicksberg (1952), Dasgupta and Maskin (1986), Robson (1994), Simon (1987) and Reny (1999) by weakening continuity conditions.

We now present the mixed strategy implications of Theorem 3.1 and Corollaries 3.2-3.3.

THEOREM 5.1 *Suppose that $G = (X_i, u_i)_{i \in I}$ is a compact, Hausdorff game. Then G has a mixed strategy Nash equilibrium if its mixed extension $\overline{G}$ is weakly transfer quasi-continuous. Moreover, $\overline{G}$ is weakly transfer quasi-continuous if it is 1) weakly transfer continuous, 2) better reply secure, 3) weakly transfer upper continuous and payoff secure, or 4) weakly transfer lower continuous and upper semicontinuous.*

REMARK 5.1 The first part of Theorem 5.1 is a particular case of Reny's theorem (Theorem 2.9 in Reny (2009)).

Reny (2009) introduced the finite deviation property and proved if the mixed extension of G has the finite deviation property, then it possesses a mixed strategy Nash equilibrium.

DEFINITION 5.2 The game G has the *finite deviation property* if whenever x^* is not a Nash equilibrium, there exist $x^1, ..., x^K \in X$, and a neighborhood $\mathcal{V}(x^*)$ of x^* so as for all $z \in \mathcal{V}(x^*)$, there exist a player i and k such that $u_i(x_i^k, z_{-i}) > u_i(z)$.

REMARK 5.2 If the game G is weakly transfer quasi-continuous, then it has the finite deviation property. Indeed, we can take $K = 1$ in Definition 5.2.

Monteiro and Page (2007) introduce the concept of uniform payoff security for games that are compact, Hausdorff, bounded and measurable. They show that if a game is compact and uniformly payoff secure, then its mixed extension $\bar{G}$ is payoff secure, but the reverse may not be true, as shown by an example in Carmona (2005).

DEFINITION 5.3 The game G is *uniformly payoff secure* if for each $i \in I$, $x_i \in X_i$, and every $\epsilon > 0$, there is a strategy $\overline{x}_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there exists a neighborhood $\mathcal{V}(y_{-i})$ of y_{-i} such that $u_i(\overline{x}_i, z_{-i}) \geq u_i(x_i, y_{-i}) - \epsilon$, for all $z_{-i} \in \mathcal{V}(y_{-i})$.

DEFINITION 5.4 The game G is said to be *uniformly transfer continuous* if for each $i \in I$, $x_i \in X_i$, and every $\epsilon > 0$, there is a strategy $\overline{x}_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there exists a neighborhood $\mathcal{V}(x_i, y_{-i})$ of (x_i, y_{-i}) such that

$$u_i(\overline{x}_i, z_{-i}) + \epsilon \geq u_i(x_i, y_{-i}) \geq u_i(z) - \epsilon, \text{ for all } z \in \mathcal{V}(x_i, y_{-i}).$$

Thus, a game G is uniformly transfer continuous if for any strategy $x_i \in X_i$, player i can choose a strategy $\overline{x}_i \in X_i$ to secure a payoff of $u_i(x_i, y_{-i}) - \epsilon$ against deviations by other players in some neighborhood of $y_{-i} \in X_{-i}$, and would be better off at (x_i, y_{-i}) even if all players deviate slightly from (x_i, y_{-i}) for all strategy profiles $y_{-i} \in X_{-i}$.

PROPOSITION 5.1 *If a game $G = (X_i,\, u_i)_{i \in I}$ is 1) uniformly payoff secure and upper semi-continuous or 2) uniformly transfer continuous, then the mixed extension $\overline{G}$ is weakly transfer continuous.*

Proposition 5.1, together with Theorem 5.1, immediately yields the following useful result.

COROLLARY 5.1 *If a game $G = (X_i, u_i)_{i \in I}$ is compact, bounded, Hausdorff, and 1) uniformly payoff secure and upper semicontinuous or 2) uniformly transfer continuous, then it possesses a mixed strategy Nash equilibrium.*

As an application of the above proposition, consider the following well-known concession game.

EXAMPLE 5.1 Let us consider $i = 1, 2$ and $x_1, x_2 \in [0, 1]$ with:

$$u_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_i), & \text{if } x_i > x_{-i}. \end{cases}$$

We make the following assumption on u_i:

ASSUMPTION 5.1

a) $\forall x \in [0,1]$, $\forall \epsilon > 0$, there exists a neighborhood $\mathcal{V}(x)$ of x such that $\phi_i(x) \geq \max(m_i(z), l_i(z)) - \epsilon$, for every $z \in \mathcal{V}(x)$.

b) $\forall x \in [0,1]$, $\forall \epsilon > 0$, there exists $y \in [0,1]$ such that $\min\{\phi_i(y), m_i(y), l_i(y)\} \geq \max\{\phi_i(x), m_i(x), l_i(x)\} - \epsilon$.

Then we have the following result.

PROPOSITION 5.2 *Suppose the concession game satisfies Assumption 5.1, and the functions $l_i(.)$, $m_i(.)$ and $\phi_i(.)$ are upper semicontinuous on $[0,1]$. Then, the game has a mixed strategy Nash equilibrium.*

6 Conclusion

In this paper, we investigate the existence of equilibria in possibly discontinuous games. We offer new existence results on Nash equilibrium for a large class of discontinuous games by introducing a new notion of very weak continuity, called weak transfer (quasi-)continuity.

These results permit us to significantly weaken the continuity condition for the existence of Nash equilibrium. We also provide examples and economic applications where our general results are applicable, but the existing theorems for pure strategy, dominant strategy, and mixed strategy Nash equilibria fail to hold. These new results help us understand the existence or non-existence of pure strategy, dominant strategy, and mixed strategy Nash equilibria in discontinuous games.

Appendix

Proof of Proposition 3.1. It is clear that a game G is weakly transfer quasi-continuous if it is weakly transfer continuous. We only need to prove either of diagonal transfer continuity and better-reply security implies weak transfer quasi-continuity.

We first consider the case of diagonal transfer continuity. Suppose $x^* \in X$ is not an equilibrium. Then, by diagonal transfer continuity of U, there exist a strategy $\bar{y} \in X$ and a neighborhood $\mathcal{V}(x^*)$ of x^* such that $U(z, \bar{y}) > U(z, z)$ for each $z \in \mathcal{V}(x^*)$, *i.e.*, $\sum_{i \in I}[u_i(\bar{y}_i, z_{-i}) - u_i(z)] > 0$ for each $z \in \mathcal{V}(x^*)$. Thus, for each $z \in \mathcal{V}(x^*)$, there exists a player $i \in I$ such as $u_i(\bar{y}_i, z_{-i}) - u_i(z) > 0$.

We now consider the case of better-reply security. Suppose, by way of contradiction, that the game is not weakly transfer quasi-continuous. Then, there exists a nonequilibrium $x^* \in X$ such that for every $\bar{y} \in X$ and every neighborhood $\mathcal{V}(x^*)$ of x^*, there exists $z \in \mathcal{V}(x^*)$ with

$$u_i(\bar{y}_i, z_{-i}) \leq u_i(z) \quad \text{for all } i \in I. \tag{6.1}$$

Letting $\bar{u}$ be the limit of the vector of payoffs corresponding to some sequence of strategies converging to x^*, and U^* be the set of all such points, which is a compact set by the boundedness of payoffs, we have $(x^*, \bar{u}) \in \text{cl}\,\Gamma$ for all $\bar{u} \in U^*$. Then, by better-reply security, for each $(x^*, \bar{u}) \in \text{cl}\,\Gamma$ with $\bar{u} \in U^*$, there exist a player i, a strategy $\bar{y}_i$, and a neighborhood $\bar{\mathcal{V}}(x^*_{-i})$ of x^*_{-i} such that $u_i(\bar{y}_i, z_{-i}) > \bar{u}_i$ for all $z_{-i} \in \bar{\mathcal{V}}(x^*_{-i})$. Then, we have $\varphi_i(\bar{y}_i, x^*_{-i}) > \bar{u}_i$ [10]. Choose $\epsilon > 0$ with $\varphi_i(\bar{y}_i, x^*_{-i}) > \bar{u}_i + \epsilon$. Since $\varphi_i(\bar{y}_i, .)$ is lower semi-continuous (cf. Reny, 1999), then

$$u_i(\bar{y}_i, z_{-i}) > \bar{u}_i + \epsilon, \text{ for each } z_{-i} \in \mathcal{V}^*(x^*_{-i}).$$

Let U^*_i be the projection of U^* to coordinate i and $u^*_i = \sup\{\bar{u}_i \in U^*_i : u_i(\bar{y}_i, z_{-i}) > \bar{u}_i + \epsilon \quad \text{for all } z_{-i} \in \bar{\mathcal{V}}(x^*_{-i})\}$. Then, for $\epsilon/2 > 0$, there is a y^*_i such that $u_i(y^*_i, z_{-i}) > (u^*_i + \epsilon) - \epsilon/2 = u^*_i + \epsilon/2$ for all $z_{-i} \in \mathcal{V}^*(x^*_{-i})$.

Now, since the game is not weakly transfer quasi-continuous, for such a securing strategy y^*_i, we can find a directed system of neighborhoods $\{\mathcal{V}^\alpha(x^*)\}_{\alpha \in \Lambda}$ and a net $z^\alpha \in \mathcal{V}^\alpha(x^*)$ such that $z^\alpha \to_\alpha x^*$ and

$$u_i(y^*_i, z^\alpha_{-i}) \leq u_i(z^\alpha) \to_\alpha \bar{u}_i \leq u^*_i.$$

Thus, for $\epsilon/2 > 0$, there exists α_1 such that whenever $\alpha > \alpha_1$, we have

$$u_i(y^*_i, z^\alpha_{-i}) \leq u^*_i + \epsilon/2 < u_i(y^*_i, z_{-i}), \text{ for each } z_{-i} \in \mathcal{V}^*(x^*_{-i}). \tag{6.2}$$

[10] The function $\varphi_i(y_i, x_{-i})$ is defined by $\varphi_i(y_i, x_{-i}) = \sup_{\mathcal{V} \in \Omega(x_{-i})} \inf_{z_{-i} \in \mathcal{V}} u_i(y_i, z_{-i})$ (cf. Reny, 1999).

Since the net $\{z^\alpha\}_{\alpha\in\Lambda}$ also converges to x^*, then for each $\mathcal{V}(x^*)$ of x^* with $Proj_{-i}(\mathcal{V}(x^*)) \subsetneqq \mathcal{V}(x^*_{-i})$, [11] there exists α_2 such that $z^\alpha \in \mathcal{V}(x^*)$ for $\alpha > \alpha_2$. Consequently, $z^\alpha_{-i} \in Proj_{-i}(\mathcal{V}^*(x^*))$ with $\alpha > \max(\alpha_1, \alpha_2)$. Thus, by (6.2), we obtain $u_i(y^*_i, z^\alpha_{-i}) \leq u^*_i + \epsilon < u_i(y^*_i, z^\alpha_{-i})$, a contradiction. Hence, the game must be weakly transfer quasi-continuous. ■

PROOF OF THEOREM 3.1. For each $y \in X$, let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x), \ \forall i \in I\}.$$

By Remark 3.1, G is weakly transfer quasi-continuous if and only if F is transfer closed-valued.

For $y \in X$, let $\bar{F}(y) = \text{cl } F(y)$. Then $\bar{F}(y)$ is closed, and by the strong diagonal transfer quasiconcavity (Remark 3.4), it is also transfer FS-convex. From Lemma 1 in Tian (1993), we deduce $\bigcap\limits_{y\in X} F(y) = \bigcap\limits_{y\in X} \bar{F}(y) \neq \emptyset$. Thus, there exists a strategy profile $\overline{x} \in X$ such that

$$u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x}), \text{ for all } y \in X \text{ and } i \in I.$$

Thus $\overline{x}$ is a pure strategy Nash equilibrium of the game G. ■

To prove Theorems 3.2-3.3, we need the following lemma.

A correspondence $C : X \to 2^Y$ is *open inverse-image* or have *lower open sections* if the set $\{x \in X : y \in C(x)\}$ is open in X, for all $y \in Y$.

LEMMA 6.1 *(See Theorem 3a, page 264 in Deguire and Lassonde (1995)) Let X_i be a nonempty, compact and convex space, $i \in J$ and $\{C_i : X \to X_i, \ i \in J\}$ be a family of correspondences such that:*

(1) for all $i \in J$, $C_i(x)$ is convex for every $x \in X$,

(2) for all $i \in J$, C_i is open inverse-image,

(3) for each $x \in X$, there exists $i \in J$ such that $C_i(x) \neq \emptyset$.

Then, there exists $\overline{x} \in X$ and $i \in J$ such that $\overline{x}_i \in C_i(\overline{x})$.

PROOF OF THEOREM 3.2. For each player $i \in I$ and every $(x_i, y) \in X_i \times X$, let

$$\varphi_i(x_i, y) = \sup_{\mathcal{V}\in\Omega(y)} \inf_{z\in\mathcal{V}} [u_i(x_i, z_{-i}) - u_i(z)]$$

where $\Omega(y)$ is the set of all open neighborhoods of y.

[11] $Proj_i(A)$ is the projection of A on space X_i.

For each i and every $x_i \in X_i$, the function $\varphi_i(x_i, .)$ is real-valued by boundedness of payoff function. We show it is also lower semicontinuous over X. Indeed, for each $i \in I$, let $x_i \in X_i$ and $\mathcal{V}$ be an open neighborhood. Consider the following function

$$g^i_{\mathcal{V}}(x_i, y) = \begin{cases} \inf\limits_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)], & \text{if } y \in \mathcal{V}, \\ -\infty, & \text{otherwise.} \end{cases}$$

We want to show that $g^i_{\mathcal{V}}(x_i, .)$ is lower semicontinuous on X, which is equivalent to showing the set

$$A(x_i) = \{y \in X : g^i_{\mathcal{V}}(x_i, y) \leq \alpha\},\ \alpha \in \mathbb{R}$$

is closed for all $x_i \in X_i$. Suppose that there exists a point $y \in X$ such that y is in the closure of $A(x_i)$, but not in $A(x_i)$. Then, there exists a net $\{y^p\}_{p\in\Lambda} \subset A(x_i)$ converging to y. Since $y \notin A(x_i)$, $\inf\limits_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)] > \alpha$. If $y \notin \mathcal{V}$, then $-\infty > \alpha$, which is impossible, and thus $y \in \mathcal{V}$ and $g^i_{\mathcal{V}}(x_i, y) > \alpha$. Thus, we have $\{y^p\}_{p\in\Lambda} \subset A(x_i)$, and then $g^i_{\mathcal{V}}(x_i, y^p) \leq \alpha$ for every $p \in \Lambda$. If there exists $\bar{p} \in \Lambda$ such that $y^{\bar{p}} \in \mathcal{V}$, then $\inf\limits_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)] \leq \alpha$, which contradicts the fact that $\inf\limits_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)] > \alpha$. Thus, for all $p \in \Lambda$, $y^p \notin \mathcal{V}$. Since the net $\{y^p\}_{p\in\Lambda}$ converges to $y \in \mathcal{V}$, there exists $\eta \in \Lambda$ such that for all $p \geq \eta$, $y^p \in \mathcal{V}$, which contradicts the fact that $y^p \notin \mathcal{V}$ for all $p \in \Lambda$. Thus, $A(x_i)$ is closed, which means that the function $g^i_{\mathcal{V}}(x_i, .)$ is lower semicontinuous over X. Since the function $\varphi_i(x_i, .)$ is the pointwise supremum of a collection of lower semicontinuous functions on X, by Lemma 2.39, page 43 in Aliprantis and Border (1994), $\varphi_i(x_i, .)$ is lower semicontinuous on X.

Let us consider the following sets:
For each $y \in X$, let

$$G(y) = \{x \in X : \varphi_i(y_i, x) \leq 0,\ \forall i \in I\},$$

and for each $x \in X$ and $i \in I$, let

$$C_i(x) = \{y_i \in X_i : \varphi_i(y_i, x) > 0\}.$$

Since $\varphi_i(x_i, .)$ is lower semicontinuous on X, then $G(y)$ is closed.

We now show $\{C_i\}_{i\in I}$ is convex and open inverse-image for all $i \in I$. Indeed, let $i \in I$, $x \in X$, $\overline{y}_i$, $\overline{\overline{y}}_i$ be two elements of $C_i(x)$ and $\theta \in [0, 1]$. Since $\overline{y}_i$ and $\overline{\overline{y}}_i$ are in $C_i(x)$, $\varphi_i(\overline{y}_i, x) > 0$ and $\varphi_i(\overline{\overline{y}}_i, x) > 0$. Then, there exist $\mathcal{V}^1(x)$ and $\mathcal{V}^2(x)$ of x such that for all $(z^1, z^2) \in \mathcal{V}^1(x) \times \mathcal{V}^2(x)$,

$$\begin{cases} u_i(\overline{y}_i, z^1_{-i}) > u_i(z^1) \\ u_i(\overline{\overline{y}}_i, z^2_{-i}) > u_i(z^2). \end{cases}$$

Thus, there exists a neighborhood $\mathcal{V}(x) = \mathcal{V}^1(x) \cap \mathcal{V}^2(x)$ such that

$$\min\{u_i(\overline{y}_i, z_{-i}), u_i(\overline{\overline{y}}_i, z_{-i})\} > u_i(z),\ \forall z \in \mathcal{V}(x).$$

Since G is quasiconcave in x_i, then $\min\{u_i(\overline{y}_i, z_{-i}), u_i(\overline{\overline{y}}_i, z_{-i})\} \leq u_i(\theta\overline{y}_i + (1-\theta)\overline{\overline{y}}_i, z_{-i})$, for each z_{-i}. Therefore, $u_i(\theta\overline{y}_i+(1-\theta)\overline{\overline{y}}_i, z_{-i}) > u_i(z),\ \forall z \in \mathcal{V}(x)$. Thus, $\theta\overline{y}_i+(1-\theta)\overline{\overline{y}}_i \in C_i(x)$. Also, let $i \in I$. Since $\varphi_i(\overline{y}_i, .)$ is lower semicontinuous, the set $\{x \in X : \varphi_i(\overline{y}_i, x) > 0\}$ is open in X, for each $y_i \in X_i$, which means C_i is open inverse-image.

Now suppose, by way of contradiction, that for each $x \in X$, there exists a player $i \in I$ such that $C_i(x) \neq \emptyset$. Then, by Lemma 6.1, there exist a point $\widetilde{x} \in X$ and $i \in I$ such that $\widetilde{x}_i \in C_i(\widetilde{x})$, *i.e.*, $\varphi_i(\widetilde{x}_i, \widetilde{x}) > 0$. Thus, by lower semicontinuity of $\varphi_i(x_i, .)$, there exists a neighborhood $\mathcal{V}(\widetilde{x})$ of $\widetilde{x}$ such that $u_i(\widetilde{x}_i, z_{-i}) > u_i(z)$, for each $z \in \mathcal{V}(\widetilde{x})$. Letting $z = \widetilde{x}$ in the last inequality, we obtain $u_i(\widetilde{x}) > u_i(\widetilde{x})$, which is impossible. Thus, there exists $\overline{x} \in X$ such that for each $i \in I$, we have $C_i(\overline{x}) = \emptyset$. Therefore, for each $i \in I$ and each $y_i \in X_i$, $\varphi_i(y_i, \overline{x}) \leq 0$. Hence,

$$\overline{x} \in \bigcap_{y\in X} G(y) \tag{6.3}$$

if $\overline{x}$ is not a Nash equilibrium. Since the game G is weakly transfer continuous, then there exists player i, $\bar{y}_i$, and a neighborhood $\mathcal{V}$ of $\overline{x}$ such that $u_i(\bar{y}_i, z_{-i}) > u_i(z)$, for all $z \in \mathcal{V}$. Then, $\varphi(\bar{y}_i, \overline{x}) > 0$, which contradicts (6.3). Therefore, $\overline{x}$ is a Nash equilibrium. ■

PROOF OF PROPOSITION 3.2. Suppose that the aggregate function $U(x, y) = \sum_{i=1}^{n} u_i(y_i, x_{-i})$ is diagonally transfer quasiconcave. Then, for any finite subset $Y^m = \{y^1, ..., y^m\} \subset X$, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\} \subset X$ such that for each $\tilde{x} = \sum_{i,j}\lambda_{i,j}x^j \in \text{co}\{x^h,\ h = 1, ..., m\}$, we have $\min_{s\in J_1} U(x, y^s) \leq U(x, x)$ where $J_1 = \{j = 1, ..., m : \sum_{i\in I}\lambda_{i,j} > 0\}$ and $\lambda \in \Delta(n, m)$. Thus, $\min_{s\in J_1}\sum_{i=1}^{n}[u_i(y_i^s, x_{-i}) - u_i(x)] \leq 0$. Therefore, there exists $(i, j) \in J = \{(i, j) :\ \lambda_{i,j} > 0\}$ such that $u_i(y_i^j, x_{-i}) - u_i(x) \leq 0$. We conclude that $\min_{(i,j)\in J}[u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0$ with $J = \{(i, j) :\ \lambda_{i,j} > 0\}$. ■

PROOF OF THEOREM 3.3. For each player $i \in I$ and every $(x_i, y) \in X_i \times X$, let

$$\varphi_i(x_i, y) = \sup_{\mathcal{V}\in\Omega(y)} \inf_{z\in\mathcal{V}}[u_i(x_i, z_{-i}) - u_i(z)]$$

where $\Omega(y)$ is the set of all open neighborhoods of y. For each i and every $x_i \in X_i$, the function $\varphi_i(x_i, .)$ is lower semicontinuous over X from the proof of Theorem 3.2.

Let us consider the following set: for each $y \in X$, let

$$G(y) = \{x \in X : \varphi_i(y_i, x) \leq 0,\ \forall i \in I\}.$$

Since the function $\varphi_i(x_i, .)$ is lower semicontinuous over X, then $G(y)$ is closed. If $\overline{x} \in \bigcap_{y\in X} G(y)$, then $\overline{x}$ is a Nash equilibrium (because G is weakly transfer continuous).

Now, suppose, by way of contradiction, that $\bigcap_{y \in X} G(y) = \emptyset$. Then, we have

$$\forall x \in X, \text{ there exists } y \in X, \ i \in I \text{ such that } \varphi_i(y_i, x) > 0. \tag{6.4}$$

Thus, X can be covered by the following subsets

$$\theta_{i,y} = \{x \in X : \varphi_i(y_i, x) > 0\}, \ i \in I \text{ and } y \in X.$$

Since $\varphi_i(y_i, .)$ is lower semicontinuous on X, the subset $\theta_{i,y}$ is open in X, for each $i \in I$ and $y \in X$. Also, since X is compact, it can be covered by a finite number of subsets $\{\theta_{i,y^j} : i = 1, ..., n \text{ and } j = 1, ..., m\}$. Consider a continuous partition of unity $\{\alpha_{i,j}\}_{\substack{i=1,...,n \\ j=1,...,m}}$ associated to the finite covering $\{\theta_{1,y^1}, ..., \theta_{n,y^m}\}$.

Since G is weakly diagonal transfer quasiconcave, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for each $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, \ h = 1, ..., m\}$ and if $J = \{(i,j) : \lambda_{i,j} > 0\}$, then

$$\min_{(i,j) \in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0. \tag{6.5}$$

Let us now consider the following function defined on X into X by

$$f(x) = \sum_{i,j} \alpha_{i,j}(x) x^j.$$

Since the functions $\alpha_{i,j}$ are continuous over the compact convex X into X, by Brouwer Fixed-Point Theorem, there exists $\tilde{x} = f(\tilde{x}) = \sum_{i,j} \alpha_{i,j}(\tilde{x}) x^j$. Let $J(\tilde{x}) = \{(i,j) : \alpha_{i,j}(\tilde{x}) > 0\}$.

If $(i,j) \in J(\tilde{x})$, then $\tilde{x} \in \text{supp}(\alpha_{i,j}) \subset \theta_{i,y^j}$. Thus, $\varphi_i(y_i^j, \tilde{x}) > 0$ for each $(i,j) \in J(\tilde{x})$. Therefore,

$$\min_{(i,j) \in J(\tilde{x})} \varphi_i(y_i^j, \tilde{x}) > 0. \tag{6.6}$$

Since $\varphi_i(y_i^j, \tilde{x}) \leq u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})$, then inequalities (6.5) and (6.6) imply $0 < \min_{(i,j) \in J(\tilde{x})} \varphi_i(y_i^j, \tilde{x}) \leq 0$, which is impossible. Therefore,

$$\emptyset \neq \bigcap_{y \in X} G(y).$$

Thus, $\overline{x} \in X$ such that $\overline{x} \in \bigcap_{y \in X} G(y)$ is a Nash equilibrium. ■

PROOF OF PROPOSITION 3.3. Suppose $\overline{x} \in X$ is not a Nash equilibrium. Then, by weak transfer upper continuity, some player i has a strategy $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(\overline{x})$ of $\overline{x}$ such that $u_i(\hat{x}_i, \overline{x}_{-i}) > u_i(z)$ for all $z \in \mathcal{V}(\overline{x})$. Choose $\epsilon > 0$ such that $u_i(\hat{x}_i, \overline{x}_{-i}) - \epsilon > \sup_{z \in \mathcal{V}(\overline{x})} u_i(z)$.

The payoff security of G implies that there exist a strategy y_i and a neighborhood $\tilde{\mathcal{V}}(\overline{x}_{-i})$ of $\overline{x}_{-i}$ such that $u_i(y_i, z_{-i}) \geq u_i(\hat{x}_i, \overline{x}_{-i}) - \epsilon$ for all $z_{-i} \in \tilde{\mathcal{V}}(\overline{x}_{-i})$. Thus, there exist $y_i \in X_i$ and a neighborhood $\hat{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(z)$ for all $z \in \hat{\mathcal{V}}(\overline{x})$. ■

PROOF OF PROPOSITION 3.4. Suppose $\overline{x} \in X$ is not a Nash equilibrium. Then, by weak transfer lower continuity, some player i has a strategy $\hat{x}_i \in X_i$ and a neighborhood $\mathcal{V}(\overline{x}_{-i})$ of $\overline{x}_i$ such that $u_i(\hat{x}_i, z_{-i}) > u_i(\overline{x})$ for all $z_{-i} \in \mathcal{V}(\overline{x}_{-i})$. Choose $\epsilon > 0$ such that $\inf\limits_{z_{-i} \in \mathcal{V}(\overline{x}_{-i})} u_i(\hat{x}_i, z_{-i}) > u_i(\overline{x}) + \epsilon$. The upper semicontinuity of G implies that there exists a neighborhood $\tilde{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(\overline{x}) + \epsilon \geq u_i(z)$ for all $z \in \tilde{\mathcal{V}}(\overline{x})$. Thus, there exist $y_i \in X_i$ and a neighborhood $\hat{\mathcal{V}}(\overline{x})$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(z)$ for all $z \in \hat{\mathcal{V}}(\overline{x})$. ■

PROOF OF PROPOSITION 3.5. Suppose x is not an equilibrium. Then some player i has a strategy y_i such that $u_i(y_i, x_{-i}) > u_i(x)$, i.e., $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$. If $(y_i, x_{-i}) \in X \backslash D_i$, then by Assumption 3, there exist a strategy profile y' and a neighborhood $\mathcal{V}(x)$ of x so that for each $z \in \mathcal{V}(x)$, there exists a player $j \in I$ such as $F_j(y'_j, S_j(y'_j, z_{-j})) > F_j(z_j, S_j(z))$, *i.e.*, $u_j(y'_j, z_{-j}) > u_j(z)$. If $(y_i, x_{-i}) \in D_i$, then by Assumption 2, there exist a player $j \in I$ and $y_j^{'}$ such that $(y'_j, x_{-j}) \in X \backslash D_j$ and $F_j(y'_j, S_j(y'_j, x_{-j})) > F_j(x_j, S_j(x))$. Thus, by Assumption 3, there exist a player $k \in I$, a strategy profile $\tilde{y}$ and a neighborhood $\mathcal{V}(x)$ of x so that for each $z \in \mathcal{V}(x)$, we have $F_k(\tilde{y}_k, S_k(\tilde{y}_k, z_{-k})) > F_j(z_k, S_k(z))$, *i.e.*, $u_k(\tilde{y}_k, z_{-k}) > u_k(z)$. Thus, the game is weakly transfer continuous. It is also convex, compact, bounded and quasiconcave, then by Theorem 3.2 it has a pure strategy Nash equilibrium. ■

PROOF OF THEOREM 4.1. For each $y \in X$, let

$$F(y) = \{x \in X : u_i(y) \leq u_i(x_i, y_{-i}),\ \forall i \in I\}.$$

We first prove that F is transfer closed valued. Let $x, y \in X$ with $x \notin F(y)$. Then x is not a dominant strategy equilibrium. By the weak dominant transfer upper quasi-continuity of the game G, there exist a strategy $y' \in X$ and a neighborhood $\mathcal{V}(x)$ of x so that for every $z \in \mathcal{V}(x)$, there exists a player i such as $u_i(y^{'}) > u_i(z_i, y^{'}_{-i})$. Therefore, for all $z \in \mathcal{V}(x)$, $z \notin F(y')$, *i.e.*, $x \notin \text{cl}\, F(y')$.

For $y \in X$, let $\bar{F}(y) = \text{cl}\, F(y)$. Then $\bar{F}(y)$ is closed, and by the strong uniform transfer quasiconcavity (Remark 4.3), it is also transfer FS-convex. From Lemma 1 in Tian (1993), we deduce $\bigcap\limits_{y \in X} F(y) = \bigcap\limits_{y \in X} \bar{F}(y) \neq \emptyset$. Thus, there exists a strategy profile $\overline{x} \in X$ such that

$$u_i(y) \leq u_i(\overline{x}_i, y_{-i}), \text{ for all } y \in X \text{ and } i \in I.$$

Thus $\overline{x}$ is a dominant strategy equilibrium of the game G. ■

PROOF OF THEOREM 4.2. For each player $i \in I$ and every $(y, x_i) \in X \times X_i$, let

$$\pi_i(y, x_i) = \sup_{\mathcal{V} \in \Omega(x_i)} \inf_{z_i \in \mathcal{V}} [u_i(y) - u_i(z_i, y_{-i})]$$

where $\Omega(x_i)$ is the set of all open neighborhoods of x_i.

For each i and every $y \in X$, the function $\pi_i(y, .)$ is both real-valued and lower semicontinuous over X_i (see the proof of Theorem 3.3).

If there exists $\bar{x} \in X$ such that for all $i \in I$,

$$\sup_{y \in X} \pi_i(y, \bar{x}_i) \leq 0,$$

then $\bar{x}$ is a dominant strategy equilibrium.

Now, suppose, by way of contradiction, that for any strategy profile $x \in X$, x is not a dominant strategy equilibrium. Then, by weak dominant transfer upper continuity, there exist a player i, a strategy $y \in X$ and a neighborhood $\mathcal{V}(x_i)$ of x_i such that $u_i(y) - u_i(z_i, y_{-i}) > 0$ for each $z_i \in \mathcal{V}(x_i)$. Thus,

$$\forall x \in X, \text{ there exists } y \in X, \ i \in I \text{ such that } \pi_i(y, x_i) > 0.$$

Thus, X can be covered by the following open subsets:

$$\theta_{i,y} = \{x_i \in X_i : \ \pi_i(y, x_i) > 0\} \times X_{-i}.$$

Since X is compact, then it can be covered by a finite number of subsets $\{\theta_{i,y^j} : \ i \in I \text{ and } j = 1, ..., m\}$. Consider a continuous partition of unity $\{\alpha_{i,j}\}_{\substack{i=1,...,n \\ j=1,...,m}}$ associated to the finite covering $\{\theta_{1,y^1}, ..., \theta_{n,y^m}\}$.

Since G is strongly uniformly transfer quasiconcave, then there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset X$ such that for each $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, \ h = 1, ..., m\}$ and if $J = \{(i, j) : \lambda_{i,j} > 0\}$, then

$$\min_{(i,j) \in J} [u_i(y^j) - u_i(\tilde{x}_i, y^j_{-i})] \leq 0. \tag{6.7}$$

Let us now consider the following function defined on X into X by

$$f(x) = \sum_{i,j} \alpha_{i,j}(x) x^j.$$

Since the functions $\alpha_{i,j}$ are continuous over the compact convex X into X, then by Brouwer Fixed-Point Theorem, there exists $\tilde{x} = f(\tilde{x}) = \sum_{i,j} \alpha_{i,j}(\tilde{x}) x^j$. Let $J(\tilde{x}) = \{(i, j) : \alpha_{i,j}(\tilde{x}) > 0\}$.

If $(i,j) \in J(\tilde{x})$, then $\tilde{x} \in \text{supp}(\alpha_{i,j}) \subset \theta_{i,y^j}$. Thus, $\pi_i(y^j, \tilde{x}_i) > 0$ for each $(i,j) \in J(\tilde{x})$. Therefore,

$$\min_{(i,j)\in J(\tilde{x})} \pi_i(y^j, \tilde{x}_i) > 0. \tag{6.8}$$

Since $\pi_i(y^j, \tilde{x}_i) \leq u_i(y^j) - u_i(\tilde{x}_i, y^j_{-i})$, then inequalities (6.7) and (6.8) imply $0 < \min_{(i,j)\in J(\tilde{x})} \pi_i(y^j, \tilde{x}_i) \leq 0$, which is impossible. ■

Proof of Proposition 5.1. Suppose $\overline{\mu} \in X$ is not a mixed strategy Nash equilibrium. Then, there exist a player i, a measure $\mu_i^* \in M_i$ and a $\epsilon > 0$ such that

$$U_i(\mu_i^*, \overline{\mu}_{-i}) - \epsilon = \int_X u_i(x) d\mu_i^*(x_i) d\overline{\mu}_{-i}(x_{-i}) - \epsilon > U_i(\overline{\mu}) = \int_X u_i(x) d\overline{\mu}(x). \tag{6.9}$$

Since the game G is uniformly transfer continuous, then the function u_i is upper semicontinuous over X and uniformly payoff secure. According to Proposition 5.1 of Reny (1999), the function $\int_X u_i(x) d\overline{\mu}(x)$ is upper semicontinuous in μ. Thus, there exists $\mathcal{V}_1(\overline{\mu})$ such that:

$$\int_X u_i(x) d\overline{\mu}(x) \geq \int_X u_i(x) d\mu(x) - \epsilon/2, \text{ for all } \mu \in \mathcal{V}_1(\overline{\mu}). \tag{6.10}$$

Also, according to the proof of Theorem 1 in Monteiro and Page (2007), there exist a measure $\tilde{\mu}_i \in M_i$ and a neighborhood $\mathcal{V}_2(\overline{\mu}_{-i})$ of $\overline{\mu}_{-i}$ such that

$$\int_X u_i(x) d\tilde{\mu}_i(x_i) d\mu_{-i}(x_{-i}) \geq \int_X u_i(x) d\mu_i^*(x_i) d\overline{\mu}_{-i}(x_{-i}) - \epsilon/2, \text{ for all } \mu_{-i} \in \mathcal{V}_2(\overline{\mu}_{-i}). \tag{6.11}$$

Combining (6.9), (6.10) and (6.11), we conclude: there exist a measure $\tilde{\mu}_i \in M_i$ and a neighborhood $\mathcal{V}(\overline{\mu})$ of $\overline{\mu}$ such that for all $\mu \in \mathcal{V}(\overline{\mu})$, we have

$$\begin{aligned} \int_X u_i(x) d\tilde{\mu}_i(x_i) d\mu_{-i}(x_{-i}) + \epsilon/2 &\geq \int_X u_i(x) d\mu_i^*(x_i) d\overline{\mu}_{-i}(x_{-i}) \\ &> \int_X u_i(x) d\overline{\mu}(x) + \epsilon \\ &\geq \int_X u_i(x) d\mu(x) + \epsilon/2 \end{aligned}$$

Thus, the mixed game $\overline{G}$ is weakly transfer continuous. ■

Proof of Proposition 5.2. Upper semicontinuity of $l_i(.)$, $m_i(.)$ and $\phi_i(.)$, together with condition a) in Assumption 5.1, implies that the concession game is upper semicontinuous. Condition b) implies that for each $x_i \in X_i$and $\epsilon > 0$, there exists a strategy $\overline{x}_i \in X_i$ such that for every $y_i \in X_{-i}$, there exists a neighborhood $\mathcal{V}(y_i)$ of y_i such that $u_i(\overline{x}_i, z_i) \geq u_i(x_i, y_i) - \epsilon$, for all $z_i \in \mathcal{V}(y_i)$. Then, it is uniformly transfer continuous. It is clear that this game G is compact, then by Corollary 5.1, we conclude that the game has a mixed strategy Nash equilibrium. ■

References

Aliprantis, C.B., Border, K.C. (1994): *Infinite Dimensional Analysis*. Springer-Verlag, New York.

Athey, S. (2001): Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information. *Econometrica*, **69**, 861–889.

Bagh, A., Jofre, A. (2006): Reciprocal Upper Semicontinuity and Better Reply Secure Games: A Comment. *Econometrica*, **74**, 1715–1721.

Barelli, P., Soza, I. (2009): On the Existence of Nash Equilibria in Discontinuous and Qualitative Games, *mimeo*.

Baye, M.R., Tian, G., Zhou, J. (1993): Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-Quasiconcave Payoffs. *The Review of Economic Studies*, **60**, 935–948.

Carmona, G. (2005): On the Existence of Equilibria in Discontinuous Games: Three Counterexamples. *International Journal of Game Theory*, **33**, 181–187.

Carmona, G. (2008): An Existence Result of Equilibrium in Discontinuous Economic Games, *mimeo*.

Dasgupta, P., Maskin, E. (1986): The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. *The Review of Economic Studies*, **53**, 1–26.

Debreu, G. (1952): A Social Equilibrium Existence Theorem. *Proceedings of the National Academy of Sciences of the U. S. A.*, 38.

Deguire, P., Lassonde, M. (1995): Familles Sélectantes. *Topological Methods in Nonlinear Analysis*, **5**, 261–269.

Gatti, R.J. (2005): A Note on the Existence of Nash Equilibrium in Games with Discontinuous Payoffs. *Cambridge Economics Working Paper No. CWPE 0510.* Available at SSRN: http://ssrn.com/abstract=678701.

Glicksberg, I.L. (1952): A Further Generalization of the Kakutani Fixed Point Theorem. *Proceedings of the American Mathematical Society*, **3**, 170–174.

Jackson, M. O. (2005): Non-existence of Equilibrium in Vickrey, Second-price, and English Auctions, working paper, Stanford University.

Karlin, S. (1959): *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. II (London: Pregamon Press).

McLennan, A., Monteiro, P. K., Tourky, R. (2009): Games with Discontinuous Payoffs: a Strengthening of Reny' s Existence Theorem, *mimeo*.

Milgrom, P., and Roberts, H. (1990): Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities, *Econometrica* 58, 1255-1277.

Milgrom, P., Weber, R. (1985): Distributional Strategies for Games with Incomplete Information, *Mathematics of Operations Research* 10, 619-632.

Monteiro, P.K., Page, F.H.Jr. (2007): Uniform Payoff Security and Nash Equilibrium in Compact Games. *Journal of Economic Theory*, **134**, 566–575.

Morgan, J., Scalzo, V. (2007): Pseudocontinuous Functions and Existence of Nash Equilibria. *Journal of Mathematical Economics*, **43**, 174–183.

Nash, J. (1950): Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences*, **36**, 48–49.

Nash, J.F. (1951): Noncooperative Games. *Annals of Maths*, **54**, 286–295.

Nishimura, K., Friedman, J. (1981): Existence of Nash Equilibrium in n-Person Games without Quasi-Concavity. *International Economic Review*, **22**, 637–648.

Roberts, J. and Sonnenschein, H. (1977): On the Existence of Cournot Equilibrium Without Concave Profit Functions, *Econometrica*, 45, 101–113.

Reny, P. J. (1999): On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica*, **67**, 1029–1056.

Reny, P. J. (2009): Further Results on the Existence of Nash Equilibria in Discontinuous Games, *mimeo*.

Robson, A. J. (1994): An Informationally Robust Equilibrium in Two-Person Nonzero-Sum Games. *Games and Economic Behavior*, **2**, 233-245.

Rosen, J.B. (1965): Existence and Uniqueness of Equilibrium Point for Concave n-Person Games. *Econometrica*, **33**, 520–534.

Rothstein, P. (2007): Discontinuous Payoffs, Shared Resources, and Games of Fiscal Competition: Existence of Pure Strategy Nash Equilibrium. *Journal of Public Economic Theory*, **9**, 335–368.

Simon, L. (1987): Games with Discontinuous Payoffs. *Review of Economic Studies*, **54**, 569–597.

Simon, L., Zame. W. (1990): Discontinuous Games and Endogenous Sharing Rules. *Econometrica*, **58**, 861–872.

Tian, G. (1992a): Generalizations of the KKM Theorem and Ky Fan Minimax Inequality, with Applications to Maximal Elements, Price Equilibrium, and Complementarity, *Journal of Mathematical Analysis and Applications*, 170, 457–471.

Tian, G. (1992b): "Existence of Equilibrium in Abstract Economies with Discontinuous Payoffs and Non-Compact Choice Spaces," *Journal of Mathematical Economics*, 21, pp. 379-388.

Tian, G. (1992c) "On the Existence of Equilibria in Generalized Games," *International Journal of Game Theory*, 20, pp.247-254.

Tian, G. (1993): Necessary and Sufficent Conditions for Maximization of a Class of Preference Relations. *Review of Economic Studies*, **60**, 949–958.

Tian, G. (2009): Existence of Equilibria in Games with Arbitrary Strategy Spaces and Payoffs: A Full Characterization, mimeo.

Tian, G., Zhou, Z. (1995): Transfer Continuities, Generalizations of the Weierstrass Theorem and Maximum Theorem: A Full Characterization. *Journal of Mathematical Economics*, **24**, 281–303.

Topkis, D. M.(1979): Equilibrium Points in Nonzero-Sum n-Person Submodular Games, *SIAM Journal on Control and Optimization* 17(6), 773–787.

Vives, X. (1990): Nash Equilibrium with Strategic Complementarities. *Journal of Mathematical Economics*, **19**, 305–321.

Yao, J.C. (1992): Nash Equilibria in n-Person Games without Convexity. *Applied Mathematics Letters*, **5**, 67–69.